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EXPLICIT CLASSIFICATION OF PARALLEL LORENTZ SURFACES IN 4D INDEFINITE SPACE FORMS WITH INDEX 3

BY

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Abstract

A Lorentz surface in an indefinite space form is called parallel if its second fundamental form is parallel. Such surfaces are locally invariant under the reflection with respect to the normal space at each point. Parallel surfaces are important in geometry as well as in general relativity since extrinsic invariants of such surfaces do not change from point to point. Parallel Lorentz surfaces in 4D *Lorentzian space forms* are classified in [16] by Chen and Van der Veken. Moreover, explicit classification of parallel Lorentz surfaces in 4D indefinite space forms with index 2 are obtained recently in a series of papers by Chen, Dillen and Van der Veken [12, 13, 14]. In this paper, we obtain the complete classification of parallel Lorentz surfaces in 4D indefinite space forms with index 3. Consequently, the complete classification of parallel Lorentz surfaces in 4D indefinite space forms are achieved.

1. Introduction

Let $\mathbb{E}^m_t$ denote the pseudo-Euclidean m-space with the canonical pseudo-Euclidean metric of index t given by

$$g_0 = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{m} dx_j^2, \tag{1.1}$$

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where $(x_1, \ldots, x_m)$ is a rectangular coordinate system of $\mathbb{E}^m_t$. We put

$$S^k_s(c) = \{x \in \mathbb{E}^{k+1}_s | \langle x, x\rangle = c^{-1} > 0\}, \tag{1.2}$$

$$H^k_s(-c) = \{x \in \mathbb{E}^{k+1}_{s+1} | \langle x, x\rangle = -c^{-1} < 0\}, \tag{1.3}$$

where $\langle\ ,\ \rangle$ is the indefinite inner product on $\mathbb{E}^{k+1}_t$.

$S^k_s(c)$ and $H^k_s(-c)$ are complete pseudo-Riemannian manifolds with index s and of constant curvature c and $-c$, which are called pseudo k-sphere and pseudo-hyperbolic k-space. These $\mathbb{E}^k_s, S^k_s$ and H^k_s are known as *indefinite space forms*, and will be denoted by R^k_s. In particular, $\mathbb{E}^k_1, S^k_1$ and H^k_1 are called Minkowski, de Sitter and anti-de Sitter spacetimes respectively in relativity theory.

Parallel surfaces are those which have parallel second fundamental form. Such surfaces are locally invariant under the reflection with respect to the normal space at each point (cf. [1, 2, 17, 23]). Moreover, extrinsic invariants of a parallel surface do no change from point to point. Hence, parallel surfaces form a natural and important family of surfaces in geometry as well as in general relativity.

For the classification of parallel surfaces in Riemannian space forms, we refer to [6, 17, 24]. Some special families of parallel surfaces in indefinite space forms were studied in [18, 19, 21]. The full classification of parallel Lorentz surfaces in 4D Lorentz space forms was achieved by B. Y. Chen and J. Van der Veken [16]. Moreover, explicit classification of parallel Lorentz surfaces in 4D indefinite space forms with index 2 are obtained recently in a series of papers by B. Y. Chen, F. Dillen and J. Van der Veken [12, 13, 14].

In this paper, we obtain the complete classification of parallel Lorentz surfaces in 4D indefinite space forms with index 3. Consequently, the complete classification of parallel Lorentz surfaces in 4D indefinite space forms are achieved.

Comparing results from [16] and the results obtained in this paper shows that Lorentz surfaces in indefinite space form $R^4_3(c)$ with index 3 are quite different from Lorentz surfaces in Lorentzian space forms $R^4_1(c)$ (see Remark 4.1 in particular).

2. Preliminaries

2.1. Basic notations, formulae and definitions

Let $R^m_s(c)$ denote an m-dimensional indefinite space form of constant sectional curvature c and with index s. The curvature tensor $\tilde{R}$ of $R^m_s(c)$ is given by

$$\tilde{R}(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\}, \tag{2.1}$$

where $\langle\ ,\ \rangle$ is the inner product associated to the metric.

Let $\psi : M^2_1 \to R^m_s(c)$ be an isometric immersion of a Lorentz surface M^2_1 into $R^m_s(c)$. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connections on M^2_1 and $R^m_s(c)$, respectively.

Let X and Y be vector fields tangent to M^2_1 and ξ a normal vector field of M^2_1 in $R^m_s(c)$. The formulae of Gauss and Weingarten give a decomposition of the vector fields $\tilde{\nabla}_X Y$ and $\tilde{\nabla}_X \xi$ into a tangent and a normal component (cf. [3, 4, 22]):

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X,Y), \tag{2.2}$$

$$\tilde{\nabla}_X \xi = -A_\xi X + D_X \xi. \tag{2.3}$$

These formulae define h, A and D, which are called the second fundamental form, the shape operator and the normal connection, respectively.

For each normal vector $\xi \in T^\perp_x M^2_1$ at $x \in M^2_1$, the shape operator A_ξ is a symmetric endomorphism of the tangent plane $T_x M^2_1$. The shape operator and the second fundamental form are related by

$$\langle h(X,Y), \xi\rangle = \langle A_\xi X, Y\rangle. \tag{2.4}$$

The mean curvature vector H of M^2_1 in $R^m_s(c)$ is defined by

$$H = \frac{1}{2}\operatorname{trace} h. \tag{2.5}$$

The equations of Gauss, Codazzi and Ricci are given respectively by

$$R(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y\} + A_{h(Y,Z)}X - A_{h(X,Z)}Y, \tag{2.6}$$

$$(\overline{\nabla}_X h)(Y,Z) = (\overline{\nabla}_Y h)(X,Z), \tag{2.7}$$
$$\langle R^D(X,Y)\xi,\eta\rangle = \langle [A_\xi, A_\eta]X, Y\rangle \tag{2.8}$$

for vector fields X, Y, Z tangent and ξ, η normal to M_1^2, where $\overline{\nabla} h$ is defined by

$$(\overline{\nabla}_X h)(Y,Z) = D_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), \tag{2.9}$$

and R^D is the curvature tensor associated to the normal connection D, i.e.,

$$R^D(X,Y)\xi = D_X D_Y \xi - D_Y D_X \xi - D_{[X,Y]}\xi. \tag{2.10}$$

A normal vector field ξ is called parallel if $D\xi = 0$ holds identically. A surface of a pseudo-Riemannian manifold is called *totally geodesic* if the second fundamental form vanishes identically. It is called *totally umbilical* if its second fundamental form satisfies $h(X,Y) = \langle X,Y\rangle H$.

By a *CMC surface* of a pseudo-Riemannian 3-manifold, we mean a surface whose mean curvature vector H satisfies $\langle H,H\rangle = constant \neq 0$.

2.2. A special coordinate system

Let M_1^2 be a Lorentz surface. We may choose a local coordinate system $\{x,y\}$ on M_1^2 such that the metric tensor is (cf. [15, 20])

$$g = -E^2(x,y)(dx\otimes dy + dy \otimes dx) \tag{2.11}$$

for some positive function E. The Levi-Civita connection of g satisfies

$$\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = \frac{2E_x}{E}\frac{\partial}{\partial x},\ \nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0,\ \nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = \frac{2E_y}{E}\frac{\partial}{\partial y} \tag{2.12}$$

and the Gaussian curvature K is given by

$$K = \frac{2EE_{xy} - 2E_xE_y}{E^4}. \tag{2.13}$$

If we put

$$e_1 = \frac{1}{E}\frac{\partial}{\partial x},\ \ e_2 = \frac{1}{E}\frac{\partial}{\partial y}, \tag{2.14}$$

then $\{e_1, e_2\}$ forms a pseudo-orthonormal frame satisfying

$$\langle e_1, e_1\rangle = \langle e_2, e_2\rangle = 0,\ \langle e_1, e_2\rangle = -1. \tag{2.15}$$

We define the connection 1-form ω by the equations:

$$\nabla_X e_1 = \omega(X)e_1, \quad \nabla_X e_2 = -\omega(X)e_2. \tag{2.16}$$

From (2.12) and (2.14) we find

$$\begin{aligned} \nabla_{e_1} e_1 &= \frac{E_x}{E^2} e_1, \quad \nabla_{e_2} e_1 = -\frac{E_y}{E^2} e_1, \\ \nabla_{e_1} e_2 &= -\frac{E_x}{E^2} e_2,\ \nabla_{e_2} e_2 = \frac{E_y}{E^2} e_2. \end{aligned} \tag{2.17}$$

By comparing (2.16) and (2.17), we get

$$\omega(e_1) = \frac{E_x}{E^2},\ \ \omega(e_2) = -\frac{E_y}{E^2}. \tag{2.18}$$

It follows from (2.5) and (2.15) that the mean curvature vector of M_1^2 is given by

$$H = -h(e_1, e_2). \tag{2.19}$$

2.3. Reduction theorem of Erbacher-Magid

Reduction Theorem. ([21]) *Let $\psi : M_i^n \to \mathbb{E}_s^m$ be an isometric immersion of a pseudo-Riemannian n-manifold M_i^n with index i into $\mathbb{E}_s^m$. If the first normal spaces are parallel, then there exists a complete $(n+k)$-dimensional totally geodesic submanifold E^* such that $\psi(M) \subset E^*$, where k is the dimension of the first normal spaces.*

The following is an easy consequence of the reduction theorem (see [9]).

Lemma 2.1. *Let $\psi : M_1^2 \to \mathbb{E}_s^m$ be an isometric immersion of a Lorentz surface M_1^2 into $\mathbb{E}_s^m$. If M_1^2 is a parallel surface, then there exists a complete $(2+k)$-dimensional totally geodesic submanifold $E^* \subset \mathbb{E}_s^m$ such that $\psi(M) \subset E^*$, where k is the dimension of the first normal spaces.*

3. Parallel Lorentz Surfaces in $\mathbb{E}^4_3$

The following theorem provides the complete classification of parallel Lorentz surfaces in $\mathbb{E}^4_3$.

Theorem 3.1. *There are seven families of parallel Lorentz surfaces in the pseudo-Euclidean* 4*-space* $\mathbb{E}^4_3$ *with index* 3*:*

(1) *a totally geodesic Lorentz plane* $\mathbb{E}^2_1$*;*

(2) *a flat minimal surface lying in a totally geodesic* $\mathbb{E}^3_2 \subseteq \mathbb{E}^4_3$ *defined by*

$$\left(0, \frac{a^2x^2}{2}, \frac{x}{2} - \frac{a^4x^2}{6} + y, \frac{x}{2} + \frac{a^4x^2}{6} - y\right),\ a > 0;$$

(3) *an anti-de Sitter space* $H^2_1(-b^2)$ *lying in a totally geodesic* $\mathbb{E}^3_2 \subseteq \mathbb{E}^4_3$ *as a totally umbilical surface via* (1.3)*;*

(4) *a non-minimal flat surface lying in a totally geodesic* $\mathbb{E}^3_2 \subseteq \mathbb{E}^4_3$ *defined by*

$$\left(0, \frac{1}{2b}\cos\left(\frac{\sqrt{2b}}{a}(a^2x+by)\right), \frac{1}{2b}\sin\left(\frac{\sqrt{2b}}{a}(a^2x+by)\right), \frac{a^2x-by}{a\sqrt{2b}}\right),\ a,b > 0;$$

(5) *a non-minimal flat surface lying in a totally geodesic* $\mathbb{E}^3_2 \subseteq \mathbb{E}^4_3$ *defined by*

$$\left(0, \frac{a^2x+by}{a\sqrt{2b}}, \frac{1}{2b}\cosh\left(\frac{\sqrt{2b}}{a}(a^2x-by)\right), \frac{1}{2b}\sinh\left(\frac{\sqrt{2b}}{a}(a^2x-by)\right)\right),\ a,b > 0;$$

(6) *a non-minimal flat surface defined by*

$$\frac{a}{\sqrt{2b}}\left(\cos\left(\frac{\sqrt{b}(a^3x+by)}{a^{5/2}}\right), \sin\left(\frac{\sqrt{b}(a^3x+by)}{a^{5/2}}\right),\right.$$
$$\left.\cosh\left(\frac{\sqrt{b}(a^3x-by)}{a^{5/2}}\right), \sinh\left(\frac{\sqrt{b}(a^3x-by)}{a^{5/2}}\right)\right)$$

with $a, b > 0$*;*

(7) *a non-minimal flat surface defined by*

$$\left(\frac{\sqrt[4]{\delta^2+\varphi^2}\cos\left(\lambda(bx+\sqrt{\delta^2+\varphi^2}y)\right)}{\sqrt{2}b\sqrt{\sqrt{\delta^2+\varphi^2}+\delta}}, \frac{\sqrt[4]{\delta^2+\varphi^2}\sin\left(\lambda(bx+\sqrt{\delta^2+\varphi^2}y)\right)}{\sqrt{2}b\sqrt{\sqrt{\delta^2+\varphi^2}+\delta}},\right.$$
$$\left.\frac{\sqrt[4]{\delta^2+\varphi^2}\cosh\left(\mu(bx-\sqrt{\delta^2+\varphi^2}y)\right)}{\sqrt{2}b\sqrt{\sqrt{\delta^2+\varphi^2}-\delta}}, \frac{\sqrt[4]{\delta^2+\varphi^2}\sin\left(\mu(bx-\sqrt{\delta^2+\varphi^2}y)\right)}{\sqrt{2}b\sqrt{\sqrt{\delta^2+\varphi^2}-\delta}}\right)$$

with $\delta, \varphi \neq 0, b > 0$ *and*

$$\lambda = \frac{\sqrt{b\sqrt{\delta^2+\varphi^2}+b\delta}}{\sqrt{\delta^2+\varphi^2}}, \quad \mu = \frac{\sqrt{b\sqrt{\delta^2+\varphi^2}-b\delta}}{\sqrt{\delta^2+\varphi^2}}.$$

Conversely, every parallel Lorentz surface M_1^2 *in* $\mathbb{E}_3^4$ *is congruent to an open portion of one of the seven families of surfaces described above.*

Proof. It follows from direct long computation that each surface described in the theorem is a parallel Lorentz surface in $\mathbb{E}_3^4$.

Conversely, assume that $L : M_1^2 \to \mathbb{E}_3^4$ is a parallel immersion of a Lorentz surface M_1^2 into $\mathbb{E}_3^4$. Then M_1^2 has constant Gauss curvature K. We choose a local coordinate system $\{x, y\}$ on M_1^2 satisfying (2.11). Then we have (2.12)-(2.19).

If M_1^2 is totally geodesic in $\mathbb{E}_3^4$, we get case (1). So, let us assume that M_1^2 is non-totally geodesic in $\mathbb{E}_3^4$.

Case (i): M_1^2 *is minimal in* $\mathbb{E}_3^4$. In this case, we get $h(e_1, e_2) = 0$ according to (2.19). So, we have

$$h(e_1, e_1) = \xi,\ h(e_1, e_2) = 0,\ h(e_2, e_2) = \eta \tag{3.1}$$

for some normal vector fields ξ, η, not both zero. Since M_1^2 is non-totally geodesic, without loss of generality we may assume that $\xi \neq 0$. Let us choose an orthonormal frame $\{e_3, e_4\}$ such that e_3 is in the direction of ξ. Hence, we obtain

$$h(e_1, e_1) = \alpha e_3,\ h(e_1, e_2) = 0,\ h(e_2, e_2) = \lambda e_3 + \mu e_4 \tag{3.2}$$

for some functions α, λ, μ with $\alpha > 0$. Let us put $\alpha = a^2$ with $a > 0$.

Since M_1^2 is a parallel surface in $\mathbb{E}_3^4$, we find from (2.9), (2.16) and (3.2) that

$$De_3 = De_4 = 0, \tag{3.3}$$

$$da = a\omega, \;\; d\lambda = -2\lambda\omega, \;\; d\mu = -2\mu\omega. \tag{3.4}$$

Since $a > 0$, the first equation in (3.4) shows that ω is exact. Hence, M_1^2 is flat. Thus, we may choose $E = 1$, which gives $\omega = 0$. Consequently, a, λ, μ are constants.

Since M_1^2 is flat, the equation of Gauss and (3.2) yield $\lambda = 0$. Also, by applying (2.4) and (3.2), we find

$$A_{e_3} = \begin{pmatrix} 0 & 0 \\ a^2 & 0 \end{pmatrix}, \; A_{e_4} = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}. \tag{3.5}$$

By using equation (2.8) of Ricci and (3.5), we find $\mu = 0$. Therefore, we obtain

$$L_{xx} = a^2 e_3, \;\; L_{xy} = 0, \;\; L_{yy} = 0. \tag{3.6}$$

Also, it follows from $De_3 = 0$ and (3.5) that

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -a^2 L_y, \;\; \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = 0. \tag{3.7}$$

After solving system (3.6)-(3.7), we get

$$L = c_0 + c_1 x + c_2 x^2 + c_3 \left(\frac{a^4 x^2}{6} - y \right).$$

Therefore, by choosing suitable initial conditions, we obtain case (2).

Case (ii): *M_1^2 is non-minimal in $\mathbb{E}_3^4$.* In this case, we have $h(e_1, e_2) \neq 0$. Thus, we may choose an orthonormal frame $\{e_3, e_4\}$ such that e_3 is in the direction of $h(e_1, e_2)$. So, we have

$$h(e_1, e_1) = \beta e_3 + \gamma e_4, \; h(e_1, e_2) = b e_3, \; h(e_2, e_2) = \delta e_3 + \varphi e_4 \tag{3.8}$$

for some functions $b, \beta, \gamma, \delta, \varphi$ with $b > 0$. Because $\bar{\nabla} h = 0$, we derive from

(2.16) and (3.8) that $De_3 = De_4 = 0$ and

$$db = 0,\ d\beta = 2\beta\omega,\ d\gamma = 2\gamma\omega,\ d\delta = -2\delta\omega,\ d\varphi = -2\varphi\omega. \tag{3.9}$$

From (2.4) and (3.8) we derive that

$$A_{e_3} = \begin{pmatrix} b & \delta \\ \beta & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & \varphi \\ \gamma & 0 \end{pmatrix} \tag{3.10}$$

Since $De_3 = 0$, the equation of Ricci and (3.10) give

$$\delta\gamma = \beta\varphi. \tag{3.11}$$

The equation of Gauss and (3.8) show that the Gauss curvature K is given by

$$K = \beta\delta + \gamma\varphi - b^2. \tag{3.12}$$

Case (ii.1): $\beta = \gamma = \delta = \varphi = 0$. Equations (3.8) and (3.10) reduce to

$$h(e_1, e_1) = 0,\ h(e_1, e_2) = be_3,\ h(e_2, e_2) = 0, \tag{3.13}$$

$$A_{e_3} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.14}$$

Since $De_3 = 0$, we see from (3.13) that the first normal bundle is a rank 3 parallel normal subbundle of the normal bundle. Therefore, Reduction Theorem shows that M_1^2 lies in a totally geodesic $\mathbb{E}_2^3 \subset \mathbb{E}_3^4$.

From $De_3 = 0$ and (3.14) we get

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x,\ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -bL_y, \tag{3.15}$$

which implies that $\tilde{\nabla}_X(L + b^{-1}e_3) = 0$. Hence, we have $L + b^{-1}e_3 = c_0$ for some vector c_0, which yields $\langle L - c_0, L - c_0\rangle = -b^{-2}$. Therefore, after applying a suitable translation, we obtain case (3).

Case (ii.2) *At least one of $\beta, \gamma, \delta, \varphi$ is nonzero.* In this case, (3.9) implies that the connection form ω is an exact 1-form. Hence, M_1^2 is flat. Thus, we may choose coordinates $\{x, y\}$ with $E = 1$, so that we have $\frac{\partial}{\partial x} = e_1$, $\frac{\partial}{\partial y} = e_2$. Hence, the metric tensor is given by

$$g = -(dx \otimes dy + dy \otimes dx). \tag{3.16}$$

Therefore, we see from (3.9) that $\beta, \gamma, \delta, \varphi$ are constant.

Case (ii.2.1): $\beta = 0$. It follows from (3.11) and (3.12) that $\varphi = b^2/\gamma$ and $\delta = 0$. Thus, (3.8) and (3.10) reduce to

$$h(e_1, e_1) = \gamma e_4,\ h(e_1, e_2) = be_3,\ h(e_2, e_2) = \frac{b^2}{\gamma} e_4, \tag{3.17}$$

$$A_{e_3} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & \frac{b^2}{\gamma} \\ \gamma & 0 \end{pmatrix}. \tag{3.18}$$

Replacing e_4 by $-e_4$ if necessary, we have $\gamma > 0$. So, we may put $\gamma = a^2$ with $a > 0$. Thus, we have

$$L_{xx} = a^2 e_4,\ \ L_{xy} = be_3,\ \ L_{yy} = \frac{b^2}{a^2} e_4. \tag{3.19}$$

Moreover, since $De_3 = 0$, we obtain from (3.18) that

$$\begin{aligned} &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x,\ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -bL_y, \\ &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = -a^2 L_y,\ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = -\frac{b^2}{a^2} L_x. \end{aligned} \tag{3.20}$$

After solving system (3.19)-(3.20) we have that

$$\begin{aligned} L(x, y) = c_0 + c_1 \cos\left(\frac{\sqrt{b}(a^3 x + by)}{a^{5/2}}\right) + c_2 \sin\left(\frac{\sqrt{b}(a^3 x + by)}{a^{5/2}}\right) \\ + c_3 \cosh\left(\frac{\sqrt{b}(a^3 x - by)}{a^{5/2}}\right) + c_4 \sinh\left(\frac{\sqrt{b}(a^3 x - by)}{a^{5/2}}\right). \end{aligned}$$

This gives case (6) after choosing suitable initial conditions.

Case (ii.2.2): $\beta \neq 0$. If $\delta = 0$, it follows from (3.11) and (3.12) that $K = b^2 = 0$, which is impossible. Thus, we must have $\delta \neq 0$.

Case (ii.2.2.1): $\gamma = 0$. It follows from (3.11) that $\varphi = 0$. Thus, we find from

$K = 0$ and (3.12) that $\beta\delta = b^2$. Therefore, (3.8) and (3.10) reduce to

$$h(e_1, e_1) = \beta e_3,\ h(e_1, e_2) = be_3,\ h(e_2, e_2) = \frac{b^2}{\beta}e_3, \tag{3.21}$$

$$A_{e_3} = \begin{pmatrix} b & \frac{b^2}{\beta} \\ \beta & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.22}$$

Thus, we have

$$L_{xx} = \beta e_3,\ \ L_{xy} = be_3,\ \ L_{yy} = \frac{b^2}{\beta}e_3. \tag{3.23}$$

Moreover, since $De_3 = 0$, we obtain from (3.22) that

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x - \beta L_y,\ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\frac{b^2}{\beta}L_x - bL_y. \tag{3.24}$$

If $\beta > 0$, we put $\beta = a^2$, $a > 0$. Then, after solving system (3.23)-(3.24), we get

$$L = c_1(a^2x - by) + c_2 \cos\left(\frac{\sqrt{2b}(a^2x+by)}{a}\right) + c_3 \sin\left(\frac{\sqrt{2b}(a^2x+by)}{a}\right),$$

which gives case (4) after choosing suitable initial conditions.

Similarly, if $\beta < 0$, then after putting $\beta = -a^2$ and solving system (3.23)-(3.24), we obtain case (5).

Case (ii.2.2.2): $\gamma \neq 0$. Since $\beta \neq 0$, we find from (3.11) that $\varphi \neq 0$. Thus, it follows from $K = 0$ and (3.12) that

$$\beta = \frac{b^2\delta}{\delta^2 + \varphi^2},\ \ \gamma = \frac{b^2\varphi}{\delta^2 + \varphi^2}. \tag{3.25}$$

Consequently, (3.8) and $E = 1$ imply that

$$L_{xx} = \frac{b^2(\delta e_3 + \varphi e_4)}{\delta^2 + \varphi^2},\ \ L_{xy} = be_3,\ \ L_{yy} = \delta e_3 + \varphi e_4. \tag{3.26}$$

Moreover, it follows from $De_3 = 0$, (3.10), and (3.25) that

$$\begin{aligned}
\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 &= -bL_x - \frac{b^2\delta}{\delta^2+\varphi^2}L_y, \ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\delta L_x - bL_y, \\
\tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 &= -\frac{b^2\varphi}{\delta^2+\varphi^2}L_y, \ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = -\varphi L_x.
\end{aligned} \tag{3.27}$$

After solving system (3.26)-(3.27), we obtain

$$\begin{aligned}
L(x,y) = &c_0 + c_1 \cos\left(\lambda(bx+\sqrt{\delta^2+\varphi^2}y\right) + c_2 \sin\left(\lambda(bx+\sqrt{\delta^2+\varphi^2}y\right) \\
&+ c_3 \cosh\left(\mu(bx-\sqrt{\delta^2+\varphi^2}y\right) + c_4 \sinh\left(\mu(bx-\sqrt{\delta^2+\varphi^2}y\right)
\end{aligned}$$

with $\delta, \varphi \neq 0, b > 0$ and

$$\lambda = \frac{\sqrt{b\sqrt{\delta^2+\varphi^2}+b\delta}}{\sqrt{\delta^2+\varphi^2}}, \quad \mu = \frac{\sqrt{b\sqrt{\delta^2+\varphi^2}-b\delta}}{\sqrt{\delta^2+\varphi^2}}.$$

This gives case (7) after choosing suitable initial conditions. □

4. Parallel Lorentz Surfaces in $S^4_3(1)$

Let $\psi : M^2_1 \to S^m_s(1)$ (resp. $\psi : M^2_1 \to H^m_s(-1)$) be an isometric immersion of a Lorentz surface M^2_1 into $S^m_s(1)$ (resp. $H^m_s(-1)$). Denote by $L = \iota \circ \psi : M^2_1 \to \mathbb{E}^{m+1}_s$ (resp. $M^2_1 \to \mathbb{E}^{m+1}_{s+1}$) the composition of ψ and $\iota : S^m_s(1) \subset \mathbb{E}^{m+1}_s$ via (1.2) (resp. $\iota : H^m_s(-1) \subset \mathbb{E}^{m+1}_{s+1}$ via (1.3)).

Denote by h and D the second fundamental form and the normal connection of M^2_1 in $S^m_s(1)$ or in $H^m_s(-1)$. Let $\tilde{h}$ and $\tilde{D}$ be the second fundamental form and the normal connections of M^2_1 in $\mathbb{E}^{m+1}_s$ or of M^2_1 in $\mathbb{E}^{m+1}_{s+1}$.

It is easy to verify that ψ *is a parallel immersion if and only if* $L = \iota \circ \psi$ *is a parallel immersion.* Moreover, it follows from Lemma 1 of [5] that the mean curvature vector H_L of L and the mean curvature vector H_ψ are related by

$$H_L = H_\psi - \epsilon L, \tag{4.1}$$

with $\epsilon = 1$ or $\epsilon = -1$ depending on $\psi : M^2_1 \to S^m_s(1)$ or $\psi : M^2_1 \to H^m_s(-1)$, respectively.

The following result was obtained in [16].

Theorem 4.1. *Let M_1^2 is a Lorentz parallel surface in $S_1^4(1) \subset \mathbb{E}_1^5$. Then M_1^2 is congruent to an open part of one of the following two types of surfaces:*

(1) *a totally umbilical de Sitter space S_1^2 given by*

$$\Big(a\sinh u, a\cosh u\cos v, a\cosh u\sin v, b, 0\Big),\ a^2+b^2=1;$$

(2) *a flat surface $S_1^1 \times S^1$ given by*

$$\Big(a\sinh u, a\cosh u, b\cos v, b\sin v, c\Big),\ \ a^2+b^2+c^2=1.$$

Conversely, each surface defined above is a Lorentzian parallel surface in $S_1^4(1)$.

Now, we give the complete classification of parallel Lorentz surfaces in the pseudo 3-sphere $S_3^4(1)$ with index 3.

Theorem 4.2. *There are twenty-one families of parallel Lorentz surfaces in $S_3^4(1) \subset \mathbb{E}_3^5$:*

(1) *a totally geodesic de Sitter space $S_1^2(1) \subset S_3^4(1)$;*

(2) *a flat minimal surface of a totally geodesic $S_2^3(1)$ given by*

$$\begin{aligned}\Bigg(0, &\cos\Big(\frac{a^2x-y}{\sqrt{2}a}\Big)\sinh\Big(\frac{a^2x+y}{\sqrt{2}a}\Big), \sin\Big(\frac{a^2x-y}{\sqrt{2}a}\Big)\sinh\Big(\frac{a^2x+y}{\sqrt{2}a}\Big),\\ &\cos\Big(\frac{a^2x-y}{\sqrt{2}a}\Big)\cosh\Big(\frac{a^2x+y}{\sqrt{2}a}\Big), \sin\Big(\frac{a^2x-y}{\sqrt{2}a}\Big)\cosh\Big(\frac{a^2x+y}{\sqrt{2}a}\Big)\Bigg),\ a>0;\end{aligned}$$

(3) *a totally umbilical flat surface of a totally geodesic $S_2^3(1)$ given by*

$$\big(0, x+xy, y-xy, x-y+xy, 1+xy\big);$$

(4) *a totally umbilical de Sitter space $S_1^2(c^2)$ lying in a totally geodesic $S_2^3(1)$*

given by

$$\left(0, \frac{xy-1}{c(x+y)}, \frac{2\sqrt{1-c^2}\,y}{c^2(x+y)}, \frac{xy+1}{c(x+y)}, 1-\frac{2y}{c^2(x+y)}\right), \quad c \in (0,1);$$

(5) *a totally umbilical anti-de Sitter space* $H_1^2(-c^2)$ *lying in a totally geodesic* $S_2^3(1)$ *given by*

$$\left(0, \frac{1}{c}\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big), \frac{1}{c}\cosh\Big(\frac{cx-cy}{\sqrt{2}}\Big)\operatorname{sech}\Big(\frac{cx+cy}{\sqrt{2}}\Big), \right.$$
$$\left. \frac{1}{c}\sinh\Big(\frac{cx-cy}{\sqrt{2}}\Big)\operatorname{sech}\Big(\frac{cx+cy}{\sqrt{2}}\Big), \frac{\sqrt{1+c^2}}{c}\right), \ c > 0;$$

(6) *a CMC flat surface lying in a totally geodesic* $S_2^3(1)$ *given by*

$$\left(0, \Big(\frac{ax}{2}+\frac{y}{a}\Big)\cos(ax), \Big(\frac{ax}{2}+\frac{y}{a}\Big)\sin(ax), \sin(ax)-\Big(\frac{ax}{2}+\frac{y}{a}\Big)\cos(ax), \right.$$
$$\left. \cos(ax)+\Big(\frac{ax}{2}+\frac{y}{a}\Big)\sin(ax)\right), \ a > 0;$$

(7) *a CMC flat surface lying in a totally geodesic* $S_2^3(1)$ *given by*

$$\left(0, \Big(\frac{ax}{2}-\frac{y}{a}\Big)\sinh(ax), \sinh(ax)-\Big(\frac{ax}{2}-\frac{y}{a}\Big)\cosh(ax), \Big(\frac{ax}{2}-\frac{y}{a}\Big)\cosh(ax), \right.$$
$$\left. \cosh(ax)-\Big(\frac{ax}{2}-\frac{y}{a}\Big)\sinh(ax)\right), \ a > 0;$$

(8) *a CMC flat surface lying in a totally geodesic* $S_2^3(1)$ *given by*

$$\left(0, \frac{\sin u \sinh v}{\sqrt{1-c^2}}, \frac{\cos u \sinh v}{\sqrt{1-c^2}}, \cos u \cosh v + \frac{b \sin u \sinh v}{\sqrt{1-c^2}}, \right.$$
$$\left. \sin u \cosh v - \frac{b \cos u \sinh v}{\sqrt{1-c^2}}\right)$$

with

$$u = \frac{\sqrt{1+c}(a^2x-(1-c)y)}{\sqrt{2}a}, \ v = \frac{\sqrt{1-c}(a^2x+(1+c)y)}{\sqrt{2}a}, \ a > 0, \ 0 < |c| < 1;$$

(9) *a non-minimal flat surface given by*

$$\left(xy-\frac{c^2y^4}{24},\frac{6x+3y-c^2y^3}{6},\frac{cy^2}{2},\frac{6x-3y-c^2y^3}{6},1+xy-\frac{c^2y^4}{24}\right),\ 0\neq c\in\mathbf{R};$$

(10) *a non-minimal flat surface given by*

$$\begin{aligned}&\frac{1}{2c^2\sqrt{c^4+p^2}}\Big(c(2c^2x+c^4y+p^2y)\cos(cy),c(2c^2x+c^4y+p^2y)\sin(cy),\\&\quad 2p\sqrt{c^4+p^2},2(c^4+p^2)\sin(cy)-c(2c^2x+c^4y+p^2y)\cos(cy),\\&\quad 2(c^4+p^2)\cos(cy)+c(2c^2x+c^4y+p^2y)\sin(cy)\Big),\ c>0,\ p\in\mathbf{R};\end{aligned}$$

(11) *a non-minimal flat surface given by*

$$\begin{aligned}&\frac{1}{2c^2\sqrt{c^4+p^2}}\Big(2p\sqrt{c^4+p^2},2(c^4+p^2)\sinh(cy)+c(2c^2x+c^4y-p^2y)\cosh(cy),\\&\quad c(2c^2x-c^4y-p^2y)\sinh(cy),c(2c^2x-c^4y-p^2y)\cosh(cy),\\&\quad 2(c^4+p^2)\cosh(cy)+c(2c^2x-c^4y-p^2y)\sinh(cy)\Big),\ c>0,\ p\in\mathbf{R};\end{aligned}$$

(12) *a non-minimal flat surface given by*

$$\left(\frac{rx^2}{2},\frac{r^2x^3-6y}{4}-\frac{x}{3},\frac{r^2x^4-24xy}{24},\frac{r^2x^3-6y}{4}+\frac{x}{3},1-\frac{r^2x^4-24xy}{24}\right),\ r\in\mathbf{R};$$

(13) *a non-minimal flat surface given by*

$$\begin{aligned}&\frac{1}{c^2}\bigg(\sqrt{1-c^4},\cos\Big(\frac{a^2cx-c^3y}{\sqrt{2}a}\Big)\sinh\Big(\frac{a^2cx+c^3y}{\sqrt{2}a}\Big),\\&\quad\sin\Big(\frac{a^2cx-c^3y}{\sqrt{2}a}\Big)\sinh\Big(\frac{a^2cx+c^3y}{\sqrt{2}a}\Big),\sin\Big(\frac{a^2cx-c^3y}{\sqrt{2}a}\Big)\cosh\Big(\frac{a^2cx+c^3y}{\sqrt{2}a}\Big),\\&\quad\cos\Big(\frac{a^2cx-c^3y}{\sqrt{2}a}\Big)\cosh\Big(\frac{a^2cx+c^3y}{\sqrt{2}a}\Big)\bigg),\ a,c>0;\end{aligned}$$

(14) *a non-minimal flat surface given by*

$$\frac{1}{\sqrt{2}c^2}\Big(\cos\Big(acx+\frac{c^3y}{a}\Big),\sin\Big(acx+\frac{c^3y}{a}\Big),\cosh\Big(acx-\frac{c^3y}{a}\Big),\sinh\Big(acx-\frac{c^3y}{a}\Big)$$
$$\sqrt{2(1+c^4)}\Big),\ a,c>0;$$

(15) *a CMC flat surface lying in a totally geodesic $S^3_2(1)$ given by*

$$\Bigg(0,\frac{\sqrt{b-\sqrt{b^2-1}}\cos\big(\sqrt{b+\sqrt{b^2-1}}\big(ax+\frac{\sqrt{b^2-1}}{a}y\big)\big)}{\sqrt[4]{4(b^2-1)}},$$
$$\frac{\sqrt{b-\sqrt{b^2-1}}\sin\big(\sqrt{b+\sqrt{b^2-1}}\big(ax+\frac{\sqrt{b^2-1}}{a}y\big)\big)}{\sqrt[4]{4(b^2-1)}},$$
$$\frac{\sqrt{b+\sqrt{b^2-1}}\cos\big(\sqrt{b-\sqrt{b^2-1}}\big(ax-\frac{\sqrt{b^2-1}}{a}y\big)\big)}{\sqrt[4]{4(b^2-1)}},$$
$$\frac{\sqrt{b+\sqrt{b^2-1}}\sin\big(\sqrt{b-\sqrt{b^2-1}}\big(ax-\frac{\sqrt{b^2-1}}{a}y\big)\big)}{\sqrt[4]{4(b^2-1)}}\Bigg)$$

with $a>0, b>1$;

(16) *a non-minimal flat surface given by*

$$\Bigg(\frac{p}{a^2},\frac{a^4+p^2+2a(2a^2x+(a^4+p^2)y)\tan(ay)}{2a^2\sqrt{2(a^4+p^2)}\sec(ay)},$$
$$\frac{(a^4+p^2)\tan(ay)-2a(2a^2x+(a^4+p^2)y)}{2a^2\sqrt{2(a^4+p^2)}\sec(ay)},$$
$$\frac{3(a^4+p^2)\tan(ay)-2a(2a^2x+(a^4+p^2)y)}{2a^2\sqrt{2(a^4+p^2)}\sec(ay)},$$
$$\frac{3a^4+3p^2+2a(2a^2x+(a^4+p^2)y)\tan(ay)}{2a^2\sqrt{2(a^4+p^2)}\sec(ay)}\Bigg)$$

with $a,p>0$;

(17) *a non-minimal flat surface given by*

$$\Bigg(\frac{p}{a^2},\frac{a^4+p^2+2a(2a^2x-(a^4+p^2)y)\tanh(ay)}{2a^2\sqrt{2(a^4+p^2)}\operatorname{sech}(ay)},$$

$$\frac{3(a^4+p^2)\tanh(ay)+2a(2a^2x-(a^4+p^2)y)}{2a^2\sqrt{2(a^4+p^2)}\,\mathrm{sech}\,(ay)},$$
$$\frac{(a^4+p^2)\tanh(ay)+2a(2a^2x-(a^4+p^2)y)}{2a^2\sqrt{2(a^4+p^2)}\,\mathrm{sech}\,(ay)},$$
$$\left.\frac{3a^4+3p^2+2a(2a^2x-(a^4+p^2)y)\tanh(ay)}{2a^2\sqrt{2(a^4+p^2)}\,\mathrm{sech}\,(ay)}\right)$$

with $a, p > 0$*;*

(18) *a non-minimal flat surface given by*

$$\left(\frac{\beta\varphi}{\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}}, \frac{\cos u\sinh v}{\sqrt{1-b^2}}, \frac{\sin u\sinh v}{\sqrt{1-b^2}},\right.$$
$$\frac{\sqrt{1-b^2}\sqrt{\delta^2+\varphi^2}\cos u\cosh v+\beta\delta\sin u\sinh v)}{\sqrt{1-b^2}\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}},$$
$$\left.\frac{\sqrt{1-b^2}\sqrt{\delta^2+\varphi^2}\sin u\cosh v-\beta\delta\cos u\sinh v)}{\sqrt{1-b^2}\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}}\right)$$

with $\beta, \varphi \neq 0$*,* $b \in (0,1)$ *and*

$$u = \frac{\sqrt{1-b^2}\left[\beta\delta x-\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}x+(\delta^2+\varphi^2)y\right]}{\sqrt{2\delta^2+2\varphi^2}\sqrt{\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}-\beta\delta}},$$
$$v = \frac{\sqrt{1-b^2}\left[\beta\delta x+\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}x+(\delta^2+\varphi^2)y\right]}{\sqrt{2\delta^2+2\varphi^2}\sqrt{\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}+\beta\delta}};$$

(19) *a non-minimal flat surface given by*

$$\left(\frac{\sqrt{\sqrt{b^2-1}(\delta^2+\varphi^2)-b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt[4]{b^2-1}\sqrt{2(\beta^2\varphi^2-\delta^2-\varphi^2)}}\cos u, \frac{\sqrt{\sqrt{b^2-1}(\delta^2+\varphi^2)-b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt[4]{b^2-1}\sqrt{2(\beta^2\varphi^2-\delta^2-\varphi^2)}}\sin u,\right.$$
$$\frac{\sqrt{\sqrt{b^2-1}(\delta^2+\varphi^2)+b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt[4]{b^2-1}\sqrt{2(\beta^2\varphi^2-\delta^2-\varphi^2)}}\cosh v, \frac{\sqrt{\sqrt{b^2-1}(\delta^2+\varphi^2)+b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt[4]{b^2-1}\sqrt{2(\beta^2\varphi^2-\delta^2-\varphi^2)}}\sinh v,$$
$$\left.\frac{\beta\varphi}{\sqrt{\beta^2\varphi^2-\delta^2-\varphi^2}}\right)$$

with $\beta,\varphi\neq 0$, $b\delta<\sqrt{(b^2-1)(\delta^2+\varphi^2)}$, $b>1$ *and*

$$u=\frac{\sqrt{\sqrt{b^2-1}\sqrt{\delta^2+\varphi^2}+b\delta}}{\sqrt{\delta^2+\varphi^2}}(\sqrt{b^2-1}x+\sqrt{\delta^2+\varphi^2}y),$$
$$v=\frac{\sqrt{\sqrt{b^2-1}\sqrt{\delta^2+\varphi^2}-b\delta}}{\sqrt{\delta^2+\varphi^2}}(\sqrt{b^2-1}x-\sqrt{\delta^2+\varphi^2}y);$$

(20) *a non-minimal flat surface given by*

$$\Bigg(\frac{\beta\varphi}{\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}},\frac{\sqrt[4]{\delta^2+\varphi^2}\sqrt{b\delta-\sqrt{(b^2-1)(\delta^2+\varphi^2)}}}{\sqrt[4]{b^2-1}\sqrt{2(\delta^2+\varphi^2-\beta^2\varphi^2)}}\cos u,$$
$$\frac{\sqrt[4]{\delta^2+\varphi^2}\sqrt{b\delta-\sqrt{(b^2-1)(\delta^2+\varphi^2)}}}{\sqrt[4]{b^2-1}\sqrt{2(\delta^2+\varphi^2-\beta^2\varphi^2)}}\sin u,\frac{\sqrt[4]{\delta^2+\varphi^2}\sqrt{b\delta+\sqrt{(b^2-1)(\delta^2+\varphi^2)}}}{\sqrt[4]{b^2-1}\sqrt{2(\delta^2+\varphi^2-\beta^2\varphi^2)}}\cos v,$$
$$\frac{\sqrt[4]{\delta^2+\varphi^2}\sqrt{b\delta+\sqrt{(b^2-1)(\delta^2+\varphi^2)}}}{\sqrt[4]{b^2-1}\sqrt{2(\beta^2\varphi^2-\delta^2-\varphi^2)}}\sin v\Bigg)$$

with $\beta,\varphi\neq 0$, $b\delta>\sqrt{(b^2-1)(\delta^2+\varphi^2)}$, $b>1$ *and*

$$u=\frac{\sqrt{b\delta+\sqrt{b^2-1}\sqrt{\delta^2+\varphi^2}}}{\sqrt{\delta^2+\varphi^2}}(\sqrt{b^2-1}x+\sqrt{\delta^2+\varphi^2}y),$$
$$v=\frac{\sqrt{b\delta-\sqrt{b^2-1}\sqrt{\delta^2+\varphi^2}}}{\sqrt{\delta^2+\varphi^2}}(\sqrt{b^2-1}x-\sqrt{\delta^2+\varphi^2}y);$$

(21) *a non-minimal flat surface given by*

$$\left(\frac{1}{2\sqrt{b^2-1}}\cos\left(\frac{\sqrt{2}((b^2-1)x-br^2y)}{\sqrt{b}r}\right),\frac{1}{2\sqrt{b^2-1}}\sin\left(\frac{\sqrt{2}((b^2-1)x-br^2y)}{\sqrt{b}r}\right),\right.$$
$$\left.\frac{((b^2-1)x-br^2y)^2}{2b\sqrt{b^2-1}\sqrt{4b^2-3}r^2},\frac{(b^2-1)x-br^2y}{\sqrt{2b}\sqrt{b^2-1}r},\frac{b(4b^2-3)r^2-((b^2-1)x-br^2y)^2}{2b\sqrt{b^2-1}\sqrt{4b^2-3}r^2}\right),$$
$$b>1,r>0.$$

Conversely, every parallel Lorentz surface M_1^2 *in* $S_3^4(1)$ *is congruent to* *an open portion of one of the* 21 *families of surfaces described above.*

Proof. It follows from direct long computation that each surface described in the theorem is a parallel Lorentz surface in $S^4_3(1)$.

Conversely, assume that $\psi : M^2_1 \to S^4_3(1)$ is an isometric immersion of a Lorentz surface M^2_1 into $S^4_3(1)$. If M is totally geodesic in $S^4_3(1)$, we obtain case (1). So, let us assume that M^2_1 is non-totally geodesic in $S^4_3(1)$.

Let us choose a local coordinate system $\{x, y\}$ on M^2_1 which satisfies (2.11). Then we have (2.12)-(2.19).

Case (i): M^2_1 *is minimal in* $S^4_3(1)$. In this case, we get $h(e_1, e_2) = 0$. So, we have

$$h(e_1, e_1) = \xi,\ h(e_1, e_2) = 0,\ h(e_2, e_2) = \eta \tag{4.2}$$

for some normal vector fields ξ, η, not both zero. Without loss of generality, we may assume that $\xi \neq 0$. Let us choose an orthonormal frame $\{e_3, e_4\}$ such that e_3 is in the direction of ξ. Hence, we obtain

$$h(e_1, e_1) = \alpha e_3,\ h(e_1, e_2) = 0,\ h(e_2, e_2) = \lambda e_3 + \mu e_4 \tag{4.3}$$

for some functions α, λ, μ with $\alpha > 0$. Let us put $\alpha = a^2$ with $a > 0$.

Since M^2_1 is a parallel surface in $S^4_3(1)$, we find from (2.9), (2.16) and (4.3) that $De_3 = De_4 = 0$ and

$$da = a\omega,\ \ d\lambda = -2\lambda\omega,\ \ d\mu = -2\mu\omega. \tag{4.4}$$

Since $a > 0$, the first equation in (4.4) shows that ω is exact. Hence, M^2_1 is flat due to the structure equation. Therefore, we may choose $E = 1$, which gives $\omega = 0$. Consequently, we see from (4.4) that a, λ, μ are constants.

It follows from (2.14), (4.3), and the formula of Gauss that the immersion $L = \iota \circ \psi : M^2_1 \to S^4_3(1) \subset \mathbb{E}^5_3$ satisfies

$$L_{xx} = a^2 e_3,\ \ L_{xy} = L,\ \ L_{yy} = \lambda e_3 + \mu e_4. \tag{4.5}$$

Since M_1^2 is flat, we find from the equation of Gauss and (4.5) that $a^2\lambda = -1$. Also, by applying (2.4) and (4.3), we find

$$A_{e_3} = \begin{pmatrix} 0 & -\frac{1}{a^2} \\ a^2 & 0 \end{pmatrix}, \; A_{e_4} = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}. \tag{4.6}$$

By using equation (2.8) of Ricci and (3.4), we obtain $\mu = 0$. Therefore, (4.5) becomes

$$L_{xx} = a^2 e_3, \;\; L_{xy} = L, \;\; L_{yy} = -\frac{e_3}{a^2}. \tag{4.7}$$

Also, it follows from $De_3 = 0$ and (4.6), we have that

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -a^2 L_y, \;\; \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = \frac{L_x}{a^2}. \tag{4.8}$$

After solving system (4.7)-(4.8), we get

$$\begin{aligned} L(x,y) = \cosh\left(\frac{a^2x+y}{\sqrt{2}a}\right)&\left(c_1\cos\left(\frac{a^2x-y}{\sqrt{2}a}\right)+c_2\sin\left(\frac{a^2x-y}{\sqrt{2}a}\right)\right) \\ +\sinh\left(\frac{a^2x+y}{\sqrt{2}a}\right)&\left(c_3\cos\left(\frac{a^2x-y}{\sqrt{2}a}\right)+c_4\sin\left(\frac{a^2x-y}{\sqrt{2}a}\right)\right). \end{aligned}$$

Hence, after choosing suitable initial conditions, we obtain case (2).

Case (ii): *M_1^2 is non-minimal in $S_3^4(1)$.* In this case, we have $h(e_1,e_2) \neq 0$. Thus, we may choose an orthonormal frame $\{e_3, e_4\}$ such that e_3 is in the direction of $h(e_1,e_2)$. So, we have

$$h(e_1,e_1) = \beta e_3 + \gamma e_4, \; h(e_1,e_2) = b e_3, \; h(e_2,e_2) = \delta e_3 + \varphi e_4 \tag{4.9}$$

for some functions $b,\beta,\gamma,\delta,\varphi$ with $b > 0$. Since $\bar{\nabla} h = 0$, we obtain from (2.16) and (4.9) that $De_3 = De_4 = 0$, and

$$db = 0, \; d\beta = 2\beta\omega, \; d\gamma = 2\gamma\omega, \; d\delta = -2\delta\omega, \; d\varphi = -2\varphi\omega. \tag{4.10}$$

From (2.4) and (4.9) we have

$$A_{e_3} = \begin{pmatrix} b & \delta \\ \beta & b \end{pmatrix}, \; A_{e_4} = \begin{pmatrix} 0 & \varphi \\ \gamma & 0 \end{pmatrix}. \tag{4.11}$$

Since $De_3 = 0$, the equation of Ricci and (4.11) give

$$\delta\gamma = \beta\varphi. \tag{4.12}$$

So, (4.9) and the equation of Gauss imply that the Gauss curvature K is given by

$$K = 1 - b^2 + \beta\delta + \gamma\varphi. \tag{4.13}$$

Case (ii.1): $\beta = \gamma = \delta = \varphi = 0$. Equations (4.9) and (4.11) reduce to

$$h(e_1, e_1) = 0,\ h(e_1, e_2) = be_3,\ h(e_2, e_2) = 0, \tag{4.14}$$

$$A_{e_3} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.15}$$

Case (ii.1.1): $b = 1$. In this case, M is flat. So, we may choose coordinates $\{x, y\}$ such that $\partial/\partial x = e_1, \partial/\partial y = e_2$. Thus, we have $g = -(dx \otimes dy + dy \otimes dx)$. Therefore, the immersion $L : M_1^2 \to S_3^4(1) \subset \mathbb{E}_2^5$ satisfies

$$L_{xx} = 0,\ \ L_{xy} = e_3 + L,\ \ L_{yy} = 0. \tag{4.16}$$

Moreover, since $De_3 = 0$, we obtain from (4.15) that

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -L_x,\ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -L_y. \tag{4.17}$$

Hence, after solving system (4.16)-(4.17), we obtain

$$L(x, y) = c_1 + c_2 x + c_3 y + c_4 xy,$$

which yields case (3).

Case (ii.1.2): $b \in (0, 1)$. Since $K = 1 - b^2 > 0$, we put $K = c^2$ with $c \in (0, 1)$. Let us choose coordinates $\{x, y\}$ such that

$$\frac{\partial}{\partial x} = \frac{\sqrt{2}e_1}{c(x + y)},\ \ \frac{\partial}{\partial y} = \frac{\sqrt{2}e_2}{c(x + y)}. \tag{4.18}$$

So, the metric tensor is given by

$$g = \frac{-2}{c^2(x+y)^2}(dx \otimes dy + dy \otimes dx), \tag{4.19}$$

and hence the Levi-Civita connection satisfies

$$\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = \frac{-2}{x+y}\frac{\partial}{\partial x}, \;\; \nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0, \;\; \nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = \frac{-2}{x+y}\frac{\partial}{\partial y}. \tag{4.20}$$

Thus, (4.9), (4.19) and (4.20) imply that

$$L_{xx} = -\frac{2L_x}{x+y}, \;\; L_{xy} = \frac{2\sqrt{1-c^2}e_3 + 2L}{c^2(x+y)^2}, \;\; L_{yy} = -\frac{2L_y}{x+y}. \tag{4.21}$$

Moreover, since $De_3 = 0$, we obtain from (4.11) that

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -\sqrt{1-c^2}L_x, \; \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\sqrt{1-c^2}L_y. \tag{4.22}$$

After solving system (4.21)-(4.22) we obtain

$$L(x,y) = c_0 + \frac{c_1 + c_2 y + c_3 xy}{x+y}.$$

Therefore, after choosing suitable initial conditions, we obtain case (4).

Case (ii.1.3): $b > 1$. Since $K = 1 - b^2$, we put $K = -c^2$ with $c > 0$. We choose coordinates $\{x, y\}$ such that

$$\frac{\partial}{\partial x} = \operatorname{sech}\left(\frac{cx+cy}{\sqrt{2}}\right)e_1, \;\; \frac{\partial}{\partial y} = \operatorname{sech}\left(\frac{cx+cy}{\sqrt{2}}\right)e_2. \tag{4.23}$$

So, the metric tensor is given by

$$g = -\operatorname{sech}^2\left(\frac{cx+cy}{\sqrt{2}}\right)(dx \otimes dy + dy \otimes dx) \tag{4.24}$$

and the Levi-Civita connection satisfies

$$\begin{aligned} &\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial x} = -\sqrt{2}c \tanh\left(\frac{cx+cy}{\sqrt{2}}\right)\frac{\partial}{\partial x}, \\ &\nabla_{\frac{\partial}{\partial x}} \frac{\partial}{\partial y} = 0, \\ &\nabla_{\frac{\partial}{\partial y}} \frac{\partial}{\partial y} = -\sqrt{2}c \tanh\left(\frac{cx+cy}{\sqrt{2}}\right)\frac{\partial}{\partial y}. \end{aligned} \tag{4.25}$$

Thus, we obtain from (4.9), (4.24) and (4.25) that

$$\begin{aligned} L_{xx} &= -\sqrt{2}c\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big)L_x, \\ L_{xy} &= \operatorname{sech}^2\Big(\frac{cx+cy}{\sqrt{2}}\Big)(\sqrt{1+c^2}e_3+L), \\ L_{yy} &= -\sqrt{2}c\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big)L_y. \end{aligned} \tag{4.26}$$

Moreover, since $De_3 = 0$, we have

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -\sqrt{1+c^2}L_x,\ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\sqrt{1+c^2}L_y. \tag{4.27}$$

After solving system (4.26)-(4.27) we obtain

$$\begin{aligned} L \;=\; & c_0 + c_1\sinh(\sqrt{2}\,cy) - c_2\cosh(\sqrt{2}\,cy) \\ & +\Big(c_3 - c_1\cosh(\sqrt{2}\,cy) + c_2\sinh(\sqrt{2}\,cy)\Big)\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big). \end{aligned}$$

Thus, after choosing suitable initial conditions, we obtain case (5).

Case (ii.2) *At least one of* $\beta, \gamma, \delta, \varphi$ *is nonzero.* In this case, (4.10) implies that ω is an exact 1-form. Hence, M_1^2 is flat. So, we may choose coordinates $\{x, y\}$ such that $E = 1$, so that we have $\frac{\partial}{\partial x} = e_1, \frac{\partial}{\partial y} = e_2$. Hence, the metric tensor is given by

$$g = -(dx \otimes dy + dy \otimes dx). \tag{4.28}$$

Therefore, (4.10) implies that $\beta, \gamma, \delta, \varphi$ are constant.

Case (ii.2.1): $\beta = 0$. It follows from (4.12) that $\gamma\delta = 0$. Thus, we have either $\gamma = 0$ or $\delta = 0$.

Case (ii.2.1.1): $\gamma = 0$. In this case, it follows from (4.13) that $b = 1$. Thus, (4.9) and (4.11) reduce to

$$h(e_1, e_1) = 0,\ h(e_1, e_2) = e_3,\ h(e_2, e_2) = \delta e_3 + \varphi e_4, \tag{4.29}$$

$$A_{e_3} = \begin{pmatrix} 1 & \delta \\ 0 & 1 \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & \varphi \\ 0 & 0 \end{pmatrix}. \tag{4.30}$$

Thus, we have

$$L_{xx}=0,\ \ L_{xy}=e_3+L,\ \ L_{yy}=\delta e_3+\varphi e_4. \tag{4.31}$$

Moreover, since $De_3=0$, we obtain from (4.30) that

$$\begin{aligned}&\tilde\nabla_{\frac{\partial}{\partial x}}e_3=-L_x,\ \ \tilde\nabla_{\frac{\partial}{\partial y}}e_3=-\delta L_x-L_y,\\ &\tilde\nabla_{\frac{\partial}{\partial x}}e_4=0,\ \ \tilde\nabla_{\frac{\partial}{\partial y}}e_4=-\varphi L_x.\end{aligned} \tag{4.32}$$

Case (ii.2.1.1.1): $\delta=0$. After solving system (4.31)-(4.32) we have

$$L(x,y)=c_0+c_1xy+c_2x+c_3y+c_4y^2-\frac{1}{6}c_2\varphi^2y^3-\frac{1}{24}c_1\varphi^2y^4,$$

which yields case (9) after choosing suitable initial conditions.

Case (ii.2.1.1.2): $\delta=c^2, c>0$. After solving system (4.31)-(4.32), we obtain

$$\begin{aligned}L(x,y)=c_0&+((2c^2x+c^4y+\varphi^2y)c_1+c_4)\cos(cy)\\&+((2c^2x+c^4y+\varphi^2y)c_2+c_3)\sin(cy)\end{aligned}$$

which yields case (10).

Case (ii.2.1.1.3): $\delta=-c^2, c>0$. After solving system (4.31)-(4.32) we obtain

$$\begin{aligned}L(x,y)=c_0&+(c_1+(2c^2x-c^4y-\varphi^2y)c_2)\cosh(cy)\\&+(c_3+(2c^2x-c^4y-\varphi^2y)c_4)\sinh(cy).\end{aligned}$$

This gives case (11).

Case (ii.2.1.2): $\gamma\neq 0$ *and* $\delta=0$. From (4.13) we get $\varphi=(b^2-1)/\gamma$. Hence, (4.9), (4.11) and (4.12) give

$$h(e_1,e_1)=\gamma e_4,\ h(e_1,e_2)=be_3,\ h(e_2,e_2)=\frac{b^2-1}{\gamma}e_4, \tag{4.33}$$

$$A_{e_3}=\begin{pmatrix}b&0\\0&b\end{pmatrix},\ A_{e_4}=\begin{pmatrix}0&\frac{b^2-1}{\gamma}\\ \gamma&0\end{pmatrix}. \tag{4.34}$$

Thus, we have

$$L_{xx} = \gamma e_4, \quad L_{xy} = be_3 + L, \quad L_{yy} = \frac{b^2-1}{\gamma} e_4. \tag{4.35}$$

Moreover, since $De_3 = 0$, we obtain from (4.34) that

$$\begin{aligned} &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x, \quad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -bL_y, \\ &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = -\gamma L_y, \quad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = \frac{1-b^2}{\gamma} L_x. \end{aligned} \tag{4.36}$$

Case (ii.2.1.2.1): $b = 1$. In this case, after solving system (4.35)-(4.36), we obtain

$$L(x,y) = c_0 + c_1 x + c_2 x^2 + c_3(\gamma^2 x^3 - 6y) + c_4 x(\gamma x^3 - 24y).$$

So, after choosing suitable initial conditions, we get case (12).

Case (ii.2.1.2.2): $b \in (0,1)$. If we put $b = \sqrt{1-c^4}$ and $a^4 = \gamma^2$, then after solving system (4.35)-(4.36), we obtain

$$\begin{aligned} L(x,y) = c_0 &+ \cos\Big(\frac{a^2cx - c^3y}{\sqrt{2}a}\Big)\Big\{c_1 \cosh\Big(\frac{a^2cx + c^3y}{\sqrt{2}a}\Big) + c_2 \sin\Big(\frac{a^2cx + c^3y}{\sqrt{2}a}\Big)\Big\} \\ &+ \sinh\Big(\frac{a^2cx - c^3y}{\sqrt{2}a}\Big)\Big\{c_3 \cosh\Big(\frac{a^2cx + c^3y}{\sqrt{2}a}\Big) + c_4 \sinh\Big(\frac{a^2cx + c^3y}{\sqrt{2}a}\Big)\Big\}, \end{aligned}$$

which gives case (13).

Case (ii.2.1.2.3): $b > 1$. If we put $b = \sqrt{1+c^4}$ and $a^4 = \gamma^2$, then after solving system (4.35)-(4.36), we obtain

$$\begin{aligned} L(x,y) = c_0 &+ c_1 \cosh\Big(acx - \frac{c^3y}{a}\Big) + c_2 \sinh\Big(acx - \frac{c^3y}{a}\Big) \\ &+ c_3 \cos\Big(acx + \frac{c^3y}{a}\Big) + c_4 \sin\Big(acx + \frac{c^3y}{a}\Big). \end{aligned}$$

Consequently, after choosing suitable initial conditions, we get case (14).

Case (ii.2.2): $\beta \neq 0$. We divide this into several cases.

Case (ii.2.2.1): $\delta = 0$. It follows from (4.12) that $\varphi = 0$. Thus, we find from

$K = 0$ and (4.13) that $b = 1$. Therefore, (4.9) reduces to

$$h(e_1, e_1) = \beta e_3 + \gamma e_4,\ h(e_1, e_2) = e_3,\ h(e_2, e_2) = 0. \tag{4.37}$$

Consequently, after interchanging x and y, this case falls into Case (ii.2.1.1).

Case (ii.2.2.2): $\delta \neq 0$ *and* $\gamma = 0$. It follows from (4.12) that $\varphi = 0$. Thus, we find from $K = 0$ and (4.13) that $\beta\delta = b^2 - 1$. Therefore, (4.9) and (4.11) reduce to

$$h(e_1, e_1) = \beta e_3,\ h(e_1, e_2) = b e_3,\ h(e_2, e_2) = \frac{b^2 - 1}{\beta} e_3, \tag{4.38}$$

$$A_{e_3} = \begin{pmatrix} b & \frac{1-b^2}{\beta} \\ \beta & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.39}$$

Thus, we have

$$L_{xx} = \beta e_3,\ \ L_{xy} = b e_3 + L,\ \ L_{yy} = \frac{b^2 - 1}{\beta} e_3. \tag{4.40}$$

Moreover, since $De_3 = 0$, we obtain from (4.39) that

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x - \beta L_y,\ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = \frac{b^2 - 1}{\beta} L_x - bL_y. \tag{4.41}$$

Case (ii.2.2.2.1): $b = 1$. If $\beta > 0$, we put $\beta = a^2, a > 0$. Then after solving system (4.40)-(4.41) we get

$$L(x, y) = \left(c_1 + c_2\left(x + \frac{2y}{a^2}\right)\right)\cos(ax) + \left(c_3 + c_4\left(x + \frac{2y}{a^2}\right)\right)\sin(ax),$$

which gives case (6).

If $\beta < 0$, we put $\beta = -a^2, a > 0$. Then after solving system (4.40)-(4.41) we obtain

$$L(x, y) = \left(c_1 + c_2\left(x - \frac{2y}{a^2}\right)\right)\cosh(ax) + \left(c_3 + c_4\left(x - \frac{2y}{a^2}\right)\right)\sinh(ax),$$

which gives case (7).

Case (ii.2.2.2.2): $b \in (0, 1)$. If we put $|\beta| = a^2, a > 0$, then after solving system (4.40)-(4.41), we get

$$\begin{aligned}L(x,y) &= c_1 \cos\left(\frac{\sqrt{1+b}(a^2x-y+by)}{\sqrt{2}a}\right)\cosh\left(\frac{\sqrt{1-b}(a^2x+y+by)}{\sqrt{2}a}\right)\\ &+ c_2 \cos\left(\frac{\sqrt{1+b}(a^2x-y+by)}{\sqrt{2}a}\right)\sinh\left(\frac{\sqrt{1-b}(a^2x+y+by)}{\sqrt{2}a}\right)\\ &+ c_3 \sin\left(\frac{\sqrt{1+b}(a^2x-y+by)}{\sqrt{2}a}\right)\cosh\left(\frac{\sqrt{1-b}(a^2x+y+by)}{\sqrt{2}a}\right)\\ &+ c_4 \sin\left(\frac{\sqrt{1+b}(a^2x-y+by)}{\sqrt{2}a}\right)\sinh\left(\frac{\sqrt{1-b}(a^2x+y+by)}{\sqrt{2}a}\right),\end{aligned}$$

which gives case (8) after choosing suitable initial conditions.

Case (ii.2.2.2.3): $b > 1$. After solving system (4.40)-(4.41), we obtain case (15).

Case (ii.2.2.3): $\delta, \gamma \neq 0$. Since $\beta \neq 0$, we find from (4.12) that $\varphi \neq 0$. Thus, it follows from $K = 0$ and (4.13) that

$$\beta = \frac{(b^2-1)\delta}{\delta^2+\varphi^2}, \quad \gamma = \frac{(b^2-1)\varphi}{\delta^2+\varphi^2}. \tag{4.42}$$

Consequently, (4.9) and $E = 1$ imply that

$$L_{xx} = \frac{b^2-1}{\delta^2+\varphi^2}(\delta e_3 + \varphi e_4), \quad L_{xy} = be_3 + L, \quad L_{yy} = \delta e_3 + \varphi e_4. \tag{4.43}$$

Moreover, it follows from $De_3 = 0$, (4.11), and (4.42) that

$$\begin{aligned}&\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x + \frac{(1-b^2)\delta}{\delta^2+\varphi^2}L_y, \quad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\delta L_x - bL_y,\\ &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = \frac{(1-b^2)\varphi}{\delta^2+\varphi^2}L_y, \quad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = -\varphi L_x.\end{aligned} \tag{4.44}$$

Case (ii.2.2.3.1): $b = 1$. We divide this into two cases.

If $\delta > 0$, we put $\delta = a^2$ with $a > 0$. Then after solving system (4.43)-

(4.44) we obtain

$$\begin{aligned} L(x,y) =& c_0 + c_1\cos(ay) + c_2\sin(ay) + c_3\{2a(2a^2x + (a^4+\varphi^2)y)\cos(ay) \\ & - 3(a^4+\varphi^2)\sin(ay)\} \\ & + c_3\{3(a^4+\varphi^2)\cos(ay) + 2a(2a^2x + (a^4+\varphi^2)y)\sin(ay)\}. \end{aligned}$$

This gives Case (16) after choosing suitable initial conditions.

Similarly, if $\delta < 0$, then after solving system (4.43)-(4.44) and choosing suitable initial conditions, we obtain case (17).

Case (ii.2.2.3.2): $b \in (0,1)$. After solving system (4.43)-(4.44), we obtain

$$L(x,y) = c_0 + \cos u(c_1\cosh v + c_2\sinh v) + \sin u(c_3\cosh v + c_4\sinh v)$$

with

$$\begin{aligned} u &= \frac{\sqrt{1-b^2}\left[\beta\delta x - \sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}x + (\delta^2+\varphi^2)y\right]}{\sqrt{2\delta^2+2\varphi^2}\sqrt{\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}-\beta\delta}}, \\ v &= \frac{\sqrt{1-b^2}\left[\beta\delta x + \sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}x + (\delta^2+\varphi^2)y\right]}{\sqrt{2\delta^2+2\varphi^2}\sqrt{\sqrt{\delta^2+\varphi^2-\beta^2\varphi^2}+\beta\delta}}. \end{aligned}$$

This gives case (18) after choosing suitable initial conditions.

Case (ii.2.2.3.3): $b > 1$. We divide this into three cases.

Case (ii.2.2.3.3.1): $b\delta < \sqrt{(b^2-1)(\delta^2+\varphi^2)}$. In this case, after solving system (4.43)-(4.44), we obtain

$$L(x,y) = c_0 + c_1\cos u + c_2\sin u + c_3\cosh v + c_4\sinh v$$

with $\beta, \delta, \varphi \neq 0, b > 1$ and

$$\begin{aligned} u &= \frac{\sqrt{\sqrt{b^2-1}\sqrt{\delta^2+\varphi^2}+b\delta}}{\sqrt{\delta^2+\varphi^2}}(\sqrt{b^2-1}x + \sqrt{\delta^2+\varphi^2}y), \\ v &= \frac{\sqrt{\sqrt{b^2-1}\sqrt{\delta^2+\varphi^2}-b\delta}}{\sqrt{\delta^2+\varphi^2}}(\sqrt{b^2-1}x - \sqrt{\delta^2+\varphi^2}y). \end{aligned}$$

Hence, after choosing suitable initial conditions, we have case (19).

Case (ii.2.2.3.3.2): $b\delta > \sqrt{(b^2-1)(\delta^2+\varphi^2)}$. In this case, after solving system (4.43)-(4.44), we obtain

$$L(x,y) = c_0 + c_1 \cos u + c_2 \sin u + c_3 \cos v + c_4 \sin v$$

with $\beta, \delta, \varphi \neq 0, b > 1$ and

$$u = \frac{\sqrt{b\delta + \sqrt{b^2-1}\sqrt{\delta^2+\varphi^2}}}{\sqrt{\delta^2+\varphi^2}}(\sqrt{b^2-1}x + \sqrt{\delta^2+\varphi^2}y),$$
$$v = \frac{\sqrt{b\delta - \sqrt{b^2-1}\sqrt{\delta^2+\varphi^2}}}{\sqrt{\delta^2+\varphi^2}}(\sqrt{b^2-1}x - \sqrt{\delta^2+\varphi^2}y).$$

Hence, after choosing suitable initial conditions, we have case (20).

Case (ii.2.2.3.3.3): $b\delta = \sqrt{(b^2-1)(\delta^2+\varphi^2)}$. In this case, we have $\delta > 0$. Let us put $\delta = r^2$ with $r > 0$. Then, after solving system (4.43)-(4.44), we obtain

$$L(x,y) = c_0 + c_1((b^2-1)x - br^2y)^2 + c_2\Big(x + \frac{br^2y}{b^2-1}\Big)$$
$$+ c_3 \cos\Big(\frac{\sqrt{2}((b^2-1)x - br^2y)}{\sqrt{b}r}\Big) + c_4 \sin\Big(\frac{\sqrt{2}((b^2-1)x - br^2y)}{\sqrt{b}r}\Big).$$

Hence, we have case (21) after choosing suitable initial conditions. □

Remark 4.1. After comparing Theorems 4.1 and 4.2, we see that there are essential difference between Lorentz surfaces in Lorentzian space forms $R^4_1(c)$ and in indefinite space forms $R^4_3(c)$ with index 3.

5. Parallel Lorentz Surfaces in $H^4_3(-1)$

Now, we provide the classification of all parallel Lorentz surfaces in the pseudo-hyperbolic 4-space $H^4_3(-1)$ with index 3.

Theorem 5.1. *There are six families of parallel Lorentz surfaces in* $H^4_3(-1) \subset \mathbb{E}^5_4$*:*

(1) *a totally geodesic anti-de Sitter space* $H^2_1(-1) \subset H^4_3(-1)$*;*

(2) *a flat minimal surface in a totally geodesic* $H^3_2(-1) \subset H^4_3(-1)$ *defined by*

$$\frac{1}{\sqrt{2}}\left(\sin\left(ax+\frac{y}{a}\right),\cos\left(ax+\frac{y}{a}\right),\cosh\left(ax-\frac{y}{a}\right),\sinh\left(ax-\frac{y}{a}\right),0\right),\ a>0;$$

(3) *a totally umbilical anti-de Sitter space* $H^2_1(-c^2)$ *in a totally geodesic*

$H^3_2(-1) \subset H^4_3(-1)$ *given by*

$$\frac{1}{c}\Bigg(0,\sqrt{c^2-1},\tanh\left(\frac{cx+cy}{\sqrt{2}}\right),\sinh(\sqrt{2}cy)\tanh\left(\frac{cx+cy}{\sqrt{2}}\right)-\cosh(\sqrt{2}cy),$$
$$\sinh(\sqrt{2}cy)-\cosh(\sqrt{2}cy)\tanh\left(\frac{cx+cy}{\sqrt{2}}\right)\Bigg),\ c>1;$$

(4) *a CMC flat surface in a totally geodesic* $H^3_2(-1)$ *given by*

$$\Bigg(\frac{\sqrt{\sqrt{1+b^2}-b}}{\sqrt{2}\sqrt[4]{1+b^2}}\cos\Big(\frac{\sqrt{\sqrt{1+b^2}+b}(a^2x+\sqrt{1+b^2}y)}{a}\Big),$$
$$\frac{\sqrt{\sqrt{1+b^2}-b}}{\sqrt{2}\sqrt[4]{1+b^2}}\sin\Big(\frac{\sqrt{\sqrt{1+b^2}+b}(a^2x+\sqrt{1+b^2}y)}{a}\Big),$$
$$\frac{\sqrt{\sqrt{1+b^2}+b}}{\sqrt{2}\sqrt[4]{1+b^2}}\cosh\Big(\frac{\sqrt{\sqrt{1+b^2}-b}(a^2x-\sqrt{1+b^2}y)}{a}\Big),$$
$$\frac{\sqrt{\sqrt{1+b^2}+b}}{\sqrt{2}\sqrt[4]{1+b^2}}\sin\Big(\frac{\sqrt{\sqrt{1+b^2}-b}(a^2x-\sqrt{1+b^2}y)}{a}\Big)\Bigg)$$

with $a,b,c>0$*;*

(5) *a non-minimal flat surface given by*

$$\frac{1}{\sqrt{2}\sqrt{1+b^2}}\Bigg(\sqrt{2}b,\cos\left(kx+\frac{k^3}{\gamma^2}y\right),\sin\left(kx+\frac{k^3}{\gamma^2}y\right),$$
$$\cosh\left(kx-\frac{k^3}{\gamma^2}y\right),\sinh\left(kx-\frac{k^3}{\gamma^2}y\right)\Bigg),\ k=\sqrt[4]{(1+b^2)\gamma^2},\ b,\gamma>0;$$

(6) *a non-minimal flat surface given by*

$$\Bigg(\frac{b\varphi}{\sqrt{\delta^2+(1+b^2)\varphi^2}},$$
$$\frac{\sqrt{\sqrt{1+b^2}(\delta^2+\varphi^2)-b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt{2}\sqrt[4]{1+b^2}\sqrt{\delta^2+(1+b^2)\varphi^2}}\cos\big(\lambda(\sqrt{1+b^2}x+\sqrt{\delta^2+\varphi^2}y\big),$$
$$\frac{\sqrt{\sqrt{1+b^2}(\delta^2+\varphi^2)-b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt{2}\sqrt[4]{1+b^2}\sqrt{\delta^2+(1+b^2)\varphi^2}}\sin\big(\lambda(\sqrt{1+b^2}x+\sqrt{\delta^2+\varphi^2}y\big),$$
$$\frac{\sqrt{\sqrt{1+b^2}(\delta^2+\varphi^2)+b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt{2}\sqrt[4]{1+b^2}\sqrt{\delta^2+(1+b^2)\varphi^2}}\cosh\big(\mu(\sqrt{1+b^2}x-\sqrt{\delta^2+\varphi^2}y\big),$$
$$\frac{\sqrt{\sqrt{1+b^2}(\delta^2+\varphi^2)+b\delta\sqrt{\delta^2+\varphi^2}}}{\sqrt{2}\sqrt[4]{1+b^2}\sqrt{\delta^2+(1+b^2)\varphi^2}}\sinh\big(\mu(\sqrt{1+b^2}x-\sqrt{\delta^2+\varphi^2}y\big)\Bigg)$$

with $\delta, \varphi \neq 0, b > 0$ *and*

$$\lambda = \frac{\sqrt{\sqrt{1+b^2}\sqrt{\delta^2+\varphi^2}+b\delta}}{\sqrt{\delta^2+\varphi^2}}, \quad \mu = \frac{\sqrt{\sqrt{1+b^2}\sqrt{\delta^2+\varphi^2}-b\delta}}{\sqrt{\delta^2+\varphi^2}}.$$

Conversely, every parallel Lorentz surface M_1^2 *in* $H_3^4(-1)$ *is congruent to an open portion of one of the six families of surfaces described above.*

Proof. It follows from direct long computation that each surface described in the theorem is a parallel Lorentz surface in $H_3^4(-1)$.

Conversely, assume that $\psi : M \to H_3^4(-1)$ is an isometric immersion of a Lorentz surface M_1^2 into $H_3^4(-1)$. If M is totally geodesic in $H_3^4(-1)$, we obtain case (1). So, let us assume that M_1^2 is non-totally geodesic in $H_3^4(-1)$.

Let us choose a local coordinate system $\{x, y\}$ on M_1^2 which satisfies (2.11). Then we have (2.12)-(2.19).

Case (i): M_1^2 *is minimal in* $H_3^4(-1)$. In this case, we get $h(e_1, e_2) = 0$. So, we have

$$h(e_1, e_1) = \xi,\ h(e_1, e_2) = 0,\ h(e_2, e_2) = \eta \tag{5.1}$$

for some normal vector fields ξ, η, not both zero. Without loss of generality, we may assume that $\xi \neq 0$. Let us choose orthonormal frame $\{e_3, e_4\}$ such that e_3 is in the direction of ξ. Hence, we obtain

$$h(e_1, e_1) = \alpha e_3,\ h(e_1, e_2) = 0,\ h(e_2, e_2) = \lambda e_3 + \mu e_4 \tag{5.2}$$

for some functions α, λ, μ with $\alpha > 0$. Let us put $\alpha = a^2$ with $a > 0$.

Since M_1^2 is a parallel surface in $H_3^4(-1)$, we find from (2.9), (2.16) and (5.2) that

$$De_3 = De_4 = 0, \tag{5.3}$$

$$da = a\omega,\ \ d\lambda = -2\lambda\omega,\ \ d\mu = -2\mu\omega. \tag{5.4}$$

Since $a > 0$, the first equation in (5.4) shows that ω is exact. Hence, M_1^2 is flat. Thus, we may choose $E = 1$, which gives $\omega = 0$. Consequently, a, λ, μ are constants.

Since M_1^2 is flat, the equation of Gauss and (5.2) yield $a^2\lambda = 1$. Also, by applying (2.4) and (5.2), we find

$$A_{e_3} = \begin{pmatrix} 0 & \frac{1}{a^2} \\ a^2 & 0 \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & \mu \\ 0 & 0 \end{pmatrix}. \tag{5.5}$$

By using Eqs. (2.8) of Ricci and (5.5), we obtain $\mu = 0$. Therefore, we obtain

$$L_{xx} = a^2 e_3,\ \ L_{xy} = -L,\ \ L_{yy} = \frac{e_3}{a^2}. \tag{5.6}$$

Also, it follows from $De_3 = 0$ and (5.5), we have

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -a^2 L_y,\ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\frac{L_x}{a^2}. \tag{5.7}$$

After solving system (5.6)-(5.7), we get

$$\begin{aligned} L(x, y) = c_1 \sin\left(ax + \frac{y}{a}\right) + c_2 \cos\left(ax + \frac{y}{a}\right) + c_3 \cosh\left(ax - \frac{y}{a}\right) \\ + c_4 \sinh\left(ax - \frac{y}{a}\right). \end{aligned}$$

Hence, after choosing suitable initial conditions, we obtain case (2).

Case (ii): *M_1^2 is non-minimal in $H_3^4(-1)$.* In this case, we have $h(e_1, e_2) \neq 0$. Thus, we may choose an orthonormal frame $\{e_3, e_4\}$ such that e_3 is in the direction of $h(e_1, e_2)$. So, we have

$$h(e_1, e_1) = \beta e_3 + \gamma e_4,\ h(e_1, e_2) = be_3,\ h(e_2, e_2) = \delta e_3 + \varphi e_4 \tag{5.8}$$

for some functions $b, \beta, \gamma, \delta, \varphi$ with $b > 0$. Since $\bar{\nabla} h = 0$, we obtain from (2.16) and (5.8) that $De_3 = De_4 = 0$, and

$$db = 0,\ d\beta = 2\beta\omega,\ d\gamma = 2\gamma\omega,\ d\delta = -2\delta\omega,\ d\varphi = -2\varphi\omega. \tag{5.9}$$

From (2.4) and (5.8) we have

$$A_{e_3} = \begin{pmatrix} b & \delta \\ \beta & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & \varphi \\ \gamma & 0 \end{pmatrix}. \tag{5.10}$$

Since $De_3 = 0$, the equation of Ricci and (5.10) give

$$\delta\gamma = \beta\varphi. \tag{5.11}$$

So, (5.8) and the equation of Gauss imply that the Gauss curvature K is given by

$$K = \beta\delta + \gamma\varphi - b^2 - 1. \tag{5.12}$$

Case (ii.1): $\beta = \gamma = \delta = \varphi = 0$. Eqs. (5.8) and (5.10) reduce to

$$h(e_1, e_1) = 0,\ h(e_1, e_2) = be_3,\ h(e_2, e_2) = 0, \tag{5.13}$$

$$A_{e_3} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.14}$$

We find from (5.12) that $K = -(b^2 + 1)$. Let us put $K = -c^2$ with $c = \sqrt{b^2 + 1}$. We choose coordinates $\{x, y\}$ such that

$$\frac{\partial}{\partial x} = \operatorname{sech}\left(\frac{cx + cy}{\sqrt{2}}\right) e_1,\ \ \frac{\partial}{\partial y} = \operatorname{sech}\left(\frac{cx + cy}{\sqrt{2}}\right) e_2. \tag{5.15}$$

So, the metric tensor is given by

$$g = -\operatorname{sech}^2\Big(\frac{cx+cy}{\sqrt{2}}\Big)(dx\otimes dy + dy\otimes dx) \tag{5.16}$$

and the Levi-Civita connection satisfies

$$\begin{aligned}
&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial x} = -\sqrt{2}c\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big)\frac{\partial}{\partial x},\\
&\nabla_{\frac{\partial}{\partial x}}\frac{\partial}{\partial y} = 0,\\
&\nabla_{\frac{\partial}{\partial y}}\frac{\partial}{\partial y} = -\sqrt{2}c\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big)\frac{\partial}{\partial y}.
\end{aligned} \tag{5.17}$$

Thus, we obtain from (5.8), (5.16) and (5.24) that

$$\begin{aligned}
&L_{xx} = -\sqrt{2}c\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big)L_x,\\
&L_{xy} = \operatorname{sech}^2\Big(\frac{cx+cy}{\sqrt{2}}\Big)(\sqrt{c^2-1}e_3 - L),\\
&L_{yy} = -\sqrt{2}c\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big)L_y.
\end{aligned} \tag{5.18}$$

Moreover, since $De_3 = 0$, we have

$$\tilde\nabla_{\frac{\partial}{\partial x}}e_3 = -\sqrt{c^2-1}L_x,\ \tilde\nabla_{\frac{\partial}{\partial y}}e_3 = -\sqrt{c^2-1}L_y. \tag{5.19}$$

After solving system (5.18)-(5.19) we obtain

$$\begin{aligned}
L \;=\;& c_0 + c_1\cosh(\sqrt{2}\,cy) + c_2\sinh(\sqrt{2}\,cy)\\
&+\Big(c_3 - c_2\cosh(\sqrt{2}\,cy) - c_1\sinh(\sqrt{2}\,cy)\Big)\tanh\Big(\frac{cx+cy}{\sqrt{2}}\Big).
\end{aligned}$$

Thus, after choosing suitable initial conditions, we obtain case (3).

Case (ii.2) *At least one of $\beta,\gamma,\delta,\varphi$ is nonzero.* In this case, (5.9) implies that ω is an exact 1-form. Hence, M_1^2 is flat. So, we may choose coordinates $\{x,y\}$ such that $E=1$, so that we have $\frac{\partial}{\partial x}=e_1, \frac{\partial}{\partial y}=e_2$. Hence, the metric tensor is given by

$$g = -(dx\otimes dy + dy\otimes dx), \tag{5.20}$$

Therefore, (5.9) implies that $\beta,\gamma,\delta,\varphi$ are constant.

Case (ii.2.1): $\beta = 0$. It follows from (5.11) and (5.12) that $\varphi = (1+b^2)/\gamma$ and $\delta = 0$. Thus, (5.8) and (5.10) reduce to

$$h(e_1, e_1) = \gamma e_4,\ h(e_1, e_2) = be_3,\ h(e_2, e_2) = \frac{1+b^2}{\gamma} e_4, \tag{5.21}$$

$$A_{e_3} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & \frac{1+b^2}{\gamma} \\ \gamma & 0 \end{pmatrix}. \tag{5.22}$$

Thus, we have

$$L_{xx} = \gamma e_4,\quad L_{xy} = be_3 - L,\quad L_{yy} = \frac{1+b^2}{\gamma} e_4. \tag{5.23}$$

Moreover, since $De_3 = 0$, we obtain from (5.22) that

$$\begin{aligned} &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x,\ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -bL_y, \\ &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = -\gamma L_y,\ \ \tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = -\frac{1+b^2}{\gamma} L_x. \end{aligned} \tag{5.24}$$

After solving system (5.23)-(5.24) we have

$$\begin{aligned} L(x,y) = c_0 &+ c_1 \cos\Big(kx + \frac{k^3}{\gamma^2} y\Big) + c_2 \sin\Big(kx + \frac{k^3}{\gamma^2} y\Big) \\ &+ c_3 \cosh\Big(kx - \frac{k^3}{\gamma^2} y\Big) + c_4 \sinh\Big(kx - \frac{k^3}{\gamma^2} y\Big) \end{aligned}$$

with $k = \sqrt[4]{(1+b^2)\gamma^2}$. This yields case (5).

Case (ii.2.2): $\beta \neq 0$. If $\delta = 0$, it follows from (5.11) and (5.12) that $K = b^2 + 1 = 0$, which is impossible. Thus, we must have $\delta \neq 0$.

Case (ii.2.2.1): $\gamma = 0$. It follows from (5.11) that $\varphi = 0$. Thus, we find from $K = 0$ and (5.12) that $\beta\delta = 1 + b^2$. Therefore, (5.8) and (5.10) reduce to

$$h(e_1, e_1) = \beta e_3,\ h(e_1, e_2) = be_3,\ h(e_2, e_2) = \frac{1+b^2}{\beta} e_3, \tag{5.25}$$

$$A_{e_3} = \begin{pmatrix} b & \frac{1+b^2}{\beta} \\ \beta & b \end{pmatrix},\ A_{e_4} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.26}$$

Thus, we have

$$L_{xx} = \beta e_3, \quad L_{xy} = be_3 - L, \quad L_{yy} = \frac{1+b^2}{\beta} e_3. \tag{5.27}$$

Moreover, since $De_3 = 0$, we obtain from (5.26) that

$$\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x - \beta L_y, \quad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\frac{1+b^2}{\beta} L_x - bL_y. \tag{5.28}$$

Thus, after solving system (5.27)-(5.28), we get

$$\begin{aligned} L(x,y) \;=\; & c_1 \cos\left(\frac{\sqrt{\sqrt{1+b^2}+b}(a^2x+\sqrt{1+b^2}y)}{a}\right) \\ & +c_2 \sin\left(\frac{\sqrt{\sqrt{1+b^2}+b}(a^2x+\sqrt{1+b^2}y)}{a}\right) \\ & +c_3 \cosh\left(\frac{\sqrt{\sqrt{1+b^2}-b}(a^2x-\sqrt{1+b^2}y)}{a}\right) \\ & +c_4 \sinh\left(\frac{\sqrt{\sqrt{1+b^2}-b}(a^2x-\sqrt{1+b^2}y)}{a}\right) \end{aligned}$$

which gives case (4) after choosing suitable initial conditions.

Case (ii.2.2.2): $\gamma \neq 0$. Since $\beta \neq 0$, we find from (5.11) that $\varphi \neq 0$. Thus, it follows from $K = 0$ and (5.12) that

$$\beta = \frac{(1+b^2)\delta}{\delta^2+\varphi^2}, \quad \gamma = \frac{(1+b^2)\varphi}{\delta^2+\varphi^2}. \tag{5.29}$$

Consequently, (5.8) and $E = 1$ imply that

$$L_{xx} = \frac{1+b^2}{\delta^2+\varphi^2}(\delta e_3 + \varphi e_4), \quad L_{xy} = be_3 - L, \quad L_{yy} = \delta e_3 + \varphi e_4. \tag{5.30}$$

Moreover, it follows from $De_3 = 0$, (5.10), and (5.29) that

$$\begin{aligned} &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_3 = -bL_x - \frac{(1+b^2)\delta}{\delta^2+\varphi^2} L_y, \quad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_3 = -\delta L_x - bL_y, \\ &\tilde{\nabla}_{\frac{\partial}{\partial x}} e_4 = -\frac{(1+b^2)\varphi}{\delta^2+\varphi^2} L_y, \quad \tilde{\nabla}_{\frac{\partial}{\partial y}} e_4 = -\varphi L_x. \end{aligned} \tag{5.31}$$

After solving system (5.30)-(5.31), we obtain

$$\begin{aligned}L(x,y) = {} & c_0+\cos\left(\lambda(\sqrt{1+b^2}x+\sqrt{\delta^2+\varphi^2}y\right)+c_2\sin\left(\lambda(\sqrt{1+b^2}x+\sqrt{\delta^2+\varphi^2}y\right) \\ & +c_3\cosh\left(\mu(\sqrt{1+b^2}x-\sqrt{\delta^2+\varphi^2}y\right)+c_4\sinh\left(\mu(\sqrt{1+b^2}x-\sqrt{\delta^2+\varphi^2}y\right)\end{aligned}$$

with $\delta,\varphi\neq 0, b>0$ and

$$\lambda=\frac{\sqrt{\sqrt{1+b^2}\sqrt{\delta^2+\varphi^2}+b\delta}}{\sqrt{\delta^2+\varphi^2}},\quad \mu=\frac{\sqrt{\sqrt{1+b^2}\sqrt{\delta^2+\varphi^2}-b\delta}}{\sqrt{\delta^2+\varphi^2}}.$$

This gives case (6) after choosing suitable initial conditions.

Remark 5.1. Parallel submanifolds in indefinite space forms have parallel mean curvature vector. The complete classification of parallel space-like and parallel Lorentz surfaces in indefinite space forms with arbitrary codimension and arbitrary index are achieved in [7, 8, 11]. For the most updated survey on submanifolds with parallel mean curvature vector, see [10].

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