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VARIATIONS OF GENERALIZED AREA FUNCTIONALS AND p-AREA MINIMIZERS OF BOUNDED VARIATION IN THE HEISENBERG GROUP

BY

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Abstract

We prove the existence of a continuous BV minimizer with C^0 boundary value for the p-area (pseudohermitian or horizontal area) in a parabolically convex bounded domain. We extend the domain of the area functional from BV functions to vector-valued measures. Our main purpose is to study the first and second variations of such a generalized area functional including the contribution of the singular part. By giving examples in Riemannian and pseudohermitian geometries, we illustrate several known results in a unified way. We show the contribution of the singular curve in the first and second variations of the p-area for a surface in an arbitrary pseudohermitian 3-manifold.

1. Introduction and Statement of the Results

In [10], Paul Yang and the authors proved the existence of a Lipschitz continuous (p-)minimizer with $C^{2,\alpha}$ boundary value for the p-area (or horizontal area) in the space $W^{1,1}$ and the uniqueness of p-minimizers in the space $W^{1,2}$ among other things. In this paper, we will prove the existence of a continuous BV minimizer with C^0 boundary value for the p-area in a

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parabolically convex bounded domain. Recall that the p-area is a special case of a more general area functional:

$$\mathcal{F}_H(u) := \int_\Omega (|\nabla u + \vec{F}| + Hu) d^m x. \tag{1.1}$$

where $\Omega \subset R^m$ is a bounded domain, $u \in W^{1,1}(\Omega), \vec{F}$ is an L^1 vector field on $\Omega, H \in L^\infty(\Omega)$, and $d^m x := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$ denotes the Euclidean volume form or the Lebesgue measure. We often denote $\mathcal{F}_H$ by $\mathcal{F}$ for the case of $H = 0:$

$$\mathcal{F}(u) := \int_\Omega |\nabla u + \vec{F}| d^m x. \tag{1.2}$$

$\mathcal{F}(\cdot)$ is called the p-area (of the graph defined by u over Ω) if $\vec{F} = -\vec{X}^*$ where $\vec{X}^* = (x_{1'}, -x_1, x_{2'}, -x_2, \ldots, x_{n'}, -x_n)$, $m = 2n$ (see [8]). In the case of a graph Σ over the R^{2n}-hyperplane in the Heisenberg group, the above definition of p-area coincides with those given in [6], [13], and [26]. In particular these notions, especially in the framework of geometric measure theory, have been used to study existence or regularity properties of minimizers for the relative perimeter or extremizers of isoperimetric inequalities (see, e.g., [13], [16], [20], [21], [23], [25], [27], [28], [29], [5]). The p-area can also be identified with the $2n+1$-dimensional spherical Hausdorff measure of Σ (see, e.g., [2], [15]). Some authors take the viewpoint of so called intrinsic graphs and obtained interesting results (see, e.g., [15], [1], and [4] which relates distributional solutions of Burgers' equation to intrinsic regular graphs). Starting from the work [8] (see also [7]), the subject was studied from the viewpoint of partial differential equations and that of differential geometry (see [10], [11], [9]; the term p-minimal is used since this is the notion of minimal surfaces in pseudohermitian geometry; "p" stands for "pseudohermitian"). In [10], one studied the situation for $u \in W^{1,1}$. To extend the domain of $\mathcal{F}$ to the space of BV functions, we define the total variation of a function $u \in L^1(\Omega)$ by

$$\int_\Omega |Du + \vec{F} d^m x| := \sup\{\int_\Omega (-u \mathrm{div} \vec{\phi} + \vec{F} \cdot \vec{\phi}) d^m x \mid \vec{\phi} \in C_0^1(\Omega), |\vec{\phi}| \leq 1\}. \tag{1.3}$$

Let $BV_{\vec{F}}(\Omega)$ denote the space of $u \in L^1(\Omega)$ such that the total variation $\int_\Omega |Du + \vec{F} d^m x| < \infty$. In this case, the notation Du (viewed as the gradient of u in the distributional sense) is in fact a vector-valued Radon signed measure (see Remark 1.5 on page 5 in [17]) and $|Du + \vec{F} d^m x|$ is the total variation measure of the measure $Du+\vec{F}d^m x$ (see the first paragraph of Section 3 for more details). When $u \in W^{1,1}(\Omega)$, we use ∇u to denote the gradient of u. Note that $BV_{\vec{F}}(\Omega)$ is reduced to the usual space of BV functions, denoted by $BV(\Omega)$, for $\vec{F}=\vec{0}$. Moreover, if $\vec{F} \in L^1(\Omega)$, it is easy to see that $u \in BV_{\vec{F}}(\Omega)$ if and only if $u \in BV(\Omega)$. For $u \in W^{1,1}$ (1.3) is the same as the one in the usual sense (in which we write $Du = (\nabla u)d^m x$).

We need to require the following condition on $\vec{F}$ (say, $\in C^1$) :

$$\partial_K F_I = \partial_I f_K, \quad I, K = 1, \ldots, m \tag{1.4}$$

for C^1-smooth functions f_K's. Denote the coordinates of R^m by $x_1, x_2, \ldots,$ x_m. We call a coordinate system orthonormal if it is obtained by a translation and a rotation from $x_1, x_2, \ldots, x_m$. We recall ([10]) the definition of a certain notion of convexity for Ω as follows.

Definition 1.1. We call $\Omega \subset R^m$ parabolically convex if for any $p \in \partial\Omega$, there exists an orthonormal coordinate system $(\tilde{x}_1,\ \tilde{x}_2, \ldots, \tilde{x}_m)$ with the origin at p and $\Omega \subset \{a\tilde{x}_1^2 - \tilde{x}_2 < 0\}$ where $a > 0$ is independent of p.

Note that a C^2-smooth bounded domain with the positively curved (positive principal curvatures) boundary is parabolically convex. On the other hand, a parabolically convex domain can be nonsmooth as shown by the following example: a planar domain defined by

$$-\sqrt{3} < x < \sqrt{3}, -\sqrt{4-x^2}+1 < y < \sqrt{4-x^2}-1.$$

For a vector field $\vec{G} = (g_1, g_2, \ldots, g_{2n})$ on $\Omega \subset R^{2n}$, we define $\vec{G}^* :=$ $(g_2, -g_1, g_4, -g_3, \ldots, g_{2n}, -g_{2n-1})$.

Theorem A. *Let Ω be a parabolically convex bounded domain in R^{2n} with $\partial\Omega \in C^{2,\alpha}(0 < \alpha < 1)$. Suppose $\vec{F} \in C^{1,\alpha}(\bar{\Omega})$ satisfies the condition* (1.4) *for $C^{1,\alpha}$-smooth and bounded f_K's in Ω and* $\mathrm{div}\vec{F}^* > 0$. *Let $\varphi \in C^0(\partial\Omega)$. Then there exists $u \in C^0(\bar{\Omega}) \cap BV(\Omega)$ such that $u = \varphi$ on $\partial\Omega$ and*

$$\int_\Omega |Du + \vec{F}d^m x| \leq \int_\Omega |Dv + \vec{F}d^m x| \tag{1.5}$$

for all $v \in C^0(\bar{\Omega}) \cap BV(\Omega)$ with $v = \varphi$ on $\partial\Omega$.

We remark that $\vec{F} = -\vec{X}^*$ satisfies the assumption in Theorem A. The idea of the proof for Theorem A goes as follows. We approximate φ by $C^{2,\alpha}$-smooth functions and apply Theorem A in [10] to get approximating Lipschitz continuous minimizers. These minimizers will converge uniformly to a continuous function u by the comparison principle (Theorem C in [10]). Then we show that u is a BV function and a minimizer in $C^0(\bar{\Omega}) \cap BV(\Omega)$ by some extra work.

On the other hand, F. Serra Cassano and D. Vittone in a recent paper ([32]) study this problem for more general domains. Let $\Omega \subset R^{2n}$ be a bounded domain with Lipschitz regular boundary. They show the functional

$$u \in BV(\Omega) \to \int_\Omega |Du - \vec{X}^* d^{2n}x| + \int_{\partial\Omega} |\ u|_{\partial\Omega} - \varphi\ |\ d\sigma \tag{1.6}$$

attains its minimum, where, for $u \in BV(\Omega)$, the trace $u|_{\partial\Omega}$ exists and lies in $L^1(\partial\Omega)$ by Theorem 2.10 in [17], $\varphi \in L^1(\partial\Omega)$ is given, and $d\sigma$ denotes the standard boundary measure. Moreover, there holds

$$\begin{aligned} &\inf\{\int_\Omega |Du - \vec{X}^* d^{2n}x| : u \in BV(\Omega), u|_{\partial\Omega} = \varphi\} \\ &= \min\{\int_\Omega |Du - \vec{X}^* d^{2n}x| + \int_{\partial\Omega} |\ u|_{\partial\Omega} - \varphi\ |\ d\sigma : u \in BV(\Omega)\} \end{aligned}$$

(see Theorem 1.4 in [32]).

Although the BV minimizers $\tilde{u}$ for (1.6) exist, the trace $\tilde{u}|_{\partial\Omega}$ may not equal φ. The BV minimizers for $\int_\Omega |Du - \vec{X}^* d^{2n}x|$ with given (even smooth) boundary value φ may not exist in general either for nonconvex domains as shown in Example 3.6 of [32]. In fact, consider $\Omega := \{1 < \sqrt{x^2+y^2} < 2\} \subset R^2$. Take the boundary value $\varphi = 0$ on $\sqrt{x^2+y^2} = 2$ while $\varphi = C$ on $\sqrt{x^2+y^2} = 1$. Then there admits no minimizer for $\int_\Omega |Du - \vec{X}^* d^{2n}x|$ with $u|_{\partial\Omega} = \varphi$ when C is large enough (see [32] for more details). The original idea comes from [14] in which R. Finn gave examples of nonexistence for the Dirichlet problem of (Euclidean) minimal surface equation.

After we have BV minimizers, we consider the variations of $\mathcal{F}$ on BV functions. Since $\mathcal{F}$ is only convex, but not strongly convex, this causes much trouble. Besides the trouble that BV functions cause, we still have trouble even for C^∞-smooth functions. For instance, let $\vec{F}=\vec{0}$, then $u\equiv 0$ is the minimizer for $\mathcal{F}(u)=\int_\Omega |\nabla u| d^m x$. Compute the first variation at $u\equiv 0$:

$$\lim_{\varepsilon\to 0}\frac{\mathcal{F}(0+\varepsilon\varphi)-\mathcal{F}(0)}{\varepsilon} = \lim_{\varepsilon\to 0}\frac{|\varepsilon|}{\varepsilon}\int_\Omega |\nabla\varphi| d^m x. \tag{1.7}$$

from which we learn that only left limit or right limit exists. However, we can still deal with the second variation of $\mathcal{F}$ (see Theorem C). Previously in [8] the second variation of $\mathcal{F}$ was studied only for C^2-smooth functions and away from the singular set $S_{\vec{F}}(u)$ $(:=\{p\in\Omega \mid \nabla u+\vec{F}=0 \text{ at } p\})$. But whether $H_m(S_{\vec{F}}(u))$ $(m=\dim\Omega)$, the m-th dimensional Hausdorff measure of $S_{\vec{F}}(u)$, vanishes is a problem. In the case of least gradient $(\vec{F}=0)$, $H_m(S_{\vec{F}}(u))$ may not be zero.

In the case of p-area, $\vec{F}=-\vec{X}^*$ where $\vec{X}^*=(x_{1'},-x_1,x_{2'},-x_2,\ldots,x_{n'},-x_n)$, $m=2n$, for $u\in C^2(\Omega)$, we have $H_m(S_{\vec{F}}(u))=0$. But for $u\in W^{1,1}(\Omega)$, $H_m(S_{\vec{F}}(u))$ may be larger than zero (see [2]). For $u\in BV(\Omega)$, we write

$$Du=(\nabla u)d^m x+dv_s^{d^m x}, d^m x\perp dv_s^{d^m x}$$

where $\nabla u\in L^1$ with respect to $d^m x$. Suppose $d^m x$ ($dv_s^{d^m x}$, resp.) is concentrated on Ω_1 (Ω_2, resp.) where $\Omega_1\cap\Omega_2=\varnothing, \Omega_1\cup\Omega_2=\Omega$. Note that $H_m(\Omega_2)=0$. Define $S_{\vec{F}}(u):=\{p\in\Omega_1 \mid \nabla u+\vec{F}=0 \text{ at } p\}$. Now whether $H_m(S_{\vec{F}}(u))=0$ (m even) for a BV minimizer u for the p-area in general is still an open problem. So we cannot neglect the role of $S_{\vec{F}}(u)$. One of the purposes of this paper is to study the second variation of $\mathcal{F}$ not avoiding $S_{\vec{F}}(u)$ even if $H_m(S_{\vec{F}}(u))\neq 0$.

The idea of computing the first and second variations is to extend the domain of $\mathcal{F}(\cdot)$ from BV functions to vector-valued measures. Then making use of the Radon-Nikodym theorem, we can easily obtain the formulas of first and second variations, which include the effect of the singular set.

Let E be a C^∞-smooth Riemannian vector bundle over a C^∞-smooth manifold X. Let $d\mu, d\nu$ be two E-valued (Radon signed) measures on X. Let $d\mu_\varepsilon=d\mu+\varepsilon d\nu$ for $\varepsilon\in R$. Define $\mathcal{F}(d\mu_\varepsilon)$ by

$$\mathcal{F}(d\mu_\varepsilon):=\int_X |d\mu_\varepsilon|.$$

(see (3.1) with Ω replaced by X) Denote $\mathcal{F}(d\mu_\varepsilon)$ by $\mathcal{F}(\varepsilon)$ for simplicity. Throughout this paper we assume that both $d\mu$ and $d\nu$ are bounded in the sense that $|d\mu|$ and $|d\nu|$ are integrable over X. By the (extended) Radon-Nikodym theorem we can write

$$\begin{aligned} d\mu_\varepsilon &= N_\varepsilon|d\mu_\varepsilon|, \\ d\nu &= A_\varepsilon|d\mu_\varepsilon| + d\nu_s^\varepsilon, d\nu_s^\varepsilon \perp |d\mu_\varepsilon| \end{aligned}$$

where N_ε, $A_\varepsilon \in L^1(|d\mu_\varepsilon|)$ with $|N_\varepsilon| = 1$ (cf. (3.2)). Recall that $\mathcal{F}'(\varepsilon_1\pm) := \lim_{\varepsilon_2\to\varepsilon_1\pm} \frac{\mathcal{F}(\varepsilon_2)-\mathcal{F}(\varepsilon_1)}{\varepsilon_2-\varepsilon_1}$. We have the following first variation formula.

Theorem B. *Suppose $d\mu$ and $d\nu$ are bounded. Then $\mathcal{F}(\varepsilon)$ is Lipschitz continuous in ε and there holds*

$$\mathcal{F}'(\varepsilon_1\pm) = \int_X N_{\varepsilon_1}\cdot A_{\varepsilon_1}|d\mu_{\varepsilon_1}| \pm \int_X |d\nu_s^{\varepsilon_1}|. \tag{1.8}$$

Let $u \in BV(\Omega)$ where $\Omega \subset R^m$ is a bounded domain with Lipschitz regular boundary. Define

$$\tilde{\mathcal{F}}_H(u) := \int_\Omega |Du + \vec{F}d^m x| + \int_\Omega Hud^m x$$

where the first term on the right side of the equality makes sense by (1.3). Recall that $\vec{F}$ is an L^1 vector field on Ω and $H \in L^\infty(\Omega)$. Recall that for $u \in BV(\Omega)$, the trace $u|_{\partial\Omega}$ exists and lies in $L^1(\partial\Omega)$ by Theorem 2.10 in [17].

Definition 1.2. Suppose $u \in BV(\Omega)$ with $u|_{\partial\Omega} = \psi$. If for all $\varphi \in BV(\Omega)$ with $\varphi|_{\partial\Omega} = 0$, there holds

$$\tilde{\mathcal{F}}_H(u) \leq \tilde{\mathcal{F}}_H(u+\varphi)$$

Then we call u a minimizer for $\tilde{\mathcal{F}}_H$ with the boundary value (trace) ψ.

Denote $\tilde{\mathcal{F}}_H(u+\varepsilon\varphi)$ by $\tilde{\mathcal{F}}_H(\varepsilon)$. We can then have the following necessary conditions for $u \in BV(\Omega)$ to be a minimizer.

Corollary B′. *Let $\Omega \subset R^m$ be a bounded domain with Lipschitz regular boundary. Suppose $u \in BV(\Omega)$ is a minimizer for $\tilde{\mathcal{F}}_H$ with $u|_{\partial\Omega} = \psi \in$*

$L^1(\partial\Omega)$. *Then there hold*

$$\tilde{\mathcal{F}}_H'(0+) = \int_\Omega N_0 \cdot A_0 |d\mu| + \int_\Omega |d\nu_s^0| + \int_\Omega H\varphi d^m x \geq 0 \tag{1.9}$$

and

$$\tilde{\mathcal{F}}_H'(0-) = \int_\Omega N_0 \cdot A_0 |d\mu| - \int_\Omega |d\nu_s^0| + \int_\Omega H\varphi d^m x \leq 0. \tag{1.10}$$

We remark that Corollary B$'$ generalizes Theorem 3.3 in [10], where $N(u) = N_0$, $(\nabla\varphi)d^m x = A_0|d\mu|$ on $\Omega\backslash S_{\vec{F}}(u)$, and $|\nabla\varphi|d^m x = |d\nu_s^0|$ on $S_{\vec{F}}(u)$. Here $A_0 = \frac{\nabla\varphi}{|\nabla u + \vec{F}|}$, $|d\mu| = |\nabla u + \vec{F}|d^m x$, and $S_{\vec{F}}(u) := \{\nabla u + \vec{F} = 0\}$ (cf. Example 3.2). Also note that (1.10) corresponds to (3.12) in [10] with φ replaced by $-\varphi$.

The singular term $\pm\int_X |d\nu_s^0|$ in (1.9) and (1.10) is not removable in general. The simplest example is that at the minimizer $u \equiv 0$ for the least gradient energy functional $\int |\nabla u| d^m x$, we have $\mathcal{F}'(0\pm) = \pm\int |d\nu_s^0| = \pm\int |\nabla\varphi| d^m x$ over $S_{\vec{F}}(u) = \Omega$ as shown in (1.7).

There are at most countably many ε's such that $|d\nu_s^\varepsilon| \neq 0$. We call ε regular if $|d\nu_s^\varepsilon| = 0$. For regular ε we have $\mathcal{F}'(\varepsilon+) = \mathcal{F}'(\varepsilon-) = \mathcal{F}'(\varepsilon)$ and $\mathcal{F}'(\varepsilon)$ is an increasing function of ε (see Proposition 3.3). Write $\mathcal{F}_+'(\varepsilon)$ ($\mathcal{F}_-'(\varepsilon)$, respectively) for $\mathcal{F}'(\varepsilon+)$ ($\mathcal{F}'(\varepsilon-)$, respectively). In Section 3 we also study the left and right continuity of $\mathcal{F}_+'$ and $\mathcal{F}_-'$ (see Proposition 3.4). We give area functionals in Riemannian and pseudohermitian geometries as examples to illustrate (1.8). For a p-area stationary surface in an arbitrary pseudohermitian 3-manifold, we obtain the "incident angle = reflected angle" condition on the singular curve (see (3.38)). The result extends previous ones in the Heisenberg group ([10], [29]).

In Section 4 we discuss the second derivative of $\mathcal{F}(\varepsilon)$. We compute the first derivatives of $\mathcal{F}_+'$ and $\mathcal{F}_-'$ in various situations.

Theorem C. *Suppose $d\mu$ and $d\nu$ are bounded, and $|A_{\varepsilon_1}|^2 \in L^1(X, |d\mu_{\varepsilon_1}|)$. Then* (1) *For ε_1 regular, there holds*

$$\lim_{\varepsilon_2\to\varepsilon_1, \varepsilon_2 \text{ regular}} \frac{\mathcal{F}'(\varepsilon_2) - \mathcal{F}'(\varepsilon_1)}{\varepsilon_2 - \varepsilon_1} = \int_X \{|A_{\varepsilon_1}|^2 - |(A_{\varepsilon_1} \cdot N_{\varepsilon_1})|^2\} |d\mu_{\varepsilon_1}| \ \ (\geq 0). \tag{1.11}$$

(2) *For ε_1 arbitrary, there holds*

$$\lim_{\varepsilon_2\to\varepsilon_1+}\frac{\mathcal{F}'_\pm(\varepsilon_2)-\mathcal{F}'_+(\varepsilon_1)}{\varepsilon_2-\varepsilon_1}=\lim_{\varepsilon_2\to\varepsilon_1-}\frac{\mathcal{F}'_\pm(\varepsilon_2)-\mathcal{F}'_-(\varepsilon_1)}{\varepsilon_2-\varepsilon_1}$$
$$=\int_X\{|A_{\varepsilon_1}|^2-|(A_{\varepsilon_1}\cdot N_{\varepsilon_1})|^2\}|d\mu_{\varepsilon_1}|\quad(\geq 0). \tag{1.12}$$

Observe that $\mathcal{F}'_-(\varepsilon_1)$ may be strictly less than $\mathcal{F}'_+(\varepsilon_1)$ (roughly speaking, $\mathcal{F}'$ is not continuous and may have a jump at ε_1). Still we have not only the existence of the left derivative of $\mathcal{F}'_-$ and the right derivative of $\mathcal{F}'_+$, but also the same value, i.e., $(\mathcal{F}'_-)'_-(\varepsilon_1)=(\mathcal{F}'_+)'_+(\varepsilon_1)$ by (1.12). This is a very special property. Note that a convex function does not have such a property in general. For instance, $f(x)=0$ for $x\leq 0, f(x)=x^2+x$ for $x>0$. We can easily check that f' has a jump at $x=0$. On the other hand, we compute $f''(x)=0$ for $x<0$ while $f''(x)=2$ for $x>0$.

A fundamental formula in deducing the second variation of $\mathcal{F}$ is (3.18) (for $|d\mu_{\varepsilon_1}|\ll|d\mu_{\varepsilon_2}|\ll|d\mu_{\varepsilon_1}|$) in Section 3:

$$(N_{\varepsilon_2}-N_{\varepsilon_1})\cdot(d\mu_{\varepsilon_2}-d\mu_{\varepsilon_1})=\frac{1}{2}|N_{\varepsilon_2}-N_{\varepsilon_1}|^2(|d\mu_{\varepsilon_2}|+|d\mu_{\varepsilon_1}|).$$

This formula generalizes (5.1) in [8]:

$$(N(u)-N(v))\cdot(\nabla u-\nabla v)=\frac{1}{2}|N(u)-N(v)|^2(|\nabla u-\vec{X}^*|+|\nabla v-\vec{X}^*|) \tag{1.13}$$

for $u,v\in C^1$. The extension (3.18) includes the case of BV functions. Also it holds for various geometries including those of Euclidean and pseudohermitian minimal surfaces. See the examples in Section 3 and the Appendix. Corresponding to (1.13), for the Riemannian mean curvature equation divTu $=H$ in R^n, where $Tu:=\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}$, we have the following structural inequality:

$$\begin{aligned}(Tu-Tv)\cdot(\nabla u-\nabla v)\ &\geq\ \frac{1}{2}|Tu-Tv|^2(\sqrt{1+|\nabla u|^2}+\sqrt{1+|\nabla v|^2})\\ &\geq\ |Tu-Tv|^2.\end{aligned}$$

The above inequality was discovered by Miklyukov [22], Hwang [19], and Collin-Krust [12] independently. The proof in [19] was obtained through the help of Shuh-Jye Chern who simplified the original proof of Hwang.

In Section 4 we give a proof of Theorem C and examples to illustrate (1.11). In particular we show that a C^2 area-stationary graph in a flat ambient space in either Riemannian or pseudohermitian geometry has the local area-minimizing property. This fact was proved individually for different situations. For the 3-dimensional Heisenberg group, it was shown by a calibration argument in [8] for the nonsingular case. Later Ritoré and Rosales ([29]) extended the result to the situation having singularities. On the other hand, using (1.11) gives a unified proof (see Example 4.1 and Example 4.2). Note that in ([29]), we are in C^2-smooth category. The singular set has no contribution to the second variation since its Lebesgue measure (in R^{2n}) vanishes according to a result of Balogh ([2]). Here Theorem C generalizes to include the singular set contribution. On the other hand, we obtained Balogh's result (for a C^2-smooth function) as Lemma 5.4 in [8] by a different argument (we used only elementary linear algebra and the implicit function theorem in the proof). Later we generalized this result to the situation of general $\vec{F}$ (see Theorem D in [10]).

When the ambient space is not flat, we know that the curvature appears in the second variation formula and the second variation is no longer nonnegative in general. This means that the way we vary by considering $|d\mu + \varepsilon dv|$ is not generic for nonflat ambient spaces. For a variational vector field with support containing a singular curve, we compute the second variation of the p-area for a stationary surface in such a direction, and cook out the contribution of the singular curve (see (4.33); the computation was completed by Hung-Lin Chiu). Note that in [8] we have done such a computation for a variational vector field with support away from the singular set.

In the Appendix, we define the notions of gradient and hypersurface area in a general formulation unifying Riemannian and pseudohermitian (horizontal or Heisenberg) structures for further development. In fact, these different geometric structures on a differentiable manifold M are better described in a unified way by assigning a nonnegative inner product $< \cdot, \cdot >$ on its cotangent bundle T^*M. The gradient $\nabla\varphi$ of a smooth function φ on M with respect to these different geometric structures can be expressed in a unified way as $\nabla\varphi := G(d\varphi)$ where $G : T^*M \to TM$ is a natural bundle morphism defined by $< G(\omega), \eta > = < \omega, \eta >$ for $\omega, \eta \in T^*M$ (cf. (A.3) and note that the first $< \cdot, \cdot >$ denotes the pairing between TM and T^*M). For instance, this $\nabla\varphi$ is nothing but the subgradient $\nabla_b\varphi$ in the pseudohermitian case.

The geometric information is hidden in G. If φ is a defining function of a hypersurface $\Sigma \subset M$, we can give a unified definition of area (or volume) element dv_Σ of Σ as follows (cf. (A.4)):

$$dv_\Sigma = \frac{d\varphi}{|d\varphi|} \rfloor dv_M$$

where dv_M is a volume form. This formula encodes the Euclidean area element, the p- (or H-) area element for a graph or an intrinsic graph in the Heisenberg group (see Example A.1, Example A.2, and Example A.3, resp.; see also Examples A.4 and A.5 for a surface in a general pseudohermitian 3-manifold). In particular, we recover the definition of Ritoré and Rosales for the p- (or H-) area element ([29]). See (A.19) in Example A.5 for more details.

We also derive a general formula for the mean curvature and give a number of examples to illustrate it. See (A.25) and Examples A.6 and A.7. When this work was being done, we received an interesting preprint (see [32]) from Francesco Serra Cassano. In [32], the authors also studied the existence (and local boundedness) of BV minimizers for the p-area (of what the authors call t-graphs and X_1-graphs). The definition (1.3) that we use here is $S_t(u)$ on page 16 of [32]. Also the boundary value in [32] is more general (see previous comments after Theorem A for more details).

Added in proof: The authors were informed of papers [18], [24] in which the second variation of the p-area was also studied and discussed.

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2. Existence and Proof of Theorem A

Take $\varphi_j^-, \varphi_k^+ \in C^\infty(\bar{\Omega})$ such that φ_j^- (φ_k^+, respectively) increasingly (decreasingly, respectively) approaches φ in C^0-norm on $\partial\Omega$. By Theorem A in [10], we can find Lipschitz continuous minimizers (for $\mathcal{F}(\cdot)$) u_j^-, u_k^+ such that $u_j^- = \varphi_j^-$ and $u_k^+ = \varphi_k^+$ on $\partial\Omega$. It follows from the maximum principle (Theorem C in [10]; here the condition $\text{div}\vec{F}^* > 0$ is used) that

$$\begin{aligned} 0 &\le u_{j_1}^- - u_{j_2}^- \le \|\varphi_{j_1}^- - \varphi_{j_2}^-\|_{C^0(\partial\Omega)}, \\ 0 &\le u_{k_1}^+ - u_{k_2}^+ \le \|\varphi_{k_1}^+ - \varphi_{k_2}^+\|_{C^0(\partial\Omega)}, \text{ and} \\ 0 &\le u_k^+ - u_j^- \le \|\varphi_k^+ - \varphi_j^-\|_{C^0(\partial\Omega)} \end{aligned} \tag{2.1}$$

in Ω for $j_1 \ge j_2$, $k_1 \le k_2$ (note that if u is a solution or a minimizer, so is u+*a constant*).Therefore in view of (2.1) u_j^- increasingly and u_k^+ decreasingly converge to the same limit $u \in C^0(\bar{\Omega})$ such that $u = \varphi$ on $\partial\Omega$.

Lemma 2.1. *Let $w_j \in L^1(\Omega) \cap BV_{\vec{F}}(\Omega)$, $w \in L^1(\Omega)$. Suppose $w_j \to w$ in L^1. Then*

$$\int_\Omega |Dw + \vec{F} d^m x| \le \lim \inf_{j\to\infty} \int_\Omega |Dw_j + \vec{F} d^m x|. \tag{2.2}$$

Moreover, if the right hand side of (2.2) *exists (finite value), then $w \in BV_{\vec{F}}(\Omega)$.*

Proof. For $\vec{\phi} \in C_0^1(\Omega), |\vec{\phi}| \le 1$, we have

$$\begin{aligned} \int_\Omega (-w\text{div}\vec{\phi} + \vec{F}\cdot\vec{\phi}) d^m x &= \lim_{j\to\infty} \int_\Omega (-w_j \text{div}\vec{\phi} + \vec{F}\cdot\vec{\phi}) d^m x \\ &\le \lim \inf_{j\to\infty} \int_\Omega |Dw_j + \vec{F} d^m x| \end{aligned}$$

Taking the supremum over all such $\vec{\phi}$, we obtain (2.2) by (1.3) (cf. Theorem 5.2.1 in [34]). If $\lim\inf_{j\to\infty} \int_\Omega |Dw_j + \vec{F} d^m x| < \infty$, then $w \in BV_{\vec{F}}(\Omega)$ by definition. □

Next we claim that $u \in BV(\Omega)$. Since u_j^- converges to u in C^0-norm (hence L^1-norm) on $\bar{\Omega}$, we have

$$\int_\Omega |Du + \vec{F} d^m x| \le \lim \inf_{j\to\infty} \int_\Omega |\nabla u_j^- + \vec{F}| d^m x. \tag{2.3}$$

by (2.2) in Lemma 2.1. We will prove that the right hand side of (2.3) exists (finite value). Let $u^-_{j,a}$ denote the solution of the following elliptic approximating equation:

$$\operatorname{div}\Big(\frac{\nabla v+\vec{F}}{\sqrt{a^2+|\nabla v+\vec{F}|^2}}\Big) = 0 \text{ in } \Omega \tag{2.4}$$
$$v = \varphi^-_j \text{ on } \partial\Omega$$

(cf. (4.1) in [10]; note that $u^-_{j,a}\in C^{2,\alpha}$ by Theorem 4.5 in [10]). From Lemma 2.1 and noting that $u^-_j=\lim_{a\to 0}u^-_{j,a}$ in C^0-norm (hence L^1-norm) on $\bar{\Omega}$, we have

$$\int_\Omega|\nabla u^-_j+\vec{F}|d^mx\leq \liminf_{a\to 0}\int_\Omega|\nabla u^-_{j,a}+\vec{F}|d^mx. \tag{2.5}$$

On the other hand, we observe that

$$\begin{aligned}|\nabla u^-_{j,a}+\vec{F}| &\leq \sqrt{a^2+|\nabla u^-_{j,a}+\vec{F}|^2}\\ &= \nabla u^-_{j,a}\cdot N^-_{j,a}+\vec{F}\cdot N^-_{j,a}+\frac{a^2}{\sqrt{a^2+|\nabla u^-_{j,a}+\vec{F}|^2}}\end{aligned} \tag{2.6}$$

where

$$N^-_{j,a}:=\frac{\nabla u^-_{j,a}+\vec{F}}{\sqrt{a^2+|\nabla u^-_{j,a}+\vec{F}|^2}}. \tag{2.7}$$

Integrating (2.6) and making use of (2.4) (to get $\nabla u^-_{j,a}\cdot N^-_{j,a}=\operatorname{div}(u^-_{j,a}N^-_{j,a})-u^-_{j,a}\operatorname{div}N^-_{j,a}=\operatorname{div}(u^-_{j,a}N^-_{j,a})$), we obtain

$$\int_\Omega|\nabla u^-_{j,a}+\vec{F}|d^mx\leq\int_{\partial\Omega}|\varphi^-_j|d\sigma+\int_\Omega\{|\vec{F}|+|a|\}d^mx \tag{2.8}$$

by noting that $|N^-_{j,a}|\leq 1$, where $d\sigma$ denotes the boundary measure. From (2.8) we have deduced the following estimate

$$\liminf_{a\to 0}\int_\Omega|\nabla u^-_{j,a}+\vec{F}|d^mx\leq\|\varphi^-_j\|_{L^\infty(\partial\Omega)}|\partial\Omega|+\|\vec{F}\|_{L^\infty(\Omega)}|\Omega| \tag{2.9}$$

where $|\partial\Omega|$ and $|\Omega|$ denote the $2n-1$ and $2n$ dimensional Hausdorff measures of $\partial\Omega$ and Ω, respectively. It now follows from (2.3), (2.5), and (2.9) that

$$\int_\Omega |Du + \vec{F} d^m x| \le \|\varphi\|_{L^\infty(\partial\Omega)} |\partial\Omega| + \|\vec{F}\|_{L^\infty(\Omega)} |\Omega| < \infty. \tag{2.10}$$

(2.10) means $u \in BV(\Omega)$. In the remaining section, we will show that u is a minimizer for $\mathcal{F}(\cdot)$ in $C^0(\bar{\Omega}) \cap BV(\Omega)$ with the same boundary value φ. Take $v \in C^0(\bar{\Omega}) \cap BV(\Omega)$ such that $v = u$ on $\partial\Omega$. Let v_τ be a mollifier of v, where $\tau > 0$. Let $\vec{G}_\tau = ((G_1)_\tau, (G_2)_\tau, \ldots)$ be a mollifier of a vector field $\vec{G} = (G_1, G_2, \ldots)$.

Lemma 2.2. *Let $\Omega' \subset\subset \Omega$ (i.e., Ω' has compact closure in Ω). For τ small enough, there holds*

$$\int_{\Omega'} |\nabla v_\tau + \vec{F}| d^m x \le \int_\Omega |Dv + \vec{F} d^m x| + \|\vec{F} - \vec{F}_\tau\|_{L^1(\Omega)}. \tag{2.11}$$

Proof. For $\vec{\phi} \in C_0^1(\Omega')$ such that $|\vec{\phi}| \le 1$ (which implies $|\vec{\phi}_\tau| \le 1$), we compute

$$\begin{aligned}\int_{\Omega'} (-v_\tau \mathrm{div}\vec{\phi} + \vec{F} \cdot \vec{\phi}) d^m x &= \int_\Omega (-v \mathrm{div}\vec{\phi}_\tau + \vec{F} \cdot \vec{\phi}_\tau + \vec{F} \cdot (\vec{\phi} - \vec{\phi}_\tau)) d^m x \\ &\le \int_\Omega |Dv + \vec{F} d^m x| + \int_\Omega (\vec{F} - \vec{F}_\tau) \cdot \vec{\phi} d^m x \\ &\le \int_\Omega |Dv + \vec{F} d^m x| + \|\vec{F} - \vec{F}_\tau\|_{L^1(\Omega)}. \end{aligned} \tag{2.12}$$

By taking the supremum of the left side of (2.12) over $\vec{\phi}$, we obtain (2.11).□

Lemma 2.3. *Let $v, \omega \in C^2(\bar{\Omega})$ satisfy $\mathrm{div} N_a(v) = \mathrm{div} N_a(\omega) = 0$ in Ω, where $N_a(\rho) := \frac{\nabla\rho + \vec{F}}{\sqrt{a^2 + |\nabla\rho + \vec{F}|^2}}$. For any $\Omega' \subset\subset \Omega$, there holds*

$$\left| \int_{\Omega'} \{ \sqrt{a^2 + |\nabla v + \vec{F}|^2} - \sqrt{a^2 + |\nabla\omega + \vec{F}|^2} \} d^m x \right| \le \int_{\partial\Omega'} |v - \omega| d\sigma. \tag{2.13}$$

Proof. Consider the following expression

$$I(s) := \int_{\Omega'} \sqrt{a^2 + |\nabla v + \vec{F} + s\nabla(\omega - v)|^2} d^m x + s \int_{\partial\Omega'} |v - \omega| d\sigma \tag{2.14}$$

for $s \in [0,1]$. Compute

$$I'(s) = \int_{\Omega'} \frac{[\nabla\upsilon + \vec{F} + s\nabla(\omega-\upsilon)]\cdot\nabla(\omega-\upsilon)}{\sqrt{a^2+|\nabla\upsilon+\vec{F}+s\nabla(\omega-\upsilon)|^2}} d^m x + \int_{\partial\Omega'} |\upsilon-\omega| d\sigma \quad (2.15)$$

and

$$\begin{aligned} &I''(s) \\ =& \int_{\Omega'} \{ \frac{|\nabla(\omega-\upsilon)|^2}{\sqrt{a^2+|\nabla\upsilon+\vec{F}+s\nabla(\omega-\upsilon)|^2}} - \\ & \frac{\{[\nabla\upsilon+\vec{F}+s\nabla(\omega-\upsilon)]\cdot\nabla(\omega-\upsilon)\}^2}{(\sqrt{a^2+|\nabla\upsilon+\vec{F}+s\nabla(\omega-\upsilon)|^2})^3} \} d^m x \\ =& \int_{\Omega'} \{ a^2|\nabla(\omega-\upsilon)|^2 + |\nabla(\omega-\upsilon)|^2 |\nabla\upsilon+\vec{F}+s\nabla(\omega-\upsilon)|^2 \\ & -\{[\nabla\upsilon+\vec{F}+s\nabla(\omega-\upsilon)]\cdot\nabla(\omega-\upsilon)\}^2\} (\sqrt{a^2+|\nabla\upsilon+\vec{F}+s\nabla(\omega-\upsilon)|^2})^{-3} d^m x \\ \geq& \ 0 \end{aligned}$$

by Cauchy's inequality for $s \in [0,1]$. It follows that

$$I'(s) \geq I'(0) \quad (2.16)$$

On the other hand, from (2.15) we compute

$$\begin{aligned} I'(0) &= \int_{\Omega'} N_a(\upsilon)\cdot\nabla(\omega-\upsilon) d^m x + \int_{\partial\Omega'} |\upsilon-\omega| d\sigma \\ &= \int_{\partial\Omega'} (\omega-\upsilon) N_a(\upsilon)\cdot\nu d\sigma + \int_{\partial\Omega'} |\upsilon-\omega| d\sigma \\ &\geq 0 \end{aligned} \quad (2.17)$$

where we have used the equation $\mathrm{div} N_a(\upsilon) = 0$ and $|N_a(\upsilon)| \leq 1$. By (2.16) and (2.17), we get $I'(s) \geq 0$, and hence $I(1) \geq I(0)$. That is

$$\int_{\Omega'} \{ \sqrt{a^2+|\nabla\upsilon+\vec{F}|^2} - \sqrt{a^2+|\nabla\omega+\vec{F}|^2} \} d^m x \leq \int_{\partial\Omega'} |\upsilon-\omega| d\sigma.$$

Switching υ and ω in the above argument, we finally reach (2.13). □

Proof of Theorem A continued. Now we consider only parabolically convex domain $\Omega' \subset\subset \Omega$ with $\partial\Omega' \in C^\infty$. For $a > 0$ let $u_{j,a}^-, v_{\tau,a}$ be the solutions to $\mathrm{div} N_a(\cdot) = 0$ in Ω', such that $u_{j,a}^- = u_j^-, v_{\tau,a} = v_\tau$ on $\partial\Omega'$, where $N_a(\rho) := \frac{\nabla\rho+\vec{F}}{\sqrt{a^2+|\nabla\rho+\vec{F}|^2}}$. We then compute

$$\begin{aligned}\int_{\Omega'} |\nabla u_j^- + \vec{F}| d^m x &\leq \int_{\Omega'} |\nabla u_{j,a}^- + \vec{F}| d^m x \quad (u_j^- \text{ is a minimizer for } \mathcal{F}(\cdot)) \\ &\leq \int_{\Omega'} \sqrt{a^2 + |\nabla u_{j,a}^- + \vec{F}|^2} d^m x \\ &\leq \int_{\Omega'} \sqrt{a^2 + |\nabla v_{\tau,a} + \vec{F}|^2} d^m x + \int_{\partial\Omega'} |u_j^- - v_\tau| d\sigma \quad (2.18)\end{aligned}$$

by (2.13). Let $\mathcal{F}_a(w) \equiv \int_{\Omega'} \sqrt{a^2 + |\nabla w + \vec{F}|^2} d^m x$. Since $v_{\tau,a}$ is a minimizer for $\mathcal{F}_a(\cdot)$ (see [10]), we estimate

$$\begin{aligned}&\int_{\Omega'} \sqrt{a^2 + |\nabla v_{\tau,a} + \vec{F}|^2} d^m x \\ &\quad\leq \int_{\Omega'} \sqrt{a^2 + |\nabla v_\tau + \vec{F}|^2} d^m x \\ &\quad\leq a|\Omega'| + \int_\Omega |\nabla v + \vec{F}| d^m x + \|\vec{F} - \vec{F}_\tau\|_{L^1(\Omega)} \qquad (2.19)\end{aligned}$$

by (2.11) (τ small enough). Combining (2.18) and (2.19) gives

$$\begin{aligned}&\int_{\Omega'} |\nabla u_j^- + \vec{F}| d^m x \\ &\quad\leq a|\Omega'| + \int_\Omega |\nabla v + \vec{F}| d^m x + \|\vec{F} - \vec{F}_\tau\|_{L^1(\Omega)} + \int_{\partial\Omega'} |u_j^- - v_\tau| d\sigma. \qquad (2.20)\end{aligned}$$

In view of (2.3) and (2.20), we conclude (1.5) by letting a go to zero and Ω' approach Ω (τ tends to zero accordingly). □

3. Extension to Measures and the First Variation

We will extend the domain of $\mathcal{F}(\cdot)$ from BV functions to vector-valued measures. Let $\Omega \subset R^m$ be a bounded domain. Let $u \in BV_{\vec{F}}(\Omega)$ where $\vec{F} := (F_i) \in L^1(\Omega)$. Then there are Radon signed measures $\lambda_1, \lambda_2, \ldots, \lambda_m$ defined in Ω such that for $i = 1, 2, \ldots, m$, (Recall that $d^m x$ denotes the

Lebesgue measure of R^m)

$$\int_\Omega u\frac{\partial\varphi}{\partial x_i}d^m x = -\int_\Omega \varphi d\lambda_i$$

for all $\varphi \in C_0^\infty(\Omega)$ (see Remark 1.5 on page 5 in [17] or see (5.1.1) in [34], and note that $u \in BV_{\vec{F}}(\Omega)$ if and only if $u \in BV(\Omega)$). Write $Du := (d\lambda_i)$. So $Du + \vec{F}d^m x$ defines a vector-valued Radon signed measure and we define its total variation (measure)

$$|Du + \vec{F}d^m x|(f) := \sup\{\int_\Omega (-u\mathrm{div}\vec{\phi} + \vec{F}\cdot\vec{\phi})d^m x \mid \vec{\phi} \in C_0^1(\Omega), |\vec{\phi}| \leq f\}$$

for f being a non-negative real-valued continuous function with compact support in Ω. By the Riesz Representation Theorem, $|Du + \vec{F}d^m x|$ is a non-negative Radon measure on Ω (mimicking the argument in Remark 5.1.2. of [34]). Similarly, for a general vector-valued measure $d\mu = (d\mu_i)$ (instead of $\mu = (\mu_i)$) on Ω, we define its total variation measure $|d\mu|$ by

$$|d\mu|(f) = \sup\{\int_\Omega \vec{\phi}\cdot d\mu \mid \vec{\phi} \in C_0^1(\Omega), |\vec{\phi}| \leq f\}.$$

We extend the domain of $\mathcal{F}(\cdot)$ to include vector-valued (Radon signed) measures $d\mu$ by defining

$$\mathcal{F}(d\mu) := \int_\Omega |d\mu| := |d\mu|(1). \tag{3.1}$$

In this section we want to compute the first variation of $\mathcal{F}(\cdot)$ in measures.

Let E be a C^∞-smooth Riemannian vector bundle over a C^∞-smooth manifold X. Let $d\mu, d\nu$ be two E-valued measures on X. We assume that both $d\mu$ and $d\nu$ are bounded in the sense that $|d\mu|$ and $|d\nu|$ are integrable over X, i.e., $\mathcal{F}(d\mu)$ and $\mathcal{F}(d\nu)$ are finite in view of (3.1) with Ω replaced by X ($\vec{\phi}$ is viewed as a C^1-smooth section of E with compact support while "$\cdot$" denotes the fibre inner product). Let $d\mu_\varepsilon := d\mu + \varepsilon d\nu$ for $\varepsilon \in R$. Since $|d\mu_\varepsilon|$ is a positive bounded measure, we can find N_ε, $A_\varepsilon \in L^1(|d\mu_\varepsilon|)$ with $|N_\varepsilon| = 1$, such that

$$\begin{aligned} d\mu_\varepsilon &= N_\varepsilon|d\mu_\varepsilon|, \\ d\nu &= A_\varepsilon|d\mu_\varepsilon| + d\nu_s^\varepsilon, d\nu_s^\varepsilon \perp |d\mu_\varepsilon| \end{aligned} \tag{3.2}$$

according to the Radon-Nikodym theorem (extending 6.9 and 6.12 in [31] to the case of vector-valued measures; see also [30]).

Proof of Theorem B. We have

$$\begin{aligned}|d\mu_{\varepsilon_2}| &= |d\mu_{\varepsilon_1} + (\varepsilon_2 - \varepsilon_1)d\nu| \\ &= |(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}||d\mu_{\varepsilon_1}| + |\varepsilon_2 - \varepsilon_1||d\nu_s^{\varepsilon_1}| \qquad (3.3)\end{aligned}$$

by (3.2). It follows from (3.3) that for $\varepsilon_2 \neq \varepsilon_1$

$$\frac{|d\mu_{\varepsilon_2}| - |d\mu_{\varepsilon_1}|}{\varepsilon_2 - \varepsilon_1} = \{|(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}| - 1\}\frac{|d\mu_{\varepsilon_1}|}{\varepsilon_2 - \varepsilon_1} + \frac{|\varepsilon_2 - \varepsilon_1|}{\varepsilon_2 - \varepsilon_1}|d\nu_s^{\varepsilon_1}|. \quad (3.4)$$

Observe that

$$\begin{aligned}|\frac{|(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}| - 1}{\varepsilon_2 - \varepsilon_1}||d\mu_{\varepsilon_1}| &= |\frac{|(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}| - |N_{\varepsilon_1}|}{\varepsilon_2 - \varepsilon_1}||d\mu_{\varepsilon_1}| \\ &\leq |A_{\varepsilon_1}||d\mu_{\varepsilon_1}| \leq |d\nu| \qquad (3.5)\end{aligned}$$

by noting that $|N_{\varepsilon_1}| = 1$. It follows from (3.4) and (3.5) that $\mathcal{F}(\varepsilon)$ is Lipschitz continuous in ε since $d\nu$ is bounded by assumption. Also observe that

$$\begin{aligned}|(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}| - 1 &= \frac{|(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}|^2 - 1}{|(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}| + 1} \qquad (3.6) \\ &= \frac{2(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} \cdot N_{\varepsilon_1} + |(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1}|^2}{|(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}| + 1}.\end{aligned}$$

Since $|A_{\varepsilon_1}||d\mu_{\varepsilon_1}|$ and $|d\nu_s^{\varepsilon_1}|$ are integrable by assumption ($d\nu$ is bounded), we can invoke the Lebesgue dominated convergence theorem to obtain

$$\lim_{\varepsilon_2 \to \varepsilon_1 \pm} \int_X \frac{|d\mu_{\varepsilon_2}| - |d\mu_{\varepsilon_1}|}{\varepsilon_2 - \varepsilon_1} = \int_X N_{\varepsilon_1} \cdot A_{\varepsilon_1}|d\mu_{\varepsilon_1}| \pm \int_X |d\nu_s^{\varepsilon_1}| \qquad (3.7)$$

by (3.4) and (3.6). □

Proof of Corollary B′. Let $d\mu = Du + \vec{F}d^m x$ denote the vector-valued measure associated to $u \in BV(\Omega)$. Let $d\nu = D\varphi$ for $\varphi \in BV(\Omega)$ with $\varphi|_{\partial\Omega} = 0$. Recall that we denote $\tilde{\mathcal{F}}_H(u + \varepsilon\varphi)$ by $\tilde{\mathcal{F}}_H(\varepsilon)$.Now it is straightforward to extend (1.8) for $X = \Omega$ to include H as below:

$$\tilde{\mathcal{F}}_H'(\varepsilon_1 \pm) = \int_\Omega N_{\varepsilon_1} \cdot A_{\varepsilon_1} |d\mu_{\varepsilon_1}| \pm \int_\Omega |d\nu_s^{\varepsilon_1}| + \int_\Omega H\varphi d^m x. \tag{3.8}$$

Letting $\varepsilon_1 = 0$ in (3.8) we have

$$\begin{aligned} &\int_\Omega N_0 \cdot A_0 |d\mu| \pm \int_\Omega |d\nu_s^0| + \int_\Omega H\varphi d^m x \\ &= \tilde{\mathcal{F}}_H'(0\pm) \\ &= \lim_{\varepsilon \to 0\pm} \frac{\tilde{\mathcal{F}}_H(u+\varepsilon\varphi) - \tilde{\mathcal{F}}_H(u)}{\varepsilon} \geq 0 \ (\leq 0, \text{ resp.}) \end{aligned}$$

for $\varepsilon \to 0+$ ($\varepsilon \to 0-$, resp.) since $\tilde{\mathcal{F}}_H(u+\varepsilon\varphi) - \tilde{\mathcal{F}}_H(u) \geq 0$ for u being a minimizer and $\varepsilon > 0$ ($\varepsilon < 0$, resp.). We have proved (1.9) and (1.10). □

Lemma 3.1. *Suppose $d\mu, d\nu$ are two bounded E-valued measures on X as described between* (3.1) *and* (3.2). *Let $d\mu_\varepsilon := d\mu + \varepsilon d\nu$ for $\varepsilon \in R$ satisfy* (3.2). *Then for $\varepsilon_1 \neq \varepsilon_2$ there holds $d\nu_s^{\varepsilon_1} \perp d\nu_s^{\varepsilon_2}$, i.e., $|\ d\nu_s^{\varepsilon_1}\ | \perp |\ d\nu_s^{\varepsilon_2}\ |$. Moreover, there exist at most countably many ε's such that $|d\nu_s^\varepsilon|(X) \neq 0$.*

Proof. Let $j = 1, 2$. Since $|d\mu_{\varepsilon_j}| \perp |d\nu_s^{\varepsilon_j}|$, we can find a measurable set E_{ε_j} such that $|d\mu_{\varepsilon_j}|$ is concentrated on E_{ε_j} and $|d\nu_s^{\varepsilon_j}|$ is concentrated on $E_{\varepsilon_j}^c$, the complement of E_{ε_j}. For any measurable set $B \subset E_{\varepsilon_1}^c \cap E_{\varepsilon_2}^c$, $|d\nu|(B) = 0$ by observing that $d\mu_{\varepsilon_1} - d\mu_{\varepsilon_2} = (\varepsilon_1 - \varepsilon_2)d\nu$. It follows that $|d\nu_s^{\varepsilon_j}|(B) = 0$ for $j = 1, 2$ since $d\nu = d\nu_s^{\varepsilon_1}$ and $d\nu = d\nu_s^{\varepsilon_2}$ on $E_{\varepsilon_1}^c \cap E_{\varepsilon_2}^c$. So $|d\nu_s^{\varepsilon_1}| = |d\nu_s^{\varepsilon_2}| = 0$ on $E_{\varepsilon_1}^c \cap E_{\varepsilon_2}^c$, and hence $|d\nu_s^{\varepsilon_1}|$ and $|d\nu_s^{\varepsilon_2}|$ are concentrated on $E_{\varepsilon_1}^c \setminus (E_{\varepsilon_1}^c \cap E_{\varepsilon_2}^c)$ and $E_{\varepsilon_2}^c \setminus (E_{\varepsilon_1}^c \cap E_{\varepsilon_2}^c)$ (the intersection of these two sets is empty), respectively. Therefore $|d\nu_s^{\varepsilon_1}| \perp |d\nu_s^{\varepsilon_2}|$.

Given a positive integer n, we can only have finitely many ε_j's such that $|d\nu|(E_{\varepsilon_j}^c) = |d\nu_s^{\varepsilon_j}|(E_{\varepsilon_j}^c) \geq \frac{1}{n}$ since

$$\begin{aligned} \sum_j |d\nu|(E_{\varepsilon_j}^c) &= \sum_j |d\nu_s^{\varepsilon_j}|(E_{\varepsilon_j}^c) \\ &= |d\nu|(\cup E_{\varepsilon_j}^c) \\ &\leq |d\nu|(X) < \infty \ \ \text{(by assumption)}. \end{aligned}$$

It follows that there are at most countably many ε's such that $|d\nu|(E_\varepsilon^c) = |d\nu_s^\varepsilon|(E_\varepsilon^c) \neq 0$. □

If ε satisfies $|d\nu_s^\varepsilon|(X) \neq 0$, we call it singular, otherwise regular. Denote $\mathcal{F}(d\mu_\varepsilon)$ by $\mathcal{F}(\varepsilon)$ for simplicity.

Lemma 3.2. *Suppose we are in the situation of Lemma 3.1. Then we have*

(1) *For* $\varepsilon_1, \varepsilon_2$ *arbitrary, there holds*

$$|d\mu_{\varepsilon_1}| + |d\nu_s^{\varepsilon_1}| \ll |d\mu_{\varepsilon_2}| + |d\nu_s^{\varepsilon_2}| \ll |d\mu_{\varepsilon_1}| + |d\nu_s^{\varepsilon_1}|; \tag{3.9}$$

(2) *For* $\varepsilon_1, \varepsilon_2$ *regular, there holds* $|d\mu_{\varepsilon_1}| \ll |d\mu_{\varepsilon_2}| \ll |d\mu_{\varepsilon_1}|$.

Proof. We may assume $\varepsilon_1 \neq \varepsilon_2$. From (3.3) we have

$$|d\mu_{\varepsilon_2}| \ll |d\mu_{\varepsilon_1}| + |d\nu_s^{\varepsilon_1}| \tag{3.10}$$

Switching ε_1 and ε_2 in (3.3) gives

$$|d\mu_{\varepsilon_1}| = |(\varepsilon_1 - \varepsilon_2)A_{\varepsilon_2} + N_{\varepsilon_2}||d\mu_{\varepsilon_2}| + |\varepsilon_1 - \varepsilon_2||d\nu_s^{\varepsilon_2}|. \tag{3.11}$$

Therefore we obtain

$$|\varepsilon_1 - \varepsilon_2||d\nu_s^{\varepsilon_2}| \ll |d\mu_{\varepsilon_1}| + |d\nu_s^{\varepsilon_1}| \tag{3.12}$$

by (3.11) and (3.10). Now it follows from (3.10) and (3.12) that

$$|d\mu_{\varepsilon_2}| + |d\nu_s^{\varepsilon_2}| \ll |d\mu_{\varepsilon_1}| + |d\nu_s^{\varepsilon_1}|.$$

By symmetry (3.9) follows. For ε_1, ε_2 regular, $|d\nu_s^{\varepsilon_1}| = |d\nu_s^{\varepsilon_2}| = 0$ and hence (2) follows from (3.9). □

Lemma 3.3. *Suppose* $d\mu$ *and* $d\mu'$ *are two bounded* E*-valued measures on* X (*see the paragraph between* (3.1) *and* (3.2)). *Assume* $|d\mu| \ll |d\mu'| \ll |d\mu|$. *Write* $d\mu = N_\mu|d\mu|, d\mu' = N_{\mu'}|d\mu'|$. *Then we have*

(1) $N_\mu \neq 0$ *a.e.* $[|d\mu'|]$ *and* $N_{\mu'} \neq 0$ *a.e.* $[|d\mu|]$ *and*

(2) *there holds*

$$(N_\mu - N_{\mu'}) \cdot (d\mu - d\mu') = \frac{1}{2}|N_\mu - N_{\mu'}|^2(|d\mu| + |d\mu'|). \tag{3.13}$$

Proof. Suppose there is a set S with $|d\mu'|(S) > 0$ and $N_\mu = 0$ on S. From the definition of N_μ, we have $|d\mu|(S) = 0$. It follows that $|d\mu'|(S) = 0$ by the assumption $|d\mu'| \ll |d\mu|$. We have reached a contradiction. Therefore $N_\mu \neq 0$ a.e. $[|d\mu'|]$. By symmetry, we also have $N_{\mu'} \neq 0$ a.e. $[|d\mu|]$. We have proved (1).

As for (2), noting that $N_\mu, N_{\mu'}$ are defined a.e. $[|d\mu|]$ and $[|d\mu'|]$, we compute

$$\begin{aligned}(N_\mu - N_{\mu'}) \cdot (d\mu - d\mu') &= (N_\mu - N_{\mu'})(N_\mu |d\mu| - N_{\mu'}|d\mu'|) \\ &= (1 - N_\mu \cdot N_{\mu'})(|d\mu| + |d\mu'|) \\ &= \frac{1}{2}|N_\mu - N_{\mu'}|^2(|d\mu| + |d\mu'|). \qquad (3.14)\end{aligned}$$

□

We remark that for general $d\mu, d\mu'$ (which may not satisfy the condition $|d\mu| \ll |d\mu'| \ll |d\mu|$), the formula (3.13) should be interpreted and modified as below. Write $d\mu = A|d\mu'| + d\nu_s'$ with $|d\mu'| \perp d\nu_s'$. Then there exists E' on which $|d\mu'|$ (hence $d\mu'$) is concentrated while $d\nu_s'$ is concentrated on $(E')^c := X \backslash E'$. Let $E_1' := E' \cap \{A = 0\}$ and $E_2' := E' \cap \{A \neq 0\}$ (note that A is defined modulo a $|d\mu'|$-measure zero set in E'). It follows that $d\mu$ is concentrated on $E := E_2' \cup (E')^c$. We extend the domain of N_μ ($N_{\mu'}$, resp.) and define N_μ ($N_{\mu'}$, resp.) to be 0 on $E^c := X \backslash E$ ($(E')^c$, resp.). Let χ_E ($\chi_{E'}$, resp.) denote the characteristic function of E (E', resp.), i.e., $\chi_E = 1$ on E and $\chi_E = 0$ on E^c. Following a similar computation in (3.14), we then have

$$\begin{aligned}(N_\mu - N_{\mu'}) \cdot (d\mu - d\mu') &= (1 - N_\mu \cdot N_{\mu'})(|d\mu| + |d\mu'|) \\ &= \frac{1}{\chi_E + \chi_{E'}}|N_\mu - N_{\mu'}|^2(|d\mu| + |d\mu'|). \quad (3.15)\end{aligned}$$

Note that since $E \cup E' = X$, we have $\chi_E + \chi_{E'} \neq 0$ on X.

For ε regular there holds

$$\mathcal{F}'(\varepsilon) = \frac{d\mathcal{F}(d\mu_\varepsilon)}{d\varepsilon} = \int_X N_\varepsilon \cdot A_\varepsilon |d\mu_\varepsilon|. \qquad (3.16)$$

Since for ε regular we have $d\nu_s^\varepsilon = 0$ and hence (3.16) follows from (3.7). Now let $\varepsilon_1, \varepsilon_2$ be regular and $\varepsilon_2 > \varepsilon_1$. Observe that $d\nu = A_{\varepsilon_2}|d\mu_{\varepsilon_2}| = A_{\varepsilon_1}|d\mu_{\varepsilon_1}|$

and hence from (3.16) we have

$$\begin{aligned}\mathcal{F}'(\varepsilon_2) - \mathcal{F}'(\varepsilon_1) &= \int_X \{N_{\varepsilon_2} \cdot A_{\varepsilon_2} |d\mu_{\varepsilon_2}| - N_{\varepsilon_1} \cdot A_{\varepsilon_1} |d\mu_{\varepsilon_1}|\} \\ &= \int_X (N_{\varepsilon_2} - N_{\varepsilon_1}) \cdot d\nu. \end{aligned} \tag{3.17}$$

On the other hand, by Lemma 3.2 and Lemma 3.3 with $d\mu = d\mu_{\varepsilon_2}, d\mu' = d\mu_{\varepsilon_1}$, (3.13) reads

$$(N_{\varepsilon_2} - N_{\varepsilon_1}) \cdot (d\mu_{\varepsilon_2} - d\mu_{\varepsilon_1}) = \frac{1}{2} |N_{\varepsilon_2} - N_{\varepsilon_1}|^2 (|d\mu_{\varepsilon_2}| + |d\mu_{\varepsilon_1}|). \tag{3.18}$$

In view of $d\nu = (\varepsilon_2 - \varepsilon_1)^{-1} \; (d\mu_{\varepsilon_2} - d\mu_{\varepsilon_1})$, we have

$$\mathcal{F}'(\varepsilon_2) - \mathcal{F}'(\varepsilon_1) = \int_X \frac{1}{2(\varepsilon_2 - \varepsilon_1)} |N_{\varepsilon_2} - N_{\varepsilon_1}|^2 (|d\mu_{\varepsilon_2}| + |d\mu_{\varepsilon_1}|) \geq 0 \tag{3.19}$$

by (3.17) and (3.18).

We remark that (3.16) generalizes Lemma 3.1 in [10]. For an arbitrary ε (regular or singular), we write $\mathcal{F}'_+(\varepsilon)$ for $\mathcal{F}'(\varepsilon+) \equiv \lim_{\tilde{\varepsilon} \to \varepsilon+} \frac{\mathcal{F}(\tilde{\varepsilon}) - \mathcal{F}(\varepsilon)}{\tilde{\varepsilon} - \varepsilon}$, the right derivative of $\mathcal{F}$ at ε. Similarly we write $\mathcal{F}'_-(\varepsilon)$ for the left derivative $\mathcal{F}'(\varepsilon-)$. Both $\mathcal{F}'_+(\varepsilon)$ and $\mathcal{F}'_-(\varepsilon)$ exist in view of (3.7). When ε is regular, $\mathcal{F}'_+(\varepsilon) = \mathcal{F}'_-(\varepsilon) = \mathcal{F}'(\varepsilon)$ (see (3.16)). We study the left and right continuity of $\mathcal{F}'_+(\varepsilon)$ and $\mathcal{F}'_-(\varepsilon)$.

Theorem 3.4. *For $\varepsilon_2 > \varepsilon_1$, we have*

$$\mathcal{F}'_+(\varepsilon_2) \geq \mathcal{F}'_-(\varepsilon_2) \geq \mathcal{F}'_+(\varepsilon_1) \geq \mathcal{F}'_-(\varepsilon_1). \tag{3.20}$$

In particular, $\mathcal{F}'(\varepsilon)$ is an increasing function of ε for ε regular. We also have the following limits:

$$\begin{aligned} &\lim_{\varepsilon_2 \to \varepsilon_1+} \mathcal{F}'_+(\varepsilon_2) = \mathcal{F}'_+(\varepsilon_1), \lim_{\varepsilon_2 \to \varepsilon_1-} \mathcal{F}'_+(\varepsilon_2) = \mathcal{F}'_-(\varepsilon_1), \\ &\lim_{\varepsilon_2 \to \varepsilon_1+} \mathcal{F}'_-(\varepsilon_2) = \mathcal{F}'_+(\varepsilon_1), \lim_{\varepsilon_2 \to \varepsilon_1-} \mathcal{F}'_-(\varepsilon_2) = \mathcal{F}'_-(\varepsilon_1). \end{aligned} \tag{3.21}$$

Moreover, $\mathcal{F}$ is convex.

Proof. That $\mathcal{F}'(\varepsilon)$ is an increasing function of ε for ε regular follows from

(3.19). From (3.2) we have

$$\begin{aligned} d\mu_{\varepsilon_2} &= d\mu_{\varepsilon_1} + (\varepsilon_2 - \varepsilon_1) d\nu \\ &= ((\varepsilon_2 - \varepsilon_1) A_{\varepsilon_1} + N_{\varepsilon_1}) |d\mu_{\varepsilon_1}| + (\varepsilon_2 - \varepsilon_1) N_s^{\varepsilon_1} |d\nu_s^{\varepsilon_1}|. \end{aligned} \tag{3.22}$$

Here we have written $d\nu_s^{\varepsilon_1} = N_s^{\varepsilon_1} |d\nu_s^{\varepsilon_1}|$. We need the following lemma to describe N_{ε_2}.

Lemma 3.5. *Let $d\lambda, d\tau$, and $d\rho$ be bounded vector-valued measures. Suppose $d\lambda = \vec{B}|d\tau| + \vec{C}|d\rho|, |d\tau| \perp |d\rho|$, and $|d\lambda| \ll |d\tau| + |d\rho| \ll |d\lambda|$. Then $\vec{B} \neq 0$ a.e. $[|d\tau|], \vec{C} \neq 0$ a.e. $[|d\rho|]$, and*

$$\vec{N}_\lambda = \frac{\vec{B}}{|\vec{B}|} \text{ a.e. } [|d\tau|]; = \frac{\vec{C}}{|\vec{C}|} \text{ a.e. } [|d\rho|] \tag{3.23}$$

where we write $d\lambda = \vec{N}_\lambda |d\lambda|$.

Proof. Since $|d\tau| \ll |d\lambda|$ ($|d\rho| \ll |d\lambda|$, respectively), we can find a vector-valued function $\vec{h}_\tau \in L^1(|d\lambda|)$ ($\vec{h}_\rho \in L^1(|d\lambda|)$, respectively) such that

$$d\tau = \vec{h}_\tau |d\lambda| \ (d\rho = \vec{h}_\rho |d\lambda|, \text{ respectively}).$$

It follows that $|d\tau| = |\vec{h}_\tau||d\lambda| = |\vec{h}_\tau|(|\vec{B}||d\tau| + |\vec{C}||d\rho|)$. So $|\vec{h}_\tau||\vec{B}| = 1$ a.e. $[|d\tau|]$ and $|\vec{h}_\tau||\vec{C}| = 0$ a.e. $[|d\rho|]$. Therefore $\vec{B} \neq 0$ a.e. $[|d\tau|]$. Similarly we have $\vec{C} \neq 0$ a.e. $[|d\rho|]$ and hence

$$\begin{aligned} &\vec{h}_\tau = 0 \text{ a.e. } [|d\rho|], |\vec{h}_\tau| = \frac{1}{|\vec{B}|} \text{ a.e. } [|d\tau|], \text{ and also} \\ &\vec{h}_\rho = 0 \text{ a.e. } [|d\tau|], |\vec{h}_\rho| = \frac{1}{|\vec{C}|} \text{ a.e. } [|d\rho|] \end{aligned} \tag{3.24}$$

by symmetry. Now we compute $\vec{N}_\lambda |d\lambda| = d\lambda = \vec{B}|d\tau| + \vec{C}|d\rho| = \vec{B}|\vec{h}_\tau||d\lambda| + \vec{C}|\vec{h}_\rho||d\lambda| = (\vec{B}|\vec{h}_\tau| + \vec{C}|\vec{h}_\rho|)|d\lambda|$. It then follows that

$$\vec{N}_\lambda = \vec{B}|\vec{h}_\tau| + \vec{C}|\vec{h}_\rho| \text{ a.e. } [|d\lambda|]. \tag{3.25}$$

Since $|d\tau| \perp |d\rho|$, we obtain (3.23) from (3.25) in view of (3.24). □

Proof of Theorem 3.4 *continued.* From Lemma 3.5 we express N_{ε_2} as follows:

$$N_{\varepsilon_2} = \begin{cases} \frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|} & \text{a.e. } [|d\mu_{\varepsilon_1}|] \\ \frac{\varepsilon_2-\varepsilon_1}{|\varepsilon_2-\varepsilon_1|}N_s^{\varepsilon_1} & \text{a.e. } [|d\nu_s^{\varepsilon_1}|]. \end{cases} \tag{3.26}$$

Now for ε_2 regular (ε_1 may not be regular) we compute

$$\begin{aligned} \mathcal{F}'(\varepsilon_2) &= \int_X N_{\varepsilon_2}\cdot d\nu \\ &= \int_X N_{\varepsilon_2}\cdot (A_{\varepsilon_1}|d\mu_{\varepsilon_1}| + d\nu_s^{\varepsilon_1}) \quad \text{(by (3.2))} \\ &= \int_X \frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}\cdot A_{\varepsilon_1}|d\mu_{\varepsilon_1}| + \frac{\varepsilon_2-\varepsilon_1}{|\varepsilon_2-\varepsilon_1|}N_s^{\varepsilon_1}\cdot d\nu_s^{\varepsilon_1} \quad (3.27) \end{aligned}$$

by (3.26). Observe that $N_s^{\varepsilon_1}\cdot d\nu_s^{\varepsilon_1} = |d\nu_s^{\varepsilon_1}|$ and the integrand in (3.27) is bounded by $|A_{\varepsilon_1}||d\mu_{\varepsilon_1}|+|d\nu_s^{\varepsilon_1}|$ (which is independent of ε_2 and integrable by assumption). We can therefore apply the Lebesgue dominated convergence theorem to get

$$\begin{aligned} \lim_{\varepsilon_2\to\varepsilon_1\pm}\mathcal{F}'(\varepsilon_2) &= \int_X N_{\varepsilon_1}\cdot A_{\varepsilon_1}|d\mu_{\varepsilon_1}| \pm |d\nu_s^{\varepsilon_1}| \\ &= \mathcal{F}'_{\pm}(\varepsilon_1) \end{aligned} \tag{3.28}$$

by (3.7). Since $\mathcal{F}'(\varepsilon)$ is increasing for ε regular and the set of regular values is dense, we can easily deduce (3.20) from (3.28). Thus we have $\lim_{\varepsilon_2\to\varepsilon_1+}\mathcal{F}'_+(\varepsilon_2) = \mathcal{F}'_+(\varepsilon_1)$ and $\lim_{\varepsilon_2\to\varepsilon_1+}\mathcal{F}'_-(\varepsilon_2) = \mathcal{F}'_+(\varepsilon_1)$. Similarly we also have $\lim_{\varepsilon_2\to\varepsilon_1-}\mathcal{F}'_+(\varepsilon_2) = \mathcal{F}'_-(\varepsilon_1)$ and $\lim_{\varepsilon_2\to\varepsilon_1-}\mathcal{F}'_-(\varepsilon_2) = \mathcal{F}'_-(\varepsilon_1)$. We have proved (3.21). That $\mathcal{F}$ is convex follows from (3.20) by elementary calculus. □

We remark that Theorem 3.4 generalizes Lemma 3.2 in [10].

Example 3.1. Consider a C^2-smooth graph $\Sigma = \{(x_1, x_2, \dots, x_m, u(x_1, x_2, \dots, x_m))\}$ in R^{m+1}. Let $d\mu := (u_{x_1}, u_{x_2}, \dots, u_{x_m}, -1)\ d^m x$ where we recall that $d^m x := dx_1 \wedge dx_2 \wedge \cdots \wedge dx_m$ be the R^{m+1}-valued measure defined on a bounded domain $X \subset R^m$, associated with the Euclidean normal to Σ. Then $|d\mu| = \sqrt{1+u_{x_1}^2+\cdots+u_{x_m}^2}\ d^m x$ is the area element of Σ with respect to the metric induced from the Euclidean metric on R^{m+1}. Let $d\nu \equiv (v_{x_1}, v_{x_2}, \dots, v_{x_m}, 0)\ d^m x$ where $v \in C_0^\infty(X)$. So from $d\mu_\varepsilon = N_\varepsilon\ |d\mu_\varepsilon|$, $d\nu =$

$A_\varepsilon\ |d\mu_\varepsilon|$, and $|d\mu_\varepsilon| = \sqrt{1+|\nabla u+\varepsilon\nabla v|^2}\ d^m x$ ($d\nu_s^\varepsilon = 0$ since $|d\mu_\varepsilon|$ is strictly positive; so each ε is regular), we obtain

$$N_\varepsilon = \frac{(\nabla u+\varepsilon\nabla v, -1)}{\sqrt{1+|\nabla u+\varepsilon\nabla v|^2}}, A_\varepsilon = \frac{(\nabla v, 0)}{\sqrt{1+|\nabla u+\varepsilon\nabla v|^2}} \tag{3.29}$$

where ∇ denotes the gradient in R^m. By Theorem B we have the first variation of the area $\mathcal{F}(0) = \mathcal{F}(d\mu)$ of Σ :

$$\begin{aligned}\mathcal{F}'(0) &= \int_X N_0\cdot A_0|d\mu| \\ &= \int_X \frac{\nabla u\cdot\nabla v}{\sqrt{1+|\nabla u|^2}} d^m x \\ &= -\int_X div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}) v d^m x\end{aligned} \tag{3.30}$$

by (3.29) and the divergence theorem. Notice that $div(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}})$ in (3.30) is the (Riemannian) mean curvature of Σ in R^{m+1}. We have recovered the classical first variation formula for the area of a graph in the Euclidean space.

Example 3.2. Consider a C^1-smooth graph $\Sigma = \{(x_1, x_{1'}, \ldots, x_n, x_{n'}, u(x_1, x_{1'}, \ldots, x_n, x_{n'}))\}$ in the Heisenberg group viewed as R^{2n+1} with the standard flat pseudohermitian structure (see [8]). Recall that $\vec{X}^* = (x_{1'}, -x_1, x_{2'}, -x_2, \ldots, x_{n'}, -x_n)$. Let ∇ denote the gradient opeator in R^{2n}. Let $d\mu := (\nabla u - \vec{X}^*)\ d^{2n}x$ where $d^{2n}x := dx_1\wedge dx_{1'}\wedge\cdots\wedge dx_n\wedge dx_{n'}$ and $d\nu \equiv (\nabla\varphi)\ d^{2n}x$ be two R^{2n}-valued measures defined on a bounded domain $\Omega\subset R^{2n}$ ($\varphi\in C_0^1(\Omega)$, say). So $|d\mu| = |\nabla u - \vec{X}^*| d^{2n}x$ is the p-area element. Denote the singular set $\{\nabla u - \vec{X}^* = 0\}$ by $S(u)$. Write $d\mu = N_0\ |d\mu|$ and $d\nu = A_0\ |d\mu| + d\nu_s$ where

$$\begin{aligned}&N_0 = \frac{\nabla u - \vec{X}^*}{|\nabla u - \vec{X}^*|}, A_0 = \frac{\nabla\varphi}{|\nabla u - \vec{X}^*|} \text{ on } \Omega\backslash S(u) \\ &d\nu_s = (\nabla\varphi) d^{2n}x \text{ on } S(u).\end{aligned} \tag{3.31}$$

Note that $|d\mu|$ is concentrated on $\Omega\backslash S(u)$ while $d\nu_s$ is concentrated on $S(u)$. By Theorem B and (3.31) we have the first variation of the p-area $\mathcal{F}(0) = \mathcal{F}(d\mu)$ of Σ :

$$\mathcal{F}'(0\pm) = \int_\Omega N_0\cdot A_0|d\mu| \pm |d\nu_s|$$

$$\begin{aligned} &= \int_{\Omega\backslash S(u)} \frac{(\nabla u - \vec{X}^*)\cdot\nabla\varphi}{|\nabla u - \vec{X}^*|} d^{2n}x \pm \int_{S(u)} |\nabla\varphi| d^{2n}x \\ &= \int_{\Omega\backslash S(u)} \mathrm{div}(\varphi \frac{\nabla u - \vec{X}^*}{|\nabla u - \vec{X}^*|}) d^{2n}x \\ &\quad - \int_{\Omega\backslash S(u)} \varphi \mathrm{div}(\frac{\nabla u - \vec{X}^*}{|\nabla u - \vec{X}^*|}) d^{2n}x \pm \int_{S(u)} |\nabla\varphi| d^{2n}x \end{aligned} \tag{3.32}$$

(cf. (3.3) in [10]). We remark that the Lebesgue measure of $S(u)$ vanishes for $u \in C^2$ (in this case, compare (3.32) with the first variation formula in [29]) or $C^{1,1}$ while there exists $u \in \cap_{0<\alpha<1} C^{1,\alpha}$ such that $S(u)$ has positive Lebesgue measure according to Balogh ([2]).

Example 3.3. For basic material in this example, the readers are referred to ([8]). Let (M, J, Θ) be a 3-dimensional oriented pseudohermitian manifold. Consider a C^2 smooth orientable surface $\Sigma \subset M$. Let $\xi \equiv \ker\Theta$ denote the contact bundle. Let $e_1 \in T\Sigma \cap \xi$ denote a characteristic vector of unit length with respect to the Levi metric $G = \frac{1}{2} d\Theta(\cdot, J\cdot)$ (at a nonsingular point). Let $e_2 \equiv Je_1$ and T denote the Reeb vector field associated to Θ. Let $\{e^1, e^2, \Theta\}$ denote the coframe field dual to the frame field $\{e_1, e_2, T\}$. The adapted (or left invariant) metric on M is defined by $h = \Theta^2 + G = \Theta^2 + (e^1)^2 + (e^2)^2$ (if restricted on the nonsingular domain). It follows that

$$\tilde{e}_1 = e_1, \tilde{e}_2 = -\frac{\alpha e_2 + T}{\sqrt{1+\alpha^2}}, N = \frac{e_2 - \alpha T}{\sqrt{1+\alpha^2}} \tag{3.33}$$

form an orthonormal basis with respect to h (recall that α is defined so that $\alpha e_2 + T \in T\Sigma$). Denote the projection of the unit normal N onto ξ by N_ξ. Denote the Riemannian area element of Σ induced from h by $d\Sigma$. Let $\tilde{e}^1 = e^1, \tilde{e}^2 = -\frac{\alpha e^2 + \Theta}{\sqrt{1+\alpha^2}}$, and $\tilde{e}^3 = \frac{e^2 - \alpha\Theta}{\sqrt{1+\alpha^2}}$ be the coframe field dual to $\tilde{e}_1, \tilde{e}_2, N$ in (3.33). We have

$$N_\xi = \frac{e_2}{\sqrt{1+\alpha^2}}, d\Sigma = \tilde{e}^1 \wedge \tilde{e}^2 \tag{3.34}$$

(assuming that Σ is oriented so that the second equality in (3.34) holds). Let $|\cdot|_h$ denote the length with respect to the metric h. From (3.34) we can now compute

$$|N_\xi|_h d\Sigma = \frac{1}{\sqrt{1+\alpha^2}} \tilde{e}^1 \wedge \tilde{e}^2$$

$$= \frac{-1}{\sqrt{1+\alpha^2}} e^1 \wedge \frac{\alpha e^2 + \Theta}{\sqrt{1+\alpha^2}}$$
$$= \Theta \wedge e^1 \tag{3.35}$$

(on the nonsingular domain; $= 0$ on the singular set) by noting that $e^1 \wedge e^2 = \alpha e^1 \wedge \Theta$ on Σ. For M being the Heisenberg group, (3.35) was pointed out in [29]. So we learn from (3.35) that the general p-area element can also be viewed as the total variation measure of a TM or ξ-valued measure $N_\xi d\Sigma$ on Σ.

By the way we will compute the first variation formula for variations having support containing the singular set (in [8] we computed it for variations having support away from the singular set). For simplicity we assume that Σ is C^1-smooth, oriented, and $\Sigma \backslash S_\Sigma$ is C^2, where S_Σ denotes the singular set consisting of a C^1-smooth curve. Suppose S_Σ divides Σ into two pieces with boundaries S_Σ^+, S_Σ^- reversely oriented on S_Σ. Let v be a C^∞ smooth vector field of M with support away from $\partial\Sigma$ when restricted to Σ. We write $v = v_1 e_1 + v_2 e_2 + fT$ (in nonsingular region). Compute the variation of the general p-area in the direction v :

$$\begin{aligned} \delta_v \int_\Sigma \Theta \wedge e^1 &= \int_{\Sigma \backslash S_\Sigma} L_v(\Theta \wedge e^1) \\ &= \int_{\Sigma \backslash S_\Sigma} d \circ i_v(\Theta \wedge e^1) + i_v \circ d(\Theta \wedge e^1) \\ &= (\int_{S_\Sigma^+} + \int_{S_\Sigma^-})(fe^1 - v_1\Theta) + \int_{\Sigma \backslash S_\Sigma} (f\alpha - v_2) H \Theta \wedge e^1 \end{aligned} \tag{3.36}$$

by (2.8') in [8], where H denotes the p-mean curvature of Σ. We say that Σ is stationary if $\delta_v \int_\Sigma \Theta \wedge e^1 = 0$ for all v. Then by (3.36) and $\Theta = 0$ on S_Σ, we learn that if Σ is stationary, then $H = 0$ by taking v with support away from S_Σ, and hence there holds

$$\int_{S_\Sigma^+} fe^1 + \int_{S_\Sigma^-} fe^1 = 0. \tag{3.37}$$

Let τ denote the positive unit vector tangent to S_Σ^+. Assume that we can extend e_1 continuously to S_Σ from both sides. Denote the extensions of e_1 and e^1 on S_Σ^+ (S_Σ^-, respectively) by e_1^+ and e_+^1 (e_1^- and e_-^1, respectively).

Then from (3.37) we have

$$0 = e^1_+(\tau) + e^1_-(-\tau) = e^+_1 \cdot \tau - e^-_1 \cdot \tau \tag{3.38}$$

where "$\cdot$" denote the inner product with respect to the adapted metric h or the Levi metric G (note that e^+_1, e^-_1, and τ are all in ξ). (3.38) is the "incident angle = reflected angle" condition on the singular curves for a p-area stationary surface. When Σ is C^2 (including the singular set S_Σ), both "angles" must be 90 degrees, i.e., $e^+_1 \cdot \tau = e^-_1 \cdot \tau = 0$ since $e^+_1 = -e^-_1$ according to (generalized) Proposition 3.5 in [8] (see a remark in Section 7 for generalizing the results in Section 3). We studied condition (3.38) for a (generalized) stationary graph in the Heisenberg group (see Theorem 6.3 in [10]). Ritoré and Rosales ([29]) obtained the same result for a C^2-smooth, oriented (immersed) surface in the Heisenberg group.

4. Second Variation and Proof of Theorem C

Recall that for ε regular we have (cf. (3.16))

$$\mathcal{F}'(\varepsilon) = \frac{d\mathcal{F}(d\mu_\varepsilon)}{d\varepsilon} = \int_X N_\varepsilon \cdot d\nu. \tag{4.1}$$

First from (4.1), we want to compute

$$\lim_{\varepsilon_2 \to \varepsilon_1} \frac{\mathcal{F}'(\varepsilon_2) - \mathcal{F}'(\varepsilon_1)}{\varepsilon_2 - \varepsilon_1} = \lim_{\varepsilon_2 \to \varepsilon_1} \int_X \left(\frac{N_{\varepsilon_2} - N_{\varepsilon_1}}{\varepsilon_2 - \varepsilon_1}\right) \cdot d\nu \tag{4.2}$$

for ε_2, ε_1 regular. From (3.22) we have

$$d\mu_{\varepsilon_2} = ((\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1})|d\mu_{\varepsilon_1}| \tag{4.3}$$

for ε_1 regular. Taking the absolute value (total variation) of both sides in (4.3) gives

$$|d\mu_{\varepsilon_2}| = |(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}||d\mu_{\varepsilon_1}| \tag{4.4}$$

or

$$|d\mu_{\varepsilon_1}| = \frac{1}{|(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}|}|d\mu_{\varepsilon_2}|. \tag{4.5}$$

Note that $(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1} \neq 0$ a.e. $[|d\mu_{\varepsilon_1}|]$ and $[|d\mu_{\varepsilon_2}|]$ as shown below.

Lemma 4.1. *Let ε_2 and ε_1 be regular. Then we have* $(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}\neq 0$ *a.e.* $[|d\mu_{\varepsilon_1}|]$ (*and hence also a.e.* $[|d\mu_{\varepsilon_2}|]$ *by Lemma* 3.2(2)).

Proof. Suppose there is a $|d\mu_{\varepsilon_1}|$-measurable set S such that $|d\mu_{\varepsilon_1}|(S)>0$ while $(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=0$. By (4.4) we have $|d\mu_{\varepsilon_2}|(S)=0$, contradicting $|d\mu_{\varepsilon_1}|\ll|d\mu_{\varepsilon_2}|$ as asserted in Lemma 3.2 (2). □

Substituting the first equality of (3.2) with $\varepsilon=\varepsilon_2$ and (4.5) into (4.3), we get

$$N_{\varepsilon_2}=\frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}. \tag{4.6}$$

From (4.6) we can write

$$N_{\varepsilon_2}-N_{\varepsilon_1}=(I)+(II) \tag{4.7}$$

where

$$\begin{aligned}(I) &= \frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}-[(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}],\\ (II) &= [(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}]-N_{\varepsilon_1}.\end{aligned}$$

So we can estimate

$$\begin{aligned}|(II)| &= |\varepsilon_2-\varepsilon_1||A_{\varepsilon_1}| \quad \text{and}\\ |(I)| &= |\frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}(1-|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|)|\\ &\leq |\varepsilon_2-\varepsilon_1||A_{\varepsilon_1}| \end{aligned} \tag{4.8}$$

by noting that $1=|N_{\varepsilon_1}|$ and making use of the triangle inequality (a.e. for $|d\mu_{\varepsilon_1}|$ and also for $|d\mu_{\varepsilon_2}|$ by Lemma 4.1 (1)). From (4.7) and (4.8) we have

$$|\frac{N_{\varepsilon_2}-N_{\varepsilon_1}}{\varepsilon_2-\varepsilon_1}|\leq 2|A_{\varepsilon_1}|. \tag{4.9}$$

Since $|A_{\varepsilon_1}||d\nu|=|A_{\varepsilon_1}|^2|d\mu_{\varepsilon_1}|$ is integrable by assumption, we can therefore apply the Lebesgue dominated convergence theorem to get

$$\lim_{\varepsilon_2\to\varepsilon_1}\int_X(\frac{N_{\varepsilon_2}-N_{\varepsilon_1}}{\varepsilon_2-\varepsilon_1})\cdot d\nu=\int_X(\lim_{\varepsilon_2\to\varepsilon_1}\frac{N_{\varepsilon_2}-N_{\varepsilon_1}}{\varepsilon_2-\varepsilon_1})\cdot d\nu \tag{4.10}$$

by (4.9). Let

$$f(t) \equiv \frac{tA_{\varepsilon_1} + N_{\varepsilon_1}}{|tA_{\varepsilon_1} + N_{\varepsilon_1}|}.$$

Recall that $tA_{\varepsilon_1} + N_{\varepsilon_1} \neq 0$ a.e. (for $|d\mu_{\varepsilon_1}|$) for $t = \varepsilon_2 - \varepsilon_1$ and 0. A straightforward computation shows that

$$f'(t) = \frac{A_{\varepsilon_1} - (A_{\varepsilon_1} \cdot N_{\varepsilon_1})N_{\varepsilon_1} + t[(A_{\varepsilon_1} \cdot N_{\varepsilon_1})A_{\varepsilon_1} - |A_{\varepsilon_1}|^2 N_{\varepsilon_1}]}{|tA_{\varepsilon_1} + N_{\varepsilon_1}|^3}. \tag{4.11}$$

It follows that

$$f'(t) \cdot A_{\varepsilon_1} = \frac{|A_{\varepsilon_1}|^2 - |(A_{\varepsilon_1} \cdot N_{\varepsilon_1})|^2}{|tA_{\varepsilon_1} + N_{\varepsilon_1}|^3} \geq 0 \tag{4.12}$$

by Cauchy's inequality (noting that $|N_{\varepsilon_1}| = 1$), and

$$\begin{aligned} \lim_{\varepsilon_2 \to \varepsilon_1} \frac{\mathcal{F}'(\varepsilon_2) - \mathcal{F}'(\varepsilon_1)}{\varepsilon_2 - \varepsilon_1} &= \int_X (\lim_{\varepsilon_2 \to \varepsilon_1} \frac{f(\varepsilon_2 - \varepsilon_1) - f(0)}{\varepsilon_2 - \varepsilon_1}) \cdot d\nu \\ &= \int_X f'(0) \cdot A_{\varepsilon_1} |d\mu_{\varepsilon_1}| \\ &= \int_X \{|A_{\varepsilon_1}|^2 - |(A_{\varepsilon_1} \cdot N_{\varepsilon_1})|^2\} |d\mu_{\varepsilon_1}| \geq 0 \end{aligned} \tag{4.13}$$

by (4.2), (4.10), (3.2) (with $d\nu_s^{\varepsilon_1} = 0$), and (4.12) for ε_2, ε_1 regular. We have proved Theorem C (1) (1.11).

Next we are going to prove Theorem C (2). Take arbitrary $\varepsilon_1, \varepsilon_2, \varepsilon_1 \neq \varepsilon_2$. First we want to express $\mathcal{F}'_{\pm}(\varepsilon_2)$ in terms of $|d\mu_{\varepsilon_1}|$ and $|d\nu_s^{\varepsilon_1}|$. Since $|d\mu_{\varepsilon_1}| \perp |d\nu_s^{\varepsilon_1}|$, there exists E_{ε_1} such that $|d\mu_{\varepsilon_1}|$ is concentrated on E_{ε_1} while $|d\nu_s^{\varepsilon_1}|$ is concentrated on $E_{\varepsilon_1}^c := X \backslash E_{\varepsilon_1}$. Moreover, $|d\nu| \ll |d\mu_{\varepsilon_1}|$ on E_{ε_1}. Recall that from (3.22) we have

$$d\mu_{\varepsilon_j} = ((\varepsilon_j - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1})|d\mu_{\varepsilon_1}| + (\varepsilon_j - \varepsilon_1) d\nu_s^{\varepsilon_1} \tag{4.14}$$

for $j = 2, 3$, where $\varepsilon_3 \neq \varepsilon_2$. By (4.14) we compute

$$\begin{aligned} \frac{|d\mu_{\varepsilon_3}| - |d\mu_{\varepsilon_2}|}{\varepsilon_3 - \varepsilon_2} &= \frac{|(\varepsilon_3 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}| - |(\varepsilon_2 - \varepsilon_1)A_{\varepsilon_1} + N_{\varepsilon_1}|}{\varepsilon_3 - \varepsilon_2} |d\mu_{\varepsilon_1}| \\ &\quad + \frac{|\varepsilon_3 - \varepsilon_1| - |\varepsilon_2 - \varepsilon_1|}{\varepsilon_3 - \varepsilon_2} |d\nu_s^{\varepsilon_1}|. \end{aligned} \tag{4.15}$$

Note that on $E_{\varepsilon_1}, N_{\varepsilon_1} \neq 0$ a.e. $[|d\mu_{\varepsilon_1}|]$ and hence both $(\varepsilon_3-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}$ and $(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}$ cannot be zero simultaneously a.e. $[|d\mu_{\varepsilon_1}|]$ since $\varepsilon_3 \neq \varepsilon_2$. Therefore we can write

$$\begin{aligned}
&\frac{|(\varepsilon_3-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|-|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}{\varepsilon_3-\varepsilon_2} \\
&= \frac{((\varepsilon_3-\varepsilon_1)^2-(\varepsilon_2-\varepsilon_1)^2)|A_{\varepsilon_1}|^2+2(\varepsilon_3-\varepsilon_2)A_{\varepsilon_1}\cdot N_{\varepsilon_1}}{(\varepsilon_3-\varepsilon_2)(|(\varepsilon_3-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|+|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|)} \\
&= \frac{((\varepsilon_3+\varepsilon_2-2\varepsilon_1)A_{\varepsilon_1}+2N_{\varepsilon_1})\cdot A_{\varepsilon_1}}{|(\varepsilon_3-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|+|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}. \qquad (4.16)
\end{aligned}$$

Suppose $(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1} \neq 0$ on E_{ε_1}(recall that $|d\mu_{\varepsilon_1}|$ is concentrated on E_{ε_1}). Then it follows from (4.16) that

$$\begin{aligned}
&\lim_{\varepsilon_3\to\varepsilon_2} \frac{|(\varepsilon_3-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|-|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}{\varepsilon_3-\varepsilon_2} \\
&= \frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}\cdot A_{\varepsilon_1}. \qquad (4.17)
\end{aligned}$$

On E_{ε_1}, if $(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=0$, then we have

$$\frac{|d\mu_{\varepsilon_3}|-|d\mu_{\varepsilon_2}|}{\varepsilon_3-\varepsilon_2} = \frac{|\varepsilon_3-\varepsilon_2|}{\varepsilon_3-\varepsilon_2}|A_{\varepsilon_1}||d\mu_{\varepsilon_1}| \qquad (4.18)$$

by observing that $(\varepsilon_3-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=(\varepsilon_3-\varepsilon_2)A_{\varepsilon_1}+(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=(\varepsilon_3-\varepsilon_2)A_{\varepsilon_1}$ in (4.15). Also from (4.15) we have

$$\frac{|d\mu_{\varepsilon_3}|-|d\mu_{\varepsilon_2}|}{\varepsilon_3-\varepsilon_2} = \frac{|\varepsilon_3-\varepsilon_1|-|\varepsilon_2-\varepsilon_1|}{\varepsilon_3-\varepsilon_2}|d\nu_s^{\varepsilon_1}|. \qquad (4.19)$$

on $E_{\varepsilon_1}^c$. Observe that

$$\begin{aligned}
&\left|\frac{|(\varepsilon_3-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|-|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}{\varepsilon_3-\varepsilon_2}\right| \\
&\leq \frac{|((\varepsilon_3-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1})-((\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1})|}{|\varepsilon_3-\varepsilon_2|} \\
&= \frac{|(\varepsilon_3-\varepsilon_2)A_{\varepsilon_1}|}{|\varepsilon_3-\varepsilon_2|} = |A_{\varepsilon_1}| \qquad (4.20)
\end{aligned}$$

by the triangle inequality. Since $d\mu$ and $d\nu$ are bounded by assumption, we obtain that $|d\mu_{\varepsilon_1}|=|d\mu+\varepsilon_1 d\nu|$ is bounded. Note that $|A_{\varepsilon_1}| \in L^1(X, |d\mu_{\varepsilon_1}|)$

since $|d\nu| = |A_{\varepsilon_1}||d\mu_{\varepsilon_1}| + |d\nu_s^{\varepsilon_1}|$ from (3.2) and hence

$$\int_X |A_{\varepsilon_1}||d\mu_{\varepsilon_1}| \leq \int_X |d\nu| < \infty$$

(note that $d\nu$ is bounded by assumption). Now from (4.15), (4.17), (4.18), (4.19), and (4.20), we can apply the Lebesgue dominated convergence theorem to conclude that

$$\begin{aligned}\mathcal{F}'_{\pm}(\varepsilon_2) &:= \lim_{\varepsilon_3\to\varepsilon_2\pm}\int_X \frac{|d\mu_{\varepsilon_3}| - |d\mu_{\varepsilon_2}|}{\varepsilon_3-\varepsilon_2} \\ &= \int_{E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}\neq 0\}} \frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}\cdot A_{\varepsilon_1}|d\mu_{\varepsilon_1}| \\ &\quad + \int_{E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}= 0\}} \pm|A_{\varepsilon_1}||d\mu_{\varepsilon_1}| \\ &\quad + \int_{E^c_{\varepsilon_1}} \frac{|\varepsilon_2-\varepsilon_1|}{\varepsilon_2-\varepsilon_1}|d\nu_s^{\varepsilon_1}|. \end{aligned} \tag{4.21}$$

Comparing (4.21) with (1.8) we obtain

$$\begin{aligned}&\mathcal{F}'_{\pm}(\varepsilon_2) - \mathcal{F}'_{+}(\varepsilon_1) \\ &= \int_{E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}\neq 0\}} \{(\frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|} - N_{\varepsilon_1})\cdot A_{\varepsilon_1}|d\mu_{\varepsilon_1}| \\ &\quad + \int_{E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}= 0\}} (\pm|A_{\varepsilon_1}| - N_{\varepsilon_1}\cdot A_{\varepsilon_1})|d\mu_{\varepsilon_1}|\} \end{aligned} \tag{4.22}$$

for $\varepsilon_2 > \varepsilon_1$ (from which terms involving $|d\nu_s^{\varepsilon_1}|$ cancel). On $E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}\neq 0\}$ there holds

$$|\frac{1}{\varepsilon_2-\varepsilon_1}(\frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|} - N_{\varepsilon_1})\cdot A_{\varepsilon_1}| \leq 2|A_{\varepsilon_1}|^2 \tag{4.23}$$

by the same estimate as in deducing (4.9) (noting that ε_1 and ε_2 are not necessarily regular in the estimate). On $E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}= 0\}$ we have $|\frac{1}{\varepsilon_2-\varepsilon_1}| = |A_{\varepsilon_1}|$ since $|N_{\varepsilon_1}| = 1$. It follows that

$$\begin{aligned}&\frac{\pm|A_{\varepsilon_1}| - N_{\varepsilon_1}\cdot A_{\varepsilon_1}}{\varepsilon_2-\varepsilon_1} \\ &= (\pm|A_{\varepsilon_1}| - N_{\varepsilon_1}\cdot A_{\varepsilon_1})(sgn(\varepsilon_2-\varepsilon_1)|A_{\varepsilon_1}|)\end{aligned}$$

$$
\begin{aligned}
&= \pm sgn(\varepsilon_2-\varepsilon_1)|A_{\varepsilon_1}|^2+|A_{\varepsilon_1}|^2\\
&= 2|A_{\varepsilon_1}|^2 \ \text{ if } \varepsilon_2>(<\text{, resp.})\varepsilon_1 \text{ for the case of } + \ (-\text{, resp.) sign}\\
(&= 0 \text{ if } \varepsilon_2>(<\text{, resp.})\varepsilon_1 \text{ for the case of } -\ (+\text{, resp.) sign}). \qquad (4.24)
\end{aligned}
$$

Here we have used the fact that $N_{\varepsilon_1}\cdot A_{\varepsilon_1}=-(\varepsilon_2-\varepsilon_1)^{-1}=sgn(\varepsilon_1-\varepsilon_2)|A_{\varepsilon_1}|$ on $E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=0\}$ (in which $|\frac{1}{\varepsilon_2-\varepsilon_1}|=|A_{\varepsilon_1}|$). From (4.22) and (4.24), we can write

$$
\begin{aligned}
\frac{\mathcal{F}'_\pm(\varepsilon_2)-\mathcal{F}'_+(\varepsilon_1)}{\varepsilon_2-\varepsilon_1} &= \int_{E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}\neq 0\}} g_{(\varepsilon_1,\varepsilon_2)}|d\mu_{\varepsilon_1}|\\
&\quad+\int_{E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=0\}} h_{(\varepsilon_1)}|d\mu_{\varepsilon_1}| \qquad (4.25)
\end{aligned}
$$

where

$$
g_{(\varepsilon_1,\varepsilon_2)}=\frac{1}{\varepsilon_2-\varepsilon_1}\Big(\frac{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}}{|(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}|}-N_{\varepsilon_1}\Big)\cdot A_{\varepsilon_1}, \qquad (4.26)
$$

and

$$
\begin{aligned}
h_{(\varepsilon_1)} &= 2|A_{\varepsilon_1}|^2 \text{ if } \varepsilon_2>(<\text{, resp.})\varepsilon_1 \text{ in the case of } +\ (-\text{, resp.) sign}\\
(&= 0 \text{ if } \varepsilon_2>(<\text{, resp.})\varepsilon_1 \text{ in the case of } -\ (+\text{, resp.) sign}). \qquad (4.27)
\end{aligned}
$$

From (4.23) and (4.27), we have

$$
\begin{aligned}
|g_{(\varepsilon_1,\varepsilon_2)}| &\le 2|A_{\varepsilon_1}|^2 \text{ on } E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}\neq 0\},\\
|h_{(\varepsilon_1)}| &\le 2|A_{\varepsilon_1}|^2 \text{ on } E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=0\}.
\end{aligned} \qquad (4.28)
$$

Now given a point $p\in E_{\varepsilon_1}$ (modulo a $|d\mu_{\varepsilon_1}|$-measure zero set), there is at most one $\varepsilon_2\neq\varepsilon_1$ such that $(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}(p)+N_{\varepsilon_1}(p)=0$. The reason is that if there are two distinct such ε_2, then $A_{\varepsilon_1}(p)=0$ and hence $N_{\varepsilon_1}(p)=0$. So all such points form a $|d\mu_{\varepsilon_1}|$-measure zero set since $N_{\varepsilon_1}\neq 0$ a.e. $[|d\mu_{\varepsilon_1}|]$. We denote such ε_2 by $\varepsilon_2(p)$. Then for any $\varepsilon,\varepsilon_1<\varepsilon<\varepsilon_2(p)$, there holds $(\varepsilon-\varepsilon_1)A_{\varepsilon_1}(p)+N_{\varepsilon_1}(p)\neq 0$. So we have

$$
\begin{aligned}
\lim_{\varepsilon\to\varepsilon_1+}\tilde{g}_{(\varepsilon_1,\varepsilon)}(p) &= \lim_{\varepsilon\to\varepsilon_1+} g_{(\varepsilon_1,\varepsilon)}(p)\\
&= f'(0+)\cdot A_{\varepsilon_1}(p)\\
&= |A_{\varepsilon_1}(p)|^2-|(A_{\varepsilon_1}\cdot N_{\varepsilon_1})(p)|^2 \qquad (4.29)
\end{aligned}
$$

by (4.11), where $\tilde{g}_{(\varepsilon_1,\varepsilon)}$ is defined on E_{ε_1} as follows:

$$\begin{aligned}\tilde{g}_{(\varepsilon_1,\varepsilon)} &= g_{(\varepsilon_1,\varepsilon)} \text{ on } E_{\varepsilon_1}\cap\{(\varepsilon-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}\neq 0\} \text{ and}\\ &= h_{(\varepsilon_1)} \text{ on } E_{\varepsilon_1}\cap\{(\varepsilon-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}= 0\}.\end{aligned}$$

In view of (4.28), (4.29), and the assumption $|A_{\varepsilon_1}|^2\in L^1(X,|d\mu_{\varepsilon_1}|)$, we can now apply the Lebesgue dominated convergence theorem to compute

$$\begin{aligned}\lim_{\varepsilon\to\varepsilon_1+}\frac{\mathcal{F}'_{\pm}(\varepsilon)-\mathcal{F}'_{+}(\varepsilon_1)}{\varepsilon-\varepsilon_1} &= \lim_{\varepsilon\to\varepsilon_1+}\int_{E_{\varepsilon_1}}\tilde{g}_{(\varepsilon_1,\varepsilon)}|d\mu_{\varepsilon_1}|\\ &= \int_{E_{\varepsilon_1}}(\lim_{\varepsilon\to\varepsilon_1+}\tilde{g}_{(\varepsilon_1,\varepsilon)})|d\mu_{\varepsilon_1}|\\ &= \int_X(|A_{\varepsilon_1}|^2-|(A_{\varepsilon_1}\cdot N_{\varepsilon_1})|^2)|d\mu_{\varepsilon_1}|\geq 0. \quad (4.30)\end{aligned}$$

In the last equality of (4.30), we have used the fact that $|d\mu_{\varepsilon_1}|(X\backslash E_{\varepsilon_1})=0$ since $|d\mu_{\varepsilon_1}|$ is concentrated on E_{ε_1}. Similarly we also have

$$\lim_{\varepsilon\to\varepsilon_1-}\frac{\mathcal{F}'_{\pm}(\varepsilon)-\mathcal{F}'_{-}(\varepsilon_1)}{\varepsilon-\varepsilon_1}=\int_X\{|A_{\varepsilon_1}|^2-|(A_{\varepsilon_1}\cdot N_{\varepsilon_1})|^2\}|d\mu_{\varepsilon_1}|\geq 0.$$

Note that $\frac{|\varepsilon_2-\varepsilon_1|}{\varepsilon_2-\varepsilon_1}|d\nu_s^{\varepsilon_1}|$ in (4.21) cancels in both cases. We have proved Theorem C (2) (1.12).

We remark that for $\varepsilon_2\neq\varepsilon_1$, $|d\nu_s^{\varepsilon_2}|$ is concentrated on $E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=0\}$. Since $|d\nu_s^{\varepsilon_2}|\perp|d\nu_s^{\varepsilon_1}|$ by Lemma 3.1 and $|d\mu_{\varepsilon_1}|\perp|d\nu_s^{\varepsilon_1}|$, we obtain that $|d\nu_s^{\varepsilon_2}|$ is concentrated on E_{ε_1} by Lemma 3.2 (1) or (3.9). Moreover, observing that $|d\nu_s^{\varepsilon_2}|\perp|d\mu_{\varepsilon_2}|$, we conclude that $|d\nu_s^{\varepsilon_2}|$ is concentrated on $E_{\varepsilon_1}\cap\{(\varepsilon_2-\varepsilon_1)A_{\varepsilon_1}+N_{\varepsilon_1}=0\}$ in view of (3.22).

Example 4.1. Continue the discussion in Example 3.1. Since every ε is regular in this case, by Theorem C (1) and (3.29) we have

$$\begin{aligned}\mathcal{F}''(0) &= \int_X\{|A_0|^2-|(A_0\cdot N_0)|^2\}|d\mu|\\ &= \int_X\frac{|\nabla v|^2+(|\nabla v|^2|\nabla u|^2-|(\nabla v\cdot\nabla u)|^2)}{(1+|\nabla u|^2)^{3/2}}d^mx\geq 0\end{aligned}$$

by Cauchy's inequality. This implies that a Riemannian minimal graph in R^{m+1} over $X\subset R^m$ has the local area-minimizing property.

Example 4.2. Continue the discussion in Example 3.2. Suppose that u and φ are in $C^{1,1}$ or C^2. Then the singular set of the graph defined by $u+\varepsilon\varphi$ has vanishing Lebesgue measure in R^{2n} according to [2]. It follows that $d\nu_s^{\varepsilon}=0$, and hence each ε is regular in this situation. By Theorem C (1) and (3.31) we have

$$\mathcal{F}''(0)=\int_{\Omega\setminus S(u)}\frac{|\nabla u-\vec{X}^*|^2|\nabla\varphi|^2-|(\nabla u-\vec{X}^*)\cdot\nabla\varphi|^2}{|\nabla u-\vec{X}^*|^3}d^{2n}x\geq 0$$

by Cauchy's inequality again. So a $C^{1,1}$ or C^2 p-area stationary graph in the Heisenberg group over $\Omega\subset R^{2n}$ has the local p- area-minimizing property. This fact was shown by a calibration argument in [8] for the nonsingular case with $n=1$. Later Ritoré and Rosales ([29]) extended the result to the situation having singularities.

Example 4.3. Continue the discussion in Example 3.3. In [8] we computed the second variation of the p-area in the direction of a vector field with support away from the singular set. Here we consider the situation of variations with support containing a singular curve as in Example 3.3. The following computation is based on a private talk given by Hung-Lin Chiu. Recall that $v=v_2e_2+fT$ (take $v_1=0$ for simplicity). Then we follow the argument in [8] to get

$$\begin{aligned}\delta_v^2\int_\Sigma\Theta\wedge e^1 &= \int_{\Sigma\setminus S_\Sigma}L_v^2(\Theta\wedge e^1)\\ &= \int_{\Sigma\setminus S_\Sigma}L_v\{(f\alpha-v_2)H+d(fe^1)\}\Theta\wedge e^1\\ &= \int_{\Sigma\setminus S_\Sigma}-(f\alpha-v_2)^2(e_2H)\Theta\wedge e^1+d\circ L_v(fe^1)\end{aligned}\quad (4.31)$$

for a p-area stationary surface Σ (hence $H=0$ on Σ). We can express the first term of the last integrand in (4.31) in terms of pseudohermitian geometric quantities (see Section 6 in [8]). The second term of the same integrand reflects the contribution of the singular curve S_Σ as shown below. By a direct computation we obtain

$$L_v(fe^1)=(vf+f^2\mathrm{Re}A_{\bar 1}^{1})e^1+(\omega(T)+\mathrm{Im}A_{\bar 1}^{1})f^2e^2-(\omega(T)+\mathrm{Im}A_{\bar 1}^{1})fv_2\Theta \quad (4.32)$$

where $A_{\bar{1}}^{1}$ and ω denote the pseudohermitian torsion and connection form, respectively. Since $\Theta(\tau) = 0$, $e_{+}^{1}(\tau) + e_{-}^{1}(-\tau) = 0$, and $e_{+}^{2}(\tau) = -e_{-}^{2}(\tau)$ by (3.38), from (4.32) we have

$$\begin{aligned}\int_{\Sigma\setminus S_\Sigma} d \circ L_v(fe^1) &= \int_{S_\Sigma^+} (\omega(T) + \mathrm{Im}A_{\bar{1}}^{1}) f^2 e_{+}^{2} + \int_{S_\Sigma^-} (\omega(T) + \mathrm{Im}A_{\bar{1}}^{1}) f^2 e_{-}^{2} \\ &= 2\int_{S_\Sigma^+} (\omega(T) + \mathrm{Im}A_{\bar{1}}^{1}) f^2 e_{+}^{2}(\tau) ds \end{aligned} \qquad (4.33)$$

where s is the unit-speed parameter for S_Σ^+.

Appendix: Generalized Heisenberg Geometry

We will discuss the notions of gradient and hypersurface area in a general formulation unifying Riemannian and pseudohermitian (horizontal or Heisenberg) structures (see, e.g., [33]).

Let M be an $m-$dimensional differentiable manifold with a nonnegative inner product $< \cdot, \cdot >$ on its cotangent bundle T^*M. Namely, $< \cdot, \cdot >$ is a symmetric bilinear form such that $< \omega, \omega > \geq 0$ for any $\omega \in T^*M$. Some authors call such a manifold M subriemannian. Clearly if $< \cdot, \cdot >$ is positive definite, $(M, < \cdot, \cdot >)$ is a Riemannian manifold. For M being the Heisenberg group H_n of dimension $m = 2n + 1$, let

$$\hat{e}_j = \frac{\partial}{\partial x_j} + y_j \frac{\partial}{\partial z}, \hat{e}_{j'} = \frac{\partial}{\partial y_j} - x_j \frac{\partial}{\partial z},$$

$1 \leq j \leq n$ be the left-invariant vector fields on H_n, in which $x_1, y_1, x_2, y_2, \ldots, x_n, y_n$, z denote the coordinates of H_n. The (contact) 1-form $\Theta \equiv dz + \sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$ annihilates $\hat{e}_j' s$ and $\hat{e}_{j'}' s$. We observe that $dx_1, dy_1, dx_2, dy_2, \ldots, dx_n, dy_n, \Theta$ are dual to $\hat{e}_1, \hat{e}_{1'}, \hat{e}_2, \hat{e}_{2'}, \ldots, \hat{e}_n, \hat{e}_{n'}, \frac{\partial}{\partial z}$. Define a nonnegative inner product by

$$\begin{aligned} &< dx_j, dx_k >= \delta_{jk}, < dy_j, dy_k >= \delta_{jk}, < dx_j, dy_k >= 0, \\ &< \Theta, dx_j >=< \Theta, dy_k >=< \Theta, \Theta >= 0. \end{aligned} \qquad (A.1)$$

We can extend the definition of the above nonnegative inner product to the situation of a general pseudohermitian manifold. Take $e_j, e_{j'} = Je_j$, $j = 1, 2, \ldots, n$ to be an orthonormal basis in the kernel of the contact form

Θ with respect to the Levi metric $\frac{1}{2}d\Theta(\cdot, J\cdot)$. Let T be the Reeb vector field of Θ (such that $\Theta(T) = 1$ and $d\Theta(T, \cdot) = 0$). Denote the dual coframe of $e_j, e_{j'}, T$ by $\theta^j, \theta^{j'}$ (and Θ). Now we can replace dx_j, dy_j by $\theta^j, \theta^{j'}$ in (A.1) to define a nonnegative inner product on a general pseudohermitian manifold:

$$\begin{aligned} &< \theta^j, \theta^k >= \delta_{jk}, < \theta^{j'}, \theta^{k'} >= \delta_{jk}, < \theta^j, \theta^{k'} >= 0, \\ &< \Theta, \theta^j >=< \Theta, \theta^{k'} >=< \Theta, \Theta >= 0. \end{aligned} \tag{A.2}$$

We use the same notation $< \cdot, \cdot >$ to denote the pairing between TM and T^*M. Define the bundle morphism $G : T^*M \to TM$ by

$$< G(\omega), \eta >=< \omega, \eta > \tag{A.3}$$

for $\omega, \eta \in T^*M$. In the Riemannian case, G is in fact an isometry. In the pseudohermitian case, $G(T^*M)$ is the contact subbundle ξ of TM, the kernel of Θ. By letting $\eta = \Theta$ in (A.3), we get $G(T^*M) \subset \xi$. On the other hand, it is easy to see that $G(\theta^j) = e_j, G(\theta^{j'}) = e_{j'}$ (and $G(\Theta) = 0$). Since $e_j, e_{j'}$, $j = 1, 2, \ldots, n$ span ξ, we have $\xi \subset G(T^*M)$. For a smooth function φ on M, we define the gradient $\nabla\varphi := G(d\varphi)$. In the pseudohermitian case, this $\nabla\varphi$ is nothing but the subgradient $\nabla_b\varphi := \sum_{j=1}^n \{e_j(\varphi)e_j + e_{j'}(\varphi)e_{j'}\}$.

Let M be a general subriemannian manifold of dimension $n+1$, i.e., an $(n+1)$-dimensional differentiable manifold with a nonnegative inner product $< \cdot, \cdot >$ on its cotangent bundle T^*M. Let φ be a defining function of a hypersurface $\Sigma \subset M$. That is, $\Sigma = \{\varphi = 0\}$. Given a volume form dv_M (independent of $< \cdot, \cdot >$), we can define an area (or volume) element dv_Σ of Σ up to sign by

$$dv_\Sigma = \frac{d\varphi}{|d\varphi|} \rfloor dv_M \tag{A.4}$$

restricted to Σ. Here for $\omega, \eta \in T^*M, |\omega| :=< \omega, \omega >^{1/2}$ and $\omega \rfloor dv_M$ is defined so that

$$\eta \wedge (\omega \rfloor dv_M) =< \eta, \omega > dv_M. \tag{A.5}$$

If we write $dv_M = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{n+1}$ for independent 1-forms ω^j's and $\omega = \lambda_j \omega^j$ (summation convention), then it is straightforward to verify that

$$\omega \rfloor dv_M = \lambda_j < \omega^j, \omega^k > (-1)^{k-1} \omega^1 \wedge \cdots \wedge \hat{\omega}^k \wedge \cdots \wedge \omega^{n+1} \tag{A.6}$$

satisfies $\omega^l \wedge (\omega \rfloor dv_M) =< \omega^l, \omega > dv_M$ for all l and hence (A.5) holds for all η. On the other hand, there is a unique n-form Φ satisfying

$$\eta \wedge \Phi =< \eta, \omega > dv_M \tag{A.7}$$

for all 1-forms η. Suppose there are two n-forms Φ_1, Φ_2 satisfying (A.7). Then it follows that $\eta \wedge (\Phi_1 - \Phi_2) = 0$ for all 1-forms η and hence $\Phi_1 - \Phi_2 = 0$. We have justified the formula (A.6). There is an intrinsic expression for $\omega \rfloor dv_M$ as follows:

$$(\omega \rfloor dv_M)(X_1, \ldots, X_n) = dv_M(G(\omega), X_1, \ldots, X_n).$$

Note that dv_Σ defined by (A.4) is independent of the choice of φ by a positive scalar multiple function, but changes sign if φ is replaced by $-\varphi$. Now we write $dv_M = \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{n+1}$ for independent 1-forms ω^j's and compute

$$\begin{aligned} d\varphi \rfloor dv_M &= d\varphi \rfloor \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^{n+1} \\ &= v_i(\varphi) < \omega^i, \omega^j > (-1)^{j-1} \omega^1 \wedge \cdots \hat{\omega}^j \cdots \wedge \omega^{n+1} \end{aligned} \tag{A.8}$$

by (A.6), where v_i's are tangent vectors dual to ω^j's and $\hat{\omega}^j$ means ω^j deleted. On $\Sigma, d\varphi = v_i(\varphi)\omega^i = 0$. Assuming $v_{n+1}(\varphi) \neq 0$, say, we have

$$\omega^{n+1} = -\frac{1}{v_{n+1}(\varphi)} \sum_{j=1}^{n} v_j(\varphi)\omega^j. \tag{A.9}$$

Substituting (A.9) into (A.8) and noting that $|d\varphi|^2 = v_i(\varphi) < \omega^i, \omega^j > v_j(\varphi)$, we obtain

$$dv_\Sigma = \frac{(-1)^n}{v_{n+1}(\varphi)} |d\varphi| \omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n. \tag{A.10}$$

Example A.1. Suppose Σ is a hypersurface of $M = R^{n+1}$. Take $< \cdot, \cdot >$ and dv_M to be the Euclidean metric and the associated volume form, respectively. Write the defining function $\varphi = z - u(x_1, x_2, \ldots, x_n)$ and $dv_M = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \wedge dz$ where $x_1, x_2, \ldots, x_n, z$ are coordinates of R^{n+1}. It follows that $|d\varphi| = (1 + u_{x^1}^2 + \cdots + u_{x^n}^2)^{1/2}$ and $d\varphi \rfloor dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n \wedge dz = (-1)^n (1 + u_{x^1}^2 + \cdots + u_{x^n}^2) dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ when restricted to Σ. So taking $\omega^j = dx_j$, $j = 1, \ldots, n, \omega^{n+1} = dz$ and noting that $v_{n+1} = \frac{\partial}{\partial z}$, $v_{n+1}(\varphi) = 1$

in (A.10), we have

$$dv_\Sigma = (-1)^n(1 + u_{x^1}^2 + \cdots + u_{x^n}^2)^{1/2} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n.$$

This is the standard area element (up to a sign) for a graph in Euclidean space.

Example A.2. For M being the Heisenberg group of dimension $2n+1$, we take the volume form $dv_M = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n \wedge \Theta$ (the volume form with respect to the left invariant metric). Let $\omega^{2j-1} = dx_j, \omega^{2j} = dy_j$, $1 \leq j \leq n, \omega^{n+1} = \Theta$ while $v_{2j-1} = \hat{e}_j, v_{2j} = \hat{e}_{j'}$,and $v_{2n+1} = \frac{\partial}{\partial z}$. For $\varphi = z - u(x_1, y_1, \ldots, x_n, y_n)$, we compute $v_{2n+1}(\varphi) = 1$ and $|d\varphi|^2 = \sum_{j=1}^n \{\hat{e}_j(\varphi)^2 + \hat{e}_{j'}(\varphi)^2\} = \sum_{j=1}^n \{(u_{x_j} - y_j)^2 + (u_{y_j} + x_j)^2\}$ by (A.1). Substituting these formulas into (A.10) gives

$$dv_\Sigma = [\sum_{j=1}^n \{(u_{x_j} - y_j)^2 + (u_{y_j} + x_j)^2\}]^{1/2} dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$$

(note that $(-1)^{2n} = 1$). This is the standard (p- or H-) area element for a graph in the Heisenberg group.

We can also recover the area element of an intrinsic graph (e.g., [1], [3]) in the Heisenberg group from (A.10). Let us explain this for the 3-dimensional case ($n = 2$).

Example A.3. Take $\omega^1 = dy, \omega^2 = \Theta = dz + xdy - ydx$, and $\omega^3 = dx$ ($x = x_1$, $y = y_1$) (so $v_1 = \hat{e}_{1'}, v_2 = \frac{\partial}{\partial z}$, and $v_3 = \hat{e}_1$). We compute $|d\varphi|^2 = \hat{e}_1(\varphi)^2 + \hat{e}_{1'}(\varphi)^2$ by (A.1) and reduce (A.10) to

$$dv_\Sigma = \sqrt{1 + (\frac{\hat{e}_{1'}(\varphi)}{\hat{e}_1(\varphi)})^2} dy \wedge \Theta. \tag{A.11}$$

An intrinsic graph is parametrized by η, τ as follows: (we have adjusted the normalization constant)

$$x = \phi(\eta, \tau), y = \eta, z = \tau + \eta\phi(\eta, \tau) \tag{A.12}$$

It follows that $\Theta = d\tau + 2\phi d\eta, dy = d\eta$, and hence $dy \wedge \Theta = d\eta \wedge d\tau$ by (A.12). In coordinates (ρ, η, τ) related to (x, y, z) by $x = \rho, y = \eta, z = \tau + \eta\rho$, we can write the defining function $\varphi = \rho - \phi(\eta, \tau)$. By the chain rule we obtain $\frac{\partial}{\partial x} = \frac{\partial}{\partial\rho} - \eta\frac{\partial}{\partial\tau}$, $\frac{\partial}{\partial y} = \frac{\partial}{\partial\eta} - \rho\frac{\partial}{\partial\tau}$, and $\frac{\partial}{\partial z} = \frac{\partial}{\partial\tau}$. It follows that $\hat{e}_1 = \frac{\partial}{\partial\rho}, \hat{e}_{1'} = \frac{\partial}{\partial\eta} - 2\rho\frac{\partial}{\partial\tau}$, and hence $\hat{e}_1(\varphi) = 1$, $\hat{e}_{1'}(\varphi) = -\phi_\eta + 2\rho\phi_\tau$. Substituting these formulas into (A.11)(and noting that $\rho = \phi(\eta,\tau)$ when restricted to Σ), we obtain

$$dv_\Sigma = \sqrt{1 + (\phi_\eta - 2\phi\phi_\tau)^2}d\eta \wedge d\tau \tag{A.13}$$

(e.g., [1], [3]).

Next we consider ω^j's to be a moving coframe such that $v_j(\varphi) = 0$ for $1 \leq j \leq n$, and $v_{n+1}(\varphi) \neq 0$ in (A.10). It follows that $|d\varphi| = |v_{n+1}(\varphi)|\,|\omega^{n+1}|$ and (A.10) is reduced to

$$dv_\Sigma = \pm|\omega^{n+1}|\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n. \tag{A.14}$$

For an (oriented) Riemannian manifold M, we take dv_M to be the associated volume form. Then we can take ω^j's to be an orthonormal basis in (A.14). Hence $|\omega^{n+1}| = 1$ and dv_Σ (up to sign) is nothing but the area form with respect to the induced metric.

Example A.4. For M being a pseudohermitian 3-manifold, let $e_1 \in T\Sigma \cap \xi$ denote the characteristic field on the nonsingular domain in [8]. Let $e_2 \equiv Je_1$ and α denote a function such that $T + \alpha e_2 \in T\Sigma$. Let e^1, e^2 (and Θ) be the coframe dual to e_1, e_2 (and T). We can take $v_1 = e_1, v_2 = (T + \alpha e_2)/\sqrt{1+\alpha^2}$, and $v_3 = (\alpha T - e_2)/\sqrt{1+\alpha^2}$ while $\omega^1 = e^1, \omega^2 = (\Theta + \alpha e^2)/\sqrt{1+\alpha^2}$ and $\omega^3 = (\alpha\Theta - e^2)/\sqrt{1+\alpha^2}$. Note that $\omega^1 \wedge \omega^2 \wedge \omega^3 = e^1 \wedge e^2 \wedge \Theta$ is the standard volume form with respect to the adapted metric $h \equiv \Theta \otimes \Theta + \frac{1}{2}d\Theta(\cdot, J\cdot)$. Observe that e^1, e^2 are orthonormal with respect to the semipositive inner product (A.2) since they are different from $\theta^1, \theta^{1'}$ by an orthogonal transformation. Thus by (A.2) we have

$$|\omega^3|^2 = \frac{< -e^2, -e^2 >}{1+\alpha^2} = \frac{1}{1+\alpha^2} \tag{A.15}$$

and

$$\omega^1 \wedge \omega^2 = e^1 \wedge (\Theta + \alpha e^2)/\sqrt{1+\alpha^2} = \sqrt{1+\alpha^2}e^1 \wedge \Theta \tag{A.16}$$

on Σ by noting that $e^1 \wedge e^2 = \alpha e^1 \wedge \Theta$ on Σ. Substituting (A.15), (A.16) into (A.14) with $n = 2$, we conclude

$$dv_\Sigma = \pm|\omega^3|\omega^1 \wedge \omega^2 = \pm e^1 \wedge \Theta. \tag{A.17}$$

The above expression first appeared in [8].

Example A.5. Let $\pi_\xi^h : TM \to \xi$ denote the projection onto ξ according to the adapted metric h. Then we have ($|\cdot|_h$ denotes the length with respect to h)

$$|\pi_\xi^h(v_3)|_h = |\frac{-e_2}{\sqrt{1+\alpha^2}}|_h = \frac{1}{\sqrt{1+\alpha^2}} = |\omega^3|. \tag{A.18}$$

In view of (A.18), (A.17) and v_3 being a unit normal with respect to h, we obtain

$$dv_\Sigma = \pm|\pi_\xi^h(N)|_h d\Sigma_h \tag{A.19}$$

where N denotes the unit normal (unique up to sign) with respect to h and $d\Sigma_h$ denotes the area element with respect to the metric induced from h. The expression (A.19) appeared in [29] for M being the 3-dimensional Heisenberg group.

Next we are going to deduce a formula for the mean curvature H viewed as the first variation of the area. Recall that in (A.10) and (A.9) φ is a defining function of a hypersurface Σ in a manifold M of dimension $n+1$ and ω^j's are independent 1-forms. We assume further $\omega^{n+1} = 0$ on Σ and

$$d\omega^j = \sum_{k=1}^{n+1} \omega^k \wedge \omega_k^j, j = 1, \ldots, n+1 \tag{A.20}$$

for some 1-forms ω_k^j. Starting from (A.10), we compute

$$\delta_{fv_{n+1}} \int_\Sigma dv_\Sigma = \int_\Sigma L_{fv_{n+1}}\{\frac{(-1)^n}{v_{n+1}(\varphi)}|d\varphi|\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n\} \tag{A.21}$$

where f is a C^∞-smooth function and $L_{fv_{n+1}}$ denotes the Lie derivative in the direction fv_{n+1}. Let $i_X\eta$ denote the interior product of the vector field X and the differential form η. Observe that $i_{fv_{n+1}}(\omega^1 \wedge \omega^2 \wedge \cdots \wedge \omega^n) = 0$ since $\omega^j(v_{n+1}) = 0$ for $1 \le j \le n$. It follows from $L_{fv_{n+1}} = i_{fv_{n+1}} \circ d + d \circ i_{fv_{n+1}}$

that

$$L_{fv_{n+1}}\{\frac{(-1)^n}{v_{n+1}(\varphi)}|d\varphi|\omega^1\wedge\omega^2\wedge\cdots\wedge\omega^n\}$$
$$= i_{fv_{n+1}}\circ d\{\frac{(-1)^n}{v_{n+1}(\varphi)}|d\varphi|\omega^1\wedge\omega^2\wedge\cdots\wedge\omega^n\}. \tag{A.22}$$

Compute

$$\begin{aligned} d(\omega^1\wedge\omega^2\wedge\cdots\wedge\omega^n) &= \sum_{j=1}^{n}(-1)^{j-1}\omega^1\wedge\cdots\wedge d\omega^j\wedge\cdots\wedge\omega^n \\ &= -(\sum_{j=1}^{n}\omega_j^j)\wedge\omega^1\wedge\omega^2\wedge\cdots\wedge\omega^n \\ &\quad +(-1)^n\sum_{j=1}^{n}\omega^1\wedge\cdots\wedge\omega_{n+1}^j\wedge\cdots\wedge\omega^n\wedge\omega^{n+1} \end{aligned} \tag{A.23}$$

where we have used (A.20). We can now obtain from (A.21), (A.22), and (A.23) that

$$\begin{aligned} \delta_{fv_{n+1}}\int_\Sigma dv_\Sigma &= \int_\Sigma f\{v_{n+1}(\frac{|d\varphi|}{v_{n+1}(\varphi)}) \\ &\quad +(-1)^n\frac{|d\varphi|}{v_{n+1}(\varphi)}\sum_{j=1}^{n}(\omega_{n+1}^j(v_j)-\omega_j^j(v_{n+1}))\}\omega^1\wedge\omega^2\wedge\cdots\wedge\omega^n. \end{aligned} \tag{A.24}$$

Here we have used $\omega^{n+1}=0$ on Σ. Comparing (A.24) with (A.10) we obtain the mean curvature

$$H=\mp\{(-1)^n\frac{v_{n+1}(\varphi)}{|d\varphi|}v_{n+1}(\frac{|d\varphi|}{v_{n+1}(\varphi)})+\sum_{j=1}^{n}(\omega_{n+1}^j(v_j)-\omega_j^j(v_{n+1}))\}. \tag{A.25}$$

Example A.6. In the Riemannian case, we can take ω^j,ω_k^j in (A.20) to be an orthonormal coframe and the associated connection forms, resp. such that $\omega^{n+1}=0$ on Σ. So from $<\omega^i,\omega^j>=\delta_{ij}$ and $v_i(\varphi)=0$ for $1\le i\le n$ we have

$$|d\varphi|^2=\sum_{i,j=1}^{n+1}v_i(\varphi)<\omega^i,\omega^j>v_j(\varphi) \ =\sum_{i=1}^{n+1}(v_i(\varphi))^2=(v_{n+1}(\varphi))^2.$$

It follows that

$$\frac{|d\varphi|}{v_{n+1}(\varphi)} = \pm 1. \tag{A.26}$$

On the other hand, the Riemannian connection forms ω_k^j's satisfy the skew-symmetric condition: $\omega_k^j + \omega_j^k = 0$. So we have

$$\omega_j^j = 0 \tag{A.27}$$

Write $\omega_i^{n+1} = h_{ij}\omega^j$ where $h_{ij} = h_{ji}$ (due to $d\omega^{n+1} = 0$ on Σ) are known to be coefficients of the second fundamental form. We then have

$$\sum_{j=1}^{n} \omega_{n+1}^j(v_j) = -\sum_{j=1}^{n} \omega_j^{n+1}(v_j) = -\sum_{j=1}^{n} h_{jj} \tag{A.28}$$

Substituting (A.26), (A.27), and (A.28) into (A.25), we obtain

$$H = \pm \sum_{j=1}^{n} h_{jj}.$$

This verifies the formula (A.25) for the Riemannian situation.

Example A.7. Consider a surface Σ in a pseudohermitian 3-manifold. We will continue to use the notations in Example 3.3. Take $\omega^1 = \Theta, \omega^2 = e^1$, and $\omega^3 = e^2 - \alpha\Theta$. That $\omega^3 = 0$ on Σ follows from $e_1 \in T\Sigma$ and $T + \alpha e_2 \in T\Sigma$. The corresponding dual vectors are $v_1 = T + \alpha e_2, v_2 = e_1$, and $v_3 = e_2$. Since $< \omega^i, \omega^j >= \delta_{ij}$ by (A.2) and $v_i(\varphi) = 0$ for $i = 1, 2$, we still have (A.26) with $n + 1 = 3$. Here φ is a defining function of Σ. From the structure equations (A.1r), (A.3r) in [8], we can take

$$\begin{aligned} &\omega_1^1 = 0, \omega_2^1 = 2e^2, \omega_3^1 = 0 \\ &\omega_2^2 = 0, \omega_3^2 = -\omega \end{aligned} \tag{A.29}$$

in (A.20), where $i\omega$ is the pseudohermitian connection form. Now by (A.25), (A.26) and (A.29), we have

$$H = \mp(0 + \sum_{j=1}^{2}(\omega_3^j(v_j) - \omega_j^j(v_3)) = \mp\omega_3^2(e_1) = \pm\omega(e_1).$$

This is an expression of the p-mean curvature in [8].

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