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VISCOUS CONSERVATION LAWS, PART I: SCALAR LAWS

BY

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Abstract

Viscous conservation laws are the basic models for the dissipative phenomena. We aim at a systematic presentation of the basic ideas for the quantitative study of the nonlinear waves for viscous conservation laws. The present paper concentrates on the scalar laws; an upcoming Part II will deal with the systems. The basic ideas for scalar viscous conservation laws originated from two sources: the theory for the hyperbolic conservation laws and the Burgers equation. We have initiated the Green's function approach. These ideas are streamlined, simplified and synthesized here. We then apply them to some new problems and raise open problems.

Quantitative understanding is necessary for further studies of the richer wave phenomena of the coupling of distinct wave types and the coupling of the boundary with the nonlinear waves. Viscous conservation laws may be viewed as the basic models for general dissipative systems.

1. Introduction

Consider the systems of hyperbolic and viscous conservation laws

$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = 0, \tag{1.1}$$

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$$\boldsymbol{u}_t + \boldsymbol{f}(\boldsymbol{u})_x = (\boldsymbol{B}(\boldsymbol{u})\boldsymbol{u}_x)_x, \ \boldsymbol{u} \in \mathbb{R}^n. \tag{1.2}$$

Important examples include the Euler equations and Navier-Stokes equations in gas dynamics:

$$\begin{aligned} &\rho_t + (\rho v)_x = 0, \\ &(\rho v)_t + (\rho v^2 + p)_x = 0, \\ &(\rho E)_t + (\rho E v + p v)_x = 0; \end{aligned} \tag{1.3}$$

$$\begin{aligned} &\rho_t + (\rho v)_x = 0, \\ &(\rho v)_t + (\rho v^2 + p)_x = (\nu v_x)_x, \\ &(\rho E)_t + (\rho E v + p v)_x = (\kappa \theta_x + \nu v v_x)_x. \end{aligned} \tag{1.4}$$

We are interested in the nonlinear behavior of solutions for the system, particularly the construction and stability of nonlinear waves such as the shock, rarefaction waves and diffusion waves. There is the *Green's function approach* for studies of nonlinear waves for dissipative systems, [30], [16], [23], [21], [22]. The approach is effective for the quantitative study of the wave behavior. The present paper concentrates on the scalar conservation laws

$$u_t + f(u)_x = 0, \tag{1.5}$$

$$u_t + f(u)_x = \kappa u_{xx}, \ u \in \mathbb{R}. \tag{1.6}$$

The *inviscid Burgers (Hopf) equation* and the *Burgers equation* are important examples:

$$u_t + u u_x = 0, \tag{1.7}$$

$$u_t + u u_x = \kappa u_{xx}, \ u \in \mathbb{R}. \tag{1.8}$$

There have been very substantial progresses toward the understanding of nonlinear waves for viscous conservation laws, due to the efforts of many people in the recent decades. The purpose of the present Part I and the upcoming Part II is to present the approaches for the quantitative understanding of the nonlinear waves. These ideas are streamlined, simplified, and synthesized here. We start with the scalar hyperbolic conservation laws

$$u_t + f(u)_x = 0, \ u \in \mathbb{R}.$$

The basic notions of compressibility, expansion, and entropy condition, and construction of elementary waves and their stability are presented using the method of *generalized characteristics*. This is a rich area and forms a necessary background for the study of viscous equations. We next consider the Burgers equation and present the detailed analysis of the construction of nonlinear waves and their *Green's function* using the *Hopf-Cole transformation*, [8], [2]. We then consider the general convex viscous conservation law

$$u_t + f(u)_x = \kappa u_{xx}, \ , \ f''(u) \neq 0, \ u \in \mathbb{R}.$$

For these conservation laws, there is the energy method for the study of the stability of nonlinear waves. The present authors have initiated the *Green's function approach*, which is effective for the quantitative study of nonlinear waves. We will present the technique combining the characteristic method, Hopf-Cole procedure, the weighted energy method, and the Green's function approach. We also consider the initial-boundary value problem, the study of the boundary effect on the propagation of the nonlinear waves.

The study for scalar laws serves as the foundation for the study of the systems in Part II. The theory for systems of conservation laws plays a central role in nonlinear analysis as many physical systems are nonlinear and dissipative. An important dissipative equation is the Boltzmann equation in the kinetic theory, [20], and [29].

Besides a new presentation of the basic ideas, the paper also contains new results and open problems are raised throughout the presentation.

2. Hyperbolic Conservation Laws

Consider scalar convex conservation laws:

$$u_t + f(u)_x = 0, \ f''(u) > 0; \tag{2.1}$$

The inviscid linear model is the transport equation and is solved by the characteristic method

$$u_t + \lambda u_x = 0, \ u(x,t) = u(x - \lambda t, 0). \tag{2.2}$$

The simplest convex conservation law (2.1) is the inviscid Burgers equation, $f(u) = u^2/2$, the *Hopf equation*,

$$h_t + \left(\frac{h^2}{2}\right)_x = 0. \tag{2.3}$$

This can be solved by *the characteristic method*:

$$\begin{cases} \frac{d}{dt}h(x(t),t) = 0, \\ \frac{d}{dt}x(t) = h(x(t),t). \end{cases} \tag{2.4}$$

The characteristic method yields the global continuous solution for all time $t > 0$ if the initial value $h(x,0)$ is a non-decreasing function. A particular solution of this kind is the *centered rarefaction wave* $h_R(x,t) = h_R(x,t;h_-,h_+),\ h_- < h_+$, a *self-similar* solution:

$$h_R(x,t) = \begin{cases} h_-, & \text{for } x < h_-t, \\ \frac{x}{t}, & \text{for } h_-t < x < h_+t, \\ h_+, & \text{for } x > h_+t. \end{cases} \tag{2.5}$$

For general initial value $h(x,0)$ the characteristic lines *compress* and eventually intersect if $h(x_1,0) > h(x_2,0)$ for some $x_1 < x_2$. The solution then becomes discontinuous, containing *shock waves*. For the general conservation law (1.5) the characteristic equation is

$$\begin{aligned} &\tfrac{d}{dt}u(x(t),t) = 0, \\ &\tfrac{d}{dt}x(t) = f'(u(x(t),t)). \end{aligned} \tag{2.6}$$

Thus the shock waves occur if the characteristic value $f'(u(x,0))$ is not an increasing function in x. The conservation law is then interpreted in the *weak sense*:

Definition 2.1. A bounded measurable function $u(x,t)$ is a weak solution of the initial value problem for the hyperbolic conservation law

$$u_t + f(u)_x = 0,\ u(x,0) = u_0(x)$$

if

$$\int_0^\infty \int_{-\infty}^\infty [\phi_t u + \phi_x f(u)](x,t)dxdt + \int_{-\infty}^\infty \phi(x,0)u_0(x)dx = 0, \tag{2.7}$$

for all smooth function $\phi = \phi(x,t)$ of compact support in $\{(x,t) : -\infty < x < \infty,\ t \geq 0.\}$

An equivalent, more physical definition is

Definition 2.2. A bounded measurable function $u(x,t)$ is a weak solution for the hyperbolic conservation law $u_t + f(u)_x = 0$ if

$$\frac{d}{dt}\int_{x_1}^{x_2} u(x,t)dx = f(u(x_1,t)) - f(u(x_2,t)), \tag{2.8}$$

for all x_1 and x_2.

The second definition has clear physical interpretation in that u is the *conserved quantity* and $f(u)$ its *flux*. It is easy to see that, as a consequence of either definition, a jump discontinuity along $x = x(t)$, with end states (u_-, u_+) and speed s:

$$u_- \equiv u(x(t) - 0),\ \ , u_+ \equiv u(x(t) + 0, t),\ \ s \equiv x'(t),$$

in a weak solution $u(x,t)$ satisfies the *Rankine-Hugoniot condition*:

$$s(u_+ - u_-) = f(u_+) - f(u_-). \tag{2.9}$$

For the Hopf equation (2.3) the shock speed is the arithmetic mean of its end states:

$$s = \frac{(h_+)^2/2 - (h_-)^2/2}{h_+ - h_-} = \frac{h_+ + h_-}{2}. \tag{2.10}$$

As we have seen, shock waves are consequence of the compression of characteristics, that is, $f'(u(x,t))$ decreases in x . This leads to the following notion of the *admissibility* of weak solutions.

Definition 2.3. A jump discontinuity (u_-, u_+) with speed s for the convex conservation law (2.1) is admissible if it satisfies the following *Lax entropy* *condition*, [12],

$$f'(u_-) > s > f'(u_+). \tag{2.11}$$

Thus *centered inviscid Burgers shock wave* from the origin for the inviscid Burgers equation (2.3) takes the form

$$h_S(x,t) = \begin{cases} h_-, \text{ for } x < st, \\ h_+, \text{ for } x > st, \\ s = \frac{h_+ + h_-}{2}, \ h_- > h_+; \end{cases} \tag{2.12}$$

and the general *centered shock wave* for the general convex conservation law (2.1) is, Figure 1:

$$u_S(x,t) = \begin{cases} u_-, \text{ for } x < st, \\ u_+, \text{ for } x > st, \\ s = \frac{f(u_+) - f(u_-)}{u_+ - u_-}, \ u_- > u_+. \end{cases} \tag{2.13}$$

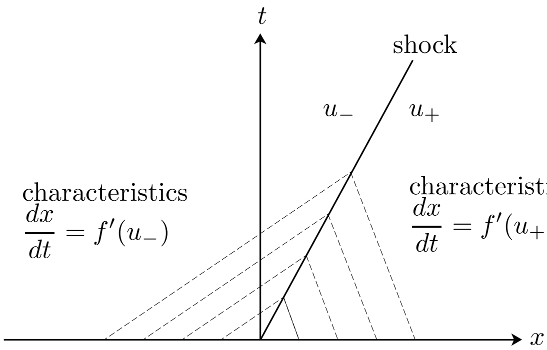

Figure 1. Centered shock wave.

Remark 2.4. The inviscid Burgers equation, the Hopf equation (2.3) has direct relation to the more general scalar conservation law, (1.5)

$$u_t + f(u)_x = 0, \ f''(u) > 0.$$

This is done by the relation

$$\lambda(x,t) \equiv f'(u(x,t)). \tag{2.14}$$

Multiplying the conservation law (1.5) by $f''(u)$, it follows easily that the function $\lambda(x,t)$ satisfies the inviscid Burgers equation

$$\lambda_t + \left(\frac{\lambda^2}{2}\right)_x = 0. \tag{2.15}$$

We have applied the chain rule. It is important to note that the notion of weak solutions does not carry over when chain rule is applied. One can easily see this by inspecting the Rankine-Hugoniot condition. Thus the equivalence of convex conservation law to the Hopf equation holds only when the solution $u(x,t)$ is *smooth*. In particular, it holds in the case of rarefaction waves and we have the *centered rarefaction wave* for the general conservation law (2.1), Figure 2:

$$f'(u(x,t)) = \begin{cases} f'(u_-), & \text{for } x < f'(u_-)t, \\ \frac{x}{t}, & \text{for } f'(u_-)t < x < f'(u_+)t, \\ f'(u_+), & \text{for } x > f'(u_+)t. \end{cases} \tag{2.16}$$

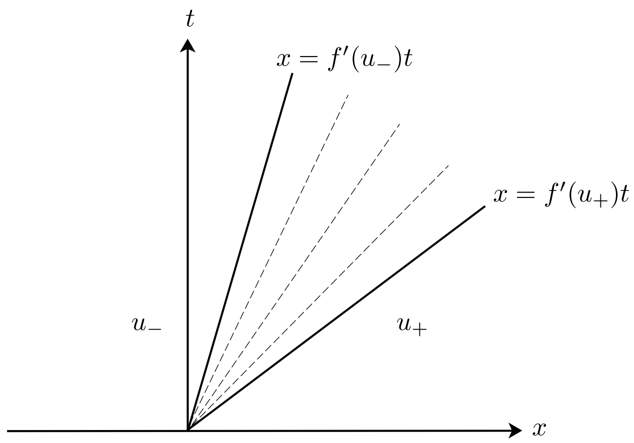

Figure 2. Centered rarefaction wave.

For a shock (u_-, u_+), the speed according to (2.9) is

$$s = \frac{f(u_+) - f(u_-)}{u_+ - u_-},$$

while for (2.3) it is

$$\hat{s} = \frac{h_+ + h_-}{2} = \frac{f'(u_+) + f'(u_-)}{2}.$$

Direct calculations show that

$$s - \hat{s} = \frac{f(u_+) - f(u_-)}{u_+ - u_-} - \frac{f'(u_+) + f'(u_-)}{2} = O(1)|u_+ - u_-|^2. \tag{2.17}$$

Thus for weak shocks, the approximation of a general convex conservation law by the inviscid Burgers equation is accurate. In fact, for general systems, it is shown that the speed of a weak shock is well approximated by the arithmetic mean of the compressing characteristic speeds of its end states, [12]. This fact was first raised and used crucially for the study of N-waves for general system of hyperbolic conservation laws in [14] and [15] using the Glimm scheme [3], and is important for the theory of viscous conservation laws, as we shall see later in this Part I and also in the forthcoming Part II for systems.

We have thus constructed the solution for the *Riemann problem*, the initial value problem for (2.1) with *Riemann data*

$$\begin{aligned} &u_t + f(u)_x = 0,\ f''(u) > 0, \\ &u(x,0) = \begin{cases} u_l, \text{ for } x < 0, \\ u_r, \text{ for } x > 0. \end{cases} \end{aligned} \tag{2.18}$$

The solution is a centered shock wave if $u_l > u_r$ and a centered rarefaction wave if $u_l < u_r$.

Besides the above elementary waves of shock and rarefaction waves, there is also the important N-waves. These are waves with compact support in x and *dissipative due to the nonlinearity of the flux function* $f(u)$. Consider the initial value problem for the inviscid Burgers equation

$$\begin{aligned} &h_t + hh_x = 0, \\ &h(x,0) = \begin{cases} 0 & \text{for } |x| > 1, \\ -p & \text{for } -1 < x < 0, \\ q & \text{for } 0 < x < 1, \end{cases} \\ &h(x,t) \equiv N(x,t;p,q). \end{aligned} \tag{2.19}$$

The construction of the N-wave $N(x,t;p,q),\ p \geq 0, q \geq 0$, is done easily by characteristic method. There is a centered rarefaction wave $h_R(x,t;-p,q)$

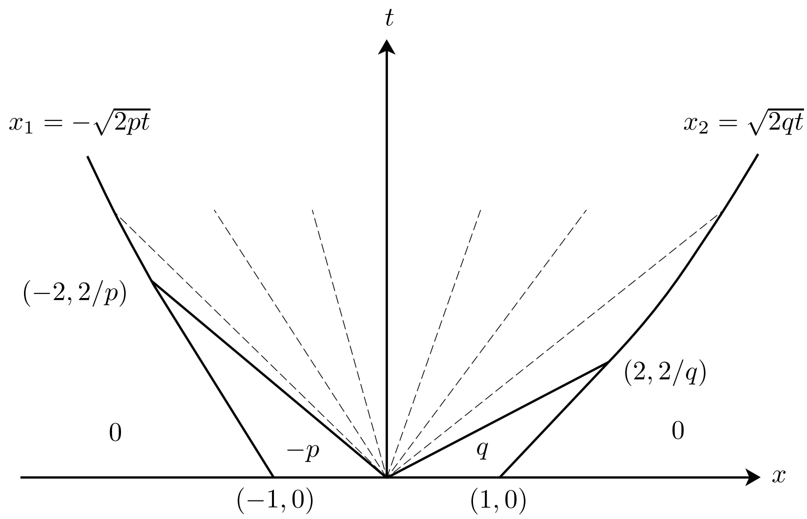

Figure 3. N-wave.

which interacts with the two shock waves $(0, -p)$ and $(q, 0)$ at later time. The solution consists of the centered rarefaction wave between two shock waves $(0, h_1(t))$, $h_1(t) \equiv h(x_1(t)+0, t)$ and $(h_2(t), 0)$, $h_2(t) \equiv h(x_2(t)-0, t)$, Figure 3. The locations $x = x_1(t)$ and $x = x_2(t)$ of the two shock are calculated using the Rankine-Hugoniot condition (2.10):

$$\frac{d}{dt}x_1(t) = \frac{h_1(t)}{2},\ \frac{d}{dt}x_2(t) = \frac{h_2(t)}{2}.$$

The starting time t_1 for the left shock is the meeting time of the shock at $x = -1 + (-p/2)t$ with the left edge $x = -pt$ of the centered rarefaction wave:

$$-1 - \frac{p}{2}t_1 = -pt_1,\ t_1 = \frac{2}{p}.$$

And the meeting location is $x_1(t_1) = -2$. Similarly we have $t_2 = \frac{2}{q}$ and $x_2(t_2) = 2$ for the right shock. After the meeting times, the values $h_1(t)$ and $h_2(t)$ are part of the rarefaction waves,

$$h_1(t) = \frac{x_1(t)}{t},\ h_2(t) = \frac{x_2(t)}{t},$$

and so the differential equations become

$$\frac{d}{dt}x_1(t) = \frac{x_1(t)}{2t},\ \text{for } t > t_1;\ \frac{d}{dt}x_2(t) = \frac{x_2(t)}{2t} \text{ for } t > t_2.$$

Solving the above differential equation for $x_1(t)$ and $x_2(t)$ with the initial values $x_1(t_1) = -2$ and $x_2(t_2) = 2$, respectively, we obtain the location of the shock waves

$$x_1(t) = -\sqrt{2pt},\ x_2(t) = \sqrt{2qt},$$

and the solution is, Figure 3,

$$N(x,t;p,q) = \begin{cases} 0,\ \text{for } x < -1 - \frac{pt}{2} \text{ and } 0 < t < \frac{2}{p};\ \text{or } x < -\sqrt{2pt} \text{ and } t > \frac{2}{p}, \\ 0,\ \text{for } x > 1 + \frac{pt}{2} \text{ and } 0 < t < \frac{2}{q};\ \text{or } x > \sqrt{2qt} \text{ and } t > \frac{2}{q}, \\ -p,\ \text{for } -1 - \frac{pt}{2} < x < -pt \text{ and } 0 < t < \frac{2}{p}, \\ q,\ \text{for } qt < x < 1 + \frac{qt}{2} \text{ and } 0 < t < \frac{2}{q}, \\ \frac{x}{t},\ \text{otherwise.} \end{cases} \tag{2.20}$$

There is of course the conservation law

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)dx = 0,\ \text{for } t \geq 0,$$

for any solution of the conservation law, viscous or inviscid. It turns out that there are *two time-invariants* p and q, which sum up to the conserved quantity above:

$$\begin{aligned} \min_x \int_{-\infty}^{x} N(y,t)dy &= -p, \qquad \max_x \int_{x}^{\infty} N(y,t)dy = q, \\ \int_{-\infty}^{\infty} N(y,t)dy &= q - p,\ \text{for } t \geq 0. \end{aligned} \tag{2.21}$$

Another remarkable thing is that the N-wave *dissipates* like heat kernel; its support $x \in (-\sqrt{2pt}, \sqrt{2qt})$ is of the order $\sqrt{t}$ and its magnitude

$$u(x,t) \in \Big(\frac{-\sqrt{2pt}}{t}, \frac{\sqrt{2qt}}{t}\Big)$$

is of the order $1/\sqrt{t}$. As we will soon see, such degree of spreading and decaying holds for general initial data with compact support. This is referred to as the *inviscid, hyperbolic dissipation.*

Consider the initial-boundary value problem for (2.1) for the quarter plane region $x, t > 0$:

$$\begin{aligned} &u_t + f(u)_x = 0,\ x, t > 0,\\ &u(x,0) = u_0(x),\\ &u(0,t) = u_1(t). \end{aligned} \tag{2.22}$$

For basic understanding of the general initial-boundary problem , we consider the elementary waves for the Riemann problem

$$\begin{aligned} &u_t + f(u)_x = 0,\ x, t > 0,\\ &u(x,0) = u_r,\ x > 0,\\ &u(0,t) = u_l,\ t > 0. \end{aligned} \tag{2.23}$$

This problem can be *over-determined*, as the hyperbolic problem needs boundary data u_l at $x = 0$ only if the characteristic speed is positive and here the characteristic speed $f'(u)$ varies with the solution u and is not known a priori. Nevertheless, there is a unique algorithm to proceed if the *boundary layer* is allowed. We first solve the corresponding Riemann problem (2.18). If it is a shock wave with positive speed, then we set the solution to be the shock wave, Figure 4. If the shock speed is negative, then we set the solution to be the state u_r that occupies the region $x > 0$ under consideration. In the latter case there is the *boundary layer* (u_l, u_r) that separate the boundary value u_l with the interior solution u_r at $x = 0+$. This boundary layer correspond to a wave (u_l, u_r) with negative speed, Figure 5. In the other case where the solution is a rarefaction wave, we do the same in that we set the solution to be the rarefaction wave that propagates into the region $x > 0$. The part of the rarefaction wave with negative speed, if exits, then forms the boundary layer, Figure 6 and Figure 7. We note that the initial-boundary value problem for the viscous conservation law is well-posed and the algorithm just mentioned, as we will see later, is consistent with the zero dissipation limit of the viscous conservation law

$$u_t + f(u)_x = \kappa u_{xx},\ \text{as } \kappa \to 0+.$$

The initial value problem for the scalar hyperbolic conservation laws is well-posed in the $L_1(x)$ norm. The above resolution of waves around the boundary is equivalent to that of [1] and also yields the $L_1(x)$ well-posedness.

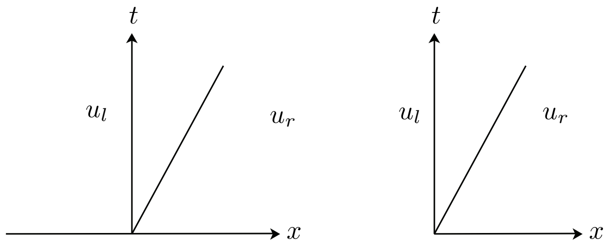

Figure 4. Shock wave with no boundary layer.

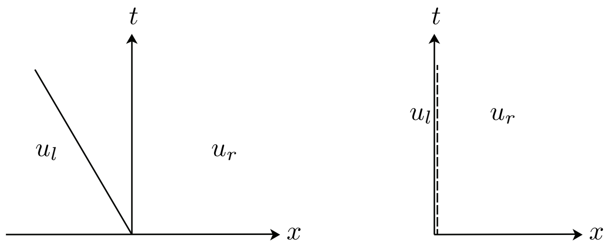

Figure 5. Shock boundary layer.

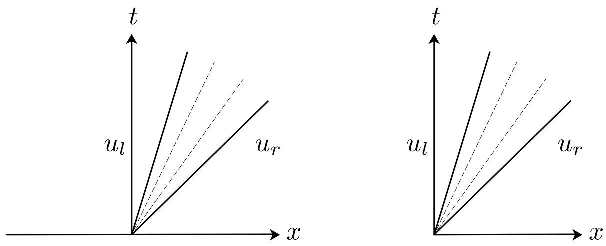

Figure 6. Rarefaction wave with no boundary layer.

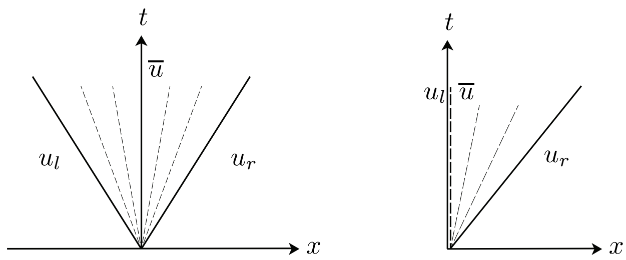

Figure 7. Rarefaction wave with boundary layer.

3. Stability of Hyperbolic Waves

We study the stability of shock, rarefaction and N waves for the hyperbolic conservation law (2.1) that have been constructed in the last section. The main tool is the notion of *generalized characteristics* of James Glimm,

[5], [13]. The entropy condition (2.11) says that the characteristic lines can impinge on the shocks only in the forward time direction. Thus a characteristic remains as a straight line in the *backward time* direction, Figure 8. A generalized characteristic is defined as a characteristic line and then the shock wave when the characteristic line impinges on the shock, Figure 8.

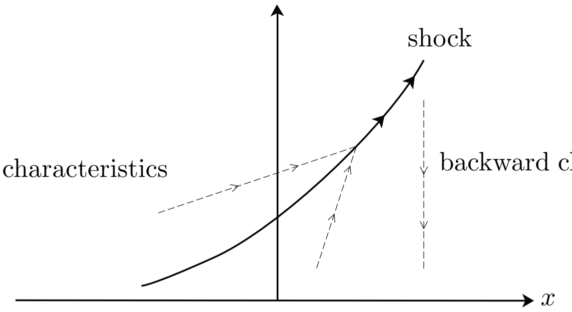

Figure 8. Generalized characteristics.

We now start the stability analysis, first with the simplest case of shock waves. Consider the centered shock wave $u_S(x,t)$, (2.13), connecting the states u_- and u_+, $u_- > u_+$. Let $u(x,t)$ be the solution of the initial value problem for the convex conservation law (2.1) with initial data a perturbation of the shock:

$$\begin{aligned} &u_t + f(u)_x = 0,\ f''(u) > 0, \\ &u(x,0) = \begin{cases} u_-, & \text{for } x < -M, \\ u_0(x), & \text{for } |x| \le M, \\ u_+, & \text{for } x > M, \end{cases} \end{aligned} \tag{3.1}$$

for some $M > 0$ and a bounded function $u_0(x)$, $|u_0(x)| < M$.

Theorem 3.1. *The solution of the initial value problem* (3.1)*,* $u_- > u_+$*, approaches the shifted centered shock* (u_-, u_+) *in finite time:*

$$u(x,t) = \begin{cases} u_-, & \textit{for } x < x_0 + st, \\ u_+, & \textit{for } x > x_0 + st,\ t > T, \end{cases}$$

and the shock formation time T *and the shock shift* x_0 *satisfy, for some*

$\beta > 0$,

$$T = O(1)\frac{M}{|u_+ - u_-|^\beta},\ x_0 = \frac{1}{u_- - u_+}\left[\int_{-\infty}^0 (u_0(x) - u_-)dx + \int_0^\infty (u_0(x) - u_+)dx\right].$$

Proof. Draw the generalized characteristics $C_1 :\ x = x_1(t)$ through $(-M, 0)$ and $C_2 :\ x = x_2(t)$ through $(M, 0)$ and set

$$D(t) \equiv x_2(t) - x_1(t)$$

the distance between them. The generalized characteristic $x = x_1(t)$ may be a characteristic line with $u = u_-$ along it; or it may be a shock $(u_-, u_1(t))$. In the former case we set also $u_1(t) = u_-$. Similarly we set $u_2(t)$ to be the left state of the solution along $x = x_2(t)$. To the left of $x = x_1(t)$ the solution takes value of u_- and to the right of $x = x_2(t)$ it takes the value of u_+. Through the generalized characteristics $(x_1(t), t)$ and $(x_2(t), t)$ at time t we draw the backward characteristics, Figure 9,

$$\begin{aligned} B_1 &\equiv \{(y, s) :\ y = x_1(t) - f'(u_1(t))(t - s),\ 0 < s < t\}, \\ B_2 &\equiv \{(y, s) :\ y = x_2(t) - f'(u_2(t))(t - s),\ 0 < s < t\}. \end{aligned}$$

The characteristics $B_1,\ B_2$ meet the initial time $t = 0$ in the interval $(-M, M)$.

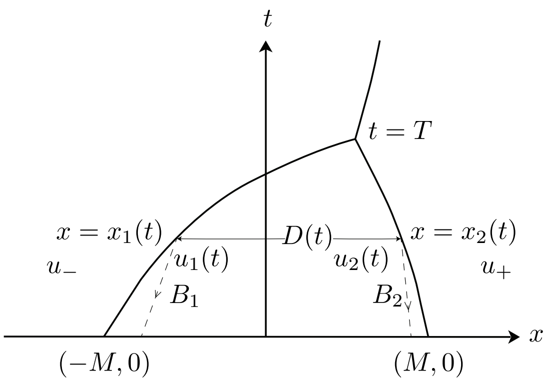

Figure 9. Stability of shock wave.

Along the characteristic line B_1, $u = u_1(t)$; and along B_2, $u = u_2(t)$. Thus

$$D(t) = O(M) + [f'(u_2(t)) - f'(u_1(t))]t,$$

where $O(M)$ is a function bounded by $2M$. On the other hand, from the Rankine-Hugoniot condition (2.9),

$$D'(t) = \frac{f(u_+) - f(u_2(t))}{u_+ - u_2(t)} - \frac{f(u_-) - f(u_1(t))}{u_- - u_1(t)}$$

We have from the entropy condition that $u_1(t) \leq u_-,\ u_2(t) \geq u_+$. There is the hypothesis $u_- > u_+$. Note that the solution $u(x,t)$ is bounded, $|u(x,t)| < M$, and so, by the convexity $f''(u) > 0$, there exists β independent of t such that $0 < \beta < 1$ and

$$D(t) \leq (1-\beta)[f'(u_+) - f'(u_-)] + \beta[f'(u_2(t)) - f'(u_1(t))].$$

We have thus obtain a differential inequality

$$D'(t) \leq \beta \frac{D(t) - O(M)}{t} + (1-\beta)[f'(u_+) - f'(u_-)]. \tag{3.2}$$

This is solved from $t = 1$ to yield

$$D(t) \leq [D(0) + O(M)]t^\beta + (1-\beta)[f'(u_+) - f'(u_-)]t,\ t > 1.$$

As $\beta < 1$ and $f'(u_+) - f'(u_-) < 0$, we have $D(t) = 0$ in finite time $t = T$ and the solution consists of a single shock after that

$$u(x,t) = \begin{cases} u_-, & \text{for } x < x_0 + st, \\ u_+, & \text{for } x > x_0 + st,\ t > T, \end{cases}$$

for some phase shift x_0. The phase shift can be determined through the conservation law

$$\frac{d}{dt}\Big[\int_{-\infty}^{st} (u(x,t) - u_-)dx + \int_{st}^{\infty} (u(x,t) - u_+)dx\Big] = 0,$$

when either the Rankine-Hugoniot condition (2.9) or the conservation law (2.8) is used. Evaluating the conserved quantity at initial time $t = 0$ and after the time $t = T$, we have

$$\int_{-\infty}^{0} (u_0(x) - u_-)dx + \int_{0}^{\infty} (u_0(x) - u_+)dx = x_0(u_- - u_+),$$

which yields the above formula for the phase shift. This completes the proof of the theorem. □

We next study the stability of centered rarefaction wave u_R, (2.16), connecting states u_- and u_+, $u_- < u_+$, and consider the initial value

$$u_t + f(u)_x = 0,\ f''(u) > 0,$$
$$u(x,0) = \begin{cases} u_-, & \text{for } x < -M, \\ u_0(x), & \text{for } |x| \le M, \\ u_+, & \text{for } x > M, \end{cases} \tag{3.3}$$

for some $M > 0$ and a bounded function $u_0(x)$, $|u_0(x)| < M$.

Theorem 3.2. *The solution of the initial value problem* (3.3), $u_- < u_+$, *approaches the centered rarefaction wave* u_R, (2.16), *at linear rate in the interior region*

$$|u(x,t) - u_R(x,t)| = O(1)\frac{1}{t+1},\ \textit{for } f'(u_-)t < x < f'(u_+)t,$$

and there is a region of width $O(1)(t+1)^{1/2}$ *around the boundary* $x = f'(u_-)t$ *and* $x = f'(u_+)t$ *where the convergence rate is sublinear*

$$|u(x,t) - u_R(x,t)| = O(1)(t+1)^{-\frac{1}{2}}.$$

Outside these finite regions in x, *the two solutions are the same, equal to* u_- *to the left and* u_+ *to the right. Consequently,*

$$\|u(x,t) - u_R(x,t)\|_{L_p(x)} = O(1)(t+1)^{-\frac{p-1}{2p}},\ p \ge 1.$$

Proof. Again we consider the generalized characteristics C_1, C_2 and use the same notations, $u_1(t)$, $u_2(t)$ as the states next to the generalized characteristics at $x = x_1(t)$, $x = x_2(t)$, respectively. Choose any location (x,t) between C_1 and C_2 and draw backward characteristic line with speed $f'(u(x,t))$ through (x,t), meeting $(\bar{x},0)$ at initial time, Figure 10:

$$x = \bar{x} + f'(u(x,t))t.$$

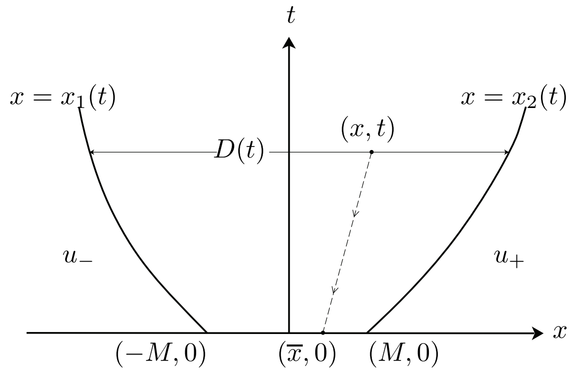

Figure 10. Stability of rarefaction wave.

As $|\bar{x}| < M$, we deduce that

$$f'(u(x,t)) = \frac{x}{t} - \frac{\bar{x}}{t} = f'(u_R)(x,t) + O(M)\frac{1}{t+1}$$

for (x,t) in the fan region of u_R and between C_1 and C_2. This proves the theorem for this region by the convexity $f''(u) \neq 0$. It remains to estimate the distance between the edges $C_1,\ C_2$ of the solution $u(x,t)$ and the edges $x_-(t) = f'(u_-)t,\ x_+(t) = f'(u_+)t$ of the centered rarefaction wave $u_R(x,t)$. Set

$$D(t) \equiv x_2(t) - x_1(t),\ E(t) \equiv D(t) - [f'(u_+) - f'(u_-)]t.$$

By the backward characteristic analysis, as in the stability analysis for the shock waves above,

$$D(t) = [f'(u_2(t)) - f'(u_1(t)]t + O(M),$$

with $O(M) \leq 2M$. From the Rankine-Hugoniot condition,

$$D'(t) = \frac{f(u_+) - f(u_2(t))}{u_+ - u_2(t)} - \frac{f(u_1(t)) - f(u_-)}{u_1(t) - u_-}.$$

From the entropy condition and our hypothesis of expansion, $u_1(t) \leq u_- < u_+ \leq u_2(t)$. From the above and the convexity of the flux $f''(u) > 0$, we have

$$E'(t) \leq \alpha[f'(u_2) - f'(u_+) + f'(u_-) - f'(u_1)] = \alpha\frac{E(t) + O(M)}{t},$$

for some constant α, $0 < \alpha < 1$. This can be solved to yield the sub-linear growth of $E(t)$ and the decay of the strength of the shocks $(u_-, u_1(t))$, $(u_2(t), u_+)$ on the generalized characteristics C_1, C_2. Next we use the Taylor expansion instead to yield

$$E'(t) = \frac{E(t) + O(1)M}{2t}\left[1 + O(1)\frac{E(t) + O(1)M}{2t}\right].$$

After large time, the value of α in the above analysis can be chosen to be close to 2, the decay rate of the shocks $(u_-, u_1(t))$, $(u_2(t), u_+)$ is close to $t^{-1/2}$, and the growth rate of $E(t)$ is close to $t^{1/2}$. And so we have

$$E'(t) = \frac{E(t) + O(1)M}{2t}\left[1 + O(1)t^{-1+\beta}\right]$$

for some β close to $\frac{1}{2}$. The equation can be solved to yield

$$E(t) = O(1)(t+1)^{\frac{1}{2}}.$$

And the shock decays as

$$|u_- - u_1(t)| + |u_2(t) - u_+| = O(1)(t+1)^{-\frac{1}{2}}.$$

This proves the pointwise estimates. The $L_p(x)$ decay follows immediately. Notice that the decay is slower around the edge of the rarefaction wave. In the interior, the decay rate in $L_p(x)$ is $(t+1)^{-(p-1)/p}$. □

We finally study the stability of N waves.

Theorem 3.3. *Suppose that the initial data $u(x,0)$ of the inviscid Burgers equation $u_t + f(u)_x = 0$ has compact support. The flux function is normalized by $f(0) = f'(0) = 0$, $f''(0) = 1$. Then the solution $u(x,t)$ satisfies*

$$-p \equiv \min_x \int_{-\infty}^{x} u(y,t)dy, \ \ q \equiv \max_x \int_{x}^{\infty} u(y,t)dy \tag{3.4}$$

are time-invariant, and the solution approaches the N-wave in the following sense: There are two relative strong shock of strength $\sqrt{2p/t}$ and $\sqrt{2q/t}$ located at $x = x_1(t) = -\sqrt{2pt} + O(1)$ and $x = x_2(t) = \sqrt{2qt} + O(1)$, respec-

tively. The solution satisfies

$$u(x,t) = \frac{x}{t} + O(1)(t+1)^{-1}, \ \textit{for } x_1(t) < x < x_2(t), \ t > 1.$$

In particular,

$$\|u(x,t) - N(x,t;p,q)\|_{L_p(x)} = O(1)(t+1)^{-\frac{1}{2}}, \ p \geq 1.$$

Proof. To prove the time invariants p, q, we first note that the minimum and maximum defining them, (3.4), take place at the continuity points of the solution, and the solution must be zero there. This is because the solution jumps down across the shocks by the entropy condition. For instance, at the infimum point $x = \hat{x}$ defining p, the solution must be non-negative at $x = \hat{x} + 0$ and non-positive at $x = \hat{x} - 0$. Otherwise the integral from $-\infty$ to $x = \hat{x}(t) + \varepsilon$, or from $-\infty$ to $x = \hat{x}(t) - \varepsilon$, $\varepsilon > 0$ and small, would reduce the integral. Thus $u(\hat{x} - 0) \leq u(\hat{x} + 0)$, which precludes the shock at $x = \hat{x}$. Also the solution is zero there. Then at a continuity point, one can draw the characteristic line C both forward and backward in time. From the conservation law (2.8), the integral from $-\infty$ to C is constant in time. As p is defined to be the infimum, we have $p(s) \leq p(t)$ for $|s - t|$ small. Thus $p'(t) \equiv 0$. Similarly $q'(t) \equiv 0$.

To obtain the pointwise estimate of the solution we again draw the left and right generalized characteristics C_1 and C_2. As in the case with the above study of the stability of the rarefaction waves, the distance $D(t)$ between the generalized characteristics expands at the rate of $t^{1/2}$ and the shock waves on them decay. Thus, for simplicity, we will make our proof easier by concentrating on the inviscid Burgers equation $f'(u) = u$. The remaining job is to estimate the location $x = x_i(t)$, $i = 1, 2$, of the generalized characteristics. As before, from the analysis using the backward characteristic lines, we have

$$u(x,t) = \frac{x}{t} + O(1)\frac{1}{t+1}, \ \text{for } x_1(t) < x < x_2(t).$$

Thus the infimum defining the time-invariant p occurs around $x = 0$ and

$$-p = \int_{x_1(t)}^{0} \left[\frac{x}{t} + O(1)\frac{1}{t+1}\right] dx + O(1)\frac{1}{t+1},$$

or $x_1(t) = -\sqrt{2pt} + O(1)$. This completes the proof of the theorem. □

Remark 3.4. The stability of the above three types of waves is indicative of the different sense of stability that we will also see in the following sections on viscous conservation laws. For the shock waves, the stability is in a strong sense. In fact, for a compact supported perturbation, the stability rate is infinite as the time asymptotic state is reached in finite time. Also it can be shown that the convergence to the shock is in $L_1(x)$ sense even for more general perturbation, so long as the conserved quantity is finite for the perturbation. The stability is orbital in that the shock location needs to be precisely located through conservation law. The approach given above is through *shock capturing*. In the study of viscous shock waves we will either determine the time-asymptotic location of the shock first or by *wave tracing*.

The stability of rarefaction wave cannot be in the $L_1(x)$ sense. This is seen by comparing the translation, say of the amount a of the portion (u_1, u_2) and of the amount b of the portion of (u_2, u_3) of the centered rarefaction wave (u_1, u_3), Figure 11. The magnitude of the difference of the translated and original rarefaction waves is of the order of $(x + a)/t - x/t = a/t$ and b/t and the region of their difference is $u_1t + a < x < u_2t$ and $u_2t < x < u_3t + b$, Figure 11. Thus the $L_p(x)$ norm of the difference is

$$[a^p(u_2 - u_1) + b^p(u_3 - u_2)]^{\frac{1}{p}} t^{-\frac{p-1}{p}}.$$

In particular, the $L_1(x)$ is non-decaying in time. As a perturbation of a rarefaction wave can represent any shift of any portion of the wave, there is no simple translation of the unperturbed wave to match the perturbed solution in order to achieve the $L_1(x)$ convergence for large time.

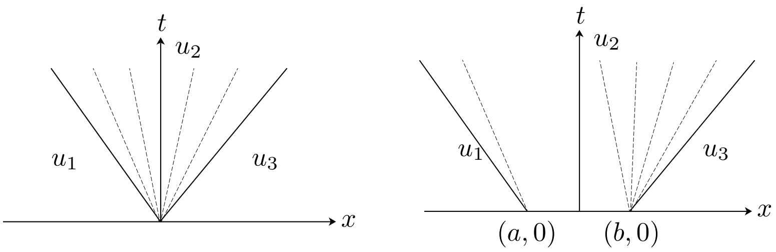

Figure 11. Translation of rarefaction wave.

The N waves are non-obvious consequence of the nonlinearity of the flux. Although the N waves and rarefactions are both dissipative, it is possible to study the more detailed dissipation of the N waves and its $L_1(x)$

behavior. For the viscous conservation laws, the viscosity will have strong time-asymptotic effect and, instead of two time invariants $p,\ q$ in the inviscid case, there is only the usual one conserved quantity that survives after large time. Nevertheless, the nonlinearity of the flux dictates that, in the $L_1(x)$ and pointwise sense, the large time behavior is nonlinear, Burgers equation type, rather than heat equation type.

4. Heat Equation

Before we turn to the study of the combined effect of the nonlinearity for hyperbolic conservation laws and the viscosity, we first consider the viscous effect. For general systems (1.2), there are rich nonlinear interactions for waves pertaining to distinct characteristic fields. The study of these interactions requires the consideration of multiple scalings. We consider in this section the linear viscous model, the heat equation

$$u_t + \lambda u_x = \kappa u_{xx}.$$

Our purpose here is to introduce the *pointwise analysis* for this simplest situation. The initial value problem is solved using the *heat kernel* $H = H(x,t) = H(x,t;\kappa)$:

$$\begin{cases} u_t + \lambda u_x = \kappa u_{xx}, \\ u(x,t) = \int_{-\infty}^{\infty} H(x-y-\lambda t, t) u(y,0) dy; \end{cases} \tag{4.1}$$

$$\begin{cases} H_t = \kappa H_{xx},\ H(x,0) = \delta(x), \\ H(x,t) = H(x,t;\kappa) \equiv (4\pi\kappa t)^{-1/2} e^{-x^2/(4\kappa t)}. \end{cases} \tag{4.2}$$

The heat kernel decays exponentially except for its essential domain of width $\sqrt{t}$. It is important to visualize the essential domain of the heat kernel, Figure 12. As aforementioned, we need to consider multiple scalings for the general study of systems. In preparation of this, we suppose that the initial data decay algebraically at $x = \pm\infty$:

$$\begin{cases} u_t = \kappa u_{xx}, \\ u(x,0) = u_0(x) = O(1)[|x|+1]^{-\alpha}. \end{cases} \tag{4.3}$$

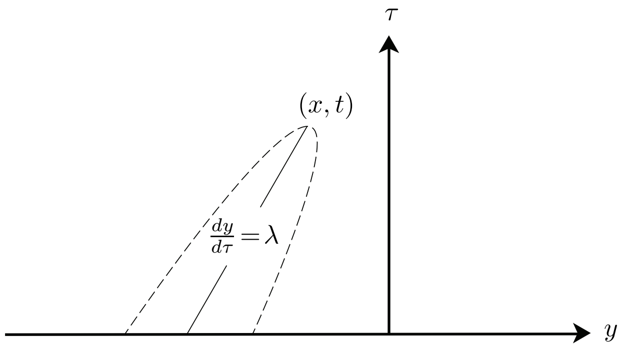

Figure 12. Heat kernal $H(x-y-\lambda(t-\tau),\tau)$.

Theorem 4.1. *Consider the solution $u(x,t)$ of (4.3). For $\alpha > 1$, the total heat*

$$c \equiv \int_{-\infty}^{\infty} u_0(x)dx, \tag{4.4}$$

is finite and the solution approaches a multiple of the heat kernel $H(x,t,\kappa)$ time asymptotically:

$$\begin{aligned} &|u(x,t) - cH(x,t+1;\kappa)| = O(1)(|x| + \sqrt{t} + 1)^{-\alpha} \\ &\quad + O(1)(t+1)^{-\frac{1}{2}} H(x,t+1;D) \cdot \begin{cases} 1, & \textit{for } \alpha > 2; \\ \log(t+1), & \textit{for } \alpha = 2, \\ (t+1)^{-\frac{\alpha-2}{2}}, & \textit{for } 1 < \alpha < 2, \end{cases} \end{aligned} \tag{4.5}$$

for any constant D greater than κ. As a consequence,

$$\begin{aligned} &\|u(x,t) - cH(x,t+1;\kappa)\|_{L_p(x)} \\ &= O(1)(t+1)^{-1+\frac{1}{2p}} \begin{cases} 1, & \textit{for } \alpha > 2; \\ \log(t+1), & \textit{for } \alpha = 2; \\ (t+1)^{-\frac{\alpha-2}{2}}, & \textit{for } 1 < \alpha < 2. \end{cases} \end{aligned} \tag{4.6}$$

Proof. As the heat kernel is of exponential type and the initial distribution is of algebraic type, we are dealing with two distinct scalings. Thus the arguments below necessarily involve consideration of cases with regard to the four regions for the target (x,t) and the source $u(y,0)$ through the heat kernel $H(x-y,t)$. In this simplest setting, we will go into some details of these cases, and it will serve as an introduction to the general *pointwise* *estimates* that will be our main emphasis.

Set

$$v(x,t) \equiv u(x,t) - cH(x,t+1),\ w(x,t) \equiv \int_{-\infty}^{x} v(y,t)dy.$$

We note that

$$v(x,0) = O(1)[|x|+1]^{-\alpha},\ w(x,0) = O(1)[|x|+1]^{-\alpha+1}. \tag{4.7}$$

By Duhamel's principle

$$v(x,t) = \int_{-\infty}^{\infty} H(x-y,t)v(y,0)dy = -\int_{-\infty}^{\infty} H_y(x-y,t)w(y,0)dy. \tag{4.8}$$

We will either use the first expression in (4.8):

$$v(x,t) = O(1)\int_{-\infty}^{\infty} t^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4\kappa t}}[|y|+1]^{-\alpha}dy \equiv O(1)I(x,t;\alpha;\kappa), \tag{4.9}$$

or the second expression in (4.8):

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{\infty} O(1)t^{-\frac{1}{2}}\frac{\partial}{\partial_x}e^{-\frac{(x-y)^2}{4\kappa t}}[|y|+1]^{-\alpha+1}dy \\ &= O(1)\int_{-\infty}^{\infty} t^{-1}e^{-\frac{(x-y)^2}{4Ct}}[|y|+1]^{-\alpha+1}dy \\ &= O(1)t^{-\frac{1}{2}}\int_{-\infty}^{\infty} t^{-\frac{1}{2}}e^{-\frac{(x-y)^2}{4Ct}}[|y|+1]^{-\alpha+1}dy \\ &\equiv O(1)t^{-\frac{1}{2}}I(x,t;\alpha-1;C), \end{aligned} \tag{4.10}$$

for any constant $C > \kappa$. Here we have used (4.7). We need to consider separately various regions for the variables (x,y,t) and use either (4.9) or (4.10). For initial time layer, $0 < t < 1$, we use (4.9); we omit the estimate in this case as it is simpler than the following ones. After the initial layer, $t > 1$, we note that

$$t^{-1} = O(1)(t+1)^{-1}.$$

We consider the following two cases:

Case 1. $|x| < M_1\sqrt{t}$.

We use (4.10) and divide the integral $I(x,t;\alpha-1,C)$ into two parts:

$$\begin{aligned} I(x,t;\alpha-1;C) &= \left[\int_{|y-x|<M_2\sqrt{t}} + \int_{|y-x|>M_2\sqrt{t}}\right] t^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{Ct}} (1+|y|)^{-\alpha+1} dy \\ &\equiv I_{11} + I_{12}. \end{aligned}$$

We take $1 \ll M_1 \ll M_2$. Then

$$\begin{aligned} I_{11} &= O(1) \int_{|y|<O(1)\sqrt{t}} t^{-\frac{1}{2}} (1+|y|)^{-\alpha+1} dy \\ &= O(1)(t+1)^{-\frac{1}{2}} \cdot \begin{cases} 1, & \text{for } \alpha > 2; \\ \log(t+1), & \text{for } \alpha = 2; \\ (t+1)^{-\frac{\alpha-2}{2}}, & \text{for } \alpha < 2. \end{cases} \end{aligned}$$

Similarly,

$$I_{12} = O(1) \int_{|y|>M_2\sqrt{t}/2} t^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{Ct}} (1+|y|)^{-\alpha+1} dy.$$

If $\alpha > 2$,

$$\begin{aligned} I_{12} &= O(1)(t+1)^{-\frac{1}{2}} \int_{|y|>M_2\sqrt{t}/2} (1+|y|)^{-\alpha+1} dy \\ &= O(1)(t+1)^{-\frac{1}{2}} (1+\sqrt{t})^{-\alpha+2} = O(1)(t+1)^{-\frac{\alpha-1}{2}}; \end{aligned}$$

and if $\alpha \le 2$,

$$I_{12} = O(1) \int_{|y|>M_2\sqrt{t}/2} t^{-\frac{\alpha}{2}} e^{-\frac{(x-y)^2}{Ct}} dy = O(1) t^{-\frac{\alpha-1}{2}} = O(1)(t+1)^{-\frac{\alpha-1}{2}}.$$

We summarize the above to conclude

$$|u(x,t) - cH(x,t+1)| = O(1)(t+1)^{-\frac{1}{2}} H(x,t+1) \cdot \begin{cases} 1, & \text{for } \alpha > 2, \\ \log(t+1), & \text{for } \alpha = 2, \\ (t+1)^{-\frac{\alpha-2}{2}}, & \text{for } \alpha < 2, \end{cases} \tag{4.11}$$

for $|x| = O(1)\sqrt{t+1}$. Note that if (4.9) is used for the estimate instead, we would yield the weaker estimate of $|u(x,t) - cH(x,t+1)| = O(1)H(x,t+1)$ for $|x| = O(1)\sqrt{t+1}$.

Case 2. $|x| > M_1\sqrt{t}$.

In this case we use (4.8). We also divide the integral into two parts:

$$v(x,t) = -\Big[\int_{|y|<|x|/M} + \int_{|y|>|x|/M}\Big] H_y(x-y,t)w(y,0)dy = I_{21} + I_{22},$$

where $M > 0$ is some constant. As in (4.10),

$$I_{21} = O(1)\int_{|y|<|x|/M} t^{-1} e^{-\frac{(x-y)^2}{4D't}} (1+|y|)^{-\alpha+1} dy,$$

where we take D' such that $D > D'' > D' > \kappa$. For the range of integration for I_{21}, we also have

$$e^{-\frac{(x-y)^2}{4D't}} = O(1)e^{-\frac{x^2}{4D''t}}$$

by choosing M large. Therefore,

$$\begin{aligned} I_{21} &= O(1)t^{-1}e^{-\frac{x^2}{4D''t}} \int_{|y|<|x|/M} (1+|y|)^{-\alpha+1} dy \\ &= O(1)(t+1)^{-\frac{1}{2}} H(x,t+1;D'') \cdot \begin{cases} 1, & \text{for } \alpha > 2, \\ \log|x|, & \text{for } \alpha = 2, \\ (|x|+1)^{-\alpha+2}, & \text{for } \alpha < 2 \end{cases} \\ &= O(1)(t+1)^{-\frac{1}{2}} H(x,t+1;D) \cdot \begin{cases} 1 & \text{for } \alpha > 2, \\ \log(t+1), & \text{for } \alpha = 2, \\ (t+1)^{-\frac{\alpha-2}{2}}, & \text{for } \alpha < 2. \end{cases} \end{aligned}$$

By integration by parts and similar to (4.9), we have

$$\begin{aligned} I_{22} &= O(1)\int_{y>|x|/M} t^{-\frac{1}{2}} e^{-\frac{(x-y)^2}{4\kappa t}} (|y|+1)^{-\alpha} dy + O(1)t^{-\frac{1}{2}} e^{-\frac{x^2}{4Dt}} (|x|+1)^{-\alpha+1} \\ &= O(1)(|x|+1)^{-\alpha} = O(1)(|x|+\sqrt{t}+1)^{-\alpha}. \end{aligned}$$

This concludes the proof of the theorem. □

The following proposition is proved similarly as above; it is listed here to contrast with the corresponding one, Proposition 7.4, for Burgers equation later.

Proposition 4.2. *For the initial value problem* (4.3) *with* $\alpha = 1$*, the solu-*

tion satisfies, for any constant $D > \kappa$,

$$u(x,t) = O(1)[H(x,t;D)\log(t+1) + (|x| + \sqrt{t} + 1)^{-1}]. \tag{4.12}$$

5. Burgers Waves

We now consider the strongly nonlinear case and start with the simple model of inviscid Burgers equation, the *Hopf equation*, and the *Burgers equation*,

$$h_t + (\frac{h^2}{2})_x \ = \ 0, \tag{5.1}$$

$$b_t + bb_x \ = \ \kappa b_{xx}. \tag{5.2}$$

The Burgers equation (5.2) can be solved explicitly by the *Hopf-Cole transformation* through the following procedure: First integrate the Burgers equation to a *Hamilton-Jacobi equation*:

$$\begin{cases} B_t + \frac{(B_x)^2}{2} = \kappa B_{xx}, \quad B_x(x,t) = b(x,t), \\ B(x,0) \equiv \int_{0-}^x b(y,0)dy. \end{cases} \tag{5.3}$$

Then introduce the *Hopf-Cole relation*

$$B(x,t) = -2\kappa \log[\phi(x,t)]. \tag{5.4}$$

Direct calculations shows that $\phi(x,t)$ satisfies the heat equation

$$\begin{cases} \phi_t = \kappa\phi_{xx}, \\ \phi(x,0) = e^{-\frac{1}{2\kappa}\int_{0-}^x b(y,0)dy}, \end{cases} \tag{5.5}$$

which is solved by convolving with the heat kernel:

$$\phi(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-y)^2}{4\kappa t} - \frac{1}{2\kappa}\int_{0-}^y b(z,0)dz} dy.$$

In fact, the Hamilton-Jacobi equation holds for $B(x,t)$ so long as $B_x(x,t) = b(x,t)$. Thus we may replace B with $\hat{B}(x,t) \equiv B(x,t) + \alpha(t)$ with the free function $\alpha(t)$. The choice of $\hat{B}$ is under the constraint $\hat{B}(0,t) = 0$ at $t = 0$.

However, if one chooses an appropriate $\alpha(t)$ so as to make

$$B(x,t)+\alpha(t)=\int_0^x b(y,t)dy,$$

then

$$\alpha(t)=2\kappa\log[\phi(0,t)]=2\kappa\log\left[\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\kappa t}}e^{-\frac{y^2}{4\kappa t}-\frac{1}{2\kappa}\int_{0-}^{y}b(z,0)dz}dy\right],$$

which is in general nonzero. In other words, the choice in (5.3) for $B(x,t)$ is particular and for simplicity in latter presentation. We summarize the above solution formula:

$$\begin{aligned}
\phi(x,t) &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\kappa t}}e^{-\frac{(x-y)^2}{4\kappa t}-\frac{1}{2\kappa}\int_{0-}^{y}b(z,0)dz}dy,\\
B_x(x,t) &= b(x,t),\ B(x,0)=\textstyle\int_{0-}^{x}b(y,0)dy,\\
B(x,t) &= -2\kappa\log[\phi(x,t)]\\
&= -2\kappa\log\Big[\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\kappa t}}e^{-\frac{(x-y)^2}{4\kappa t}-\frac{1}{2\kappa}\int_{0-}^{y}b(z,0)dz}dy\Big]\\
b(x,t) &= \frac{\int_{-\infty}^{\infty}\frac{x-y}{t}e^{-\frac{(x-y)^2}{4\kappa t}-\frac{1}{2\kappa}\int_{0-}^{y}b(z,0)dz}dy}{\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4\kappa t}-\frac{1}{2\kappa}\int_{0-}^{y}b(z,0)dz}dy}
\end{aligned}\tag{5.6}$$

This procedure is used to find various interesting particular solutions of the Burgers equation. The *Burgers kernel*, the *nonlinear diffusion wave* $b_D(x,t)=b_D(x,t;A)$ is

$$\begin{aligned}
&(b_D)_t+b_D(b_D)_x=\kappa(b_D)_{xx},\\
&b_D(x,0)=A\delta(x),\\
&\phi_D(x,t)=e^{-\frac{A}{2\kappa}}+\int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty}\frac{1}{\sqrt{\pi}}(1-e^{-\frac{A}{2\kappa}})e^{-y^2}dy,\\
&b_D(x,t)=\frac{\frac{\sqrt{\kappa}}{\sqrt{t}}(e^{\frac{A}{2\kappa}}-1)e^{-\frac{x^2}{4\kappa t}}}{\sqrt{\pi}+\int_{\frac{x}{\sqrt{4\kappa t}}}^{\infty}(e^{\frac{A}{2\kappa}}-1)e^{-y^2}dy}.
\end{aligned}\tag{5.7}$$

Unlike the heat kernel, which is symmetric in x, the nonlinearity makes the Burgers kernel to lean toward the right when the mass A is positive, and to the left when A is negative, Figure 13. The Burgers kernel can also be found by the self-similarity argument

$$b(x,t) = \sqrt{\frac{\kappa}{t}}\psi\Big(\frac{x}{\sqrt{\kappa t}}, \frac{A}{\kappa}\Big).$$

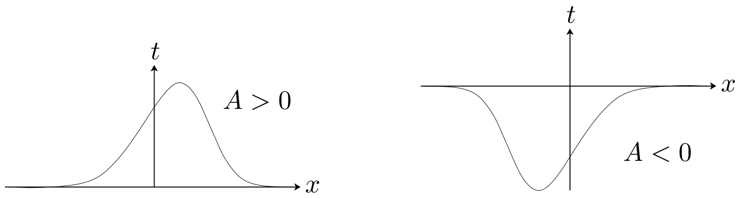

Figure 13. Burgers kernal $b_D(x,1;A)$.

The *Burgers rarefaction wave* $b_R(x,t)=b_R(x,t;\lambda_0), \lambda_0>0$ is the solution with the Riemann data, here taken to be symmetric in x, for simplicity,

$$\begin{aligned}
&(b_R)_t + b_R(b_R)_x = \kappa(b_R)_{xx},\\
&b_R(x,0) = \begin{cases} -\lambda_0, & \text{for } x<0,\\ \lambda_0, & \text{for } x>0.\end{cases}\\
&b_R(x,t) = \lambda_0 \frac{\mathrm{Erfc}(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) - e^{\frac{\lambda_0 x}{\kappa}}\mathrm{Erfc}(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}{\mathrm{Erfc}(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}}) + e^{\frac{\lambda_0 x}{\kappa}}\mathrm{Erfc}(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})}\\
&B_R(x,t) = -2\kappa\log[\phi_R(x,t)]\\
&\phi_R(x,t) = e^{-\frac{\lambda_0 x}{2\kappa}+\frac{(\lambda_0)^2 t}{4\kappa}}\mathrm{Erfc}(\frac{-x+\lambda_0 t}{\sqrt{4\kappa t}})\frac{1}{\sqrt{\pi}} + e^{\frac{\lambda_0 x}{2\kappa}+\frac{(\lambda_0)^2 t}{4\kappa}}\mathrm{Erfc}(\frac{x+\lambda_0 t}{\sqrt{4\kappa t}})\frac{1}{\sqrt{\pi}}.
\end{aligned} \tag{5.8}$$

Here Erfc is the error function:

$$\mathrm{Erfc}(z) \equiv \int_z^\infty e^{-y^2}dy.$$

The *Burgers shock wave* $b_S(\xi) = b_S(\xi;\lambda_0) = b_S(x-st;\lambda_0;\kappa),\ \xi = x-st$, is a traveling wave solution with speed s, taken to be zero, for simplicity, and strength $2\lambda_0$:

$$\begin{aligned}
&(b_S)_t + b_S(b_S)_x = \kappa(b_S)_{xx},\\
&b_S(-\infty) = \lambda_0,\ b_S(\infty) = -\lambda_0,\ \lambda_0 > 0.
\end{aligned} \tag{5.9}$$

The shock $b_S(x,\lambda_0)$ can be obtained as the time-asymptotic state of the solution of the Riemann problem:

$$\begin{aligned}&(u_S)_t + u_S(u_S)_x = \kappa(u_S)_{xx},\\&u_S(x,0) = \begin{cases}\lambda_0, & \text{for } x<0,\\ -\lambda_0, & \text{for } x>0,\end{cases}\\&b_S(x) = \lim_{t\to\infty} u_S(x,t).\end{aligned} \tag{5.10}$$

From (5.6),

$$u_S(x,t) = -\lambda_0 \frac{\mathrm{Erfc}(\frac{-x-\lambda_0 t}{\sqrt{4\kappa t}}) - e^{-\frac{\lambda_0 x}{\kappa}}\mathrm{Erfc}(\frac{x-\lambda_0 t}{\sqrt{4\kappa t}})}{\mathrm{Erfc}(\frac{-x-\lambda_0 t}{\sqrt{4\kappa t}}) + e^{\frac{-\lambda_0 x}{\kappa}}\mathrm{Erfc}(\frac{x-\lambda_0 t}{\sqrt{4\kappa t}})}. \tag{5.11}$$

There is also the corresponding function ϕ as in the case of the rerafefaction case, and the time-asymptotic state $b_S(x) = \lim_{t\to\infty} u_S(x,t)$ satisfies

$$\begin{aligned}&(b_S)_t + b_S(b_S)_x = \kappa(b_S)_{xx},\\&b_S(x) = -\lambda_0 \tanh(\tfrac{\lambda_0 x}{2\kappa}) = -\lambda_0 \frac{e^{\frac{\lambda_0 x}{2\kappa}} - e^{-\frac{\lambda_0 x}{2\kappa}}}{e^{\frac{\lambda_0 x}{2\kappa}} + e^{-\frac{\lambda_0 x}{2\kappa}}},\\&B_S(x) = -2\kappa \log\Big[(e^{\frac{\lambda_0 x}{2\kappa}} + e^{-\frac{\lambda_0 x}{2\kappa}})e^{\frac{(\lambda_0)^2 t}{2\kappa}}\Big],\\&\phi_S(x) = \Big[e^{\frac{\lambda_0 x}{2\kappa}} + e^{-\frac{\lambda_0 x}{2\kappa}}\Big]e^{\frac{(\lambda_0)^2 t}{2\kappa}}.\end{aligned} \tag{5.12}$$

Remark 5.1. There is the *initial layer* of the formation of the Burgers shock profile $b_S(x)$ from the Riemann data $u_S(x,0)$ in (5.10), Figure 14. The *shock formation time* is the time it takes for the solution $u_S(x,t)$ to become close to the shock profile $b_S(x)$. From (5.11) it is the values

$$\frac{|\pm x - \lambda_0 t|}{\sqrt{4\kappa t}}$$

that need to be greater than certain fixed large number. As the shock is stationary, we can consider any given location x and the shock formation time T is therefore determined by

$$\frac{\lambda_0 T}{\sqrt{4\kappa T}} = O(1), \text{ or } T = O(1)\frac{\kappa}{(\lambda_0)^2}. \tag{5.13}$$

Thus the shock formation time is proportional to the strength κ of viscosity and inverse proportional to the square of the shock strength λ_0. The reason

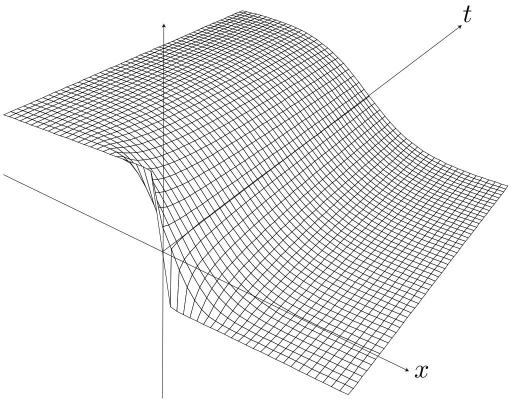

Figure 14. Burgers shock formation.

for the dependence on the viscosity is clear; the formation time is longer for weaker shock as the resulting weak compressibility delays the shock formation. As the Burgers equation is a good approximation to nonlinear waves in dissipation systems, the above shock layer consideration is useful for the study of similar problems for the general system, e.g. [7].

For the rarefaction wave $b_R(x,t)$, the formation time is the time it takes for $b_R(x,t)$ to be close to the self-similar solution (2.5):

$$|b_R(x,t) - \frac{x}{t}| \ll 1, \text{ for } -\lambda_0 < \lambda \equiv \frac{x}{t} < \lambda_0.$$

This can be quantified from the expression for $b_R(x,t)$ in (5.8). For this, we use the easily verified estimate:

$$\mathrm{Erfc}(z) \equiv \int_z^\infty e^{-y^2} dy = e^{-z^2}\left[\frac{1}{2z} - \frac{1}{4z^3} + \frac{3}{8z^5} + O(1)\frac{1}{z^7}\right], \text{ for } z > 0, \quad (5.14)$$

to obtain from (5.8):

$$|b_R(x,t) - \frac{x}{t}| = O(1)\Big[\frac{1}{|\lambda_0 x - t|} + \frac{1}{|\lambda_0 x + t|}\Big], \text{ for } -\lambda_0 t + \sqrt{t} < x < \lambda_0 t - \sqrt{t}. \quad (5.15)$$

From (5.15), the convergence to the self-similar inviscid solution is of the order of $(t+1)^{-1}$ uniformly in the viscosity κ for the region strictly inside the rarefaction wave. Inside the rarefaction wave, the rate of $(t+1)^{-1}$ is the same as the perturbation within the inviscid model, Figure 15. Around

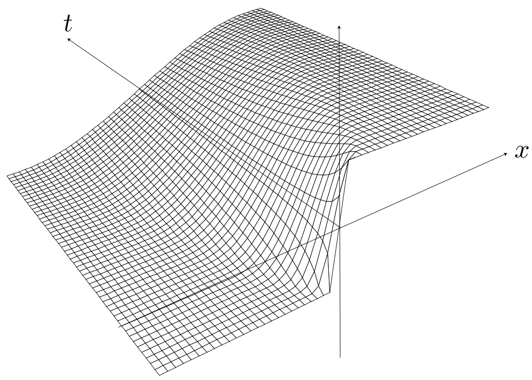

Figure 15. Burgers rarefaction wave.

the edges of the rarefaction waves $x = \pm\lambda_0 t$, the convergence is less obvious and the inviscid and viscous models differ. We will elaborate on these later when we study the stability of rarefaction waves.

6. Burgers Green's Functions

Suppose that $\bar{u}$ is a given solution of the Burgers equation with corresponding function $\bar{\phi}$ in the Hopf-Cole relation (5.4):

$$\begin{cases} \bar{U}(x,t) = \int_{0-}^{x} \bar{u}(y,t)dy, \\ \bar{U}(x,t) = -2\kappa \log \bar{\phi}(x,t). \end{cases} \tag{6.1}$$

Consider the Burgers equation linearized around the particular solution:

$$\begin{cases} v_t + (\bar{u}v)_x = \kappa v_{xx} \\ V_t + \bar{U}_x V_x = \kappa V_{xx}. \end{cases} \tag{6.2}$$

We can use the Hopf-Cole procedure to find the solution formula for this equation. This is done by linearizing the Hopf-Cole relation (5.4)

$$V + \bar{U} = -2\kappa \log[\bar{\phi} + \zeta]$$

around $\bar{\phi}$:

$$V = -2\kappa \frac{\zeta}{\bar{\phi}}. \tag{6.3}$$

From the Hopf-Cole relation (6.1) and the linearized Hopf-Cole relation (6.3), the linearized Burgers equation (6.2) yields the heat equation

$$\zeta_t = \kappa \zeta_{xx}.$$

This gives us the explicit solution formula for the initial value problem of (6.2):

$$\begin{cases} V(x,t) = \dfrac{\displaystyle\int_{-\infty}^{\infty} \Big[\frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-y)^2}{4\kappa t}} \bar{\phi}(y,0) V(y,0)\Big] dy}{\bar{\phi}(x,t)}, \\ \zeta(x,t) = -\frac{1}{2\kappa} \displaystyle\int_{-\infty}^{\infty} \Big[\frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-y)^2}{4\kappa t}} \bar{\phi}(y,0) V(y,0)\Big] dy. \end{cases} \tag{6.4}$$

We can apply this procedure to find the Green's function $\bar{G}(x,t;x_0,t_0)$ for the Burgers equation linearized around a given solution $\bar{u}(x,t)$:

$$\begin{aligned} &\bar{G}_t + \bar{u}\bar{G}_x = \kappa(\bar{G})_{xx}, \\ &\bar{G}(x,t_0;x_0,t_0) = \delta(x-x_0), \end{aligned} \tag{6.5}$$

to obtain from (6.4) the explicit representation

$$\bar{G}(x,t;x_0,t_0) = \frac{1}{\sqrt{4\pi\kappa(t-t_0)}} e^{-\frac{(x-x_0)^2}{4\kappa(t-t_0)}} \frac{\bar{\phi}(x_0,t_0)}{\bar{\phi}(x,t)} \tag{6.6}$$

The shock wave $b_S(x)$ is a function of the space function only, and so the Green's function $G_S(x,t;x_0,t_0) = G_S(x,t-t_0;x_0,0) \equiv G_S(x,t-t_0;x_0)$:

$$\begin{aligned} &(G_S)_t + b_S(G_S)_x = \kappa(G_S)_{xx}, \\ &G_S(x,0;x_0) = \delta(x-x_0). \end{aligned} \tag{6.7}$$

Apply (6.6) to obtain

$$\begin{aligned} G_S(x,t;x_0) &= \frac{\int_{-\infty}^{\infty} \Big[\frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-y)^2}{4\kappa t}} \phi_S(y,0) G_S(y,0;x_0)\Big] dy}{\phi_S(x,t)} \\ &= \frac{1}{\sqrt{4\pi\kappa t}} e^{-\frac{(x-x_0)^2}{4\kappa t}} \frac{\phi_S(x_0,0)}{\phi_S(x,t)}, \end{aligned}$$

and so, from the last identity in (5.12),

$$G_S(x,t;x_0)=\frac{1}{\sqrt{4\pi\kappa t}}e^{-\frac{(x-x_0)^2}{4\kappa t}}\frac{e^{\frac{\lambda_0 x_0}{2\kappa}}+e^{-\frac{\lambda_0 x_0}{2\kappa}}}{e^{\frac{\lambda_0 x}{2\kappa}}+e^{-\frac{\lambda_0 x}{2\kappa}}}e^{-\frac{(\lambda_0)^2 t}{4\kappa}}. \tag{6.8}$$

This can be rewritten in terms of the heart kernel $H(x,t)$, (4.2):

$$G_S(x,t;x_0)=\frac{1+e^{-\frac{\lambda_0|x_0|}{\kappa}}}{1+e^{-\frac{\lambda_0|x|}{\kappa}}}\cdot\begin{cases}H(x+\lambda_0 t,t), & \text{for } x>0,\ x_0>0;\\ e^{-\frac{\lambda_0|x|}{\kappa}}H(x-\lambda_0 t,t), & \text{for } x>0,\ x_0<0;\\ H(x-\lambda_0 t,t), & \text{for } x<0,\ x_0<0;\\ e^{-\frac{\lambda_0|x|}{\kappa}}H(x+\lambda_0 t,t), & \text{for } x<0,\ x_0>0.\end{cases} \tag{6.9}$$

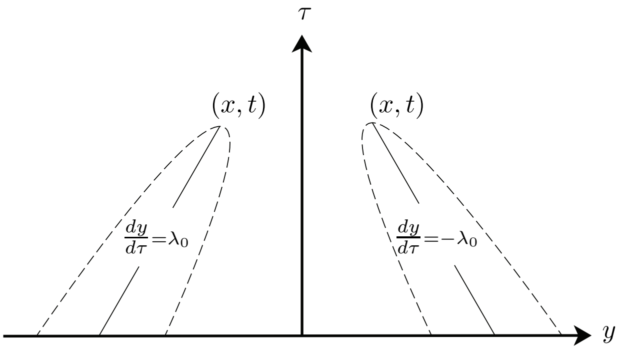

Figure 16. Green's function for Burgers shock wave $G_S(y,t-\tau;x)$.

Notice that, within the accuracy of exponential decaying term $e^{-\lambda_0|x_0|}$, the Green's function equals the heat kernel propagating with speed $\pm\lambda_0$, Figure 16. This basic fact has been used by many authors, including the present ones, see, for instance, [27].

The Green's function $G_R(x,t;x_0,t_0);\ t\geq t_0$ satisfies

$$\begin{aligned}(G_R)_t+b_R(G_R)_x &= \kappa(G_R)_{xx},\\ G_R(x,t_0;x_0,t_0) &= \delta(x-x_0).\end{aligned} \tag{6.10}$$

As the equation coefficient $b_R(x,t)$, (5.8), depends on both variables (x,t), the Green's function is of the form $G_R(x,t;x_0,t_0),\ t\geq t_0$. We have

$$G_R(x,t;x_0,t_0) = \frac{1}{\phi_R(x,t)}\int_{-\infty}^{\infty}\frac{1}{\sqrt{4\pi\kappa(t-t_0)}}e^{-\frac{(x-y)^2}{4\kappa(t-t_0)}}\phi_R(y,t_0)G_R(y,t_0;x_0,t_0)dy.$$

And so

$$\begin{aligned} G_R(x,t;x_0,t_0) &= \frac{1}{\sqrt{4\pi\kappa(t-t_0)}}e^{-\frac{(x-x_0)^2}{4\kappa(t-t_0)}}e^{-\frac{(\lambda_0)^2(t-t_0)}{4\kappa}} \\ &\quad\times\frac{e^{-\frac{\lambda_0 x_0}{2\kappa}}\mathrm{Erfc}(\frac{\lambda_0 t_0-x_0}{\sqrt{4\kappa t_0}})+e^{\frac{\lambda_0 x_0}{2\kappa}}\mathrm{Erfc}(\frac{\lambda_0 t_0+x_0}{\sqrt{4\kappa t_0}})}{e^{-\frac{\lambda_0 x}{2\kappa}}\mathrm{Erfc}(\frac{\lambda_0 t-x}{\sqrt{4\kappa t}})+e^{\frac{\lambda_0 x}{2\kappa}}\mathrm{Erfc}(\frac{\lambda_0 t+x}{\sqrt{4\kappa t}})}. \end{aligned} \tag{6.11}$$

The above Green's functions $G_S(x,t;x_0,t_0),\ G_R(x,t;x_0,t_0)$ are for the antiderivative variables. From these we can easily construct the Green's function $g_S(x,t;x_0,t_0),\ g_R(x,t;x_0,t_0)$ for the original variables by $g_{x_0} = -G_x$:

$$\begin{aligned} (g_S)_t + (b_S g_S)_x &= \kappa(g_S)_{xx},\ g_S(x,t_0;x_0,t_0) = \delta(x-x_0), \\ g_S(x,t;x_0,t_0) &= \int_{x_0}^{\infty}(G_S)_x(x,t;y,t_0)dy; \end{aligned} \tag{6.12}$$

$$\begin{aligned} (g_R)_t + (b_R g_R)_x &= \kappa(g_R)_{xx},\ g_R(x,t_0;x_0,t_0) = \delta(x-x_0), \\ g_R(x,t;x_0,t_0) &= \int_{x_0}^{\infty}(G_R)_x(x,t;y,t_0)dy. \end{aligned} \tag{6.13}$$

The Green's function $G_R(x,t;y,s)$ exhibits both the hyperbolic expansion and parabolic diffusion. We have from the estimate on the error function, (5.14), that the Green's function, , (6.11), has the following estimates after the initial time, $t,s \geq {\lambda_0}^{-2}$, see [10] and [22], with the viscosity coefficient $\kappa = 1$ here:

$$
\begin{aligned}
&G_R(x,t;y,s)\\
&= O(1)\begin{cases}
\dfrac{e^{-\frac{(x-y+\lambda_0(t-s))^2}{4(t-s)}}}{\sqrt{t-s}} \qquad \text{for } x\le -\lambda_0 t+\sqrt{t}, y\le -\lambda_0 s+\sqrt{s},\\[2ex]
\left(\frac{\sqrt{s}}{y+\lambda_0 s}+\frac{\sqrt{s}}{\lambda_0 s-y}\right)\dfrac{e^{-\frac{t(y-sx/t)^2}{4s(t-s)}}e^{-\frac{(x+\lambda_0 t)^2}{4t}}}{\sqrt{(t-s)}}\\
\text{for } x\le -\lambda_0 t+\sqrt{t}, y\in(-s\lambda_0+\sqrt{s},\lambda_0 s-\sqrt{s}),\\[2ex]
\dfrac{e^{-\frac{(x+\lambda_0 t)^2}{4t}}e^{-\frac{s(x-\lambda_0 t)^2}{4t(t-s)}}e^{-\frac{(y-\lambda_0 s)^2}{4(t-s)}}e^{\frac{(x-\lambda_0 t)(y-\lambda_0 s)}{2(t-s)}}}{\sqrt{(t-s)}}\\
\text{for } x\le -\lambda_0 t+\sqrt{t},\ y\ge \lambda_0 s-\sqrt{s},\\[2ex]
\dfrac{1}{\frac{\sqrt{t}}{x+\lambda_0 t}+\frac{\sqrt{t}}{\lambda_0 t-x}}\dfrac{e^{\frac{(x+\lambda_0 t)(y+\lambda_0 s)}{2(t-s)}}e^{-\frac{(y+\lambda_0 s)^2}{4(t-s)}}e^{\frac{-s(x+\lambda_0 t)^2}{4t(t-s)}}}{\sqrt{t-s}}\\
\text{for } x\in(-\lambda_0 t+\sqrt{t},\lambda_0 t-\sqrt{t}),\ y\le -\lambda_0 s+\sqrt{s},\\[2ex]
\dfrac{\frac{\sqrt{s}}{y+\lambda_0 s}+\frac{\sqrt{s}}{\lambda_0 s-y}}{\frac{\sqrt{t}}{x+\lambda_0 t}+\frac{\sqrt{t}}{\lambda_0 t-x}}\dfrac{e^{-\frac{t(y-sx/t)^2}{4s(t-s)}}}{\sqrt{(t-s)}}\\
\text{for } x\in(-\lambda_0 t+\sqrt{t},\lambda_0 t-\sqrt{t}), y\in(-\lambda_0 s+\sqrt{s},\lambda_0 s-\sqrt{s}),\\[2ex]
\dfrac{1}{\frac{\sqrt{t}}{x+\lambda_0 t}+\frac{\sqrt{t}}{\lambda_0 t-x}}\dfrac{e^{\frac{-s(x-\lambda_0 t)^2}{4t(t-s)}}e^{-\frac{(y-\lambda_0 s)^2}{4(t-s)}}e^{\frac{(x-\lambda_0 t)(y-\lambda_0 s)}{2(t-s)}}}{\sqrt{t-s}}\\
\text{for } x\in(-\lambda_0 t+\sqrt{t},\lambda_0 t-\sqrt{t}),\ y\ge \lambda_0 s-\sqrt{s},\\[2ex]
\dfrac{e^{\frac{(x+\lambda_0 t)(y+\lambda_0 s)}{2(t-s)}}e^{\frac{-(y+\lambda_0 s)^2}{4s}}e^{\frac{-s(x+\lambda_0 t)^2}{4t(t-s)}}e^{-\frac{(x-\lambda_0 t)^2}{4t}}}{\sqrt{t-s}}\\
\text{for } x\ge \lambda_0 t-\sqrt{t}, y< -\lambda_0 s+\sqrt{s},\\[2ex]
\left(\frac{\sqrt{s}}{y+\lambda_0 s}+\frac{\sqrt{s}}{\lambda_0 s-y}\right)\dfrac{e^{\frac{-t(y-sx/t)^2}{4s(t-s)}}e^{-\frac{(x-\lambda_0 t)^2}{4t}}}{\sqrt{t-s}}\\
\text{for } x\ge \lambda_0 t+\sqrt{t},\ y\in(-\lambda_0 s+\sqrt{s},\lambda_0 s-\sqrt{s}),\\[2ex]
\dfrac{e^{-\frac{(x-y-\lambda_0(t-s))^2}{4(t-s)}}}{\sqrt{t-s}} \qquad \text{for } x\ge \lambda_0 t-\sqrt{t},\ y\ge \lambda_0 s-\sqrt{s}.
\end{cases}
\end{aligned}
\tag{6.14}
$$

The rich structure in the above expressions is due to the coupling of different scaling as we mentioned before. Take, for instance, when both the source (y,s) and the target (x,t) are in the wave region $-\lambda_0 t+\sqrt{t}<x<\lambda_0 t-\sqrt{t},\ -\lambda_0 s+\sqrt{s}<y<\lambda_0 s-\sqrt{s}$. In this region, the hyperbolic scaling

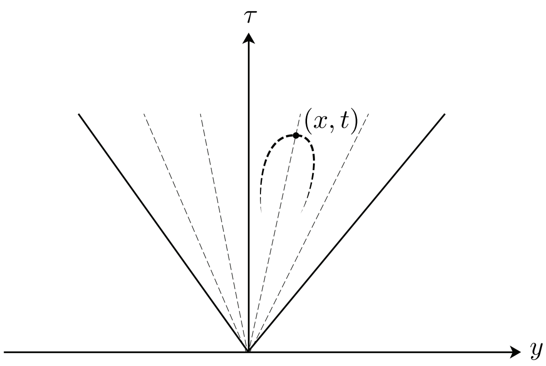

Figure 17. Green's function for Burgers rarefaction wave $G_R(y,\tau;x,t)$.

is strong and we have, from above,

$$G_R(x,t;ys)=O(1)\frac{\frac{\sqrt{s}}{y+\lambda_0 s}+\frac{\sqrt{s}}{\lambda_0 s-y}}{\frac{\sqrt{t}}{x+\lambda_0 t}+\frac{\sqrt{t}}{\lambda_0 t-x}}\frac{e^{-\frac{t(y-sx/t)^2}{4s(t-s)}}}{\sqrt{(t-s)}}.$$

For small time, t close to s, the Green's function is close to the heat kernel, because the hyperbolic effect of expansion has not yet asserted itself. On the other hand, for larger time its essential support moves around the hyperbolic characteristics $y/s=x/t$, Figure 17. Similar estimates holds for the Green's function

$$g_R(x,t;y,s)=-\int_{-\infty}^{y}G_R(x,t;z,s)dz.$$

Note that the nonlinearity in the Burgers equation is a critical one. For instance, the difference of the Burgers kernel and the heat kernel decays no faster than the heat kernel. This can be seen by simple scaling analysis on the level of the heat and Burgers kernels: $(b_D)_t=O(1)t^{-3/2}$, $(b_D)_{xx}=O(1)t^{-3/2}$ and that the nonlinear term is also of the same order $b_D(b_D)_x=O(1)t^{-1/2}\cdot O(1)t^{-1}=O(1)t^{-3/2}$. We have seen in Remark 2.4 the relevance of the inviscid Burgers equation for general convex hyperbolic conservation laws. For the viscous conservation law (1.6) we have, for $\lambda(u)=f'(u)$,

$$\lambda_t+\lambda\lambda_x=\kappa\lambda_{xx}-\kappa\frac{f'''(u)}{(f''(u))^2}(\lambda_x)^2. \tag{6.15}$$

Thus there is the truncation error of

$$-\frac{f'''(u)}{(f''(u))^2}(\lambda_x)^2.$$

Whether or not this is a small truncation error depends on the situation. For the rarefaction waves and diffusion waves the truncation error is small for large time, as $\lambda_x \to 0$ as $t \to \infty$ for these waves. For instance, for the diffusion waves, $(\lambda_x)^2$ tends to zero at the rate of t^{-2}, faster than the rate of $t^{-3/2}$ for $\lambda_t + \lambda\lambda_x$ and for λ_{xx}.Thus the Green's function G_R we have studied is useful for the time-asymptotic analysis of the propagation of diffusion waves and the viscous rarefaction waves, as we will see in Section 7 and Section 9.

7. Stability of Diffusion Waves

We now study the nonlinear stability of the viscous waves that have been constructed. We aim at the pointwise description of the perturbation of the waves. To illustrate the basic idea, we start with the diffusion waves for the Burgers equation.

Theorem 7.1. *Consider the solution $u(x,t)$ of the initial value problem for the Burgers equation*

$$\begin{cases} u_t + uu_x = \kappa u_{xx}, \\ u(x,0) = u_0(x) = O(1)[|x|+1]^{-\alpha}, \ \alpha > 1. \end{cases} \tag{7.1}$$

Then for

$$c \equiv \int_{-\infty}^{\infty} u_0(x)dx,$$

the solution approaches the Burgers kernel $b_D(x,t;c)$, (5.7), time asymptotically:

$$|u(x,t)-b_D(x,t+1;c)| = O(1)(|x|+\sqrt{t}+1)^{-\alpha}+O(1)(t+1)^{-\frac{1}{2}}H(x,t+1;D)$$

$$\cdot\begin{cases} 1, & \text{for } \alpha > 2; \\ \log(t+1), & \text{for } \alpha = 2, \\ (t+1)^{-\frac{\alpha-2}{2}}, & \text{for } 1 < \alpha < 2, \end{cases} \tag{7.2}$$

for any constant D greater than κ. As a consequence,

$$\|u(x,t)-b_D(x,t+1;c)\|_{L_p(x)} = O(1)(t+1)^{-1+\frac{1}{2p}} \begin{cases} 1, & \text{for } \alpha > 2; \\ \log(t+1), & \text{for } \alpha = 2; \\ (t+1)^{-\frac{\alpha-2}{2}}, & \text{for } 1<\alpha<2. \end{cases} \tag{7.3}$$

The theorem can be proved directly using the Hopf-Cole transformation. Instead of doing this, we present the *weakly nonlinear analysis* here. The reason for doing this is that the weakly nonlinear analysis for stability can be applied in the general situation of systems. However, for this we need to assume that the initial value is the small perturbation of the Burgers kernel:

$$u(x,0) = b_D(x,t+1) + O(1)\varepsilon(1+|x|)^{-\alpha}, \tag{7.4}$$

for sufficiently small ε. In fact, we will carry out the analysis for the perturbation of the constant state $\bar{u}$ solution for the general convex conservation laws. Consider convex conservation law

$$\begin{cases} u_t + f(u)_x = \kappa u_{xx}, \\ u(x,0) = u_0(x) = \bar{u} + O(1)\varepsilon[|x|+1]^{-\alpha}, \alpha > 1. \end{cases} \tag{7.5}$$

For the diffusion waves considered here, the characteristic value $f'(u)$ is well-approximated by the Burgers solution time-asymptotically, (6.15). We thus expect $f'(u)(x,t) - f'(\bar{u})(x,t)$ to tend to the solution of

$$\lambda_t + f'(\bar{u})\lambda_x + \lambda\lambda_x = \kappa\lambda_{xx}.$$

The diffusion wave is therefore $b_D(x - f'(\bar{u})t, t; c_0 f''(\bar{u}))$ with the constant c_0 the total mass of the perturbation:

$$c_0 \equiv \int_{-\infty}^{\infty} [u(x,0) - \bar{u}]dx.$$

In the following theorem we express the convergence of basic variable $u(x,t) - \bar{u}(x,t)$.

Theorem 7.2. *Consider the solution $u(x,t)$ of (7.5). Suppose that in (7.5) ε is sufficiently small and $1 < \alpha < 2$. Then the solution approaches the* *constant state plus the Burgers kernel,* (5.7)*, time asymptotically:*

$$\left|u(x,t)-\bar{u}-\frac{1}{f''(\bar{u})}b_D(x-f'(\bar{u})(t+1),t+1;c_0f''(\bar{u}))\right|$$
$$= O(1)\varepsilon(|x-f'(\bar{u})(t+1)|+\sqrt{t}+1)^{-\alpha}. \tag{7.6}$$

As a consequence,

$$\left\|u(x,t)-\bar{u}-\frac{1}{f''(\bar{u})}b_D(x-f'(\bar{u})(t+1),t+1;c_0f''(\bar{u}))\right\|_{L_p(x)}$$
$$= O(1)\varepsilon(t+1)^{-\frac{\alpha}{2}+\frac{1}{2p}}. \tag{7.7}$$

Proof. To simplify our notation, let

$$b(x,t)\equiv\frac{1}{f''(\bar{u})}b_D(x-\bar{\lambda}(t+1),t+1;c_0f''(\bar{u})), \qquad \bar{\lambda}\equiv f'(\bar{u}).$$

From (5.7) we have

$$\begin{cases} b_t+\bar{\lambda}b_x+f''(\bar{u})(\frac{b^2}{2})_x=\kappa b_{xx}, \\ b(x,0)=\frac{1}{f''(\bar{u})}b_D(x-\bar{\lambda},1;c_0f''(\bar{u})) \\ \qquad\quad =\frac{1}{f''(\bar{u})}\frac{\sqrt{\kappa}(e^{c_0f''(\bar{u})/(2\kappa)}-1)e^{-\frac{(x-\bar{\lambda})^2}{4\kappa}}}{\sqrt{\pi}+\int_{(x-\bar{\lambda})/\sqrt{4\kappa}}^{\infty}(e^{c_0f''(\bar{u})/(2\kappa)}-1)e^{-y^2}dy}. \end{cases} \tag{7.8}$$

Let

$$v(x,t)\equiv u(x,t)-\bar{u}-b(x,t).$$

From (7.5) and (7.8) v satisfies

$$\begin{cases} v_t+\bar{\lambda}v_x=\kappa v_{xx}-g_x(x,t), \\ v(x,0)=u(x,0)-\bar{u}-b(x,0), \end{cases} \tag{7.9}$$

where

$$\begin{aligned} g(x,t) &= \Big[f(u)-f(\bar{u})-f'(\bar{u})(u-\bar{u})-\frac{1}{2}f''(\bar{u})b^2\Big](x,t) \\ &= \Big[\frac{1}{2}f''(\bar{u})(b+v)^2+O(1)(b+v)^3-\frac{1}{2}f''(\bar{u})b^2\Big](x,t) \\ &= \Big[f''(\bar{u})bv+\frac{1}{2}f''(\bar{u})v^2+O(1)|b|^3+O(1)|v|^3\Big](x,t). \end{aligned} \tag{7.10}$$

Recalling the definition of c_0, and using the fact that b_D is conserved, by (5.7) we have

$$\begin{aligned}\int_{-\infty}^{\infty} v(x,0)dx &= \int_{-\infty}^{\infty}[u(x,0)-\bar{u}]dx - \int_{-\infty}^{\infty}\frac{1}{f''(\bar{u})}b_D(x-\bar{\lambda},1;c_0f''(\bar{u}))dx\\ &= c_0 - \frac{1}{f''(\bar{u})}\int_{-\infty}^{\infty} b_D(x-\bar{\lambda},0;c_0f''(\bar{u}))dx\\ &= c_0 - \frac{1}{f''(\bar{u})}c_0f''(\bar{u}) = 0.\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} v(x,t)dx = \int_{-\infty}^{\infty} v(x,0)dx = 0.$$

As in the proof of Theorem 4.1 we introduce the antiderivative

$$w(x,t) = \int_{-\infty}^{x} v(y,t)dy = -\int_{x}^{\infty} v(y,t)dy.$$

From (7.5) it is clear that

$$c_0 = \int_{-\infty}^{\infty}[u(x,0)-\bar{u}]dx = O(1)\varepsilon\int_{-\infty}^{\infty}(|x|+1)^{-\alpha}dx = O(1)\varepsilon.$$

Therefore, with (7.8),

$$\begin{aligned}v(x,0) &= u(x,0)-\bar{u}-b(x,0) = O(1)\varepsilon(|x|+1)^{-\alpha} + O(1)c_0e^{-\frac{(x-\bar{\lambda})^2}{4\kappa}}\\ &= O(1)\varepsilon(|x|+1)^{-\alpha}, \qquad (7.11)\\ w(x,0) &= O(1)\varepsilon(|x|+1)^{-\alpha+1}.\end{aligned}$$

We now perform a priori estimate on v. Let

$$M(t) = \sup_{0\le\tau\le t}\|v(x,\tau)(|x-\bar{\lambda}(\tau+1)|+\sqrt{\tau}+1)^{\alpha}\|_{L_\infty(x)}.$$

Then

$$|v(x,t)| \le M(t)(|x-\bar{\lambda}(t+1)|+\sqrt{t}+1)^{-\alpha}, \quad -\infty<x<\infty, \quad t\ge 0. \quad (7.12)$$

Applying Duhamel's principle to (7.9), we have

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{\infty} H(x-y-\bar{\lambda}t,t)v(y,0)dy \\ &\quad - \int_0^t \int_{-\infty}^{\infty} H(x-y-\bar{\lambda}(t-\tau),t-\tau)g_y(y,\tau)dyd\tau \\ &= I_1 + I_2. \end{aligned} \tag{7.13}$$

Because of (7.11), the estimate of I_1 is the same as the proof of Theorem 4.1, and the result is stated in (4.5). That is,

$$I_1 = O(1)\varepsilon(|x-\bar{\lambda}t|+\sqrt{t}+1)^{-\alpha} = O(1)\varepsilon(|x-\bar{\lambda}(t+1)|+\sqrt{t}+1)^{-\alpha}. \tag{7.14}$$

To estimate I_2, from (5.7) we note that

$$b(x,t) = O(1)\varepsilon(t+1)^{-\frac{1}{2}}e^{-\frac{(x-\bar{\lambda}(t+1))^2}{4\kappa(t+1)}}. \tag{7.15}$$

Applying (7.12) and (7.15) to (7.10), we obtain

$$\begin{aligned} g(x,t) &= O(1)[\varepsilon M(t)+\varepsilon^3](t+1)^{-\frac{1+\alpha}{2}}e^{-\frac{(x-\bar{\lambda}(t+1))^2}{4\kappa(t+1)}} \\ &\quad +O(1)M(t)^2(|x-\bar{\lambda}(t+1)|+\sqrt{t}+1)^{-2\alpha}. \end{aligned}$$

Therefore,

$$\begin{aligned} I_2 &= \int_0^t\int_{-\infty}^{\infty}\frac{\partial}{\partial y}H(x-y-\bar{\lambda}(t-\tau),t-\tau)g(y,\tau)dyd\tau \\ &= O(1)[\varepsilon M(t)+\varepsilon^3]\int_0^t\int_{-\infty}^{\infty}(t-\tau)^{-1}e^{-\frac{(x-y-\bar{\lambda}(t-\tau))^2}{4\kappa(t-\tau)}}(\tau+1)^{-\frac{1+\alpha}{2}}e^{-\frac{(y-\bar{\lambda}(\tau+1))^2}{4\kappa(\tau+1)}}dyd\tau \\ &\quad +O(1)M(t)^2\int_0^t\int_{-\infty}^{\infty}(t-\tau)^{-1}e^{-\frac{(x-y-\bar{\lambda}(t-\tau))^2}{4\kappa(t-\tau)}}(|y-\bar{\lambda}(\tau+1)|+\sqrt{\tau}+1)^{-2\alpha}dyd\tau \\ &= I_{21}+I_{22}. \end{aligned} \tag{7.16}$$

Here I_{21} is the leading term. To estimate I_{21} we complete the square:

$$\begin{aligned} &-\frac{(x-y-\bar{\lambda}(t-\tau))^2}{4\kappa(t-\tau)} - \frac{(y-\bar{\lambda}(\tau+1))^2}{4\kappa(\tau+1)} \\ &= -\frac{t+1}{4\kappa(t-\tau)(\tau+1)}\left[y-\frac{(\tau+1)x}{t+1}\right]^2 - \frac{(x-\bar{\lambda}(t+1))^2}{4\kappa(t+1)}. \end{aligned}$$

Thus

$$\begin{aligned}I_{21} &= O(1)[\varepsilon M(t)+\varepsilon^3]\int_0^t (t-\tau)^{-\frac{1}{2}}(\tau+1)^{-\frac{\alpha}{2}}(t+1)^{-\frac{1}{2}}e^{-\frac{(x-\bar\lambda(t+1))^2}{4\kappa(t+1)}}d\tau\\ &= O(1)[\varepsilon M(t)+\varepsilon^3](t+1)^{-\frac{1}{2}}e^{-\frac{(x-\bar\lambda(t+1))^2}{4\kappa(t+1)}}\\ &\quad\times\left[\int_0^{\frac{t}{2}}(t+1)^{-\frac{1}{2}}(\tau+1)^{-\frac{\alpha}{2}}d\tau+\int_{\frac{t}{2}}^t (t-\tau)^{-\frac{1}{2}}(t+1)^{-\frac{\alpha}{2}}d\tau\right]\\ &= O(1)[\varepsilon M(t)+\varepsilon^3](t+1)^{-\frac{1}{2}}e^{-\frac{(x-\bar\lambda(t+1))^2}{4\kappa(t+1)}}(t+1)^{-\frac{\alpha-1}{2}}\\ &= O(1)[\varepsilon M(t)+\varepsilon^3][|x-\bar\lambda(t+1)|+\sqrt{t}+1]^{-\alpha}. \qquad (7.17)\end{aligned}$$

To estimate I_{22} we divide the space integral into two parts:

$$\begin{aligned}I_{22} =&O(1)M(t)^2\left[\int_0^t\int_{|y-\bar\lambda(\tau+1)|\le\sqrt{\tau+1}}+\int_0^t\int_{|y-\bar\lambda(\tau+1)|\ge\sqrt{\tau+1}}\right](t-\tau)^{-1}\\ &\times e^{-\frac{(x-y-\bar\lambda(t-\tau))^2}{4\kappa(t-\tau)}}(|y-\bar\lambda(\tau+1)|+\sqrt{\tau}+1)^{-2\alpha}dyd\tau\\ =&I_{221}+I_{222}. \qquad (7.18)\end{aligned}$$

Here the estimate of I_{221} is completely parallel to I_{21}:

$$\begin{aligned}I_{221} &= O(1)M(t)^2\int_0^t\int_{|y-\bar\lambda(\tau+1)|\le\sqrt{\tau+1}}(t-\tau)^{-1}\\ &\quad\times e^{-\frac{(x-y-\bar\lambda(t-\tau))^2}{4\kappa(t-\tau)}}(\tau+1)^{-\alpha}e^{-\frac{(y-\bar\lambda(\tau+1))^2}{4\kappa(\tau+1)}}dyd\tau\\ &= O(1)M(t)^2e^{-\frac{(x-\bar\lambda(t+1))^2}{4\kappa(t+1)}}(t+1)^{-\frac{1}{2}}\int_0^t(t-\tau)^{-\frac{1}{2}}(\tau+1)^{\frac{1}{2}-\alpha}d\tau\\ &= O(1)M(t)^2e^{-\frac{(x-\bar\lambda(t+1))^2}{4\kappa(t+1)}}\begin{cases}(t+1)^{-1}, & \text{for } \alpha>\frac{3}{2};\\ (t+1)^{-1}\log(t+1), & \text{for } \alpha=\frac{3}{2};\\ (t+1)^{\frac{1}{2}-\alpha}, & \text{for } \alpha<\frac{3}{2}.\end{cases}\end{aligned}$$

Noting $1<\alpha<2$, we have

$$\begin{aligned}I_{221} &= O(1)M(t)^2(t+1)^{-\frac{\alpha}{2}}e^{-\frac{(x-\bar\lambda(t+1))^2}{4\kappa(t+1)}}\\ &= O(1)M(t)^2(|x-\bar\lambda(t+1)|+\sqrt{t}+1)^{-\alpha}. \qquad (7.19)\end{aligned}$$

For I_{222} we change the variable from $y-\bar{\lambda}(\tau+1)$ to y and set

$$\tilde{x}=x-\bar{\lambda}(t+1)$$

to simplify our notations. Then

$$I_{222}=O(1)M(t)^2\int_0^t\int_{|y|\geq\sqrt{\tau+1}}(t-\tau)^{-1}e^{-\frac{(\tilde{x}-y)^2}{4\kappa(t-\tau)}}|y|^{-2\alpha}dyd\tau.$$

We also consider $t\geq 1$ since the case $t<1$ is trivial.

Case 1. $|\tilde{x}|<\sqrt{t+1}$.

$$\begin{aligned}
I_{222} &= O(1)M(t)^2\Big[\int_0^{\frac{t}{2}}(t+1)^{-1}\int_{\sqrt{\tau+1}}^{\infty}y^{-2\alpha}dyd\tau \\
&\quad+\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}(\tau+1)^{-\alpha}\int_{|y|\geq\sqrt{\tau+1}}(t-\tau)^{-\frac{1}{2}}e^{-\frac{(\tilde{x}-y)^2}{4\kappa(t-\tau)}}dyd\tau\Big]\\
&= O(1)M(t)^2\Big[\int_0^{\frac{t}{2}}(t+1)^{-1}(\tau+1)^{\frac{1}{2}-\alpha}d\tau+\int_{\frac{t}{2}}^t(t-\tau)^{-\frac{1}{2}}(t+1)^{-\alpha}d\tau\Big]\\
&= O(1)M(t)^2\begin{cases}(t+1)^{-1}, & \text{for } \alpha>\frac{3}{2}\\ (t+1)^{-1}\log(t+1), & \text{for } \alpha=\frac{3}{2}\\ (t+1)^{\frac{1}{2}-\alpha}, & \text{for } \alpha<\frac{3}{2}\end{cases}\\
&= O(1)M(t)^2(t+1)^{-\frac{\alpha}{2}}=O(1)M(t)^2(|\tilde{x}|+\sqrt{t}+1)^{-\alpha}\\
&= O(1)M(t)^2(|x-\bar{\lambda}(t+1)|+\sqrt{t}+1)^{-\alpha}. \qquad (7.20)
\end{aligned}$$

Case 2. $|\tilde{x}|>\sqrt{t+1}$.

$$\begin{aligned}
I_{222}=&O(1)M(t)^2\int_0^t\int_{\sqrt{\tau+1}\leq|y|\leq|\tilde{x}|/2}(t-\tau)^{-1}e^{-\frac{(\tilde{x}-y)^2}{8\kappa(t-\tau)}}e^{-\frac{\tilde{x}^2}{32\kappa(t-\tau)}}|y|^{-2\alpha}dyd\tau\\
&+O(1)M(t)^2\int_0^t\int_{|y|>|\tilde{x}|/2}(t-\tau)^{-1}e^{-\frac{(\tilde{x}-y)^2}{4\kappa(t-\tau)}}|\tilde{x}|^{-2\alpha}dyd\tau.
\end{aligned}$$

For the first integral we repeat the estimate of (7.20). Thus

$$I_{222}=O(1)M(t)^2e^{-\frac{\tilde{x}^2}{32\kappa(t+1)}}\begin{cases}(t+1)^{-1}, & \text{for } \alpha>\frac{3}{2}\\ (t+1)^{-1}\log(t+1), & \text{for } \alpha=\frac{3}{2}\\ (t+1)^{\frac{1}{2}-\alpha}, & \text{for } \alpha<\frac{3}{2}\end{cases}$$

$$
\begin{aligned}
&+ O(1)M(t)^2\int_0^t (t-\tau)^{-\frac{1}{2}}|\tilde{x}|^{-2\alpha}d\tau \\
=&O(1)M(t)^2\big[e^{-\frac{\tilde{x}^2}{32\kappa(t+1)}}(t+1)^{-\frac{\alpha}{2}} + (t+1)^{\frac{1}{2}}|\tilde{x}|^{-2\alpha}\big] = O(1)M(t)^2|\tilde{x}|^{-\alpha} \\
=&O(1)M(t)^2(|\tilde{x}| + \sqrt{t}+1)^{-\alpha} \\
=&O(1)M(t)^2(|x-\bar{\lambda}(t+1)| + \sqrt{t}+1)^{-\alpha}. \qquad (7.21)
\end{aligned}
$$

Combining (7.13), (7.14) and (7.16) to (7.21), we have

$$v(x,t) = O(1)[\varepsilon + M(t)^2](|x-\bar{\lambda}(t+1)| + \sqrt{t}+1)^{-\alpha}.$$

That is,

$$\|v(x,t)(|x-\bar{\lambda}(t+1)| + \sqrt{t}+1)^{\alpha}\|_{L_\infty(x)} \le C[\varepsilon + M(t)^2]$$

for some constant $C > 0$. This implies

$$M(t) \le C[\varepsilon + M(t)^2].$$

Therefore, if $M(t)$ is small, we have

$$M(t) \le 2C\varepsilon. \qquad (7.22)$$

By a continuity argument, (7.22) is true provided ε is sufficiently small. Substituting this into (7.12) we have

$$v(x,t) = O(1)\varepsilon(|x-\bar{\lambda}(t+1)|+\sqrt{t}+1)^{-\alpha} = O(1)\varepsilon(|x-f'(\bar{u})(t+1)|+\sqrt{t}+1)^{-\alpha}.$$

□

Remark 7.3. Theorem 7.2 is for $1 < \alpha < 2$. If $\alpha \ge 2$, we replace the initial condition in (7.5) by

$$u(x,0) = \bar{u} + O(1)\varepsilon(|x|+1)^{-\alpha} = \bar{u} + O(1)\varepsilon(|x|+1)^{-\beta}$$

with any $\beta < 2 \le \alpha$, and apply Theorem 7.2. Therefor, the L_∞ rate is arbitrarily close to but slower than $(t+1)^{-1}$. This is a contrast to the Burgers equation, where in (7.2) the optimal rate is $(t+1)^{-1}$ for $\alpha > 2$, and $(t+1)^{-1}\log(t+1)$ for $\alpha = 2$ (with $\bar{u} = 0$). These optimal rates are obtained by way of Hopf-Cole transformation. For a general scalar conservation law we use Duhamel's principle to perform a priori estimates. It is clear that the leading term is I_{21} in (7.16), which comes from the term $f''(\bar{u})bv$ in the source g in (7.10). Suppose $\alpha > 2$ in (7.5). If we imitate the result for the Burgers equation and replace (7.12) by

$$|v(x,t)| \le M(t)\big[(|x-\bar{\lambda}(t+1)|+\sqrt{t}+1)^{-\alpha}+(t+1)^{-\frac{1}{2}}H(x-\bar{\lambda}(t+1),t+1;D)\big],$$

we will have

$$\begin{aligned}
I_{21} =& O(1)[\varepsilon M(t)+\varepsilon^3]\int_0^t\int_{-\infty}^{\infty}(t-\tau)^{-1}e^{-\frac{(x-y-\bar{\lambda}(t-\tau))^2}{4\kappa(t-\tau)}}(\tau+1)^{-\frac{3}{2}}e^{-\frac{(y-\bar{\lambda}(\tau+1))^2}{4\kappa(\tau+1)}}dyd\tau\\
&= O(1)[\varepsilon M(t)+\varepsilon^3]\int_0^t(t-\tau)^{-\frac{1}{2}}(\tau+1)^{-1}(t+1)^{-\frac{1}{2}}e^{-\frac{(x-\bar{\lambda}(t+1))^2}{4\kappa(t+1)}}d\tau\\
&= O(1)[\varepsilon M(t)+\varepsilon^3]H(x-\bar{\lambda}(t+1),t+1)\int_0^t(t-\tau)^{-\frac{1}{2}}(\tau+1)^{-1}d\tau\\
&= O(1)[\varepsilon M(t)+\varepsilon^3]H(x-\bar{\lambda}(t+1),t+1)(t+1)^{-\frac{1}{2}}\log(t+1).
\end{aligned}$$

Because of the extra $\log(t+1)$, the a priori estimate cannot be closed. The same situation happens to the case $\alpha = 2$ as well: After the iteration, the term $(t+1)^{-\frac{1}{2}}H(x-\bar{\lambda}(t+1),t+1;D)\log(t+1)$ will induce the term $(t+1)^{-\frac{1}{2}}H(x-\bar{\lambda}(t+1),t+1;D)\log^2(t+1)$.

The following proposition is compared with the one for heat equation, Proposition 4.2.

Proposition 7.4. *The solution of*

$$\begin{aligned}
&u_t+uu_x=\kappa u_{xx},\\
&u(x,0)=u_0(x)=\begin{cases} C[|x|+1]^{-1}, & \textit{for } x>0,\\ 0, & \textit{for } x\le 0.,\end{cases}
\end{aligned}\tag{7.23}$$

satisfies

$$u(x,t)=O(1)C(x^2+t)^{-\frac{1}{2}}.$$

Proof. From the solution formula (5.6),

$$u(x,t) \;=\; \frac{\int_{-\infty}^{\infty}\frac{x-y}{t}e^{-\frac{(x-y)^2}{4\kappa t}-\frac{1}{2\kappa}\int_{0-}^{y}u_0(z)dz}dy}{\int_{-\infty}^{\infty}e^{-\frac{(x-y)^2}{4\kappa t}-\frac{1}{2\kappa}\int_{0-}^{y}u_0(z)}dy}$$

$$= \frac{\int_{-\infty}^{0} \frac{x-y}{t} e^{-\frac{(x-y)^2}{4\kappa t}} dy + \int_0^{\infty} \frac{x-y}{t} (1+|y|)^{-\frac{C}{2\kappa}} e^{-\frac{(x-y)^2}{4\kappa t}} dy}{\int_{-\infty}^{0} e^{-\frac{(x-y)^2}{4\kappa t}} dy + \int_0^{\infty} (1+|y|)^{-\frac{C}{2\kappa}} e^{-\frac{(x-y)^2}{4\kappa t}} dy}. \tag{7.24}$$

The remaining of the proof is to estimate these integrals for the cases $x > 0$ and $x < 0$ and use the error function estimate (5.14), e.g. [22]. We omit the details. □

8. Stability of Shock Waves

Consider the viscous convex conservation law corresponding to the hyperbolic scalar law (2.1):

$$u_t + f(u)_x = \kappa u_{xx},\ f''(u) > 0. \tag{8.1}$$

A *viscous shock wave* is a traveling wave, $\phi((x-st)/\kappa) \equiv \phi(\xi/\kappa)$, with speed s and end states $u_\pm$:

$$-s\phi' + f(\phi)' = \phi'',\ \phi(\pm\infty) = u_\pm.$$

Integrating this from $x = -\infty$ to get

$$\phi' = f(\phi) - f(u_-) - s(\phi - u_-),\ \phi(\infty) = u_+. \tag{8.2}$$

For the wave to exist, we need to require that the R.H.S. of the equation is zero at $\phi = u_+$:

$$f(u_+) - f(u_-) = s(u_+ - u_-),$$

which is the same as the Rankine-Hugoniot condition (2.9) for hyperbolic conservation law. We also need the R.H.S. to be of the same sign as $u_+ - u_-$:

$$[f(u) - f(u_-) - s(u - u_-)](u_+ - u_-) > 0.$$

For convex conservation law, $f''(u) > 0$, considered here, the above is satisfied if and only if $u_- > u_+$ and in fact we have

$$f'(u_-) > s > f'(u_+),\ \frac{d}{d\xi} f'(\phi(\xi)) < 0,\ \text{for} -\infty < \xi < \infty. \tag{8.3}$$

The first is the entropy condition (2.11), and the second the *compressibility* of the shock waves.

It also results in

$$\phi(\xi) \in (u_+, u_-). \tag{8.4}$$

We will study the nonlinear stability of the shock wave. First we note that the translation of the shock induces the flux:

$$\int_{-\infty}^{\infty} [\phi((x+x_0)/\kappa) - \phi(x/\kappa)]dx = x_0[u_+ - u_-], \tag{8.5}$$

which can be seen easily by considering the differentiation with respect to x_0 of the identity. For simplicity the speed of the shock is taken to be zero $s = 0,\ \phi(\xi) = \phi(x)$,

$$f(\phi(x/\kappa))_x = \phi(x/\kappa)_{xx}. \tag{8.6}$$

Theorem 8.1. *Consider the perturbation of the shock wave $\phi(x/\kappa)$:*

$$u(x,0) \equiv \phi(x/\kappa) + \bar{v}(x,0), \tag{8.7}$$

$$\bar{v}(x,0) = O(1)\varepsilon(|x|+1)^{-\alpha}. \tag{8.8}$$

Set

$$c_0 \equiv \int_{-\infty}^{\infty} \bar{v}(x,0)dx,\ \ x_0 \equiv \frac{c_0}{u_+ - u_-}.$$

Then, for ε sufficiently small and the constant $\alpha > \frac{3}{2}$, the solution of the initial value problem for (8.1) *tends to the traveling wave time-asymptotically:*

$$\lim_{t\to\infty} |u(x,t) - \phi((x+x_0)/\kappa)| = 0.$$

Proof. From (8.5)

$$\int_{-\infty}^{\infty} [\phi((x+x_0)/\kappa) - u(x,0)]dx = 0,\ \ x_0 \equiv \frac{c_0}{u_+ - u_-}.$$

The perturbation induces a phase shift x_0 of the shock if the *total mass* c_0 of the perturbation is nonzero. By replacing $\phi(x)$ in (8.9) with $\phi((x+x_0)/\kappa)$

the perturbation has zero total mass.

$$\begin{cases} u_t + f(u)_x = \kappa u_{xx},\ f''(u) > 0, \\ u(x,t) \equiv \phi((x+x_0)/\kappa) + v(x,t), \\ f(\phi)_x = \phi_{xx},\ \phi(\pm\infty) = u_\pm, \\ v(x,0) = O(1)\varepsilon(|x|+1)^{-\alpha}, \\ \int_{-\infty}^{\infty} v(x,0)dx = 0, \end{cases} \tag{8.9}$$

here, for simplicity, we still write the translated shock as $\phi(x/\kappa)$. By the zero total mass property, we may consider the anti-derivative of v:

$$w(x,t) \equiv \int_{-\infty}^{x} v(y,t)dy,\ w(\pm\infty,t) = 0.$$

The equations for the perturbation are:

$$\begin{cases} v_t + [f'(\phi(x/\kappa))v + O(1)v^2]_x = \kappa v_{xx}, \\ w_t + f'(\phi(x/\kappa))w_x + O(1)(w_x)^2 = \kappa w_{xx},\ w_x = v. \end{cases} \tag{8.10}$$

The simplest method for stability is the energy method, e.g. [6], [24], here integrating the second equation in (8.10) times w:

$$\int_{-\infty}^{\infty} \frac{1}{2} w^2(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \Big[-(f'(\phi(x/\kappa)))_x \frac{w^2}{2} + \kappa(w_x)^2 + O(1)w(w_x)^2 \Big] dxdt$$
$$= \int_{-\infty}^{\infty} \frac{1}{2} w^2(x,0)dx. \tag{8.11}$$

The crucial *compressibility* property, (8.3),

$$f'(\phi(x))_x < 0$$

makes the first two terms in the double integral in (8.11) positive definite. The third term $O(1)w(w_x)^2$ can be dominated by the second term $\kappa(w_x)^2$ provided that w is small. This is so, by Sobolev calculus, if we can estimate the $L_2(x)$ norm of w and w_x. The above is for the $L_2(x)$ estimate for w. And similarly, starting with integrating the first equation in (8.10) times $v = w_x$, and using the above energy estimate, one obtains the $L_2(x)$ estimate for

$v = w_x$. Thus, in conclusion, we obtain the desired energy estimates

$$\int_{-\infty}^{\infty} \frac{1}{2} w^2(x,t)dx + \int_0^t \int_{-\infty}^{\infty} \kappa (w_x)^2 dxdt = O(1) \int_{-\infty}^{\infty} \frac{1}{2} w^2(x,0)dx,$$

and similar one for v. This proves the stability of the shock profile ϕ provided that the strength of the perturbation ε is small. □

In order for the above straightforward energy method to work, the decay rate $(|x|+1)^{-\alpha}$ of the initial perturbation need to be sufficiently large $\alpha > \frac{3}{2}$. This is not necessary if one adopts a combination of pointwise and energy method. It is necessary for the initial data to be in $L_1(x)$ if the time-asymptotic shift x_0 can be calculated a priori.

Remark 8.2. The linearized equation

$$v_t + (f'(\phi(x/\kappa))v)_x = \kappa v_{xx}$$

has a trivial solution $v(x,t) = \phi'(x/\kappa)/\kappa$ induced by the translation of the shock $\phi(x/\kappa) \to \phi((x+x_0)/\kappa)$. In other words, the linearized operator has a kernel $\phi'(x)$. After the phase shift of the shock is determined by conservation law, (8.9), this kernel is screened out. One way is to consider the anti-derivative and obtain the linearized equation

$$w_t + f(\phi(x/\kappa))w_x = \kappa w_{xx}.$$

This linear equation does not have non-dissipative solution, and the stability analysis is done easily by energy method as in the above.

One can take advantage of one aspect of the scalar equation, namely the maximum principle, for the stability analysis to obtain results for large, even non-integrable initial perturbation, [9], [4].

The stability of the shock profiles can be studied more quantitatively be either the Green's function approach or by the weighted energy method. We will illustrate the Green's function approach when we study the stability of the rarefaction waves. For shock waves, the weighted energy method is illustrative in exhibiting the relation of the entropy condition and the decay of the perturbation. Consider the convex conservation law

$$u_t + f(u)_x = \kappa u_{xx},\ \ f''(u) > 0.$$

A shock, its speed taken to be zero, $\phi(x)$ is compressive in that

$$\phi'(x) < 0,\ \ \phi(\pm\infty) = u_\pm,\ \ f'(u_-) > 0 > f'(u_+).$$

In fact, it is easy to see from the Rankine-Hugoniot condition that

$$f'(u_-) > 2C_1|u_+ - u_-|,\ \ f'(u_+) < -2C_1|u_+ - u_-|,$$

for some positive constant C_1 depending on the strength α of convexity $f''(u) > \alpha$. This form of compressibility says that at far field x close to $\pm\infty$ a perturbation propagates toward the center of the shock with speed $|f'(u_\pm)|$ comparable to the strength $|u_+ - u_-|$ of the shock. Due to dissipation term κu_{xx} the perturbation cancelled out and resulting in a phase shift x_0 of the shock. To see this, we use the weighted energy method, integrating the second equation in (8.10) times $wA(x)$:

$$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}w^2 A dx + \int_{-\infty}^{\infty} w^2 A\left(-\frac{1}{2}f'(\phi(x/\kappa))_x - \frac{f'(\phi(x/\kappa))}{2}\frac{A'}{A} - \frac{\kappa}{2}\frac{A''}{A}\right)dx$$
$$+\int_{-\infty}^{\infty} A(1+O(1)w)|w_x|^2 dx = 0. \tag{8.12}$$

Let

$$\begin{cases} b \in (1/4, 1), \\ A(x) \equiv \exp\left(-\dfrac{b}{2\kappa}\displaystyle\int_0^x f'(\phi(\tau/\kappa))d\tau\right). \end{cases} \tag{8.13}$$

With (8.12) and (8.13), one has

$$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}w^2 A dx + \int_{-\infty}^{\infty} w^2 A\left(\frac{-\frac{1}{2}(1-\frac{b}{2})f''(\phi(\frac{x}{\kappa}))\phi'(\frac{x}{\kappa}) + \frac{b^2}{8}|f'(\phi(\frac{x}{\kappa}))|^2}{\kappa}\right)dx$$
$$+\int_{-\infty}^{\infty} A(1+O(1)w)|w_x|^2 dx = 0. \tag{8.14}$$

By the property that $f''>0$, there exists $\alpha>0$ such that

$$f''(u)>\alpha \text{ for } u\in(u_+,u_-). \tag{8.15}$$

This, (8.2), and (8.4) yield that there exists $\gamma>0$ such that

$$-\frac{1}{2}(1-b/2)f''(\phi(\eta))\phi'(\eta)\geq -\gamma\alpha(f(\phi(\eta))-f(u_-))>0. \tag{8.16}$$

Then, by the convexity of f to yield

$$\gamma_0\equiv \min_{u\in(u_+,u_-)}\big(-\gamma\alpha(f(u)-f(u_-))+|f'(u)|^2\big)>0. \tag{8.17}$$

The estimates (8.16) and (8.17) together with the identity (8.14) give that

$$\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}w^2Adx+\int_{-\infty}^{\infty}w^2A\frac{\gamma_0}{\kappa}dx+\int_{-\infty}^{\infty}A(1+O(1)w)|w_x|^2dx=0. \tag{8.18}$$

This yields the following estimate, c.f. (8.11),

$$e^{\gamma_0 t/\kappa}\int_{-\infty}^{\infty}A(x)w(x,t)^2dx+\int_0^t\int_{-\infty}^{\infty}e^{\gamma_0 s/\kappa}A(x)(1+O(1)w)|w_x(x,s)|^2dxds$$
$$=\int_{-\infty}^{\infty}A(x)w(x,0)^2dx. \tag{8.19}$$

Note that in order for the last integral to be bounded, the initial perturbation needs to decay exponentially because of the exponential growth of the weighted function $w(x)$. With such an initial data, (8.19) yields the exponential decay, in both x and t, of the perturbation. Algebraic decay rates of the perturbation can be obtained similarly when its initial values decays algebraically. For this, different weighted functions of algebraic rates should be chosen; details are omitted.

For the stability of viscous shock waves corresponding to non-convex flux $f(u)$, see [25]. For non-convex flux, there are wave pattern with complex combination of shock and rarefaction waves. The stability of such wave patterns is not explored nearly sufficiently, see [28].

Both of the weighted energy estimate and the elementary energy estimate (8.11) yield the nonlinear stability of shock waves. The weighted energy method yields the pointwise decay rates and more geometric understanding of the wave propagation. It is also clear from the Burgers Green's function $G_S(x,t;x_0)$ around the shock profile, (6.9), that a perturbation around the shock profile propagates toward the shock, except for an exponential decaying factor. The above estimate (8.19) says that this is so for general convex conservation law. Thus we can also use the Green's function approach for the pointwise estimates. We will carry out the Green's function approach in Section 10 when we study the stability of rarefaction waves.

The Green's functions for Burgers waves have been constructed. In the next section we show that the energy method allows us to estimate the Green's function for weak shock waves for more general convex conservation laws.

9. Estimates of the Green's Function

In this section, we normalized $\kappa = 1$ and consider the initial value problem:

$$\begin{cases} \partial_t g + f'(\phi(x))\partial_x g - \partial_{xx} g = 0,\ (x,t) \in \mathbb{R}\times\mathbb{R}^+, \\ |g(x,0)| \le \sigma_0 e^{-\sigma_0|x|/2}, \end{cases} \tag{9.1}$$

where

$$\begin{cases} \sigma_0 \equiv \min(1, |\lambda_-|, |\lambda_+|), \\ \lambda_\pm \equiv f'(u_\pm). \end{cases}$$

One introduces an initial approximation solution $g_0(x,t)$ to the problem (9.1) as follows:

$$g_0(x,t) \equiv \chi_-(x)g_-(x,t) + \chi_+(x)g_+(x,t), \tag{9.2}$$

where $\chi_\pm(x)$ is a partition of unit satisfying

$$\begin{cases} \chi_-(x) + \chi_+(x) = 1, \\ 0 \le \chi_+'(x) \le O(1)\sigma_0 e^{-4\sigma_0|x|}, \\ |\chi_+''(x)| \le O(1)\sigma_0^2 e^{-4\sigma_0|x|}, \end{cases}$$

and the functions $g_\pm(x,t)$ are given by

$$g_\pm(x,t) \equiv \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y-\lambda_\pm t)^2}{4t}}}{\sqrt{4\pi t}} g(y,0)dy.$$

The function $g_0(x,t)$ satisfies

$$|g_0(x,t)| \le O(1)\sigma_0 e^{-\frac{\sigma_0^2 t}{4}-\frac{\sigma_0|x|}{2}}. \tag{9.3}$$

The truncation error of g_0,

$$g_e \equiv -\left(\partial_t + f'(\phi)\partial_x - \partial_x^2\right) g_0(x,t),$$

satisfies

$$\begin{aligned}|g_e(x,t)| \le{}& |\chi_-(x)(f'(\phi)-f'(u_-))\partial_x g_-(x,t) + \chi_+(x)(f'(\phi)-f'(u_+))\partial_x g_+(x,t)| \\ &+2|\chi_-'(x)|\left(|\partial_x(g_-(x,t)-g_+(x,t))| + |g_-(x,t)-g_+(x,t)|\right) \\ &+2|\chi_-''(x)||g_-(x,t)-g_+(x,t)|. \end{aligned}\tag{9.4}$$

This yields the following estimate with the weighted function $A(x)$ given in (8.13) with $\kappa = 1$ and $b = 1/4$

$$\int_{-\infty}^{\infty} g_e(x,t)^2 A(x)dx < O(1)\sigma_0^4 \frac{e^{-\frac{\sigma_0^2 t}{2}}}{\sqrt{t}} \text{ for } t > 0. \tag{9.5}$$

Denote

$$w(x,t) \equiv g(x,t) - g_0(x,t)$$

and the function $w(x,t)$ will satisfy

$$\begin{cases} \partial_t w + f'(\phi)\partial_x w - \partial_x^2 w = g_e(x,t), \\ w(x,0) \equiv 0. \end{cases} \tag{9.6}$$

By applying the energy estimates (8.12)-(8.19) with (9.5), there exist K_0 and $K_1 > 0$ such that

$$e^{\gamma_0 t}\int_{-\infty}^{\infty} A(x)w(x,t)^2 dx + \frac{1}{2}\int_0^t\int_{-\infty}^{\infty} e^{\gamma_0 s}A(x)|w_x(x,s)|^2 dxds$$

$$\leq \frac{K_0}{\sigma_0^2}\int_0^t\int_{-\infty}^{\infty} A(x)e^{\gamma_0 s}g_e(x,s)^2dxds \leq K_1\sigma_0. \tag{9.7}$$

Remark 9.1. *Here, the constants* $\sqrt{\gamma_0}$ *and* σ_0 *are of the same order as* $\min(1,|u_- - u_+|)$.

Thus,

$$\int_{-\infty}^{\infty} A(x)w(x,t)^2dx+\frac{1}{2}\int_0^t\int_{-\infty}^{\infty} e^{-\gamma_0(t-s)}A(x)|w_x(x,s)|^2dxdsds \leq K_1\sigma_0e^{-\gamma_0 t}. \tag{9.8}$$

One uses the following bootstrap procedure to obtain the pointwise estimate of $w(x,t)$ for $x\leq 0$:

$$\begin{aligned}|w(x,t)| &\leq \left|\int_0^t\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y-\lambda_-(t-s))^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\big(g_e(y,s)-(f'(\phi(y))-\lambda_-)w_y(y,s)\big)\,dyds\right|\\ &\leq \left|\int_0^t\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y-\lambda_-(t-s))^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}g_e(y,s)dyds\right|\\ &\quad+\left|\int_0^t\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y-\lambda_-(t-s))^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}(f'(\phi)-\lambda_-)w_y(y,s)dyds\right|. \end{aligned}\tag{9.9}$$

By Hölder inequality and (9.8), one has

$$\begin{aligned}&\left|\int_0^t\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y-\lambda_-(t-s))^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}(f'(\phi)-\lambda_-)w_y(y,s)dyds\right|\\ &\leq O(1)\left(\int_0^t\int_{-\infty}^{\infty}\frac{e^{-\frac{(x-y-\lambda_-(t-s))^2}{2(t-s)}}}{(t-s)}\frac{(f'(\phi)-\lambda_-)^2e^{\gamma_0(t-s)}}{A(y)}dyds\right)^{1/2}\\ &\quad\times\left(\int_0^t\int_{-\infty}^{\infty} e^{-\gamma_0(t-s)}A(y)|w_y|^2dyds\right)^{1/2}\\ &\leq O(1)\left(\int_0^t\left(\frac{\sigma_0^2e^{-2\sigma_0|x|-2\sigma_0^2(t-s)}}{\sqrt{t-s}}\right)e^{\gamma_0(t-s)}ds\right)^{1/2}\left(\sigma_0e^{-\gamma_0 t}\right)^{1/2}\end{aligned}$$

$$\leq O(1)\sigma_0 e^{-\sigma_0|x|-\gamma_0 t}. \tag{9.10}$$

This and (9.9) result in, for $x \leq 0$,

$$|w(x,t)| \leq O(1)\sigma_0 e^{-\sigma_0|x|-\gamma_0 t}. \tag{9.11}$$

With a symmetric argument, one has

$$|w(x,t)| \leq O(1)\sigma_0 e^{-\sigma_0|x|-\gamma_0 t} \text{ for } x \geq 0. \tag{9.12}$$

By (9.3) and (9.12) together with Remark 9.1, it follows that, for some $D_0 > 0$,

$$|g(x,t)| \leq O(1)\sigma_0 e^{-\frac{\sigma_0}{2}|x|-\frac{\sigma_0^2 t}{D_0}}. \tag{9.13}$$

This estimate (9.13) establishes a semi-group property for the initial value problem (9.1):
There exist A_0 and $E_0 > 0$ such that for $t > 0$

$$\begin{aligned} &G_\phi^t[g(\cdot,0)](x) \equiv g(x,t),\\ &\begin{cases} |g(x,0)| \leq \sigma_0 e^{-\sigma_0|x|}, \\ |G_\phi^t[g(\cdot,0)](x)| \leq A_0\sigma_0 e^{-\sigma_0|x|-\frac{\sigma_0^2}{E_0}t}. \end{cases}\end{aligned} \tag{9.14}$$

By the semi-group property, the Green's function $G(x,t;x_0,t_0)$ of $(\partial_t + f'(\phi)\partial_x - \partial_x^2)f = 0$ is $G(x,t-t_0;x_0,0)$. The function $G(x,t;x_0,0)$ is the solution of an initial value problem

$$\begin{cases} (\partial_t + f'(\phi)\partial_x - \partial_x^2)G(x,t;x_0,0) = 0 \text{ for } x \in \mathbb{R}, t > 0, \\ G(x,0;x_0,0) = \delta(x-x_0). \end{cases} \tag{9.15}$$

Similar to the introduction of the initial approximate solution $g_0(x,t)$, we denote the initial approximate Green's function $G_0(x,t;x_0,0)$ and its truncation error $G_e(x,t;x_0,0)$ as follows:

$$\begin{cases} G_0(x,t;x_0,0) \equiv \chi_-(x)\dfrac{e^{-\frac{(x-x_0-\lambda_- t)^2}{4t}}}{\sqrt{4\pi t}} + \chi_+(x)\dfrac{e^{-\frac{(x-x_0-\lambda_+ t)^2}{4t}}}{\sqrt{4\pi t}}, \\ G_e(x,t;x_0,0) \equiv -(\partial_t + f'(\phi)\partial_x - \partial_x^2)G_0(x,t;x_0,0), \\ W(x,t) \equiv G(x,t;x_0,0) - G_0(x,t;x_0,0). \end{cases} \tag{9.16}$$

There exists $E_1 > 0$ such that for any $\tau > 0$ the function $G_e(x, \tau; x_0, 0)$

$$\frac{G_e(x,\tau;x_0,0)}{\sigma_0 e^{-\sigma_0|x|}} \le O(1)\begin{cases} e^{-\frac{(|x_0|-|\lambda_-|\tau)^2}{E_1\tau}}\left(\frac{1}{\tau}+\frac{\sigma_0}{\sqrt{\tau}}\right) \text{ for } x_0<0, \\ e^{-\frac{(|x_0|-|\lambda_+|\tau)^2}{E_1\tau}}\left(\frac{1}{\tau}+\frac{\sigma_0}{\sqrt{\tau}}\right) \text{ for } x_0>0. \end{cases} \tag{9.17}$$

The equation for $W(x, t)$ is

$$\begin{cases} (\partial_t + f'(\phi)\partial_x - \partial_x^2)W = G_e, \text{ for } t > 0, \\ W(x,0) \equiv 0. \end{cases} \tag{9.18}$$

With the property (9.17), one can apply operator $G_\phi^{t-\tau}$ to the element $G_e(\cdot, \tau)$ and the Duhamel's principle to (9.18) to yield that

$$\begin{aligned} |W(x,t)| &= \left|\int_0^t G_\phi^{t-\tau}[G_e(\cdot,\tau)](x)d\tau\right| \\ &\le O(1)\sigma_0 e^{-\sigma_0|x|}\begin{cases} \int_0^t e^{-\frac{\sigma_0^2}{E_0}(t-\tau)-\frac{(|x_0|-|\lambda_-|\tau)^2}{E_1\tau}}\left(\frac{1}{\tau}+\frac{\sigma_0}{\sqrt{\tau}}\right)d\tau \text{ for } x_0<0, \\ \int_0^t e^{-\frac{\sigma_0^2}{E_0}(t-\tau)-\frac{(|x_0|-|\lambda_+|\tau)^2}{E_1\tau}}\left(\frac{1}{\tau}+\frac{\sigma_0}{\sqrt{\tau}}\right)d\tau \text{ for } x_0>0. \end{cases} \end{aligned} \tag{9.19}$$

Then, by (9.16) and the estimate (9.19) one obtains the global estimate of the Green's function $G(x, t; x_0, 0)$. There exists $E > 0$ such that

$$\begin{aligned} &\left|G(x,t;x_0,0) - \left(\chi_-(x)\frac{e^{-\frac{(x-x_0-\lambda_-t)^2}{4t}}}{\sqrt{4\pi t}} + \chi_+(x)\frac{e^{-\frac{(x-x_0-\lambda_+t)^2}{4t}}}{\sqrt{4\pi t}}\right)\right| \\ &\qquad \le O(1)\sigma_0|\log\sigma_0|e^{-\sigma_0|x|}\begin{cases} e^{-\frac{(|x_0|-|\lambda_-|t)^2}{4Et}} \text{ for } x_0<0, \\ e^{-\frac{(|x_0|-|\lambda_+|t)^2}{4Et}} \text{ for } x_0>0. \end{cases} \end{aligned} \tag{9.20}$$

Thus the Green's function for general convex conservation laws is similar to the one for the Burgers equation, (6.9).

10. Stability of Rarefaction Waves

Rarefaction waves are expansive. Consequently, for an inviscid rarefaction wave $u(x,t)$, when parts of the wave is translated to become $v(x,t)$, then they are different in a region with width $O(1)t$ and the magnitude of their difference is $O(1)t^{-1}$. Thus we have

$$|u(\cdot,t)-v(\cdot,t)|_{L_p(x)}=[O(1)t(O(1)t^{-1})^p]^{\frac{1}{p}}=O(1)t^{-\frac{p-1}{p}},\ p\geq 1. \tag{10.1}$$

In particular, it decays in $L_p(x)$ for $p>1$ and does not decay for $L_1(x)$. Note that there are continuum parameters of translations. Thus, unlike shock waves, for a given perturbation of a rarefaction wave, it is not possible to have a simple translation of the rarefaction wave in order for the perturbation to decay in $L_1(x)$ norm. This arguments were made for the hyperbolic conservation laws in Section 3. Nevertheless, it holds also for the viscous conservation laws. For viscous rarefaction waves, the dissipation term works as usual outside the rarefaction fan. Inside the rarefaction fan there are interesting combined effects of the hyperbolic expansion and the dissipation, see Figure 17 in Section 6. As we will see in Part II for systems, there is an additional strong coupling effect with other characteristic families. There is no exact rarefaction waves for the system. In fact, even for scalar conservation laws we need to use the explicit Burgers rarefaction wave $b_R(x,t)$, (5.8), as an approximation, and study the stability of rarefaction wave accordingly:

$$\begin{cases} u_t+f(u)_x=\kappa u_{xx},\ f''(u)>0,\ \lambda(u)\equiv f'(u),\\ \lambda(u(x,t))=b_R(x,t+T_0)+v(x,t),\\ |v(x,0)|+|v_x(x,0)|=O(1)\varepsilon(|x|+1)^{-\alpha},\ \alpha>\frac{1}{2},\\ v_t+[b_Rv+\frac{1}{2}v^2]_x=\kappa v_{xx}-\kappa\frac{f'''(u)}{f''(u)^2}[v_x+(b_R)_x]^2. \end{cases} \tag{10.2}$$

The last identity comes directly from (5.8), (6.15). Here we have chosen the starting time T_0 for the rarefaction wave to be large so that the wave has already expanded sufficiently. This choice is made for convenience, as we are focusing on the time-asymptotic behavior. Set $u_\pm$ to be the limiting state of the rarefaction wave. In (5.8), we have $u_\pm=\pm\lambda_0$.

We now study the stability of rarefaction wave, first using the energy method. The key property is the expansion of the wave, c.f. (5.8):

$$\frac{\partial}{\partial x} b_R(x,t) > 0.$$

Theorem 10.1. *There exists a global solution to* (10.2), *and, for sufficiently small* $|u_+ - u_-|(T_0)^2 + \varepsilon^2$,

$$\sup_x |f'(u(x,t)) - b_R(x,t)| \to 0, \text{ as } t \to \infty. \tag{10.3}$$

Proof. The equation (10.2) is of the form

$$v_t + (b_R v)_x = \kappa v_{xx} - \frac{1}{2}(v^2)_x + O(1)[(v_x)^2 + ((b_R)_x)^2].$$

Multiply this by v and integrate to obtain

$$\begin{aligned}\frac{d}{dt}\int_{-\infty}^{\infty}\frac{1}{2}v^2(x,t)dx &+ \int_{-\infty}^{\infty}\Big[\frac{1}{2}(b_R)_x v^2 + \kappa(v_x)^2\Big](x,t)dx \\ &= \int_{-\infty}^{\infty}[O(1)v(v_x)^2 + O(1)v((b_R)_x)^2](x,t)dx.\end{aligned}$$

The last term can be dominated by

$$\begin{aligned}\int_{-\infty}^{\infty} O(1)v((b_R)_x)^2 dx &\le \frac{1}{4}\int_{-\infty}^{\infty}[v^2(b_R)_x + O(1)((b_R)_x)^3]dx \\ &= \frac{1}{4}\int_{-\infty}^{\infty} v^2(b_R)_x dx + O(1)|u_+ - u_-|(T_0)^{-2},\end{aligned}$$

where we have used the expansion of the Burgers rarefaction wave $(b_R)_x$, (5.8), which says basically that it decays with rate $(t+T_0)^{-1}$ with essential width $(\lambda(u_+) - \lambda(u_-))(t+T_0)$. With the above, we close the first energy estimate under the assumption that $v(x,t)$ is sufficiently small:

$$\begin{aligned}&\int_{-\infty}^{\infty} v^2(x,t)dx + \int_0^t\int_{-\infty}^{\infty}[|(b_R)_x|v^2 + \kappa(v_x)^2](x,s)dxds \\ &= O(1)[(T_0)^{-2} + \int_{-\infty}^{\infty} v^2(x,0)dx = O(1)[|u_+ - u_-|(T_0)^2 + \varepsilon^2].\end{aligned}$$

The smallness of $v(x,t)$ is assured by the next level of energy estimate for $v_x(x,t)$ and the smallness of $|u_+ - u_-|(T_0)^{-2} + \varepsilon^2$. Note that the smallness of

$|u_+ - u_-|(T_0)^{-2}$ is ensured either for weak rarefaction wave, $|u_+ - u_-| \ll 1$, or for the wave sufficiently expanded, $T_0 \gg 1$. This finishes the energy estimate and the time-asymptotic stability of the rarefaction wave. $\square$

We next study the quantitative behavior of the perturbation of the rarefaction wave using the Green's function approach. Consider the initial perturbation of algebraic type:

$$\begin{cases} u_t + f(u)_x = \kappa u_{xx}, \ f''(u) > 0, \ \lambda(u) \equiv f'(u), \\ \lambda(u(x,t)) = b_R(x, t+T_0) + v(x,t), \\ v(x,0) = \varepsilon(|x|+1)^{-\alpha}, \ 1 < \alpha < 2, \\ v_t + [b_R v + \frac{1}{2}v^2]_x = \kappa v_{xx} - \kappa \frac{f'''(u)}{f''(u)^2}[v_x + (b_R)_x]^2. \end{cases} \tag{10.4}$$

Note that, except for the case of Burgers equation $f(u) = u^2/2$, $b_R(x,t)$ is not an exact solution of the convex conservation law and so the perturbation will be non-zero for $t > 0$. In the following theorem, it is interesting to see the difference and similarity between the viscous and the inviscid cases, Theorem 3.2.

Theorem 10.2. *There exists a global solution to* (10.4)*, and for sufficiently small* $|u_+ - u_-|(T_0)^2 + \varepsilon^2$*,*

$$\begin{aligned} &|f'(u(x,t)) - b_R(x,t)| \\ &= O(1) \begin{cases} \frac{e^{-\frac{(x+\lambda_0 t)^2}{Dt}}}{\sqrt{T_0+t}} + \varepsilon(|x+\lambda_0(t+1)| + \sqrt{t}+1)^{-\alpha}, \\ \qquad\qquad \textit{for } x < -\lambda_0 t + \sqrt{t}, \\ \dfrac{1}{|x+\lambda_0 t + T_0|} + \dfrac{1}{|x-\lambda_0 t| + T_0} \\ \qquad\qquad \textit{for } x \in (-\lambda_0 t + \sqrt{t}, \lambda_0 t - \sqrt{t}), \\ \dfrac{e^{-\frac{(x-\lambda_0 t)^2}{5t}}}{\sqrt{1+t}} + \varepsilon(|x-\lambda_0(t+1)| + \sqrt{t}+1)^{-\alpha} \\ \qquad\qquad \textit{for } x > \lambda_0 t - \sqrt{t}. \end{cases} \end{aligned} \tag{10.5}$$

In particular

$$\|f'(u(x,t)) - b_R(x,t)\|_{L_p(x)} = O(1)(t+1)^{-\frac{p-1}{2p}}, \ p \geq 1.$$

Proof. We have, by integrating the equation (10.2) times g_R, the Duhamel's principle, (6.11), The result,

$$\begin{aligned} v(x,t) &= \int_{-\infty}^{\infty} v(y,0) g_B(y,0;x,t) dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \Big[(g_R)_y \frac{1}{2} v^2 - \kappa \frac{f'''(u)}{f''(u)^2} [v_y + (b_R)_y]^2 g_R \Big] (y,s) dy ds. \quad (10.6) \end{aligned}$$

There are two known terms in the above expression. The first known term is the convolution in y of the Green's function with the initial perturbation $v(y,0)$. The first yields the ansatz as stated in the theorem. The far field is the same as for the diffusion waves in Theorem 7.2, because the influence of the rarefaction wave is weak there. Around the rarefaction wave region, it is similar to the difference of the viscous rarefaction wave and the inviscid diffusion wave in (5.15), as both are under both the dissipation and the nonlinear hyperbolictiy. The second known term is the convolution, both in space and time (y,s), with the known source $O(1)[(b_R)_y]^2 = O(1)[(b_R)_y(y,s+T_0)]^2$. The computations and estimates of these convolutions are done using the estimates of the Green's function in Section 6. This yields the same ansatz as the first known term but without the terms involving ε. After these computations, we plug the ansatz into the double integral in the above expression and show that the ansatz can be closed, a standard procedure, as in the study of the stability of nonlinear diffusion waves. For the details, see Section 4 of [22]. □

11. Initial-Boundary Value Problem

In this section we study the boundary effect on the propagation of the nonlinear waves. We first study the stability of shock wave propagating away from the boundary using the Green's function approach. We do this for the Burgers equation, as the Green's function is explicitly known, (6.9). The analysis applies to general convex conservation law using the explicit construction of accurate approximate Green's function, (9.20). There are two cases for the shock waves. The situation is simpler when the viscous shock is propagating toward the boundary and becomes the boundary layer, c.f. Figure 5 in Section 2. We consider the case when the viscous shock is propagating away from the boundary, c.f. Figure 4 in Section 2. Instead of taking the shock with positive speed, we take the boundary to move toward the left and the shock to be stationary. Thus we consider the initial-boundary value problem:

$$\begin{cases} u_t + uu_x = u_{xx}, \ -L-t < x < \infty, \\ u(x,0) = \phi(x) + \bar{u}_0(x), \ x > 0, \\ u(-L-t,t) = u_-, \ u(+\infty,t) = u_+ = -u_-, \end{cases} \tag{11.1}$$

where $L > 0$ is a large constant, $\phi(x) = -u_- \tanh \frac{(u_- x)}{2}$ is a stationary Burgers shock wave, and the initial perturbation $\bar{u}_0(x)$ satisfies the assumptions:

$$\begin{cases} \bar{u}_0(-L) = u_- - \phi(-L), \\ \bar{u}_0(x) = O(1)e^{-L}(1+x+L)^{-\alpha}, \quad \alpha > 1, \\ \bar{u}_{0x}(x) = O(1)e^{-L}(1+x+L)^{-\alpha-1}. \end{cases} \tag{11.2}$$

The solution $u(x,t)$ for (11.1) is expected to tend to the Burgers shock wave $\phi(x-x_0)$ with a shift x_0 to be determined. Unlike the initial value problem, (8.5), there is no a priori explicit expression of the shift in terms of the initial data $\bar{u}_0(x)$ and the boundary data u_-:

$$\begin{aligned} x_0 &\equiv \frac{\lim_{t\to\infty} \int_{-L-t}^{\infty} (u(x,t) - \phi(x))\, dx}{u_- - u_+} \\ &= \frac{\displaystyle\int_{-L}^{\infty} \bar{u}_0(x)dx + \lim_{t\to\infty} \int_0^t (u_- - \phi(-L-\sigma) - u_x(-L-\sigma,\sigma))d\sigma}{u_- - u_+}. \end{aligned} \tag{11.3}$$

So the shift depends also on the boundary flux $u_x(-L-t,t)$ as well as the initial perturbation.

To obtain the boundary estimates, we first consider the linearization of $u(x,t)$ around $\phi(x)$. Without loss of generality, we assume $u_- = 1$, and set

$$v(x,t) = u(x,t) - \phi(x) - \Psi(x,t), \ \Psi(x,t) \equiv (1-\phi(-L-t))(x+L+t+1)^{-\alpha}.$$

Then the small perturbation $v(x,t)$ satisfies

$$\begin{cases} v_t + (\phi v)_x - v_{xx} = -\left(\frac{v^2}{2} + \Psi v\right)_x - \Psi_t - \left(\phi\Psi + \frac{\Psi^2}{2}\right)_x + \Psi_{xx}, \\ \qquad\qquad -L - t < x < \infty,\ t > 0, \\ v(-L-t,t) = 0, \qquad v(\infty,t) = 0, \\ v(x,0) = O(1)e^{-L}(1+x+L)^{-\alpha}, \qquad v(-L,0) = 0, \\ v_x(x,0) = O(1)e^{-L}(1+x+L)^{-\alpha-1}. \end{cases} \tag{11.4}$$

In the following, we are interested in the time-asymptotic propagation of the shock wave and so we take the initial shock location L to be large.

Theorem 11.1. (Boundary Estimates) *Suppose that L is sufficiently large. Then the solution $v(x,t)$ for* (11.4) *satisfies*

$$v(x,t) = O(1)e^{-\frac{L}{2r}} \begin{cases} e^{-\frac{|x|}{2}}|x+L+t| & \text{for } x \in [-L-t, -L-t+1], \\ e^{-\frac{|x|}{2}} & \text{for } x \in [-L-t, 0), \\ e^{-\frac{|x|}{2}} + (1+x+L+t)^{-\alpha} & \text{for } x \geq 0, \end{cases} \tag{11.5}$$

and

$$v_x(-L-t,t) = O(1)e^{-\frac{L}{2r}} e^{\frac{-L-t}{2}}, \quad r \gtrsim 1.$$

From Theorem 11.1, we can obtain the boundary estimate for u

$$\begin{aligned} u_x(-L-t,t) &= v_x(-L-t,t) + \phi'(-L-t) + \Phi_x(-L-t,t) \\ &= O(1)e^{-\frac{L}{2r}} e^{\frac{-L-t}{2}}. \end{aligned} \tag{11.6}$$

Then the time-asymptotical shift x_0 of the Burgers shock can be determined by (11.3)

$$x_0 = O(1)e^{\frac{-2}{3}L}.$$

With the correction of the Burgers shock location, the convergent rates of $u(x,t)$ to $\phi(x-x_0)$ can be obtained. Let

$$\begin{aligned} &\bar{v}(x,t) \equiv u(x,t) - \phi(x-x_0), \\ &w(x,t) \equiv -\int_x^\infty \bar{v}(y,t)dy, \qquad (\, w_x = \bar{v}\,). \end{aligned}$$

Theorem 11.2. *Suppose that L is sufficiently large. Then $w(x,t)$ satisfies*

$$|w(x,t)| \le C_0 \begin{cases} e^{\frac{-|x|}{4}} e^{\frac{-t}{8}} e^{\frac{-L}{4}} + e^{\frac{-|x|}{2}} (x+t+L+1)^{-\alpha+1} e^{\frac{-L}{4}}, & -L-t \le x \le 0, \\ (x+t+L+1)^{-\alpha+1} e^{\frac{-L}{4}}, & x > 0, \end{cases} \tag{11.7}$$

$$|w_x(x,t)| \le C_0 \begin{cases} e^{\frac{-|x|}{4}} e^{\frac{-t}{8}} e^{\frac{-L}{4}} + e^{\frac{-|x|}{2}} (x+t+L+1)^{-\alpha+1} e^{\frac{-L}{4}}, & -L-t \le x \le 0, \\ (x+t+L+1)^{-\alpha} e^{\frac{-L}{4}} + e^{\frac{-|x|}{3}} (x+t+L+1)^{-\alpha+1} e^{\frac{-L}{4}}, & x > 0, \end{cases} \tag{11.8}$$

where $C_0 > 0$ is a constant.

There are two mechanisms which govern the solution behavior for this initial- boundary problem: the compressibility of the shock and the presence of the boundary. To prove Theorem 11.1, we will introduce an iteration scheme which can separate these two effects. We divide the $x-t$ domain into two regions:

$$\begin{aligned} I &= \Big\{(x,t) : -L-t < x < \frac{-L-t}{2},\ t > 0\Big\}, \\ II &= \Big\{(x,t) : \frac{-L-t}{2} \le x < \infty,\ t > 0\Big\}. \end{aligned}$$

In region I, we use the Green's function $K^B(x,t;y,s)$ for

$$\begin{cases} w_t + w_x = w_{xx}, \\ w(-L-t,t) = 0, \qquad w(\infty,t) = 0, \end{cases} \tag{11.9}$$

and focus on the boundary effect, ignoring the nonlinearity of the shock. The Green's function $K^B(x,t;y,s)$ satisfies the forward equation

$$\begin{cases} K^B_s - K^B_y + K^B_{yy} = 0, \\ K^B(x,t;-L-s,s) = 0,\ K^B(x,t;\infty,s) = 0, \\ K^B(x,t;y,t) = \delta(x-y). \end{cases} \tag{11.10}$$

By odd reflection, K^B can be expressed in terms of heat kernel $H(x,t;\kappa = 1)$,(4.2),

$$K^B(x,t;y,s) = H(x-y-(t-s),t-s;1) - H(x+y-t+3s+2L,t-s;1)e^{-2(y+L+s)}. \tag{11.11}$$

Multiply K^B with the equation of v in (11.4) and integrate to yield the solution representation in region I

$$\begin{aligned} v(x,t) &= \int_{-L}^{\infty} K^B(x,t;y,0)v(y,0)dy \\ &+ \int_0^t \int_{-L-\sigma}^{\infty} K^B_y(x,t;y,\sigma)\left[(\phi(y)-1+\Psi(y,\sigma))v(y,\sigma)+\frac{v(y,\sigma)^2}{2}\right]dyd\sigma \\ &+ \int_0^t \int_{-L-\sigma}^{\infty} K^B(x,t;y,\sigma)\left\{-\Psi_\sigma-\left(\phi\Psi+\frac{\Psi^2}{2}\right)_y+\Psi_{yy}\right\}(y,\sigma)dyd\sigma. \end{aligned} \tag{11.12}$$

In region II, we will focus on the nonlinearity of the Burgers shock. Let $g_S(x,t;y,s)$ be the Green's function for

$$z_t + (\phi(x)z)_x - z_{xx} = 0, \quad -\infty < x < \infty, \tag{11.13}$$

and let $G_S(x,t;y,s)$ be the Green's function for the anti-derivative variable $\tilde{z}$

$$\tilde{z}_t + \phi(x)\tilde{z}_x - \tilde{z}_{xx} = 0, \quad -\infty < x < \infty. \tag{11.14}$$

From section 6, we have

$$G_S(x,t;y,s) = \frac{\cosh(\frac{y}{2})}{\cosh(\frac{x}{2})}H(x-y,t-s;1)e^{-\frac{t-s}{4}} \tag{11.15}$$

and

$$\begin{aligned} g_S(x,t;y,s) &= \int_y^{\infty}(G_S)_x(x,t;\xi,s)d\xi \\ &= G_S(x,t;y,s) - \frac{\int_y^{\infty}\sinh(\frac{x-\xi}{2})H(x-\xi,t-s;1)e^{-\frac{t-s}{4}}d\xi}{2\cosh^2\frac{x}{2}} \end{aligned} \tag{11.16}$$

Multiply g_S with the equation of v in (11.4) and integrate to yield the solution representation in region II

$$\begin{aligned} v(x,t) &= \int_{-L}^{\infty} g_S(x,t;y,0)v(y,0)dy - \int_0^t g_S(x,t;-L-\sigma,\sigma)v_y(-L-\sigma,\sigma)d\sigma \\ &+ \int_0^t \int_{-L-\sigma}^{\infty}(g_S)_y(x,t;y,\sigma)\left(\Psi(y,\sigma)v(y,\sigma)+\frac{v(y,\sigma)^2}{2}\right)dyd\sigma \end{aligned}$$

$$+\int_0^t\int_{-L-\sigma}^{\infty} g_S(x,t;y,\sigma)\left\{-\Psi_\sigma-\left(\phi\Psi+\frac{\Psi^2}{2}\right)_y+\Psi_{yy}\right\}(y,\sigma)dyd\sigma. \quad (11.17)$$

By (11.12) and (11.17), we introduce the following iteration scheme to construct a sequence of function $\{v^n\}_{n=0}^{\infty}$ to get the boundary estimate: for $x\in[-L-t,-(L+t)/2)$,

$$\begin{aligned} v^0(x,t) &= \int_{-L}^{\infty} K^B(x,t;y,0)v(y,0)dy \\ &+\int_0^t\int_{-L-\sigma}^{\infty} K^B(x,t;y,\sigma)\left\{-\Psi_\sigma-\left(\phi\Psi+\frac{\Psi^2}{2}\right)_y+\Psi_{yy}\right\}(y,\sigma)dyd\sigma, \quad (11.18)\end{aligned}$$

for $x\geq-(L+t)/2$,

$$\begin{aligned} v^0(x,t) &= \int_{-L}^{\infty} g_S(x,t;y,0)v(y,0)dy \\ &-\int_0^t g_S(x,t;-L-\sigma,\sigma)v_y^0(-L-\sigma,\sigma)d\sigma \\ &+\int_0^t\int_{-L-\sigma}^{\infty} g_S(x,t;y,\sigma)\left\{-\Psi_\sigma-\left(\phi\Psi+\frac{\Psi^2}{2}\right)_y+\Psi_{yy}\right\}(y,\sigma)dyd\sigma; \quad (11.19)\end{aligned}$$

for $x\in[-L-t,-(L+t)/2)$, $n\geq 1$,

$$\begin{aligned} v^n(x,t) &= \int_{-L}^{\infty} K^B(x,t;y,0)v(y,0)dy \\ &+\int_0^t\int_{-L-\sigma}^{\infty} K_y^B(x,t;y,\sigma)\left[(\phi(y)-1+\Psi(y,\sigma))v^{n-1}(y,\sigma)+\frac{v^{n-1}(y,\sigma)^2}{2}\right]dyd\sigma \\ &+\int_0^t\int_{-L-\sigma}^{\infty} K^B(x,t;y,\sigma)\left\{-\Psi_\sigma-\left(\phi\Psi+\frac{\Psi^2}{2}\right)_y+\Psi_{yy}\right\}(y,\sigma)dyd\sigma; \quad (11.20)\end{aligned}$$

for $x\geq-(L+t)/2$, $n\geq 1$,

$$\begin{aligned} v^n(x,t) &= \int_{-L}^{\infty} g_S(x,t;y,0)v(y,0)dy-\int_0^t g_S(x,t;-L-\sigma,\sigma)v_y^n(-L-\sigma,\sigma)d\sigma \\ &+\int_0^t\int_{-L-\sigma}^{\infty} (g_S)_y(x,t;y,\sigma)\left(\Psi(y,\sigma)v^{n-1}(y,\sigma)+\frac{v^{n-1}(y,\sigma)^2}{2}\right)dyd\sigma \\ &+\int_0^t\int_{-L-\sigma}^{\infty} g_S(x,t;y,\sigma)\left\{-\Psi_\sigma-\left(\phi\Psi+\frac{\Psi^2}{2}\right)_y+\Psi_{yy}\right\}(y,\sigma)dyd\sigma. \quad (11.21)\end{aligned}$$

Lemma 11.3 *There exist constants $C_0 > 0$ and $r > 1$ such that*

$$v^0(x,t) \le C_0 \begin{cases} e^{-\frac{L}{2r}} e^{-\frac{|x|}{2}} |x+L+t| & \text{for } x \in [-L-t, -L-t+1], \\ e^{-\frac{L}{2r}} e^{-\frac{|x|}{2}} & \text{for } x \in [-L-t, 0), \\ e^{-\frac{L}{2r}} e^{-\frac{|x|}{2}} + e^{-\frac{L}{2}} (1+x+L+t)^{-\alpha} & \text{for } x \ge 0, \end{cases} \quad (11.22)$$

$$v^0_x(-L-t,t) \le C_0 e^{-\frac{L}{2r}} e^{-\frac{L+t}{2}}.$$

Proof. Let $X = x+L+t$ and $Y = y+L+\sigma$. From (11.11), K^B can be rewritten as

$$\begin{aligned} K^B(x,t;y,\sigma) &= (\, H(Y-X, t-\sigma; 1) - H(Y+X, t-\sigma;1)\,)\, e^{(X-Y)-(t-\sigma)} \\ &= -X \int_{-1}^{1} \partial_Y H(Y-\theta X, t-\sigma; 1) d\theta \, e^{(X-Y)-(t-\sigma)}. \end{aligned} \quad (11.23)$$

Then for $x \in [-L-t, -L-t+1]$,

$$\begin{aligned} & \left| \int_{-L}^{\infty} K^B(x,t;y,0) v(y,0) dy \right| \\ &= \left| \int_0^{\infty} \int_{-1}^{1} (-X) \partial_Y H(Y-\theta X, t; 1) d\theta e^{(X-Y)-t} v(Y,0) dY \right| \\ &= \left| (-X) \int_{-1}^{1} \int_0^{\infty} \partial_Y H(Y-\theta X, t;1) e^{(X-Y)-t} v(Y,0) dY d\theta \right| \\ &= \left| X \int_{-1}^{1} \int_0^{\infty} H(Y-\theta X, t; 1) e^{(X-Y)-t} (-v(Y,0) + v_Y(Y,0)) dY d\theta \right| \\ &\le CX e^{-L} e^{-t}; \end{aligned} \quad (11.24)$$

and by $|-\Psi_\sigma - \left(\phi\Psi + \frac{\Psi^2}{2}\right)_y + \Psi_{yy}| = O(1) e^{-L-\sigma} (Y+1)^{-\sigma}$, we can obtain

$$\begin{aligned} & \left| \int_0^t \int_{-L-\sigma}^{\infty} K^B(x,t;y,\sigma) \left\{ -\Psi_\sigma - \left(\phi\Psi + \frac{\Psi^2}{2} \right)_y + \Psi_{yy} \right\} (y,\sigma) dy d\sigma \right| \\ &= \Bigg| \int_0^t \int_0^{\infty} (-X) \int_{-1}^{1} \partial_Y H(Y-\theta X, t-\sigma; 1)\, d\theta \\ &\qquad \times e^{(X-Y)-(t-\sigma)} [\Psi_\sigma - (\phi\Psi + \frac{\Psi^2}{2})_Y + \Psi_{YY}] dY d\sigma \Bigg| \\ &= O(1) X \int_0^t \int_{-1}^{1} \int_0^{\infty} \int_{-1}^{1} \frac{1}{\sqrt{t-\sigma}} H(\frac{Y-\theta X}{D}, t-\sigma; 1) d\theta e^{(X-Y)-(t-\sigma)} \end{aligned}$$

$$\times e^{-L-\sigma}(Y+1)^{-\alpha}dY\,d\theta d\sigma$$
$$= O(1)Xe^{-s(L+t)}, \quad s \gtrsim 1. \tag{11.25}$$

From (11.24) and (11.25), we have

$$\begin{cases} v^0(-L-t,t)=0, \\ v_x^0(-L-t,t)=O(1)e^{-s(L+t)}. \end{cases} \tag{11.26}$$

For $-L-t<x<\frac{-L-t}{2}$, we have, by straightforward computation,

$$\begin{aligned} |v^0(x,t)| \le O(1)\Big[&\int_{-L}^{\infty} H(x-y-t,t;1)e^{-L}(1+y+L)^{-\alpha}dy \\ &+\int_0^t\int_{-L-\sigma}^{\infty} H(x-y-t-\sigma,t-\sigma;1)e^{-L-\sigma}(1+y+L+\sigma)^{-\alpha}dyd\sigma\Big] \\ \le O(1)&e^{\frac{-L}{2}}e^{\frac{-|x|}{2}}e^{\frac{-t}{4}}. \end{aligned}$$

For $x\ge\frac{-L-t}{2}$, by (11.15) and (11.26), we have

$$\begin{aligned} &\left|\int_0^t g_S(x,t;-L-\sigma,\sigma)v_y^0(-L-\sigma,\sigma)d\sigma\right| \\ \le\ & O(1)\int_0^t\Big[e^{\frac{-|x|}{2}}e^{\frac{L+\sigma}{2}}H(x+L+\sigma,t-\sigma;1)e^{\frac{-(t-\sigma)}{4}}+e^{-|x|}\Big]\cdot e^{\frac{-L}{2r}}e^{\frac{-(L+\sigma)}{2}}d\sigma \\ =\ & O(1)e^{\frac{-|x|}{2}}e^{\frac{-L}{2r}} \end{aligned} \tag{11.27}$$

$$\begin{aligned} &\left|\int_0^t\int_{-L-\sigma}^{\infty} g_S(x,t;y,\sigma)\left\{-\Psi_\sigma-\left(\phi\Psi+\frac{\Psi^2}{2}\right)_y+\Psi_{yy}\right\}(y,\sigma)dyd\sigma\right| \\ \le\ & O(1)\Big[\int_0^t\int_{-L-\sigma}^{\infty} G_S\cdot e^{-L-\sigma}(1+y+L+\sigma)^{-\alpha}dyd\sigma \\ &+\int_0^t\int_{-L-\sigma}^{\infty} e^{-|x|}\cdot e^{-L-\sigma}(1+y+L+\sigma)^{-\alpha}dyd\sigma\Big] \\ \equiv\ & E_1(x,t)+E_2(x,t). \end{aligned} \tag{11.28}$$

It is easy to get

$$|E_2(x,t)|=O(1)e^{-|x|}e^{-L}. \tag{11.29}$$

By (6.9), we have for $\frac{-L-t}{2} \le x \le 0$,

$$
\begin{aligned}
&|E_1(x,t)| \\
&= O(1)\Big(\int_0^t \int_{-L-\sigma}^0 H(x-y-(t-\sigma), t-\sigma; 1) e^{-L-\sigma} (1+y+L+\sigma)^{-\alpha} dy d\sigma \\
&\quad + \int_0^t \int_0^\infty e^{-|x|} H(x-y+(t-\sigma), t-\sigma; 1) e^{-L-\sigma} (1+y+L+\sigma)^{-\alpha} dy d\sigma \Big) \\
&= O(1)(e^{\frac{-|x|}{2}} e^{\frac{-t}{4}} e^{\frac{-L}{2}} + e^{-|x|} e^{-L}),
\end{aligned} \tag{11.30}
$$

and for $x > 0$,

$$
\begin{aligned}
&|E_1(x,t)| \\
&= O(1)\Big(\int_0^t \int_{-L-\sigma}^0 e^{-|x|} H(x-y-(t-\sigma), t-\sigma; 1) e^{-L-\sigma} (1+y+L+\sigma)^{-\alpha} dy d\sigma \\
&\quad + \int_0^t \int_0^\infty H(x-y+(t-\sigma), t-\sigma; 1) e^{-L-\sigma} (1+y+L+\sigma)^{-\alpha} dy d\sigma \Big) \\
&= O(1)\Big\{ e^{-|x|} e^{-L} + \int_0^t \Big(\int_0^{\frac{x+t-\sigma}{2}} + \int_{\frac{x+t-\sigma}{2}}^\infty \Big) H(x-y+(t-\sigma), t-\sigma; 1) \\
&\quad \times e^{-L-\sigma} (1+y+L+\sigma)^{-\alpha} dy d\sigma \Big\} \\
&= O(1)\Big(e^{-|x|} e^{-L} + \int_0^t \int_0^{\frac{x+t-\sigma}{2}} \frac{1}{\sqrt{4\pi(t-\sigma)}} e^{\frac{-(x+t-\sigma)^2}{16(t-\sigma)}} e^{-L-\sigma} (1+y+L+\sigma)^{-\alpha} dy d\sigma \\
&\quad + \int_0^t \int_{\frac{x+t-\sigma}{2}}^\infty H(x-y+(t-\sigma), t-\sigma; 1) e^{-L-\sigma} (1+\frac{x+t-\sigma}{2}+L+\sigma)^{-\alpha} dy d\sigma \Big) \\
&= O(1)(e^{\frac{-|x|}{8}} e^{\frac{-t}{16s}} e^{-L} + (1+x+t+L)^{-\alpha} e^{-L}), \quad s \gtrsim 1.
\end{aligned} \tag{11.31}
$$

(11.27), (11.29), (11.30) and (11.31) imply that $v^0(x,t)$ satisfies the estimate (11.22) for $x \ge \frac{-L-t}{2}$. □

Proof for Theorem 11.1

The function v^n can be written as

$$
v^n = v^0 + (v^1 - v^0) + (v^2 - v^1) + (v^3 - v^2) + \cdots + (v^n - v^{n-1}). \tag{11.32}
$$

Let

$$\begin{cases}\delta^0(x,t)\equiv v^0(x,t),\\ \delta^n(x,t)\equiv v^n(x,t)-v^{n-1}(x,t),\quad n\geq 1.\end{cases}\tag{11.33}$$

We claim by induction that for $n\geq 1$

$$\begin{cases}|||\delta^n|||\leq C|||\delta^{n-1}|||e^{-\frac{L}{2r}},\\ |\delta^n(x,t)|\leq C|x+L+t||||\delta^{n-1}|||e^{-\frac{L}{2r}}e^{-\frac{|x|}{2}},\quad \text{for } x\in[-L-t,-L-t+1),\\ |\delta^n_x(-L-t,t)|\leq C|||\delta^{n-1}|||e^{-\frac{L}{2r}}e^{-\frac{L+t}{2}},\end{cases}\tag{11.34}$$

under the weighted super norm

$$|||h|||\equiv\sup_{\substack{t\geq 0\\ -L-t\leq x<0}}\frac{|h(x,t)|}{e^{-\frac{|x|}{2}}}+\sup_{\substack{t\geq 0\\ x\geq 0}}\frac{|h(x,t)|}{e^{-\frac{|x|}{2}}+(1+x+L+t)^{-\alpha}}.\tag{11.35}$$

Similar to (11.23), we can rewrite K^B_y as

$$\begin{aligned}K^B_y=K^B_Y&=-K^B+(H_Y(Y-X,t-\sigma;1)-H_Y(Y+X,t-\sigma;1))e^{(X-Y)-(t-\sigma)}\\ &=-X\int_{-1}^{1}(-\partial_Y H+\partial_{YY}H)(Y-\theta X,t-\sigma;1)d\theta\, e^{(X-Y)-(t-\sigma)}.\end{aligned}\tag{11.36}$$

where $X=x+L+t$ and $Y=y+L+\sigma$. For $n=1$ and $x\in[-L-t,-L-t+1)$,

$$\begin{aligned}\delta^1(x,t)&=\int_0^t\int_{-L-\sigma}^{\infty}K^B_y(x,t;y,\sigma)\left[(\phi-1+\Psi)v^0+\frac{(v^0)^2}{2}\right](y,\sigma)dyd\sigma\\ &=\left(\int_0^{t-1}+\int_{t-1}^{t}\right)\int_{-L-\sigma}^{\infty}\cdots\ dyd\sigma\\ &\equiv J_1+J_2.\end{aligned}\tag{11.37}$$

By (11.36), Lemma 11.3 and

$$\Psi(x)\leq(1-\phi(x))\leq\begin{cases}e^x,\ -L-t\leq x\leq 0,\\ 2,\quad x>0,\end{cases}$$

we can get

$$
\begin{aligned}
&|J_1(x,t)| \\
\le\ & O(1)\int_0^{t-1}\int_{-L-\sigma}^{\infty}|x+L+t|\int_{-1}^{1}\frac{e^{\frac{-(y+L+\sigma-\theta(x+L+t))^2}{4d(t-\sigma)}}}{\sqrt{t-\sigma}}e^{x-y}d\theta \\
&\times\Big|\Big[(\phi-1+\Psi)v^0+\frac{(v^0)^2}{2}\Big](y,\sigma)\Big|dyd\sigma \\
=\ & O(1)|x+L+t|\int_0^{t-1}\int_{-1}^{1}(\int_{-L-\sigma}^{0}+\int_0^{\infty})\cdots\ dyd\sigma \\
=\ & O(1)|x+L+t|\|\|v^0\|\|\Bigg\{\int_0^{t-1}\int_{-1}^{1}\int_{-L-\sigma}^{0}\frac{e^{\frac{-(y+L+\sigma-\theta(x+L+t))^2}{4d(t-\sigma)}}}{\sqrt{t-\sigma}}e^{x-y} \\
&\times(e^{-|y|}e^{\frac{-|y|}{2}}+e^{-|y|}e^{\frac{-L}{2r}})dyd\theta d\sigma \\
&+\int_0^{t-1}\int_{-1}^{1}\int_0^{\infty}\frac{e^{\frac{-(y+L+\sigma-\theta(x+L+t))^2}{4d(t-\sigma)}}}{\sqrt{t-\sigma}}e^{x-y}(e^{\frac{-|y|}{2}}+(1+y+L+\sigma)^{-\alpha})dyd\theta d\sigma\Bigg\} \\
\le\ & O(1)|x+L+t|\|\|v^0\|\|e^{\frac{-|x|}{2}}e^{-\frac{L}{2r}}. \qquad (11.38)
\end{aligned}
$$

$$
\begin{aligned}
J_2(x,t) &= \int_{t-1}^{t}\int_{-L-\sigma}^{\infty}K_y^B(x,t;y,\sigma)\left[(\phi-1+\Psi)v^0+\frac{(v^0)^2}{2}\right](y,\sigma)dyd\sigma \\
&= \int_{t-1}^{t}(\int_{-L-\sigma}^{-L-\sigma+4}+\int_{-L-\sigma+4}^{0}+\int_0^{\infty})\cdots\ dyd\sigma \\
&\equiv J_{21}+J_{22}+J_{23}. \\
J_{21}(x,t) &= \int_{t-1}^{t}\int_0^4 K_Y^B\left[(\phi-1+\Psi)v^0+\frac{(v^0)^2}{2}\right](Y-L-\sigma,\sigma)dYd\sigma \\
&= \int_{t-1}^{t}(\int_0^{2X}+\int_{2X}^{4})\cdots\ dYd\sigma \qquad (0\le X<1) \\
&\equiv J_{211}+J_{212}.
\end{aligned}
$$

By lemma 11.3, (11.36) and

$$|\phi(x)-1+\Psi(x,t)|=O(1)|x+L+t|e^{-|x|}\quad \text{for } -L-t\le x<0, \qquad (11.39)$$

we have

$$|J_{211}(x,t)|\ \le\ O(1)\int_{t-1}^{t}\int_0^{2X}\Big(H(Y-X,t-\sigma)+\frac{H(\frac{Y-X}{D},t-\sigma)}{\sqrt{t-\sigma}}\Big)e^{X-Y-(t-\sigma)}$$

$$\times|||v^0|||(Ye^{\frac{3}{2}(Y-L-\sigma)}+Ye^{Y-L-\sigma}e^{\frac{-L}{2r}})dYd\sigma$$
$$\leq\ O(1)|x+L+t||||v^0|||e^{-|x|}e^{-\frac{L}{2r}} \tag{11.40}$$

For $0\leq 2X\leq Y\leq 4$ and $0\leq\theta\leq 1$, we have $(Y-\theta X)\geq\frac{Y}{2}$. Then

$$\begin{aligned}
&|J_{212}(x,t)|\\
&=\Big|\int_{t-1}^{t}\int_{2X}^{4}(-X)\int_{-1}^{1}(-\partial_Y H+\partial_{YY}H)(Y-\theta X,t-\sigma;1)d\theta e^{(X-Y)-(t-\sigma)}\\
&\quad\times\Big[(\phi-1+\Psi)v^0+\frac{(v^0)^2}{2}\Big](Y-L-\sigma,\sigma)dYd\sigma\Big|\\
&\leq O(1)X\int_{t-1}^{t}\int_{2X}^{4}\int_{-1}^{1}(\frac{1}{\sqrt{t-\sigma}}+\frac{1}{t-\sigma})H(Y-\theta X,t-\sigma;1)e^{(X-Y)-(t-\sigma)}d\theta\\
&\quad\cdot|||v^0|||(Ye^{\frac{3}{2}(Y-L-\sigma)}+Ye^{Y-L-\sigma}e^{\frac{-L}{2r}})dYd\sigma\\
&\leq O(1)X|||v^0|||\int_{t-1}^{t}\int_{-1}^{1}\int_{2X}^{4}\frac{Y}{t-\sigma}H(\frac{Y}{2},t-\sigma;1)dYd\theta d\sigma e^{-t}e^{(\frac{-1}{2r}-1)L}\\
&=O(1)|x+L+t||||v^0|||e^{\frac{-|x|}{s}}e^{-\frac{L}{2r}},\quad s\gtrsim 1.
\end{aligned} \tag{11.41}$$

For $0\leq X\leq 1$ and $Y\geq 4$, we have from (11.36)

$$|K_y^B|\leq O(1)XH(\frac{Y-\theta X}{D},t-\sigma;1)e^{(X-Y)-(t-\sigma)}. \tag{11.42}$$

Then it is easy to get

$$|J_{22}(x,t)|,|J_{23}(x,t|\leq O(1)|x+L+t||||v^0|||e^{\frac{-|x|}{2}}e^{-\frac{L}{2r}}. \tag{11.43}$$

Thus, from (11.40), (11.41) and (11.43), we have

$$|J_2(x,t)|\leq O(1)|x+L+t||||v^0|||e^{\frac{-|x|}{2}}e^{-\frac{L}{2r}}. \tag{11.44}$$

Combining (11.38) and (11.44), we have

$$\begin{cases}|\delta^1(x,t)|\leq O(1)|x+L+t||||v^0|||e^{\frac{-|x|}{2}}e^{-\frac{L}{2r}}, & -L-t\leq x<-L-t+1,\\ |\delta_x^1(x,t)|\leq O(1)|||v^0|||e^{\frac{-|x|}{2}}e^{-\frac{L}{2r}}.\end{cases} \tag{11.45}$$

With (11.36) and the fact $(g_S)_y=-(G_S)_x$, we can obtain, by straightforward computation,

$$|\delta^1(x,t)| \le O(1)|||v^0|||e^{\frac{-|x|}{2}}e^{-\frac{L}{2r}}, \quad -L-t \le x < \frac{-L-t}{2},$$

$$|\delta^1(x,t)| \le O(1)|||v^0|||e^{-\frac{L}{2r}} \begin{cases} e^{-\frac{|x|}{2}} & \text{for } x \in [-(L+t)/2, 0), \\ e^{-\frac{|x|}{2}} + (1+x+L+t)^{-\alpha} & \text{for } x \ge 0. \end{cases}$$

Thus (11.34) is true for $n = 1$. It can be proved that (11.34) also holds for $n = k + 1$ by similar analysis with $n = 1$,and the details is omitted.

With the claim (11.34), we have that $\{|||\delta^n|||\}$ and $\{|\delta_x^{n+1}(-L-t,t)|\}$ are geometric sequences when L is sufficiently large. Then

$$v(x,t) = \lim_{n\to\infty} v^n(x,t), \tag{11.46}$$

and by Lemma 11.3 and (11.34)

$$\begin{aligned} |v(x,t)| &= |\sum_{n=0}^{\infty} \delta^n(x,t)| \\ &\le |v^0(x,t)| + Ce^{-\frac{L}{2r}}(|||\delta^0||| + |||\delta^1||| + |||\delta^2||| + \cdots) \\ &\quad \times \begin{cases} e^{-\frac{|x|}{2}}|x+L+t| & \text{for } x \in [-L-t, -L-t+1), \\ e^{-\frac{|x|}{2}} & \text{for } x \in [-L-t, 0), \\ e^{-\frac{|x|}{2}} + (1+x+L+t)^{-\alpha}, & \text{for } x > 0, \end{cases} \\ &\le \bar{C}e^{-\frac{L}{2r}} \begin{cases} e^{-\frac{|x|}{2}}|x+L+t| & \text{for } x \in [-L-t, -L-t+1), \\ e^{-\frac{|x|}{2}} & \text{for } x \in [-L-t, 0), \\ e^{-\frac{|x|}{2}} + (1+x+L+t)^{-\alpha}, & \text{for } x > 0. \end{cases} \end{aligned}$$

□

The proof of Theorem 11.2 is analogous to Theorem 11.1, the details are omitted, [11].

The stability of the shock propagating away form the boundary can also be done using the energy method, [18], [26], without the rate of convergence. There is the interesting case of the propagating of shock waves with the same speed as the boundary. In this case, the shock still propagates away from the boundary due to the boundary effect. However, the speed is then of the order of $1/t$ and the distance from the boundary is of the order of $\log t$. The situation is more subtle and has been studied by the pointwise method for the Burgers equation, [19].

For the stability of rarefaction waves, straightforward energy method applies. We consider the interesting case of portion of the rarefaction wave become a boundary layer and another portion propagating away from the boundary, c.f. Figure 7 in Section 2. Assume that the flux function is convex $f''(u) > 0$ and $f(0) = f'(0) = 0$. For $u_- < 0 < u_+$ $(f'(u_-) < 0 < f'(u_+))$, we consider the following initial boundary value problem

$$\begin{cases} u_t + f(u)_x = u_{xx}, \quad x > 0,\ t > 0, \\ u(0,t) = u_-, \\ u(x,0) = u_0(x) = \begin{cases} u_- & \text{at } x = 0, \\ \to u_+ & \text{at } x \to \infty. \end{cases} \end{cases} \tag{11.47}$$

Thus we expect to have the time-asymptotic configuration of the rarefaction boundary layer $\phi(x)$ connectingu_- and u_0 plus the rarefaction wave $(0, u_+)$ propagating into the region $x > 0$. We construct the approximate rarefaction wave $(0, u_+)$ with smooth initial values for the inviscid Burgers equation, see Remark 2.4, (2.15),

$$\begin{cases} h_t + hh_x = 0, \quad x \in R,\ t > 0, \\ h(x,0) = h_0(x) = f'(u_+) \cdot \kappa_q \int_0^x (1+y^2)^{-q} dy, \quad q > \frac{1}{2}, \end{cases} \tag{11.48}$$

where $\kappa_q \int_0^\infty (1+y^2)^{-q} dy = 1$. Then $\psi(x,t) := (f')^{-1}(h(x,t))$ is a smooth function and satisfies

$$\begin{cases} \psi_t + f(\psi)_x = 0, \\ \psi(0,t) = 0, \\ \psi(x,0) = \psi_0(x) = (f')^{-1}(h_0(x)) \begin{cases} 0 & \text{at } x = 0, \\ \to u_+ & \text{at } x \to \infty. \end{cases} \end{cases} \tag{11.49}$$

Let $\phi(x)$ be a stationary solution which satisfies the ordinary differential equation

$$\begin{cases} f'(\phi) = \phi_{xx}, \\ \phi(0) = u_-, \quad \phi(\infty) = 0. \end{cases} \tag{11.50}$$

For the initial boundary value problem (11.47), we have the following stability result, [17].

Theorem 11.4. *Suppose that* $u_0 - \phi(\cdot) - \psi(\cdot, 0) \in H^1$. *Then there exists a unique global solution* $u(x,t)$ *of* (11.47) *such that*

$$u - \phi - u_R \in C([0,\infty), H^1)$$

$$(u - \phi - u_R)_x,\ (u-\phi)_{xx} \in L^2(R_+ \times R_+)$$

and

$$\sup_{R_+} |u(x,t) - \phi(x,t) - u_R(x,t)| \to 0 \quad \text{as } t \to \infty,$$

where u_R *is the centered rarefaction wave* $(0, u_+)$, (2.16), *and* $\phi(x)$ *is the boundary rarefaction wave,* (11.50).

Set

$$v(x,t) = u(x,t) - \Phi(x,t) = u(x,t) - \phi(x) - \psi(x,t).$$

Then $v(x,t)$ satisfies the reformulated problem

$$\begin{cases} v_t + (f(\Phi + v) - f(\Phi))_x - v_{xx} = F, \\ v(0,t) = 0, \\ v(x,0) = v_0(x) := u_0(x) - \phi(x) - \psi(x,0), \end{cases} \tag{11.51}$$

where

$$F = -(f'(\phi+\psi) - f'(\phi))\phi_x - (f'(\phi+\psi) - f'(\psi))\psi_x + \psi_{xx} \tag{11.52}$$

Theorem 11.5. *If* $v_0 \in H^1$, *then there exists a unique solution* $v(x,t)$ *of* (11.51) *which satisfies*

$$v \in C([0,\infty), H^1), \quad v_x \in L^2([0,\infty), H^1),$$

and

$$\sup_{R_+} |v(x,t)| \to 0 \quad \text{as } t \to \infty.$$

Lemma 11.6 *The solution* $\psi(x,t)$ *for* (11.49) *satisfies*

(i) $0 \le \psi(x,t) < u_+$, $\psi_x(x,t) > 0$, $(x,t) \in R_+ \times (0,\infty)$.

(ii) *For any* $1 \le p \le \infty$ *there exists a constant* $C_{p,q}$ *such that*

$$\begin{aligned} \|\psi_x(t)\|_{L^p} &\le C_{p,q} \min(u_+, u_+^{1/p} t^{-1+\frac{1}{p}}), \\ \|\psi_{xx}(t)\|_{L^p} &\le C_{p,q} \min(u_+, u_+^{-\frac{(p-1)}{2pq}} t^{-1-\frac{(p-1)}{2pq}}). \end{aligned}$$

(iii) $\lim_{t\to\infty} |\psi(x,t) - u_R(x,t)| = 0$.

Theorem 11.5 and Lemma 11.6 imply Theorem 11.4. Theorem 11.5 is proved by the local existence of the solution in the solution space

$$X(0,T) = \{v \in C([0,T]; H^1),\ v_x \in L^2(0,T; H^1)$$

and

$$\partial_x^m v(x,t)|_{x=0} < +\infty \text{ for } t \in (0,T] \text{ and } m \in Z_+\}.$$

and the following priori estimates

Proposition 11.7. *Suppose that* $v \in X(0,T)$ *is a solution of* (11.51). *Then there exists a positive constant* C, *independent of* T, *satisfying*

$$\|v(t)\|_1^2 + \int_0^t (\|\sqrt{\Phi_x(\tau)}v(\tau)\|^2 + \|v_x(\tau)\|_1^2) d\tau \le C(\|v_0\|_1^2 + 1). \qquad (11.53)$$

Proof. Multiplying $(11.51)_1$ by v and integrating the equation over R_+, we have by $v(0,t) = 0$

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \int_0^\infty (f(\Phi+v) - f(\Phi) - f'(\Phi)v)\Phi_x dx + \|v_x(t)\|^2 = \int_0^\infty Fv dx. \qquad (11.54)$$

Since $\Phi_x = \psi_x + \psi_x > 0$ and $f''(\Phi+v) \ge c_0 > 0$, (11.54) gives

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2 + \|\sqrt{\Phi_x(t)}v(t)\|^2 + \|v_x(t)\|^2 \le C\left|\int_0^\infty Fv dx\right|. \qquad (11.55)$$

We estimate the last term of (11.55) using (11.52). First,

$$\left|C - \int_0^\infty (f'(\phi+\psi) - f'(\phi))\phi_x v dx\right|$$

$$\leq\ C\int_0^\infty \psi\phi_x|v|dx = \int_0^{f^{'}(u_+)t} + \int_{f^{'}(u_+)t}^\infty := I_1 + I_2. \tag{11.56}$$

By $\phi < 0$, $\psi > 0$, Lemma 11.6 and the fact $|\phi(x)| \leq C(1+x)^{-1}$, we have

$$\begin{aligned} I_1 &\leq\ C\sup_{R_+}|v|\cdot\{[\phi\psi]_0^{f^{'}(u_+)t} + \int_0^{f^{'}(u_+)t}(-\phi)\psi_x dx\} \\ &\leq\ C\|v(t)\|^{1/2}\|v(t)_x\|^{1/2}(1+t)^{-1}\int_0^{f^{'}(u_+)t}\frac{dx}{1+x} \\ &\leq\ \frac{1}{8}\|v_x(t)\|^2 + C\{(1+t)^{-1}\log(2+t)\}^{4/3}(\|v(t)\|^2+1), \end{aligned} \tag{11.57}$$

and

$$\begin{aligned} I_2 &\leq\ C\sup_{R_+}|v|\cdot u_+\int_{f^{'}(u_+)t}^\infty \phi_x(x)dx \leq C\|v(t)\|^{1/2}\|v(t)_x\|^{1/2}(1+t)^{-1} \\ &\leq\ \frac{1}{8}\|v_x(t)\|^2 + C(1+t)^{-4/3}(\|v(t)\|^2+1). \end{aligned} \tag{11.58}$$

Secondly, in a similar fashion to (11.57) and (11.58),

$$\begin{aligned} &\Big|C - \int_0^\infty (f^{'}(\phi+\psi) - f^{'}(\psi))\psi_x v dx\Big| \\ &\leq \frac{1}{4}\|v_x(t)\|^2 + C\{(1+t)^{-4/3} + ((1+t)^{-1}\log(2+t))^{4/3}\}(\|v(t)\|^2+1). \end{aligned} \tag{11.59}$$

Thirdly,

$$\Big|\int_0^\infty \psi_{xx} v dx\Big| \leq \|\psi_{xx}\|\|v(t)\|. \tag{11.60}$$

Substituting (11.57)- (11.60) into (11.55), we have

$$\begin{aligned} &\frac{d}{dt}\|v(t)\|^2 + \|\sqrt{\Phi_x(t)}v(t)\|^2 + \frac{1}{2}\|v_x(t)\|^2 \\ &\leq C\{(1+t)^{-4/3} + ((1+t)^{-1}\log(2+t))^{4/3} + (1+t)^{-1-\frac{1}{4q}}\}(\|v(t)\|^2+1). \end{aligned} \tag{11.61}$$

Then integrating (11.61) over $(0,t)$ and using Gronwall inequality, we have

$$\|v(t)\|^2 + \int_0^t (\|\sqrt{\Phi_x(\tau)}v(\tau)\|^2 + \|v_x(\tau)\|^2)d\tau \leq C(\|v_0\|_1^2 + 1). \tag{11.62}$$

The estimates of higher order derivatives can be obtained by the similar fashion to the above and the details is omitted. □

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