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ORIENTED CIRCUIT DOUBLE COVER AND CIRCULAR FLOW AND COLOURING

BY

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Abstract

For a set $\mathcal{C}$ of directed circuits of a graph G that form an oriented circuit double cover, we denote by $I_{\mathcal{C}}$ the graph with vertex set $\mathcal{C}$, in which two circuits C and C' are connected by k edges if $|\underline{C} \cap \underline{C'}| = k$. Let $\Phi_c^*(G) = \min\chi_c(I_{\mathcal{C}})$, where the minimum is taken over all the oriented circuit double covers of G. It is easy to show that for any graph G, $\Phi_c(G) \leq \Phi_c^*(G)$. On the other hand, it follows from well-known results that for any integer $2 \leq k \leq 4$, $\Phi_c^*(G) \leq k$ if and only if $\Phi_c(G) \leq k$; for any integer $k \geq 1$, $\Phi_c^*(G) \leq 2 + \frac{1}{k}$ if and only if $\Phi_c(G) \leq 2 + \frac{1}{k}$. This papers proves that for any rational number $2 \leq r \leq 5$ there exists a graph G for which $\Phi_c^*(G) = \Phi_c(G) = r$. We also show that there are graphs G for which $\Phi_c(G) < \Phi_c^*(G)$.

1. Introduction

Graphs in this paper may have parallel edges but no loops. Suppose $G = (V, E)$ is a bridgeless graph. Replace each edge e of G by two opposite arcs, we obtain a symmetric digraph, whose arc set is denoted by $D(G)$. For an arc x in $D(G)$, denote by x^{-1} the opposite arc of x (i.e., x and x^{-1} correspond to the same edge of G). For a subset S of $D(G)$, let $S^{-1} = \{x^{-1} : x \in S\}$. A subset S of $D(G)$ is called *asymmetric* if $S \cap S^{-1} = \emptyset$. For an

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asymmetric subset S of $D(G)$, denote by $\underline{S}$ the set of edges obtained from S by omitting the orientations.

For each subset A of V, denote by $\partial^+(A)$ the set of arcs with heads in A and tails in $V - A$. A *cut* of G is a set X of arcs in $D(G)$ such that $X = \partial^+_G(A)$ for some $A \subset V$. A *bond* of G is a minimal cut of G. A *directed circuit* of G is a subset C of $D(G)$ such that $C \cap C^{-1} = \emptyset$ and C induces a connected digraph (also denoted by C) with $d^+_C(v) = d^-_C(v) = 1$ for each vertex $v \in V(C)$. Unless explicitly specified, a circuit means a directed circuit. A *cycle* C is the union of a set of circuits $C_1, C_2, \cdots, C_k$ such that $\underline{C_i} \cap \underline{C_j} = \emptyset$ for $i \neq j$. Here k could be 0, in which case $C = \emptyset$. An *oriented circuit double cover* (respectively, an *oriented cut double cover*) of G is a family $\mathcal{C}$ of circuits (respectively, a family $\mathcal{B}$ of bonds) which form a partition of $D(G)$. Note that cuts and cycles of G are asymmetric subsets of $D(G)$.

A *chain* in G is a mapping $f : D(G) \to R$ such that $f(x^{-1}) = -f(x)$ for all $x \in D(G)$. If f is a chain in G and S is a subset of $D(G)$, then $f(S) = \sum_{x \in S} f(x)$. A *flow* in G is a chain f such that for each cut X, $f(X) = 0$. A flow f in G is called an r-flow if for every arc $x \in D(G)$, $1 \leq |f(x)| \leq r - 1$. The *circular flow number* $\Phi_c(G)$ of G is the least r for which G admits an r-flow.

A *tension* in G is a chain f such that for each circuit C, $f(C) = 0$. An *r-tension* is a tension f such that $1 \leq |f(x)| \leq r - 1$ for all $x \in D(G)$. The *circular chromatic number* $\chi_c(G)$ is the least r such that G admits an r-tension.

As circuit and cut are dual concepts, tension and flow, and consequently, the circular chromatic number and the circular flow number, are dual concepts. In particular, for planar graphs G, $\chi_c(G) = \Phi_c(G^*)$, where G^* is the geometrical dual of G.

It is well-known [24] and not difficult to prove that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, and $\Phi(G) - 1 < \Phi_c(G) \leq \Phi(G)$, where $\chi(G)$ is the *chromatic number* of G, and $\Phi(G)$ is the *flow number* of G, i.e., the least positive integer k such that G admits a nowhere zero k-flow. So $\chi_c(G)$ is a refinement of $\chi(G)$, and $\Phi_c(G)$ is a refinement of $\Phi(G)$.

For a graph G, the circular chromatic number $\chi_c(G)$ of G has an alternate (and more commonly used) definition. Given a real number $r \geq 1$, an r-colouring of G is a mapping $c : V \to [0, r)$ such that for each edge $e = uv$ of G, $|c(u) - c(v)|_r \geq 1$, here $|a|_r = \min\{|a|, r - |a|\}$. The circular chromatic number of G is $\chi_c(G) = \inf\{r : G$ admits an r-colouring $\}$. It is known [24] that the infimum in the definition is always attained. If G is finite then $\chi_c(G) = p/q$ is a rational number. In case $r = p/q$ is a rational number then the existence of an r-colouring of G is equivalent to the existence of a mapping $f : V \to \{0, 1, \cdots, p-1\}$ such that for each edge $e = uv$, $q \leq |f(u) - f(v)| \leq p - q$. Such a mapping is called a (p, q)-colouring of G.

There is no dual concept for the concept of a vertex of a graph. By observing that the set of vertices corresponds to a family of cuts that form an oriented cut double cover, we give an alternate definition of the circular chromatic number of a graph in terms of cuts. This definition has a natural dual form. First we define the concept of intersection graph. Suppose $\mathcal{X}$ is a family of asymmetric subsets of $D(G)$. The *intersection graph induced by* $\mathcal{X}$, denoted by $I_{\mathcal{X}}$, has vertex set $\mathcal{X}$ in which two vertices A and B are connected by $k(A, B)$ parallel edges, where $k(A, B) = |\underline{A} \cap \underline{B}|$. We write $A \sim B$ if $k(A, B) \neq 0$, i.e., A and B are connected by at least one edge.

For a graph G, let $\chi_c^*(G) = \min\{\chi_c(I_{\mathcal{X}}) : \mathcal{X}$ is an oriented cut double cover of $G\}$.

Lemma 1. *For any graph G, $\chi_c(G) = \chi_c^*(G)$.*

Proof. Let $\mathcal{X} = \{\partial^+(v) : v \in V(G)\}$. Then $\mathcal{X}$ is an oriented cut double cover of G and $I_{\mathcal{X}}$ is isomorphic to G. So $\chi_c(G) \geq \chi_c^*(G)$. On the other hand, if $\mathcal{X}$ is an oriented cut double cover of G and c is a (p, q)-colouring of $I_{\mathcal{X}}$, then we can obtain a p/q-tension of G as follows: for each arc x of $D(G)$, let B and B' be the cuts in $\mathcal{X}$ containing x and x^{-1}, respectively. Let $f(x) = (c(B) - c(B'))/q$. As c is a (p, q)-colouring of $I_{\mathcal{X}}$, we know that $1 \leq |f(x)| \leq p/q - 1$. Observe that

$$f(x) = \frac{1}{q}\left(\sum_{B \in \mathcal{B}, x \in B} c(B) - \sum_{B \in \mathcal{B}, x^{-1} \in B} c(B)\right).$$

For any circuit C of G, for each cut B in $\mathcal{X}$, $|C \cap B| = |C^{-1} \cap B|$. So

$$f(C) = \frac{1}{q} \sum_{B \in \mathcal{B}} (\sum_{x \in C \cap B} c(B) - \sum_{x^{-1} \in C \cap B} c(B)) = 0.$$

Hence f is a p/q-tension of G. Therefore $\chi_c(G) \leq \chi_c^*(G)$, and hence equality holds. $\square$

For those graphs that have an oriented circuit double cover, the parameter $\chi_c^*(G)$ has a natural dual form. Suppose $\mathcal{C}$ is an oriented circuit double cover of G. Let $I_{\mathcal{C}}$ be the intersection graph induced by $\mathcal{C}$. In case G is a planar graph and $\mathcal{C}$ is the set of facial circuits of G, then $I_{\mathcal{C}}$ is just the planar dual of G. In this case, we have $\Phi_c(G) = \chi_c(I_{\mathcal{C}})$. Thus for any graph G which has an oriented circuit double, we define

$$\Phi_c^*(G) = \min\{\chi_c(I_{\mathcal{C}}) : \mathcal{C} \text{ is an oriented circuit double cover of } G\}.$$

If the oriented circuit double cover conjecture is true, this definition applies to all bridgeless graphs G. In other words, $\Phi_c^*(G)$ is defined if and only if $\Phi_c(G)$ is defined.

Lemma 2. *For any graph G which has an oriented circuit double cover, $\Phi_c(G) \leq \Phi_c^*(G)$.*

Proof. Let $\mathcal{C}$ be an oriented circuit double cover of G with $\chi_c(I_{\mathcal{C}}) = \Phi_c^*(G) = p/q$. Let c be a (p,q)-colouring of $I_{\mathcal{C}}$. For each arc x of $D(G)$, let C and C' be the circuits containing x and x^{-1}, respectively. Let $f(x) = (c(C) - c(C'))/q$. Similarly as in the proof of Lemma 1, f is a p/q-flow of G. So $\Phi_c(G) \leq \Phi_c^*(G)$. $\square$

Compare to Lemma 1, a natural question is whether or not the equality $\Phi_c(G) = \Phi_c^*(G)$ holds for all graphs G for which $\Phi_c^*(G)$ are defined. In other words, the question is as follows:

Question 1. *Suppose G is a bridgeless graph which has an oriented circuit double cover. Suppose $\Phi_c(G) \leq p/q$. Is it true that there exists an oriented circuit double cover $\mathcal{C}$ of G with $\chi_c(I_{\mathcal{C}}) \leq p/q$?*

If $q = 1$, i.e., $p/q = p$ is an integer, then we are considering flow number (instead of circular flow number) and circuit double cover. There seems to be a mysterious connection between flow and circuit double cover. A *k-cycle double cover* is a circuit double cover in which the circuits can be partitioned into k parts, each part is a cycle (i.e., edge disjoint union of circuits). Equivalently, a k-cycle double cover is a circuit double cover $\mathcal{C}$

with $\chi(I_{\mathcal{C}}) \leq k$. It was proved by Tutte [16] that a graph admits a nowhere zero 3-flow if and only if G has an oriented 3-cycle double cover, and proved by Jaeger [5] and Archdeacon [1] that G admits a nowhere zero 4-flow if and only if G has an oriented 4-cycle double cover. Of course, a graph G admits a nowhere zero 2-flow if and only if G itself is a cycle, and hence has an oriented 2-cycle double cover. In other words, for $k = 2, 3, 4$, $\Phi_c(G) \leq k$ if and only if $\Phi_c^*(G) \leq k$. For $k = 5$, the question remains open. However, it was conjectured by Tutte [17] that every bridgeless graph G admits a nowhere zero 5-flow, and conjectured by Archdeacon [1] and Jaeger [6] that every bridgeless graph has an oriented 5-cycle double cover. If both conjectures are true, then we would have $\Phi_c(G) \leq k$ if and only if $\Phi_c^*(G) \leq k$ for any integer k.

It was proved by Jaeger [6] that a graph G admits an integer flow f with $|f(e)| \in \{k, 2k+1\}$ if and only if G has an oriented cycle double cover $\mathcal{C}$ consisting of $2k+1$ cycles $C_0, C_1, \cdots, C_{2k}$ such that $C_i \cap C_j \neq \emptyset$ only if $|i-j| = 1$. This is equivalent to say that $\Phi_c(G) \leq (2k+1)/k$ if and only if $\Phi_c^*(G) \leq (2k+1)/k$.

A common feature of these results and conjectures is to assert that for some rational numbers r, $\Phi_c^*(G) \leq r$ if and only if $\Phi_c(G) \leq r$. As the inequality $\Phi_c(G) \leq \Phi_c^*(G)$ always hold, this implies that for these rational numbers r, the answer to Question 1 is "yes", i.e., $\Phi_c^*(G) = r$ if and only if $\Phi_c(G) = r$. Is this true for every rational number? If not, then for which rational numbers, such equivalence exists?

As mentioned above, if G is a planar graph, then $\Phi_c^*(G) = \Phi_c(G)$. Instead of embedding a graph on the plane (or sphere), one may embed a graph on other orientable surfaces. An embedding of a graph G in a surface Σ is called a 2-cell embedding if each face of the embedded graph is homeomorphic to the open unit disk. A 2-cell embedding of a connected graph in a surface Σ is *circular* if the boundary of each face is a circuit. If G has a circular 2-cell embedding in an orientable surface Σ, then the faces $\mathcal{F}$ of G form an oriented circuit double cover of G. The graph $I_{\mathcal{F}}$ is the surface dual of G. It is known [16] that if a circular 2-cell embedding of G in an orientable surface is k-face colourable, then $\Phi(G) \leq k$. The same proof shows that if such an embedding is circular r-face colourable, then $\Phi_c(G) \leq r$. The converse is not true. There are graphs G with $\Phi_c(G) \leq r$

such that no circular 2-cell embedding of G in an orientable surface which is circular r-colourable.

Our question is of similar flavor, but is different. Instead of considering all possible circular 2-cell embedding of G in an orientable surface, we consider all possible oriented circuit double covers of G. The latter corresponds to consider all possible circular 2-cell embedding of G in an orientable 'pseudo' face or 'pinched' face, a topological space obtained from an orientable surface by repeatedly identifying some points into a single point (that is why it is called a pinched surface). So we are choosing among a wider range of embeddings. However, the answer is still negative. In this paper, we shall prove that for each rational number $2 \le p/q \le 5$ there is a graph G with $\Phi_c(G) = \Phi_c^*(G) = p/q$. On the other hand, there are graphs G for which $\Phi_c^*(G) > \Phi_c(G)$.

2. Edge Rooted Graphs and Series-Parallel Joins of Graphs

Let $G = (V, E)$ be a multi-graph and $\mathcal{C}$ an oriented circuit double cover of G. Let $I_{\mathcal{C}}$ be the intersection graph induced by $\mathcal{C}$. Note that each edge of G corresponds to an edge of $I_{\mathcal{C}}$. We shall denote by e^* the edge in $I_{\mathcal{C}}$ corresponding to the edge e in G.

Let e be an edge of G. The pair (G, e) is called a *rooted graph*, or a graph with *root edge* e. Then $(I_{\mathcal{C}}, e^*)$ is called the *rooted dual graph* (induced by $\mathcal{C}$).

Suppose (G, e) and (G', e') are vertex disjoint rooted graphs, and $e = uv$, $e' = u'v'$.

- The *parallel join* $P((G, e), (G', e'))$ of (G, e) and (G', e') is the rooted graph (G'', e''), where G'' is obtained from the disjoint union of G and G' by deleting e, e', identifying u and u' into a vertex u'', identifying v and v' into a vertex v'', and adding an edge joining u'' and v''. The root edge e'' of G'' is the edge joining u'' and v''.
- The *series join* $S((G, e), (G', e'))$ of (G, e) and (G', e') is the rooted graph (G'', e''), where G'' is obtained from the disjoint union of G and G' by deleting the edges e, e', identifying u and u', one adding an edge joining v and v'. The root edge e'' is the edge joining v and v'.

Note that given (G,e) and (G',e'), the graphs $P((G,e),(G',e'))$ and $S((G,e),(G',e'))$ are not uniquely determined. In $P((G,e),(G',e'))$, instead of identifying u and u', v with v', one could also identify u with v', v with u'. However, for our purpose, the difference between the resulting graphs is insignificant. Indeed, no matter which corresponding vertices are identified, they have isomorphic cyclic matroids, and hence have the same circular flow number as well as the same circular chromatic number.

For simplicity, when the roots are clear from the context, we may simply write $S(G,G')$ and $P(G,G')$. It should be understood that $S(G,G')$ and $P(G,G')$ are also rooted graphs, where the root edges are as specified in the previous paragraph.

Given an oriented circuit double cover $\mathcal{C}$ of G and an oriented circuit double cover $\mathcal{C}'$ of G', one obtain an oriented circuit double cover of $S(G,G')$ and an oriented circuit double cover of $P(G,G')$ as follows: Suppose $e=uv$ and $e'=u'v'$ and $S(G,G')$ and $P(G,G')$ are defined as before. We may assume that $A\in\mathcal{C}$ contains the arc uv, $B\in\mathcal{C}$ contains vu, $A'\in\mathcal{C}'$ contains $v'u'$ and $B'\in\mathcal{C}'$ contains $u'v'$. Let X be the union of $A-e$, $A'-e'$ and $v'v$, Y be the union of $B-e$, $B'-e'$ and vv'. Then $(\mathcal{C}\cup\mathcal{C}'-\{A,B,A',B'\})\cup\{X,Y\}$ is an oriented circuit double cover of $S(G,G')$. Let X' be the union of $A-e$ and $A'-e'$, Y' be the union of $B-e$ and $v''u''$, Z be the union of $B'-e'$ and $u''v''$. Then $(\mathcal{C}\cup\mathcal{C}'-\{A,B,A',B'\})\cup\{X',Y',Z\}$ is an oriented circuit double cover of $P(G,G')$. We shall denote by $S(\mathcal{C},\mathcal{C}')$ and $P(\mathcal{C},\mathcal{C}')$ the above constructed oriented circuit double covers of $S(G,G')$ and $P(G,G')$, respectively.

Lemma 3. *Suppose G,G' are rooted graphs, and $\mathcal{C},\mathcal{C}'$ are oriented circuit double covers of G and G', respectively. Then $S(I_{\mathcal{C}},I_{\mathcal{C}'})$ is isomorphic to $I_{P(\mathcal{C},\mathcal{C}')}$ and $P(I_{\mathcal{C}},I_{\mathcal{C}'})$ is isomorphic to $I_{S(\mathcal{C},\mathcal{C}')}$.*

Proof. This is just the duality of the operations of series join and parallel join. Compare the graphs $S(I_{\mathcal{C}},I_{\mathcal{C}'})$ and $I_{P(\mathcal{C},\mathcal{C}')}$. When we form the circuit cover $P(\mathcal{C},\mathcal{C}')$ of $P(G,G')$, we take the union of A and A' to obtain a new circuit X'. This corresponds to the operation of identifying the two vertices A,A' into a single vertex in $S(I_{\mathcal{C}},I_{\mathcal{C}'})$. The circuits B,B' are replaced by Y' and Z, which are actually the same as B,B', except the edge e is replaced by a new edge which is common to Y' and Z. This corresponds to the operation of deleting the root edges of $I_{\mathcal{C}}$ and $I_{\mathcal{C}'}$ and adding an edge joining the two vertices B and B' in $S(I_{\mathcal{C}}, I_{\mathcal{C}'})$. The other half of the lemma is proved analogously. $\square$

Suppose (G, e) is a rooted graph and $r \geq 2$ is a real number. For a real number a, let $a \pmod r$ be the unique real number a' such that $0 \leq a' < r$ and $a - a'$ is a multiple of r. Let x be any of the two arcs corresponding to e. We introduce a few sets as follows:

- $\mathcal{F}_r(G, e)$ is the set of flows f in G such that for each edge $e' \neq e$, $1 \leq |f(e')| \leq r - 1$.
- $\mathcal{T}_r(G, e)$ the set of tensions f in G such that for each edge $e' \neq e$, $1 \leq |f(e')| \leq r - 1$.
- $L_r^F(G, e) = \{f(x) \pmod r : f \in \mathcal{F}_r(G, e)\}$.
- $L_r^T(G, e) = \{f(x) \pmod r : f \in \mathcal{T}_r(G, e)\}$.

If u, v are the end vertices of e, then $x = uv$ or $x = vu$ does not make any difference in the definition above. This is so because $f(uv) = -f(vu)$ and for any $f \in \mathcal{F}_r(G, e)$, $-f \in \mathcal{F}_r(G, e)$; for any $f \in \mathcal{T}_r(G, e)$, $-f \in \mathcal{T}_r(G, e)$.

Note that $\mathcal{F}_r(G, e)$ (respectively, $\mathcal{T}_r(G, e)$) could be empty, in which case $L_r^F(G, e)$ (respectively, $L_r^T(G, e)$) is also empty. Indeed, it is easy to see that $L_r^F(G, e) \neq \emptyset$ if and only if $\Phi_c(G/e) \leq r$, $0 \in L_r^F(G, e)$ if and only if $\Phi_c(G - e) \leq r$, and $L_r^F(G, e) \cap [1, r-1] \neq \emptyset$ if and only if $\Phi_c(G) \leq r$. Here G/e is the graph obtained from G by contracting e, and $G - e$ is obtained from G by deleting e. Similarly, $L_r^T(G, e) \neq \emptyset$ if and only if $\chi_c(G - e) \leq r$, $0 \in L_r^T(G, e)$ if and only if $\chi_c(G/e) \leq r$, and $L_r^T(G, e) \cap [1, r-1] \neq \emptyset$ if and only if $\chi_c(G) \leq r$.

Suppose A and B are subsets of $[0, r)$. Let $A + B = \{a + b \pmod r : a \in A, b \in B\}$. We view the interval $[0, r)$ as forming a circle, by identifying 0 and r. For $a, b \in [0, r)$, we denote by $[a, b]_r$ the interval from a to b in this circle, along the increasing direction. To be precise, if $a < b$, then $[a, b]_r = \{x : a \leq x \leq b\}$, if $a > b$, then $[a, b]_r = \{x : a \leq x < r\} \cup \{x : 0 \leq x \leq b\}$. As a convention, we denote by $[a, a]_r$ the set consisting the single element a and denote by $[0, r]_r$ the set of all real number $0 \leq x < r$. The *length* of an interval $[a, b]_r$ is equal to $b - a$ if $a \leq b$ and equal to $r - a + b$ if $a > b$.

The following lemmas are easy (and an equivalent form appeared in [11, 12]).

Lemma 4. *Suppose* (G, e) *and* (G', e') *are rooted graphs and* $r \geq 2$ *is a real number. Then*

$$L_r^F(S((G,e),(G',e'))) = L_r^F(G,e) \cap L_r^F(G',e')$$

and

$$L_r^F(P((G,e),(G',e'))) = L_r^F(G,e) + L_r^F(G',e').$$

$$L_r^T(S((G,e),(G',e'))) = L_r^T(G,e) + L_r^T(G',e')$$

and

$$L_r^T(P((G,e),(G',e'))) = L_r^T(G,e) \cap L_r^T(G',e').$$

Proof. Assume $e = uv$, $e' = u'v'$. If $f \in \mathcal{F}_r(G,e)$ is a flow with $f(uv) = t$ and $g \in \mathcal{F}_r(G',e')$ is a flow with $g(v'u') = t$, then the mapping h defined as $h(z) = f(z)$ if $z \in D(G) - \{uv, vu\}$, $h(z) = g(z)$ if $z \in D(G') - \{u'v', v'u'\}$ and $h(v'v) = f(uv) = g(v'u') = t$ is flow in $\mathcal{F}_r(S(G,G'),e'')$. Conversely, from a flow $h \in \mathcal{F}_r(S(G,G'),e'')$ is a flow with $h(v'v) = t$, one obtains a flow $f \in \mathcal{F}_r(G,e)$ with $f(uv) = t$ and $g \in \mathcal{F}_r(G',e')$ with $g(v'u') = t$, which are the restrictions of h to $G - e$ and $G' - e'$, respectively, plus the obvious definition of flows on the arcs of e and e'. Therefore

$$L_r^F(S((G,e),(G',e'))) = L_r^F(G,e) \cap L_r^F(G',e').$$

The other equalities are proved similarly. $\square$

The following lemma is easy and well-known [11, 8].

Lemma 5. *Suppose* $A = [a,b]_r$ *and* $B = [c,d]_r$. *If the sum of the lengths of* A *and* B *is less than* r, *then* $A + B = [a+c, b+d]_r$ *(here the addition is modulo* r*). If one of* A, B *is an empty set, then* $A + B = \emptyset$. *Otherwise,* $A + B = [0, r]_r$.

We call a rooted dual $(I_{\mathcal{C}}, e^*)$ of (G,e) *perfect* if for any $2 \leq r$, $L_r^F(G,e) = L_r^T(I_{\mathcal{C}}, e^*)$. When the roots are clear from the context or is of no significance, we simply say that $I_{\mathcal{C}}$ is a perfect dual of G. It is easy to verify that for a planar graph G, the facial circuits form an oriented circuit double cover which induces a perfect dual with any edge as the root edge. Thus we have the following lemma.

Lemma 6. *If G is a planar graph then for any edge e of G, (G, e) has a perfect dual.*

Lemma 7. *For any oriented circuit double cover $\mathcal{C}$ of G, for any edge e of G, for any $r \geq 2$, $L_r^T(I_{\mathcal{C}}, e^*) \subseteq L_r^F(G, e)$.*

Proof. Suppose $\mathcal{C}$ is an oriented circuit double cover of G. Let e be an edge of G and e^* the corresponding edge in $I_{\mathcal{C}}$. For any tension $g \in L_r^T(I_{\mathcal{C}}, e^*)$, we obtain a flow $f \in L_r^F(G, e)$ as follows: For each circuit $C \in \mathcal{C}$, let φ_C be the characteristic flow on C, i.e., $\varphi_C(x) = 1$ if x is an arc of C, $\varphi_C(x) = -1$ if x^{-1} is an arc of C, and $\varphi_C(x) = 0$ otherwise. Let $f = \sum_{C \in \mathcal{C}} g(C)\varphi_C$. Then it is straightforward to verify that $f \in L_r^F(G, e)$ and $f(e) = g(e^*)$. □

Lemma 8. *If G has a perfect dual , then $\Phi_c^*(G) = \Phi_c(G)$.*

Proof. By definition, $\Phi_c(G) = \min\{r : L_r^F(G, e) \cap [1, r-1] \neq \emptyset\}$. Let $I_{\mathcal{C}}$ be a perfect dual of G. Then $\chi_c(I_{\mathcal{C}}) = \min\{r : L_r^T(I_{\mathcal{C}}, e^*) \cap [1, r-1] \neq \emptyset\}$. As $L_r^F(G, e) = L_r^T(I_{\mathcal{C}}, e^*)$ for any $r \geq 2$, we conclude that $\chi_c(I_{\mathcal{C}}) = \Phi_c(G)$. Hence $\Phi_c^*(G) \leq \chi_c(I_{\mathcal{C}}) = \Phi_c(G)$. As $\Phi_c^*(G) \geq \Phi_c(G)$ for any graph G, the equality holds. □

Lemma 9. *If $I_{\mathcal{C}}$ is a perfect dual of G, $I_{\mathcal{C}'}$ is a perfect dual of G', then $G_{P(\mathcal{C},\mathcal{C}')}$ is a perfect dual of $S(G, G')$ and $G_{S(\mathcal{C},\mathcal{C}')}$ is a perfect dual of $P(G, G')$.*

Proof. This follows easily from Lemma 4 and Lemma 3. □

3. Graphs G with $\Phi_c(G) = \Phi_c^*(G)$

Let $\mathcal{C}$ be the oriented circuit double cover of the Petersen graph G as shown in Figure 1 below. Then $I_{\mathcal{C}}$ is a copy of K_5 with a few edges duplicated.

Let e be an edge of G as depicted in Figure 1 such that e^* is an edge of K_5 with no parallel edges, i.e., the two circuits containing e intersect at e only. Assume $e' = xy$ is an edge of K_5. It is easy to verify that for any $r < 4$,

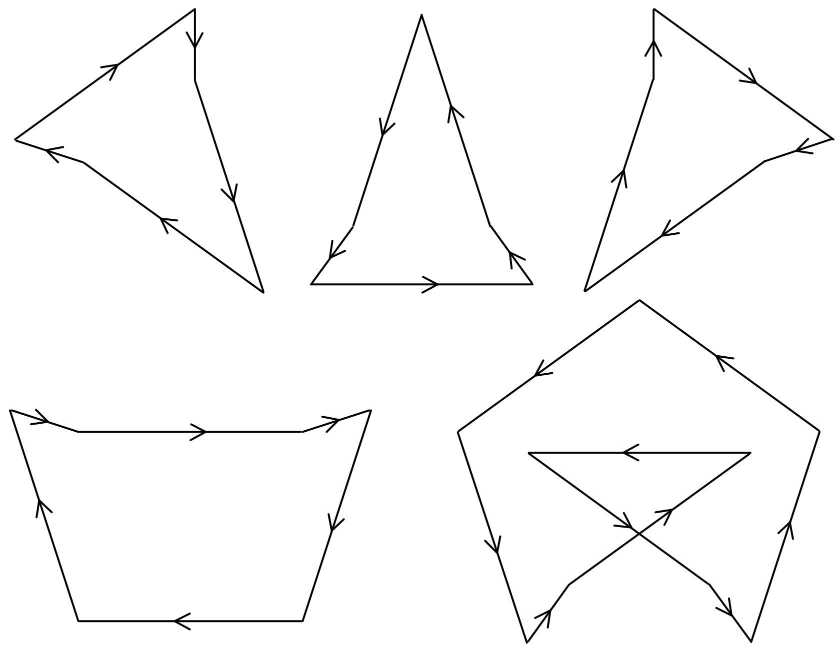

Figure 1: Oriented circuit double cover of the Petersen graph.

$K_5 - e'$ is not r-colourable, and for any $4 \le r < 5$, for any $\delta \in [4, r-4]$, $K_5 - e'$ has an r-colouring c with $|c(x) - c(y)| = \delta$ and moreover, for any r-colouring c of $K_5 - e'$, we have $|c(x) - c(y)| \le r - 4$. It follows that if $r < 4$ then $L_r^T(K_5, e') = \emptyset$, and if $4 \le r < 5$ then $L_r^T(K_5, e') = [4, r-4]_r$. Note that for a rooted graph (G, e), by duplicating an edge other than the root edge e will not change the set $L_r^T(G, e)$. Therefore the $L_r^T(I_{\mathcal{C}}, e^*)$ is equal to $L_r^T(K_5, e')$ for all r.

Lemma 10. *For any edge e of the Petersen graph G, the rooted graph (G, e) has a perfect dual.*

Proof. It was shown in [13] that for $r < 4$, $L_r^F(G, e) = \emptyset$ and for $4 \le r < 5$, $L_r^F(G, e) = [4, r-4]_r$. So for any $2 \le r < 5$, $L_r^T(I_{\mathcal{C}}, e^*) = L_r^F(G, e)$. I.e., $(I_{\mathcal{C}}, e^*)$ is a perfect dual of (G, e). □

Theorem 1. *For any rational number $2 \le p/q \le 5$, there is a graph G with $\Phi_c(G) = p/q$ and moreover G has an edge e such that (G, e) has a perfect dual. Consequently, $\Phi_c^*(G) = \Phi_c(G)$.*

Proof. For $2 \le p/q \le 4$, it is known [22] that there is a planar graph G with $\chi_c(G) = p/q$. Hence the dual graph G^* has $\Phi_c(G^*) = p/q$, and the conclusion follows from Lemma 6. Thus we only need to consider rational numbers $p/q \in (4,5]$. Assume $4 < p/q \le 5$. We shall construct, by induction on q, a rooted graph $(G_{p/q}, e)$ with $\Phi_c(G_{p/q}) = p/q$ which has a perfect dual.

It is well-known (cf. [22]) that for each fraction p/q with $p, q > 1$ (and $(p,q) = 1$), there are unique integers $0 < a < p$ and $0 < b < q$ such that $pb - aq = 1$. We call the fraction a/b the *lower parent* of p/q, denote it by $p_l(p/q) = a/b$. Let $a' = p - a$ and $b' = q - b$. The fraction a'/b' is called the *upper parent* of p/q, and denoted by $p_u(p/q) = a'/b'$. Observe that

$$p_l(p/q) < p/q < p_u(p/q).$$

Moreover, since $a'b - ab' = (p-a)b - a(q-b) = pb - qa = 1$, so if $a < a'$ then $p_l(a'/b') = a/b$. It is also easy to see that in case $a > a'$ then $p_l(a'/b') < a/b$. So in any case, $p_l(a'/b') \le a/b$. Similarly, $p_u(a/b) \ge a'/b'$. □

Lemma 11. *Suppose $4 < p/q < 5$ and that $a/b = p_l(p/q), a'/b' = p_u(p/q)$. There exists a rooted graph $(G_{p/q}, e)$ which has a perfect dual. Moreover, for $a/b \le r < a'/b'$,*

$$L_r^F(G_{p/q}) = [p - 1 - (q-1)r,\ qr - p + 1],$$

for $r < a/b$,

$$L_r(G_{p/q}) = \emptyset.$$

Proof. Let $G_{5/1}$ be the Petersen graph. It follows from Lemma 10 that $G_{5/1}$ has a perfect dual and for $4 \le r < 5$,

$$L_r^F(G_{5/1}) = [4, r-4]_r,$$

and for $r < 4$,

$$L_r(G_{5/1}) = \emptyset.$$

If $p/q = (4k+1)/k$ and $k \ge 2$, then $a/b = 4/1$ and $a'/b' = (4k-3)/(k-1)$. Let $G_{(4k+1)/k}$ be the parallel join of $G_{(4(k-1)+1)/(k-1)}$ and $G_{5/1}$. By

Lemma 9, $G_{(4k+1)/k}$ has a perfect dual. Moreover, by Lemma 4 and Lemma 5,

$$\begin{aligned} L_r^F(G_{(4k+1)/k}) &= [4(k-1)-(k-2)r,\ (k-1)r-4(k-1)]_r + [4, r-4]_r \\ &= [4k-(k-1)r,\ kr-4k]_r. \end{aligned}$$

(Because $r < (4(k-1)+1)/(k-1)$, the sum of the lengths of the two intervals is less than r. Also note that $4k-(k-2)r \pmod r = 4k-(k-1)r$). For $r < a/b = 4$,

$$L_r^F(G_{(4k+1)/k}) = \emptyset.$$

Assume that $4 < p/q < 5$ and $p/q \neq (4k+1)/k$. Then $a/b \neq 4$ and $b < q$, $a'/b' \leq 5$ and $b' < q$. By induction hypothesis, both graphs $G_{a/b}$ and $G_{a'/b'}$ have been constructed. Let Q be the graph consisting of two vertices and two parallel edges. Then $L_r^F(Q) = [1, r-1]_r$. We construct the graph $G_{p/q}$ as follows:

Let X be the series join of $G_{a/b}$ and Q. Let Y be the parallel join of two copies of X. Let Z be the series join of Y and $G_{5/1}$. Finally, let $G_{p/q}$ be the parallel join of Z and $G_{a'/b'}$. It follows from Lemma 9 that $G_{p/q}$ has a perfect dual. It remains to calculate the label set $L_r^F(G_{p/q})$.

If $r < a/b$, then $L_r^F(G_{a/b})$ is either empty, or equal to $[a-1-(b-1)r, br-a+1]_r$. In the former case, $L_r^F(X) = \emptyset$. In the latter case, $L_r^F(X) = [a-1-(b-1)r, br-a+1]_r \cap [1, r-1]_r = \emptyset$. So in any case $L_r^F(X) = \emptyset$ and hence $L_r^F(G_{p/q}) = \emptyset$. Now consider the case that $a/b \leq r < a'/b'$.

$$\begin{aligned} L_r^F(X) &= [a-1-(b-1)r, br-a-1]_r \cap [1, r-1]_r \\ &= [1, br-a+1]_r \cup [a-1-(b-1)r, r-1]_r. \end{aligned}$$

$$\begin{aligned} L_r^F(Y) = L_r^F(X) + L_r^F(X) &= ([1,\ br-a+1] \cup [a-1-(b-1)r, r-1]) \\ &\quad +([1,\ br-a+1] \cup [a-1-(b-1)r, r-1]) \\ &= [2, 2br-2a+2] \cup [2a-2-(2b-1)r, r-2] \\ &\quad \cup [a-(b-1)r, br-a]. \end{aligned}$$

$$L_r^F(Z) = L_r^F(Y) \cap L_r(G_{5/1})$$

$$\begin{aligned} &= ([2, 2br-2a+2] \cup [2a-2-(2b-1)r, r-2] \\ &\quad \cup [a-(b-1)r, br-a]) \cap [4, r-4] \\ &= [a-(b-1)r, br-a]_r. \end{aligned}$$

(See Figure 2 below. for a possible position of the points).

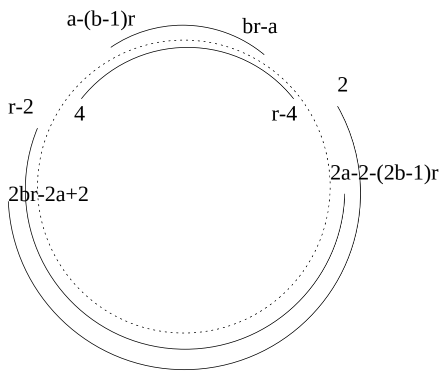

Figure 2: Illustration of the involved intervals

$$\begin{aligned} L_r^F(G_{p/q}) &= L_r^F(Z) + L_r^F(G_{a'/b'}) \\ &= [a-(b-1)r, br-a] + [a'-1-(b'-1)r, b'r-a'+1] \\ &= [(a+a')-1-(b+b'-1)r, (b+b')r-(a+a')+1] \\ &= [p-1-(q-1)r, qr-p+1]. \end{aligned}$$

This completes the proof of Lemma 11. □

Since $[p-1-(q-1)r, qr-p+1]_r \cap [1, r-1]_r \neq \emptyset$ if and only if $r \geq p/q$, we conclude that $\Phi_c(G_{p/q}) = p/q$.

3. Graphs with $\Phi_c^*(G) > \Phi_c(G)$

In this section, we present a class of graphs G for which $\Phi_c^*(G) > \Phi_c(G)$. Let J_{2k+1} be the *flower snark* which has vertex set $V(J_{2k+1}) = \{a_i, b_i, c_i, d_i : i = 0, 1, \cdots, 2k\}$ and edge set $E(J_{2k+1}) = \{b_ia_i, b_ic_i, b_id_i, a_ia_{i+1}, c_id_{i+1}, d_ic_{i+1} : i = 0, 1, \cdots, 2k\}$, where the summation in indices are modulo $2k+1$. The figure 3 is a example of flower snark for $k = 3$. It is known [15] and [9] that $\Phi_c(J_{2k+1}) = (4k+1)/k$.

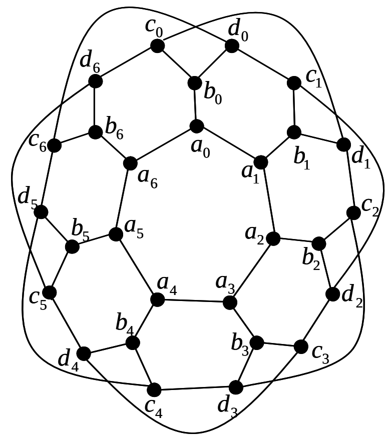

Figure 3: The graph of flower snark J_7.

Theorem 2. *If $k \geq 4$, then $\Phi_c^*(J_{2k+1}) > \Phi_c(J_{2k+1})$.*

Proof. Assume to the contrary that $\Phi_c^*(J_{2k+1}) = \Phi_c(J_{2k+1}) = (4k+1)/k$. Then J_{2k+1} has an oriented circuit double cover $\mathcal{C}$ which contains at least $p \geq 4k+1$ circuits. When $k \geq 4$, all circuits of J_{2k+1} is of length at least 6. Let H_i be the subgraph of J_{2k+1} induced by $\{a_i, b_i, c_i, d_i, a_{i+1}, b_{i+1}, c_{i+1}, d_{i+1}\}$. Each circuit of length 6 is contained in H_i for some i. Moreover, there are only three circuits of length 6 contained in H_i. These circuits are

$$\begin{aligned} A_i &= \{a_i, b_i, d_i, c_{i+1}, b_{i+1}, a_{i+1}\}, \\ B_i &= \{a_i, b_i, c_i, d_{i+1}, b_{i+1}, a_{i+1}\}, \\ C_i &= \{c_i, b_i, d_i, c_{i+1}, b_{i+1}, d_{i+1}\}. \end{aligned}$$

Note that any two circuits of these three circuits share three consecutive edges. If $\mathcal{C}$ contains two of these three circuits, say $\mathcal{C}$ contains two circuits A_i, B_i, then there is no circuit in $\mathcal{C}$ that can cover the edge $a_{i-1}a_i$. So $\mathcal{C}$ contains at most one circuit of length 6 of H_i. Therefore, $\mathcal{C}$ contains at most $2k+1$ circuits of length 6. Let s be the number of circuits of length 6 of $\mathcal{C}$. Then the sum of the lengths of the circuits in $\mathcal{C}$ is at least

$$6s + 7(4k+1-s) \geq 6(4k+1) + 2k.$$

So if $k \geq 4$, then the sum is at least $6(4k+1) + 8 > 6(4k+2)$. On the other hand, each edge of J_{2k+1} is covered by two circuits, so the sum of the lengths of circuits in $\mathcal{C}$ is equal to $6(4k+2)$, which is a contradiction. $\square$

The proof above actually shows something more: let $\Phi_c^{\#}(G)$ be the least r such that there is a cycle double cover (not necessarily oriented) $\mathcal{C}$ of G such that $\chi_c(I_{\mathcal{C}}) = r$. Then for flower snarks J_{2k+1}, if $k \geq 4$ then either $\Phi_c^{\#}(J_{2k+1}) \leq 4$ or $\Phi_c(J_{2k+1}) < \Phi_c^{\#}(J_{2k+1})$. Thus we conclude that $\Phi_c(J_{2k+1}) \neq \Phi_c^{\#}(J_{2k+1})$. But the possibility that $\Phi_c^{\#}(J_{2k+1}) < \Phi_c(J_{2k+1})$ is not ruled out. Actually, for many other graphs G, we do have $\Phi_c^{\#}(J_{2k+1}) < \Phi_c(J_{2k+1})$. For example, it is known [19] that if G admits a nowhere zero 4-flow then G has a 3-cycle double cover. So if $\Phi_c(G) \leq 4$, then $\Phi_c^{\#}(J_{2k+1}) \leq 3$.

As observed before, for planar graphs G we have $\Phi_c^*(G) = \Phi_c(G)$. Peterson minor free graphs share many properties with planar graphs. A natural question is whether or not the equality $\Phi_c^*(G) = \Phi_c(G)$ holds for Peterson minor free graphs. In the following, we give a negative answer to this question.

Suppose G is a graph. An orientation D of G is an asymmetric subset of $D(G)$ such that $\underline{\mathrm{D}} = E$. The following lemma is proved in [3].

Lemma 12. *For any graph G, $\Phi_c(G) \leq r$ if and only if G has an orientation D such that for any bond B of G,*

$$\frac{1}{r-1} \leq \frac{|B \cap D|}{|B^{-1} \cap D|} \leq r-1.$$

Theorem 3. *Let G be the graph defined as Figure 4. Then $\Phi_c^*(G) > \Phi_c(G)$.*

First we show that $\Phi_c(G) = 7/2$. Let D be the orientation of G in which the cycle $(v_0, v_1, \ldots, v_7)$ is a directed cycle, and for $i = 0, 1, 2, 3$, v_{2i} has out-degree 2, v_{2i+1} has in-degree 2. The edge ab is arbitrarily oriented. It can be verified that for any bond B of G, $2/5 \leq |B \cap D|/|B^{-1} \cap D| \leq 5/2$. By Lemma 12, $\Phi_c(G) \leq 7/2$. On the other hand, let D be an orientation of G. Assume $\Phi_c(G) < 7/2$. By Lemma 12, G has an orientation D such that for any bond B, $2/5 < \frac{|B\cap D|}{|B^{-1}\cap D|} < 5/2$. If for some i, both v_i and v_{i+1} has out-degree (or in-degree) at least 2, then for $B = \partial^+(\{v_i, v_{i+1}\})$, we have $|B \cap D|/|B^{-1} \cap D| = 3$ (or $|B \cap D|/|B^{-1} \cap D| = 1/3$), which is a contradiction. So we may assume that for $i = 0, 1, 2, 3$, v_{2i} has out-degree 2, v_{2i+1} has in-degree 2. By considering the bond $\partial^+(\{a\})$, we know that a has either in-degree 3 and out-degree 2 or in-degree 2 and out-degree 3. Assume a has

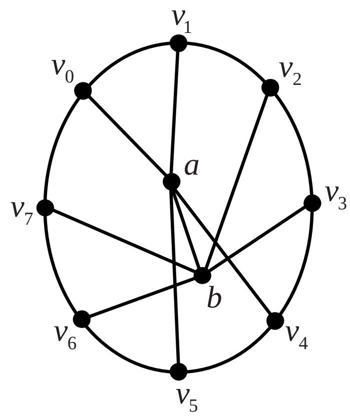

Figure 4: The graph G.

in-degree 3 and out-degree 2. Then for the bond $B = \partial^+(\{v_0, a, v_4\})$, we have $|B \cap D|/|B^{-1} \cap D| = 5/2$, a contradiction. This proves that $\Phi_c(G) = 7/2$.

Lemma 13. *There is no oriented circuit double cover $\mathcal{C}$ of G such that $I_{\mathcal{C}}$ is $(7,2)$-colourable.*

Proof. Assume that $\mathcal{C}$ is an oriented circuit double cover of G and $I_{\mathcal{C}}$ is $(7,2)$-colourable. Let $\underline{\mathcal{C}} = \{\underline{C} : C \in \mathcal{C}\}$.

Since $\Phi_c(G) = 7/2$, we know $\chi_c(I_{\mathcal{C}}) = 7/2$, and hence any colour class is non-empty. For $i = 0, 1, \ldots, 6$, let C_i be a circuit of colour i. Then for any circuit $C \in \mathcal{C}$ of colour i, $\underline{C}$ is disjoint from $\underline{C}_{i-1}$ and $\underline{C}_{i+1}$. It is obvious that $C_{i-1} \neq C_{i+1}^{-1}$ (see Claim 1 below). So for any $C \in \mathcal{C}$, $G - \underline{C}$ contains at least two distinct undirected circuits.

Let D be the undirected circuit $(v_0, v_1, \ldots, v_7)$. As $G - D$ is acyclic, for any $0 \leq i \leq 6$, $\underline{C}_i \cap D \neq \emptyset$.

Because $\mathcal{C}$ is a circuit double cover of the G, we have the following

Claim 1. *If $C, C' \in \mathcal{C}$ are two circuits that both $\underline{C}, \underline{C}'$ contain edge v_iv_{i+1}, then exactly one of $\underline{C}, \underline{C}'$ contains the edge $v_{i+1}v_{i+2}$ and exactly one of them contains the edge $v_{i-1}v_i$.*

Assume C is a circuit of G such that $\underline{C} \cap D = \{v_{i-1}v_i, v_iv_{i+1}\}$. If $ab \notin \underline{C}$, then $\underline{C}$ must contains the other edge of D, a contracdition. So by the observation above, $ab \in \underline{C}$ Thus we have the following claim.

Claim 2. *If $C \in \mathcal{C}$, and $\underline{C} \cap D = \{v_{i-1}v_i, v_iv_{i+1}\}$, then $ab \in \underline{C}$.*

Assume C is a circuit of G such that $|\underline{C} \cap D| \geq 5$ or $|\underline{C} \cap D| = 4$ but $\underline{C} \cap D$ is not a subpath of D, then it can be verified that $G - \underline{C}$ contains at most one circuit. Therefore we have the following

Claim 3. *If $C \in \mathcal{C}$, then $|\underline{C} \cap D| \leq 4$. Moreover, if $C \in \mathcal{C}$ and $|\underline{C} \cap D| = 4$, then $\underline{C} \cap D$ is a subpath of D.*

Since $(G-D) \cup \{v_7v_0\}$ contains a unique circuit (v_7, v_0, a, b), we conclude that if $C \in \mathcal{C}$ and $\underline{C} \cap D = \{v_7v_0\}$ then $ab \in \underline{C}$. Similarly, we have the following

Claim 4. *If $C \in \mathcal{C}$, and $\underline{C} \cap D = \{v_0v_7\}, \{v_1v_2\}, \{v_3v_4\}$ or $\{v_5v_6\}$, then $ab \in \underline{C}$.*

Claim 5. *For $C \in \mathcal{C}$, $|\underline{C} \cap D| \leq 3$.*

Proof. Assume $|\underline{C}_1 \cap D| = 4$. By Claim 3, $\underline{C}_1 \cap D$ is a subpath of D. Without loss of generality, assume $\underline{C}_1 \cap D = \{v_0v_1, v_1v_2, v_2v_3, v_3v_4\}$. Then C_0 and C_2 are two circuits contained in $G - \{v_0, v_1, \ldots, v_4\}$.

If $|\underline{C}_0 \cap D| = 2$, then $\underline{C}_0 = (a, v_5, v_6, v_7, b)$, and $\underline{C}_2 = (a, v_5, v_6, b)$ or (v_6, v_7, b). Then no other circuit in $\mathcal{C}$ contains v_4v_5 or v_0v_7, which is a contradiction.

Thus we assume that $|\underline{C}_0 \cap D| = 1$ and similarly $|\underline{C}_2 \cap D| = 1$. So $\underline{C}_0, \underline{C}_2$ must be circuits (a, b, v_6, v_5) and (a, v_6, v_7). Since $\mathcal{C}$ is a circuit double cover of G, we conclude that for some $C' \in \mathcal{C}$, $\underline{C}' = (v_0, a, v_4, v_5, v_6, v_7)$.

Since $|\underline{C}' \cap D| = 4$, the same argument as above shows that (v_1, v_2, b, a) and (v_2, v_3, b) belongs to $\mathcal{C}$. This implies that for some $C'' \in \mathcal{C}$, $\underline{C}'' = (v_0, v_1, a, v_5, v_4, v_3, b, v_7)$. But $|\underline{C}'' \cap D| = 4$ and $\underline{C}'' \cap D$ is not a subpath of D, in contrary to Claim 3. $\square$

Claim 6. *At least one of the triangles $(v_0, v_1, a), (v_2, v_3, b), (v_4, v_5, a)$ and (v_6, v_7, b) belongs to $\underline{\mathcal{C}}$.*

Proof. Assume that none of the circuits $(v_0, v_1, a), (v_2, v_3, b), (v_4, v_5, a)$ and (v_6, v_7, b) belongs to $\underline{\mathcal{C}}$. Since $|\underline{C} \cap D| \leq 3$ for all $C \in \mathcal{C}$, and $\sum_{C \in \mathcal{C}} |\underline{C} \cap D| = 16$, there are at least 3 circuits C of $\mathcal{C}$ such that $|\underline{C} \cap D| \leq 2$. By Claims 2 and 4, there are at least 3 circuits of $\mathcal{C}$ containing the edge ab, which is a contradiction. □

We assume $(v_0, v_1, a) \in \underline{\mathcal{C}}$. By Claims 1 and 5, $(v_7, v_0, v_1, v_2, b) \in \underline{\mathcal{C}}$. Without loss of generality, assume $\underline{C}_1 = (v_7, v_0, v_1, v_2, b)$. Then $\underline{C}_0, \underline{C}_2$ are distinct circuits of $G - \{v_7, v_0, v_1, v_2\}$.

Assume first that $(v_4, v_5, a) \notin \underline{\mathcal{C}}$. If $\underline{C}_0 \cap \underline{C}_2 \cap D = \emptyset$, then we may assume that $\underline{C}_0 = (v_5, v_6, b, a)$. Then $\underline{C}_2 = (v_3, v_4, a, b)$ or (v_3, v_4, v_5, a, b). It follows from Claim 1 that the other circuit C' in $\mathcal{C}$ containing edge v_5v_6 also contains v_4v_5 and v_6v_7. As $|C' \cap D| \leq 3$, we have $C' = (v_7, v_6, v_5, v_4, a, b)$, but then ab is covered three times by $\mathcal{C}$, a contradiction. Thus $\underline{C}_0, \underline{C}_2$ must be (v_4, v_5, v_6, b, a) and (v_5, v_4, v_3, b, a). It is easy to see that either (b, a) or (v_4, v_5) are in the same direction in C_0 and C_2. This is contrary to the assumption that $\mathcal{C}$ is an oriented circuit double cover of G.

Assume next that $(v_4, v_5, a) \in \underline{\mathcal{C}}$. By claim 1, $(v_3, v_4, v_5, v_6, b) \in \underline{\mathcal{C}}$. It is easy to verify that if one of (v_2, v_3, b), (v_6, v_7, b), (v_1, v_2, v_3, v_4, a) and (v_5, v_6, v_7, v_0, a) belongs to $\underline{\mathcal{C}}$, then all of them belong to $\underline{\mathcal{C}}$. But then there is no circuit in $\mathcal{C}$ that can cover the edge ab, a contradiction. So, none of (v_2, v_3, b), (v_6, v_7, b),(v_1, v_2, v_3, v_4, a) and (v_5, v_6, v_7, v_0, a) belongs to $\underline{\mathcal{C}}$. This implies that (v_1, v_2, v_3, b, a) and (v_4, v_3, v_2, b, a) must belong to $\underline{\mathcal{C}}$. It is easy to verify that either (b, a) or (v_2, v_3) in the same direction in these two circuits, a contradiction.

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