Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **5** (2010), No. 2, pp. 201-214

MULTIPLY WARPED PRODUCT SUBMANIFOLDS IN KENMOTSU SPACE FORMS

BY

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Abstract

Recently, B. Y. Chen and F. Dillen established general sharp inequalities for multiply warped product submanifolds in arbitrary Riemannian manifolds. As applications, they obtained obstructions to minimal isometric immersions of multiply warped products into Riemannian manifolds.
Later, the authors proved similar inequalities for multiply warped products isometrically immersed in Sasakian space forms.
In this paper, the authors obtain inequalities for multiply warped products isometrically immersed in Kenmotsu space forms together with derivation of some applications.

1. Introduction

Let $N_1, \ldots, N_k$ be Riemannian manifolds and let $N = N_1 \times \cdots \times N_k$ be the Cartesian product of $N_1, \ldots, N_k$. For each i, denote by $\pi_i : N \to N_i$ the canonical projection of N onto N_i. When there is no confusion, we identify N_i with the horizontal lift of N_i in N via π_i.

If $\sigma_2, \ldots, \sigma_k : N_1 \to R_+$ are positive-valued functions, then

$$< X, Y > = < \pi_{1*}X, \pi_{1*}Y > + \sum_{i=2}^{k} (\sigma_i \circ \pi_1)^2 < \pi_{i*}X, \pi_{i*}Y > \qquad (1.1)$$

Received March 3, 2009 and in revised form May 12, 2009.

AMS Subject Classification: Primary 53C40; Secondary 53C25.

Key words and phrases: multiply warped product, multiply CR-warped product, second fundamental form, warping functions, mean curvature, Kenmotsu space form.

defines a Riemannian metric g on N called a *multiply warped product metric.* The product manifold N endowed with this metric is denoted by $N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$.

For a multiply warped product manifold $N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$, let $\mathcal{D}_i$ denote the distributions obtained from the vectors tangent to N_i (or more) precisely, vectors tangent to the horizontal lifts of N_i.

Assume that

$$x : N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k \to \widetilde{M}$$

is an isometric immersion of a multiply warped product $N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ into a Riemannian manifold $\widetilde{M}$. Denote by h is the second fundamental form of x. Then the immersion x is called *mixed totally geodesic* if $h(\mathcal{D}_i, \mathcal{D}_j) = \{0\}$ holds for distinct $i, j \in \{1, \ldots, k\}$.

Let $\Psi : N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k \to \widetilde{M}$ denote an isometric immersion of a multiply warped product $N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ into an arbitrary Riemannian manifold $\widetilde{M}$.

Denote by trace h_i the trace of h restricted to N_i, that is

$$\text{trace } h_i = \sum_{\alpha=1}^{n_i} h(e_\alpha, e_\alpha)$$

for some orthonormal frame fields $e_1, \ldots, e_{n_i}$ of $\mathcal{D}_i$.

In [4], B. Y. Chen and F. Dillen established the following general inequality for arbitrary isometric immersions of multiply warped product manifolds in arbitrary Riemannian manifolds.

Theorem 1.1. *Let $x : N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k \to \widetilde{M}^m$ be an isometric immersion of a multiply warped product $N = N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ into an arbitrary Riemannian m-manifold. Then we have*

$$\sum_{j=2}^{k} n_j \frac{\Delta \sigma_j}{\sigma_j} \leq \frac{n^2}{4} ||H||^2 + n_1 (n - n_1) \max \widetilde{K}, \quad n = \sum_{j=1}^{n} n_j, \tag{1.2}$$

where $\max \widetilde{K}(p)$ *denotes the maximum of the sectional curvature function of $\widetilde{M}^m$ restricted to 2-planes sections of the tangent space T_pN of N at $p = (p_1, \ldots, p_k)$.*

The equality of (1.2) holds identically if and only if the following two statements hold:

(1) *x is a mixed totally geodesic immersion satisfying*

$$\text{trace } h_1 = \cdots = \text{trace } h_k$$

(2) *at each point $p \in N$, the sectional curvature function $\widetilde{K}$ of $\widetilde{M}^m$ satisfies $\widetilde{K}(u, v) = \max \widetilde{K}(p)$ for each unit vector u in $T_{p_1}(N_1)$ and each unit vector v in $T_{(p_2,\ldots,p_k)}(N_2 \times \cdots \times N_k)$.*

We prove a similar inequality for multiply warped product submanifolds of a Kenmotsu space form.

In the following, a multiply warped product $N_{\mathsf{T}} \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ in a Kenmotsu space form $\widetilde{M}(c)$ is called a *multiply CR-warped product.* If N_{T} is an invariant submanifold and $N_\perp :=_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ is an anti-invariant submanifold of $\widetilde{M}(c)$.

In [4], B. Y. Chen and F. Dillen also proved that for any multiply CR-warped product $N_{\mathsf{T}} \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$in an arbitrary Kaehler manifold $\widetilde{M}$ the second fundamental form h and the warping functions $\sigma_2, \ldots, \sigma_k$ satisfy:

$$||h||^2 \geq 2 \sum_{i=2}^{k} n_i ||\nabla(\ln \sigma_i)||^2, \tag{1.3}$$

where $n_i = \dim N_i$ $(i = 2, \ldots, k)$ and $\nabla(\ln \sigma_i)$ is the gradient of $\ln \sigma_i$ $(i = 2, \ldots, k)$.

The second purpose of this article is to obtain a similar inequality for multiply CR-warped products in Kenmotsu space forms.

2. Preliminares

In this section, we recall some definitions and basic formulas which we will use later.

A $(2m+1)$-dimensional Riemannian manifold $\left(\widetilde{M}, g\right)$ is said to be a *Kenmotsu manifold* if it admits an endomorphism ϕ of its tangent bundle

$T\widetilde{M}$, a vector field ξ and a 1-form η satisfying:

$$\begin{gathered}\phi^2 = -Id + \eta \otimes \xi,\ \eta(\xi) = 1,\ \phi\xi = 0,\ \eta \circ \phi = 0, \\ g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),\ \eta(X) = g(X, \xi), \\ \left(\widetilde{\nabla}_X \phi\right) Y = -g(X, \phi Y)\xi - \eta(Y)\phi X,\ \widetilde{\nabla}_X \xi = X - \eta(X)\xi,\end{gathered} \tag{2.1}$$

for any vector fields X, Y on $\widetilde{M}$, where $\widetilde{\nabla}$ denotes the Riemannian connection with respect to g.

We denote by ω the fundamental 2-form of $\widetilde{M}$, i.e.,

$$\omega(X, Y) = g(\phi X, Y),\ \forall\ X,\ Y \in \Gamma\left(T\widetilde{M}\right). \tag{2.2}$$

A plane section π in $T_p\widetilde{M}$ is called a *ϕ-section* if it is spanned by X and ϕX, where X is a unit tangent vector orthogonal to ξ. The sectional curvature of a ϕ-section is called a *ϕ-sectional curvature.* A Kenmotsu manifold with constant ϕ-holomorphic sectional curvature c is said to be a *Kenmotsu space form* and is denoted by $\widetilde{M}(c)$.

The curvature tensor $\widetilde{R}$ of a Kenmotsu space form is given by [5]

$$\begin{aligned}\widetilde{R}(X, Y)Z \;=\; & \frac{c-3}{4}\{g(Y, Z)X - g(X, Z)Y\} \\ & + \frac{c+1}{4}\{[\eta(X)Y - \eta(Y)X]\eta(Z) \\ & + [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)]\xi \\ & + \omega(Y, Z)\phi X - \omega(X, Z)\phi Y - 2\omega(X, Y)\phi Z\}.\end{aligned} \tag{2.3}$$

Let $\widetilde{M}$ be a Kenmotsu manifold and M an n-dimensional submanifold tangent to ξ. For any vector field X tangent to M, we put

$$\phi X = PX + FX,$$

where PX (resp. FX) denotes the tangential (resp. normal) component of ϕX. Then P is an endomorphism of tangent bundle TM and F is a normal bundle valued 1-form on TM.

The equation of Gauss is given by

$$\widetilde{R}(X,Y,Z,W)=R(X,Y,Z,W)+g(h(X,W),h(Y,Z))-g(h(X,Z),h(Y,W)) \tag{2.4}$$

for any vectors X, Y, Z, W tangent to M.

Let $p \in M$ and $\{e_1,\ldots,e_n,e_{n+1},\ldots,e_{2m+1}\}$ an orthonormal basis of the tangent space $T_p\widetilde{M}$, such that $e_1,\ldots,e_n$ are tangents to M at p. We denote by H the mean curvature vector, that is

$$H(p)=\frac{1}{n}\sum_{i=1}^{n}h(e_i,e_i). \tag{2.5}$$

As is known, M is said to be minimal if H vanishes identically.

Also, we set

$$h_{ij}^{r}=g(h(e_i,e_j),e_r),\ i,j\in\{1,\ldots,n\},\ r\in\{n+1,\ldots,2m+1\} \tag{2.6}$$

the coefficients of the second fundamental form h with respect to $\{e_1,\ldots,e_n, e_{n+1},\ldots,e_{2m+1}\}$, and

$$||h||^2=\sum_{i,j=1}^{n}g(h(e_i,e_j),h(e_i,e_j)). \tag{2.7}$$

By analogy with submanifolds in a Kaehler manifold, different classes of submanifolds in a Kenmotsu manifold were considered (see, for example [12]).

A submanifold M tangent to ξ is called an *invariant* (resp. *anti-invariant*) submanifold if $\phi(T_pM)\subset T_pM,\ \forall\ p\in M$ (resp. $\phi(T_pM)\subset T_p^{\perp}M,\ \forall\ p\in M$).

A *warped product immersion* is defined as follows: Let $M_1\times_{\rho_2}M_2\times\cdots\times_{\rho_k}M_k$ be a warped product and let $\Psi_i:N_i\to M_i$, $i=1,\ldots,k$, be isometric immersions, and define $\sigma_i=\rho_i\circ\Psi_1:N_1\to R_+$ for $i=2,\ldots,k$. Then the map

$$\Psi:N_1\times_{\sigma_2}N_2\times\cdots\times_{\sigma_k}N_k\to M_1\times_{\rho_2}M_2\times\cdots\times_{\rho_k}M_k$$

given by

$$\Psi(x_1,\ldots,x_k)=(\Psi_1(x_1),\ldots,\Psi_k(x_k))$$

is an isometric immersion, which is called a *warped product immersion* [10].

Let n be a natural number ≥ 2 and let $n_1,\ldots,n_k$ be k natural numbers. If $n_1+\cdots+n_k=n$, then $(n_1,\ldots,n_k)$ is called a partition of n.

We recall the following general algebraic lemma from [3] for later use.

Lemma 2.1. *Let $a_1,\ldots,a_n$ be n real numbers and let k be an integer in $[2,n-1]$. Then, for any partition $(n_1,\ldots,n_k)$ of n, we have*

$$\sum_{1\leq i_1<j_1\leq n_1} a_{i_1}a_{j_1}+\sum_{n_1+1\leq i_2<j_2\leq n_1+n_2} a_{i_2}a_{j_2}+\cdots+\sum_{n_1+\cdots+n_{k-1}+1\leq i_1<j_1\leq n} a_{i_k}a_{j_k}$$
$$\geq \frac{1}{2k}\left[(a_1+\cdots+a_n)^2-k\left(a_1^2+\cdots+a_n^2\right)\right],$$

with the equality holding if and only if

$$a_1+\cdots+a_{n_1}=\cdots=a_{n_1+\cdots+n_{k-1}+1}+\cdots+a_n.$$

3. Anti-invariant Multiply Warped Product Submanifolds in Kenmotsu Space Forms

Recently, Bang-Yen Chen and Franki Dillen established a sharp relationship between the warping functions $\sigma_2,\ldots,\sigma_k$ of a multiply warped product $N_1\times_{\sigma_2}N_2\times\cdots\times_{\sigma_k}N_k$ isometrically immersed in an arbitrary Riemannian manifold and the squared mean curvature $||H||^2$ (see [4], [6]).

Following that, the present author obtained a similar relationship for doubly warped products isometrically immersed in arbitrary Riemannian manifolds (see [8]).

We prove a similar inequality for multiply warped product submanifolds of a Kenmotsu space form.

In this section, we investigate anti-invariant multiply warped product submanifolds in a Kenmotsu space form $\widetilde{M}(c)$.

Theorem 3.1. *Let x be an anti-invariant isometric immersion of an n-dimensional multiply warped product $N_1\times_{\sigma_2}N_2\times\cdots\times_{\sigma_k}N_k$ into an $(2m+1)$-dimensional Kenmotsu space form $\widetilde{M}(c)$. Then:*

$$\sum_{j=2}^{k} n_j\frac{\Delta\sigma_j}{\sigma_j}\leq\frac{n^2}{4}||H||^2+n_1\left(n-n_1\right)\frac{c-3}{4},\quad n=\sum_{j=1}^{n} n_j. \tag{3.1}$$

The equality sign of (3.1) holds identically if and only if x is a mixed totally geodesic immersion and

$$\text{trace } h_1=\cdots=\text{trace } h_k \tag{3.2}$$

holds, where trace h_i denotes the trace of h restricted to N_i.

Proof. Let $N=N_1\times_{\sigma_2}N_2\times\cdots\times_{\sigma_k}N_k$ be the Riemannian product of the Riemannian manifolds $N_1,\ldots,N_k$.

We know (see [4]) that the sectional curvature function of the multiply warped product $N_1\times_{\sigma_2}N_2\times\cdots\times_{\sigma_k}N_k$ satisfies

$$K(X_1\wedge X_i) = \frac{1}{\sigma_i}\left((\nabla_{X_1}X_1)\sigma_i-X_1^2\sigma_i\right), \tag{3.3}$$

$$K\left(X_i\wedge X_j\right) = -\frac{g(\nabla\sigma_i,\nabla\sigma_i)}{\sigma_i\sigma_j},\ i,j=2,\ldots,k, \tag{3.4}$$

for each unit vector X_i tangent to N_i, where $\nabla\sigma$ denotes the gradient of σ.

In particular, (3.3) implies that, for each $i=2,\ldots,k$, we have

$$\Delta\sigma_i=\sigma_i\sum_{j=1}^{n_1}K\left(e_j\wedge X_i\right), \tag{3.5}$$

for any unit vector X_i tangent to N_i, where $\{e_1,\ldots,e_{n_1}\}$ is an orthormal basis of $T_{\pi_1(p)}N_1$.

From the equation of Gauss, we have

$$2\tau=n^2||H||^2-||h||^2+n\left(n-1\right)\frac{c-3}{4}, \tag{3.6}$$

where $n_i=\dim N_i$, $n=n_1+\cdots+n_k$, τ is the scalar curvature, H is the mean curvature and h is the second fundamental form of N in $\widetilde{M}(c)$.

Let us put

$$\eta = 2\tau - n\,(n-1)\,\frac{c-3}{4} - n^2\left(1-\frac{1}{k}\right)||H||^2. \tag{3.7}$$

Then it follows from (3.6) and (3.7) that

$$n^2||H||^2 = k\left(\eta + ||h||^2\right). \tag{3.8}$$

Let us also put

$$\Delta_1 = \{1,\ldots,n_1\},\ldots,\Delta_k = \{n_1+\cdots+n_{k-1}+1,\ldots,n_1+\cdots+n_k\}.$$

For a given point $p \in N$ we choose an orthonormal basis $e_1,\ldots,e_{2m+1}$ at p such that, for each $j \in \Delta_i$, e_j is tangent to N_i for $i = 1,\ldots,k$. Moreover, we choose the normal vector e_{n+1} in the direction of the mean curvature vector at p (when the mean curvature vanishes at p, e_{n+1} can be chosen to be any unit normal vector at p) and $e_{2m+1} = \xi$.

Then we get from (3.8) that

$$\Big(\sum_{A=1}^{n} a_A\Big)^2 - k\sum_{A=1}^{n}(a_A)^2 = k\Big[\eta + \sum_{A\neq B}\left(h_{AB}^{n+1}\right)^2 + \sum_{r=n+2}^{2m}\sum_{A,B=1}^{n}(h_{AB}^r)^2\Big], \tag{3.9}$$

where $a_A = h_{AA}^{n+1}$ and $h_{AB}^r =< h\,(e_A, e_B)\,, e_r >$ with $1 \le A, B \le n$ and $n+1 \le r \le 2m$.

Because $(n_1,\ldots,n_k)$ is a partition of n, we may apply Lemma 2.1 to (3.9). From this we obtain

$$\begin{aligned}\sum_{\alpha_1<\beta_1} a_{\alpha_1}a_{\beta_1} + \sum_{\alpha_2<\beta_2} a_{\alpha_2}a_{\beta_2} + \cdots + \sum_{\alpha_k<\beta_k} a_{\alpha_k}a_{\beta_k}\\ \ge \frac{\eta}{2} + \sum_{A<B}\left(h_{AB}^{n+1}\right)^2 + \frac{1}{2}\sum_{r=n+2}^{2m}\sum_{A,B=1}^{n}(h_{AB}^r)^2,\end{aligned} \tag{3.10}$$

where α_i, $\beta_i \in \Delta_i$, $i = 1,\ldots,k$.

On the other hand, from the equation of Gauss and (3.5), we find

$$\begin{aligned}\sum_{i=2}^{k} n_i \frac{\Delta\sigma_i}{\sigma_i} &= \sum_{j\in\Delta_1}\sum_{\beta\in\Delta_2\cup\cdots\cup\Delta_k} K\left(e_j\wedge e_\beta\right)\\ &= \tau - \sum_{1\le j_1<j_2\le n_1} K\left(e_{j_1}\wedge e_{j_2}\right) - \sum_{n_1+1\le\alpha<\beta\le n} K\left(e_\alpha\wedge e_\beta\right)\\ &= \tau - \frac{n_1\left(n_1-1\right)\left(c-3\right)}{8} - \sum_{r=n+1}^{2m}\sum_{1\le j_1<j_2\le n_1}\left(h^r_{j_1j_1}h^r_{j_2j_2} - \left(h^r_{j_1j_2}\right)^2\right)\\ &\quad -\frac{n_1(n-n_1-1)(c-3)}{8} - \sum_{r=n+1}^{2m}\sum_{n_1+1\le\alpha<\beta<n}\left(h^r_{\alpha\alpha}h^r_{\beta\beta} - \left(h^r_{\alpha\beta}\right)^2\right). \quad (3.11)\end{aligned}$$

Therefore, by combining (3.7), (3.10) and (3.11), we obtain

$$\begin{aligned}\sum_{i=2}^{k} n_i \frac{\Delta\sigma_i}{\sigma_i} &\le \tau - \frac{n\left(n-1\right)\left(c-3\right)}{8} + \frac{n_1\left(n-n_1\right)\left(c-3\right)}{4} - \frac{\eta}{2}\\ &\quad -\frac{1}{2}\sum_{r=n+2}^{2m}\sum_{A,B=1}^{n}\left(h^r_{AB}\right)^2 - \sum_{\substack{1\le j\le n_1\\ n_1+1\le\alpha<n}}\left(h^{n+1}_{j\alpha}\right)^2\\ &\quad + \sum_{r=n+2}^{2m}\sum_{1\le j_1<j_2\le n_1}\left(\left(h^r_{j_1j_2}\right)^2 - h^r_{j_1j_1}h^r_{j_2j_2}\right)\\ &\quad + \sum_{r=n+2}^{2m}\sum_{n_1+1\le\alpha<\beta<n}\left(\left(h^r_{\alpha\beta}\right)^2 - h^r_{\alpha\alpha}h^r_{\beta\beta}\right)\\ &= \tau - \frac{n\left(n-1\right)\left(c-3\right)}{8} + \frac{n_1\left(n-n_1\right)\left(c-3\right)}{4} - \frac{\eta}{2}\\ &\quad - \sum_{r=n+1}^{2m}\sum_{1\le j\le n_1}\sum_{n_1+1\le\alpha\le n}\left(h^r_{j\alpha}\right)^2\\ &\quad -\frac{1}{2}\sum_{r=n+2}^{2m}\Big(\sum_{1\le j\le n_1} h^r_{jj}\Big)^2 - \frac{1}{2}\sum_{r=n+2}^{2m}\Big(\sum_{n_1+1\le\alpha\le n} h^r_{\alpha\alpha}\Big)^2\\ &\le \tau - \frac{n\left(n-1\right)\left(c-3\right)}{8} + \frac{n_1\left(n-n_1\right)\left(c-3\right)}{4} - \frac{\eta}{2}\\ &= \frac{n^2}{4}||H||^2 + \frac{n_1\left(n-n_1\right)\left(c-3\right)}{4}, \quad (3.12)\end{aligned}$$

which proves the inequality (3.1).

If the equality sign of (3.1) holds, then all of inequalities in (3.10) and (3.12) become equalities. Hence, by applying Lemma 2.1, we know that the equality sign of (3.1) holds if and only if the immersion is mixed totally geodesic and trace $h_1 = \cdots =$ trace h_k hold identically.

The converse statement is straightforward. □

As applications, we derive certain obstructions to the existence of minimal anti-invariant multiply warped product submanifolds in Kenmotsu space forms.

Corollary 3.2. *If $\sigma_2, \ldots, \sigma_k$ are harmonic functions on N_1, then $N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ admits no minimal anti-invariant immersion into a Kenmotsu space form $\widetilde{M}(c)$ with $c < 3$.*

Proof. Assume $\sigma_2, \ldots, \sigma_k$ are harmonic functions on N_1 and $N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ admits a minimal anti-invariant immersion into a Kenmotsu space form $\widetilde{M}(c)$.

Then, the inequality (3.1) becomes $c \geq 3$. □

Corollary 3.3. *If $\sigma_2, \ldots, \sigma_k$ are eigenfunctions of the Laplacian Δ on N_1 with nonnegative eigenvalues, then $N_1 \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ admits no minimal anti-invariant immersion into a Kenmotsu space form $\widetilde{M}(c)$ with $c \leq 3$.*

4. An inequality for the squared norm of the second fundamental form

Recently, Bang-Yen Chen and Franki Dillen established a sharp relationship between the warping functions $\sigma_2, \ldots, \sigma_k$ of a multiply warped product $N_\top \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ and the squared norm of the second fundamental form $||h||^2$ (see [4]).

Following that, the author obtained a similar inequality for doubly CR-warped products isometrically immersed in Kenmotsu space forms (see [9]).

In the present section, we will give an intersting inequality for the squared norm of the second fundamental form (an extrinsic invariant) in terms of the warping functions (intrinsic invariants) for multiply CR-warped products isometrically immersed in Kenmotsu manifolds.

Theorem 4.1. *Let $N = N_{\mathsf{T}} \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ be an n-dimensional multiply CR-warped product in a Kenmotsu manifold $\widetilde{M}$, such that N_{T} is tangent to ξ. Then the second fundamental form h and the warping functions $\sigma_2, \ldots, \sigma_k$ satisfy*

$$||h||^2 \geq 2 \sum_{i=2}^{k} n_i [||\nabla (\ln \sigma_i)||^2 - 1], \tag{4.1}$$

where $n_i = \dim N_i$ $(i = 2, \ldots, k)$ and $\nabla (\ln \sigma_i)$ is the gradient of $\ln \sigma_i$ $(i = 2, \ldots, k)$.

The equality sign of (4.1) holds identically if and only if the following statements hold:

1. *N_{T} is a totally geodesic submanifold of $\widetilde{M}$;*
2. *For each $i \in \{2, \ldots, k\}$, N_i is a totally umbilical submanifold of $\widetilde{M}$;*
3. *$N_{\perp} :=_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ is immersed as mixed totally geodesic submanifold in $\widetilde{M}$;*
4. *For each $p \in N$, the first normal space Imh_p is a subspace of $\phi (T_p N_{\perp})$;*
5. *N is a minimal submanifold of $\widetilde{M}$.*

Proof. Let $N = N_{\mathsf{T}} \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ be an n-dimensional multiply CR-warped product of a Sasakian manifold $\widetilde{M}$, such that N_{T} is an invariant submanifold tangent to ξ and $N_{\perp} :=_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ is an anti-invariant submanifold of $\widetilde{M}$.

Let $\mathcal{D}_{\mathsf{T}}, \mathcal{D}_2, \ldots, \mathcal{D}_k, \mathcal{D}_{\perp}$ denote the distributions obtained from vectors tangent to $N_{\mathsf{T}}, N_2, \ldots, N_k, N_{\perp}$, respectively.

Let $\widehat{\nabla}$, ∇ denote the Levi-Civita connections of the Riemannian product $N_{\mathsf{T}} \times N_2 \times \cdots \times N_k$ and of the multiply warped product $N_{\mathsf{T}} \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$.

If we put $H_i = -\nabla((\ln \sigma_i) \circ \pi_1)$, then we have (cf. [10])

$$\nabla_X Y - \widehat{\nabla}_X Y = \sum_{i=2}^{k} \left(g\left(X^i, Y^i\right) H_i - g\left(H_i, X\right) Y^i - g\left(H_i, Y\right) X^i \right), \tag{4.2}$$

where X^i denotes the N_i-component of X.

Since $N_\mathsf{T} \times_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ is a multiply warped product, (4.2) implies that N_T is totally geodesic in N. Thus we have

$$g\left(\nabla_X Z, Y\right) = g\left(\nabla_X Y, Z\right) = 0,$$

for any vector fields X, Y in $\mathcal{D}_\mathsf{T}$ and Z in $\mathcal{D}_\perp$.

(4.2) also implies that

$$\nabla_X Z = \sum_{i=2}^{k} \left(X\left(\ln \sigma_i\right)\right) Z^i, \tag{4.3}$$

for any vector fields X in $\mathcal{D}_\mathsf{T}$ and Z in $\mathcal{D}_\perp$, where Z^i denotes the N_i-component of Z.

By applying (4.3) we find

$$g\left(h\left(\phi X, Z\right), \phi W\right) = g\left(\widetilde{\nabla}_Z \phi X, \phi W\right) = g\left(\phi \widetilde{\nabla}_Z X, \phi W\right) = \tag{4.4}$$

$$= g\left(\widetilde{\nabla}_Z X, W\right) = g\left(\nabla_Z X, W\right) = \sum_{i=2}^{k} \left(X\left(\ln \sigma_i\right)\right) g(Z^i, W^i),$$

for any vector fields X in $\mathcal{D}_\mathsf{T}$ and Z, W in $\mathcal{D}_\perp$.

On the other hand, since the ambient manifold $\widetilde{M}$ is Kenmotsu, it is easily seen that

$$h\left(\xi, Z\right) = 0. \tag{4.5}$$

For a given point $p \in N$ we may choose an orthonormal basis $e_1, \ldots, e_n$ at p such that e_α is tangent to N_i for each $\alpha \in \Delta_i$, $i = 2, \ldots, k$. For each $i \in \{2, \ldots, k\}$, (4.4) implies that

$$\sum_{\alpha \in \Delta_i} g\left(h\left(\phi X, e_\alpha\right), \phi e_\alpha\right) = n_i \sum_{i=2}^{k} X\left(\ln \sigma_i\right). \tag{4.6}$$

Now, inequality (4.1) follows from (4.5) and (4.6).

It follows from (4.6) that the equality sign of (4.1) holds identically if and only if we have

$$h\left(\mathcal{D}_{\mathsf{T}},\mathcal{D}_{\mathsf{T}}\right)=\{0\},\ h\left(\mathcal{D}_{\perp},\mathcal{D}_{\perp}\right)=\{0\},\ h\left(\mathcal{D}_{\mathsf{T}},\mathcal{D}_{\perp}\right)\subset\phi\mathcal{D}_{\perp}. \tag{4.7}$$

Because N_{T} is totally geodesic in $N=N_{\mathsf{T}}\times_{\sigma_2}N_2\times\cdots\times_{\sigma_k}N_k$ the first condition in (4.7) implies that N_{T} is totally geodesic in $\widetilde{M}$. This gives statement *1.*

From (4.2) we know that, for any $2\leq i\neq j\leq k$, and any vector field Z_i in $\mathcal{D}_i$ and Z_j in $\mathcal{D}_j$ we have $\nabla_{Z_i}Z_j=0$. This yields

$$g\left(\nabla_{Z_i}W_i,Z_j\right)=0.$$

Thus, if $\widehat{h}_i$ denotes the second fundamental form of N_i in N we have

$$\widehat{h}_i\left(\mathcal{D}_i,\mathcal{D}_i\right)\subset\mathcal{D}_{\mathsf{T}}. \tag{4.8}$$

From (4.4) and (4.8) we find

$$\widehat{h}_i\left(Z_i,W_i\right)=-\left(X\left(\ln\sigma_i\right)\right)g\left(Z_i,W_i\right),$$

for Z_i,W_i tangent to N_i. Therefore, by combining the first condition in (4.7) and (4.8) is obtained statement 2.

Statement 3 follows immediately from (4.2) and the second condition in (4.7).

Statement 4 follows from (4.7).

Moreover, by (4.7), it follows that N is a minimal submanifold of $\widetilde{M}$. □

Corollary 4.2. *Let $\widetilde{M}(c)$ be a $(2m+1)$-dimensional Kenmotsu space form of constant ϕ-sectional curvature c and $N=N_{\mathsf{T}}\times_{\sigma_2}N_2\times\cdots\times_{\sigma_k}N_k$ an n-dimensional non-trivial multiply warped product submanifold, such that N_{T} is an invariant submanifold tangent to ξ and $N_{\perp}:=_{\sigma_2}N_2\times\cdots\times_{\sigma_k}N_k$ is an anti-invariant submanifold of $\widetilde{M}(c)$ satisfying*

$$||h||^2=2\sum_{i=2}^{k}n_i[||\nabla\left(\ln\sigma_i\right)||^2-1].$$

Then, we have:

1. *N_{T} is a totally geodesic invariant submanifold of $\widetilde{M}(c)$. Hence N_{T} is a Kenmotsu space form of constant ϕ-sectional curvature c.*

2. *$N_{\perp} :=_{\sigma_2} N_2 \times \cdots \times_{\sigma_k} N_k$ is a totally umbilical anti-invariant submanifold of $\widetilde{M}(c)$. Hence $N_{\perp}$ is a real space form of sectional curvature $\varepsilon \geq \frac{c-3}{4}$.*

Proof. Statement 1 follows from Theorem 4.1.

Also, we know that $N_{\perp}$ is a totally umbilical submanifold of $\widetilde{M}(c)$. Gauss equation implies that $N_{\perp}$ is a real space form of constant sectional curvature $\varepsilon \geq \frac{c-3}{4}$. □

References

1. K. Arslan, R. Ezentas, I. Mihai and C. Murathan, Contact CR-warped product submanifolds in Kenmotsu space forms, *J. Korean Math. Soc.*, **42** (2005), No. 5, 1101-1110.

2. B. Y. Chen, On isometric minimal immersions from warped products into real space forms, *Proc. Edinburgh Math. Soc.*, **45** (2002), 579-587.

3. B. Y. Chen, Ricci curvature of real hypersurfaces in complex hyperbolic space, *Arch. Math.* (Brno) **38**(2002), 73-80.

4. B. Y. Chen and F. Dillen, Optimal inequalities for multiply warped product submanifolds, *Int. Electron. J. Geom.*, **1** (2008), 1-11.

5. K. Kenmotsu, A class of almost contact Riemannian Manifolds, *Tohoku Math. J.*, **24** (1972), 93-103.

6. A. Mihai, I. Mihai and R. Miron (Eds.), *Topics in Differential Geometry*, Ed. Academiei Romane, Bucharest, 2008.

7. A. Olteanu, CR-doubly warped product submanifolds in Sasakian space forms, *Bull. Transilv. Univ. Brasov, Ser. III*, **1** (50), (2008), 269-278.

8. A. Olteanu, A general inequality for doubly warped product submanifolds, to appear in *Math. J. Okayama Univ.*

9. A. Olteanu, Contact CR-doubly warped product submanifolds in Kenmotsu space forms, Preprint.

10. S. Nölker, Isometric immersions of warped products, *Differential Geom. Appl.*, **6** (1996), 1-30.

11. B. O'Neill, *Semi-Riemannian Geometry with Applications to Relativity*, Academic Press, New York, 1983.

12. K. Yano and M. Kon, *Structures on Manifolds*, World Scientific, Singapore, 1984.

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