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VISCOUS RAREFACTION WAVES

BY

TAI-PING LIU AND SHIH-HSIEN YU

Abstract

A rarefaction wave for system of viscous conservation laws induces strong coupling of waves pertaining to different characteristic fields. For the characteristic field pertaining to the rarefaction wave, there are the nonlinear hyperbolic, linear in time, and the parabolic, sub-linear in time, dissipations. We study the quantitative properties of the waves propagating around the rarefaction wave. Our analysis depends on the explicit informations on the wave coupling obtained through the Hopf-Cole transformation for the Burgers equation, and an inner-outer expansion for waves propagating between the characteristic directions.

1. Introduction

Consider the general system of viscous conservation laws

$$\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = \boldsymbol{u}_{xx},\ \ \boldsymbol{u} \in \mathbb{R}^n. \tag{1.1}$$

The main goal of this paper is to understand the strong coupling of waves pertaining different characteristic fields due to the presence of a rarefaction wave in the background. We show that this coupling is stronger than for shock wave. The main reason being that shock waves have exact form, and are compressive and highly stable; while there is no exact, time-invariant form for the rarefaction waves which induces strong coupling with the other, transversal characteristic fields. Our analysis makes use of the explicit solu-

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tions to the Burgers and linearized Burgers equations through the Hopf-Cole transformation. We devise a scheme of inner-outer expansion to study the nonlinear interaction of waves pertaining to the rarefaction characteristic field and other transversal fields. It is assumed that the associated hyperbolic conservation laws

$$\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0, \ \boldsymbol{u} \in \mathbb{R}^n, \tag{1.2}$$

is strictly hyperbolic

$$\begin{cases} \boldsymbol{F}'(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) = \lambda_i(\boldsymbol{u})\boldsymbol{r}_i(\boldsymbol{u}) \\ \boldsymbol{l}_i(\boldsymbol{u})\boldsymbol{F}'(\boldsymbol{u}) = \lambda_i(\boldsymbol{u})\boldsymbol{l}_i(\boldsymbol{u}), \\ \boldsymbol{l}_i(\boldsymbol{u})\boldsymbol{r}_j(\boldsymbol{u}) = \delta_i^j, \end{cases} \\ \lambda_1(\boldsymbol{u}) < \cdots < \lambda_n(\boldsymbol{u}), \tag{1.3}$$

and that each characteristic field is either genuinely nonlinear or linearly degenerate, [5]. An important physical system with both genuinely nonlinear and linear degenerate characteristic fields is the Euler equations in gas dynamics:

$$\begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho(u^2+e) \end{pmatrix}_t + \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \frac{1}{2}\rho u(u^2+e) + pu \end{pmatrix}_x = 0. \tag{1.4}$$

Here the pressure p as the function of the density ρ and the internal energy e is given as $p(\rho, e) = (\gamma - 1)\rho e, \ \gamma > 1$, for polytropic gases. We normalize the eigenvectors so that, for each $i \in 1, \cdots, n$, either

$$\begin{cases} \nabla\lambda_i(\boldsymbol{u}) \cdot \boldsymbol{r}_i(\boldsymbol{u}) = 1; \text{or}, \\ \nabla\lambda_i(\boldsymbol{u}) \cdot \boldsymbol{r}_i(\boldsymbol{u}) = 0. \end{cases} \tag{1.5}$$

The genuine nonlinearity is needed only for the characteristic field, say $\lambda_i(\boldsymbol{u})$, of the rarefaction wave $(\boldsymbol{u}_-, \boldsymbol{u}_+)$. For other characteristic fields, no assumption is needed. It is the values of $\nabla\lambda_j(\boldsymbol{u}) \cdot \boldsymbol{r}_j(\boldsymbol{u})$ at the end states $\boldsymbol{u}_-$ (or $\boldsymbol{u}_+$) for $j < i$ (or $j > i$) that matters. For each genuinely nonlinear characteristic field, there are the shock waves and rarefaction waves. These two types of waves are different in their stability property. The shock waves are compressive and have permanent exact form. Consequently, a shock wave is strongly nonlinear stable. On the other hand, a shock wave is orbital stable and its location needs to be identified exactly for the perturbation to decay around it. Thus the stability of a shock wave is necessarily locally in $L_1(x)$. The shock location can either be a priori determined through global conservation laws or be traced using local conservation laws, see [9], [10], [11], [13], and references therein. A rarefaction wave is expansive and remains stable when locally translated. The translation can be arbitrary. Thus a rarefaction wave is not stable locally in $L_1(x)$. There is no exact explicit representation of viscous rarefaction waves, though there is an accurate approximation of rarefaction waves for the system using the scalar Burgers equation. From the explicit approximation, it is clear that a rarefaction wave in general induces a strong coupling with other transversal fields. The basic coupling for the general systems can be expressed in terms of the coefficients

$$C^j_{kl}(\boldsymbol{u}) \equiv \frac{1}{2}\boldsymbol{l}_j \boldsymbol{F}''(\boldsymbol{u})(\boldsymbol{r}_k, \boldsymbol{r}_l).$$

The effective way to understand the strong coupling is through the explicit, accurate approximation of the rarefaction wave using the Burgers equation. There is a truncation error in the transversal j-characteristic field if the coupling coefficient $C^j_{ii} \neq 0$. As a consequence, the coupling of waves pertaining to different characteristics is strong. Another important feature is that the viscous rarefaction wave contains both the nonlinear hyperbolic expansion and the parabolic diffusion; the formal is linear while the later is sub-linear in time. There are many studies of nonlinear stability of rarefaction waves using the energy method, see [3], [14], [4], [1] and references therein. On the other hand, more quantitative description of the propagation of waves over a rarefaction wave requires deeper understanding of the coupling of waves. It is our goal to present the quantitative analysis for the intricate linear-nonlinear couplings that are induced by the presence of a rarefaction wave.

A basic understanding for the rarefaction waves is provided by the Burgers and the inviscid Burgers (Hopf) equations

$$u_t + uu_x = u_{xx}, \tag{1.6}$$

$$u_t + uu_x = 0. \tag{1.7}$$

Consider the rarefaction wave with Riemann initial data (u_-, u_+), $u_- < u_+$,

$$u(x,0) = \begin{cases} u_-, & \text{for } x < 0, \\ u_+, & \text{for } x > 0. \end{cases}$$

By a transformation of the independent variables (x,t), we may centralize the rarefaction wave, for some $\varepsilon > 0$,

$$u(x,0) = \begin{cases} -\varepsilon, & \text{for } x < 0, \\ \varepsilon, & \text{for } x > 0. \end{cases} \tag{1.8}$$

The inviscid rarefaction wave $B^i(x,t)$, the solution of (1.7) and(1.8), is self-similar:

$$B^i(x,t) \equiv \begin{cases} \varepsilon & \text{for } x > \varepsilon t, \\ x/t & \text{for } x \in (-\varepsilon t, \varepsilon t), \\ -\varepsilon & \text{for } x < -\varepsilon t. \end{cases} \tag{1.9}$$

The Burgers rarefaction wave B^v, the solution of (1.6) and (1.8) can be constructed by the Hopf-Cole transformation and has the explicit form:

$$\begin{aligned} B^v(x,t) \; &= \; \varepsilon \left(\int_{\frac{\varepsilon t - x}{2\sqrt{t}}}^{\infty} e^{-z^2} dz - e^{\varepsilon x} \int_{\frac{\varepsilon t + x}{2\sqrt{t}}}^{\infty} e^{-z^2} dz \right) \\ &\qquad \left(\int_{\frac{\varepsilon t - x}{2\sqrt{t}}}^{\infty} e^{-z^2} dz + e^{\varepsilon x} \int_{\frac{\varepsilon t + x}{2\sqrt{t}}}^{\infty} e^{-z^2} dz \right)^{-1}. \end{aligned} \tag{1.10}$$

In Section 2 we study the difference of these two rarefaction waves. These rarefaction waves are used for the construction of approximate rarefaction waves for the system. To highlight the primary coupling of waves pertaining the different characteristic directions, we thus derive in Section 2 a reduced, primary nonlinear system as coupling of the scalar heat and Burgers equations with sources.

The Green's function for the Burgers equation around its rarefaction wave can also be constructed explicitly by the Hopf-Cole transformation. The analysis of the dual nonlinear hyperbolic and linear parabolic dissipative properties of the Green's function, done in Section 3, is essential for the study of the general systems. The Green's function for the transverse field is accurately and explicitly constructed based on the characteristic method. The main part of the present paper, the study of the coupling of waves for the primary nonlinear system, is done in Section 4. Due to the strong coupling effects of the rarefaction wave on other, transversal characteristic fields, it is necessary to distinguish the two cases: The transversal characteristic field is linear degenerate or genuine nonlinear. In the case of linear degeneracy, the coupling effect give rise to a wave of the order of $(t+1)^{-1/2}\log(t+1)$. For this the usual Duhamel's principle suffices for the analysis. In the case of genuine nonlinearity, more precise information on the coupling effect using Hopf-Cole transformation is obtained. The rate is then $(t+1)^{-1/2}$. Direct analysis immediately suggests these two rates, see, for instance, [12]. However, the complete analysis for the genuinely nonlinear transversal field requires intricate analysis, see Remark 4.7 and also Remark 2.1. This paper presents the first definitive result on the pointwise estimates on the propagation of perturbation over a rarefaction wave. This faster rate is similar to that for the Burgers solution with the initial data

$$u(x,0) = \frac{1}{|x|+1}.$$

In Section 4 we devise an inner-outer scheme to extract the main part of the coupling of the rarefaction wave with a transversal genuinely nonlinear characteristic field. The main part is reduced to an initial value problem for the Burgers equation similar to the one just mentioned, (4.34), and can then be calculated explicitly using the Hopf-Cole transformation. For simplicity in presentation, we will carry out the analysis in Section 4 for 2×2 genuinely nonlinear systems. The complete analysis of the primary nonlinear system requires an interesting combination of the Hopf-Cole analysis and the Duhamel's principle. Identifying the above exact rates is necessary for closing the analysis when the other nonlinearities are put into consideration. This last step is done in the remaining sections. In Section 5, we consider the genuinely nonlinear systems of Section 4, starting with lemmas on some integrations that will arise later in the nonlinear coupling. Finally, in Section 6, we consider systems with both linearly degenerate and genuinely nonlinear transversal characteristic fields. For simplicity, we consider a system, whose associated hyperbolic conservation laws are the Euler equations in gas dynamics, (1.4).

Let $R_i(\lambda)$ be the integral curve of the right eigenvector $\boldsymbol{r}_i$ connecting the inviscid i-rarefaction wave $(\boldsymbol{u}_-,\boldsymbol{u}_+)$ of the system (1.2) and parametrized by the eigenvalue λ_i:

$$\begin{cases} \frac{d}{d\lambda}R_i(\lambda) \parallel \boldsymbol{r}_i(R_i(\lambda)), \\ \lambda_i(R_i(\lambda)) = \lambda, \\ \boldsymbol{u}_- = R_i(-\varepsilon),\ \boldsymbol{u}_+ = R_i(\varepsilon),\ \varepsilon > 0. \end{cases}$$

The inviscid rarefaction wave for the system (1.2) and the approximate viscous rarefaction wave for the system (1.1) are constructed based on the inviscid Burgers and Burgers rarefaction waves, (1.9), (1.10):

$$R_i(B^i(x,t));\ \boldsymbol{v}^a(x,t) \equiv R_i(B^v(x,t)).$$

For the i-rarefaction wave $(\boldsymbol{u}_-,\boldsymbol{u}_+)$, we set the relevant end states for the transversal characteristic set:

$$\boldsymbol{u}_j^0 \equiv \begin{cases} \boldsymbol{u}_-, & \text{for } j < i, \\ \boldsymbol{u}_+, & \text{for } j > i; \end{cases}$$

and the corresponding characteristic values

$$\lambda_j^0 \equiv \lambda_j(\boldsymbol{u}_j^0).$$

We divide the set of transversal characteristic fields $j \neq i$ into the disjoint union of Λ_g and Λ_l, with Λ_g the genuinely nonlinear set and Λ_l the linearly degenerate fields:

$$\begin{cases} \Lambda_g \equiv \{j|\ j \neq i,\ \lambda_j \quad \text{is genuinely nonlinear at } \boldsymbol{u}_i^0; \\ \Lambda_l \equiv \{j|\ j \neq i,\ \lambda_j \quad \text{is linearly degenerate at } \boldsymbol{u}_i^0\}. \end{cases}$$

Theorem 1.1. *Suppose that the strength ε of the inviscid i-rarefaction $(\boldsymbol{u}_-,\boldsymbol{u}_+)$ is sufficiently small. Then there is a solution $\boldsymbol{u}(x,t)$ of the system* (1.2) *satisfying*

$$|\boldsymbol{u}(x,t) - R_i(B^i(x,t))| = O(1)\varepsilon\left[\frac{1}{\sqrt{(x+\varepsilon t)^2+t}} + \frac{1}{\sqrt{(x-\varepsilon t)^2+t}}\right]$$

$$+O(1)\sum_{j<i}\frac{1}{\sqrt{(x-\lambda_j^0t)^2+t}}\chi_{[\lambda_j^0t,\varepsilon t]}+O(1)\sum_{j>i}\frac{1}{\sqrt{(x-\lambda_j^0t)^2+t}}\chi_{[-\varepsilon t,\lambda_j^0t]}$$
$$+O(1)\varepsilon\left(\frac{1}{\sqrt{t}}e^{-\frac{(x+\varepsilon t)^2}{5t}}+\frac{1}{\sqrt{t}}e^{-\frac{(x-\varepsilon t)^2}{5t}}+\sum_{j\in\Lambda_l}\frac{\log t}{\sqrt{t}}e^{-\frac{(x-\lambda_j^0t)^2}{5t}}+\sum_{j\in\Lambda_g}\frac{1}{\sqrt{t}}e^{-\frac{(x-\lambda_j^0t)^2}{5t}}\right).\tag{1.11}$$

Moreover, there exists $C>0$ *such that, for each* $j\in\Lambda_l$ *with* $C_{ii}^j\neq 0$,

$$|\boldsymbol{u}(x,t)-R_i(B^i(x,t))|\geq\varepsilon\frac{\log t}{C\sqrt{t}}\ \textit{for}\ |x-\lambda_j^0t|\leq\frac{\sqrt{t}}{C};\tag{1.12}$$

and, for each $j\in\Lambda_g$ *with* $C_{ii}^j\neq 0$,

$$|\boldsymbol{u}(x,t)-R_i(B^i(x,t))|\geq\varepsilon\frac{1}{C\sqrt{t}}\ \textit{for}\ |x-\lambda_j^0t|\leq\frac{\sqrt{t}}{C}.\tag{1.13}$$

Here, the function $\chi_{[a,b]}$ is the characteristic function for a given interval $[a,b]$, i.e.

$$\chi_{[a,b]}(x)=\begin{cases}1 & \text{if } x\in[a,b],\\ 0 & \text{else.}\end{cases}$$

For the theorem, we construct a solution of the system (1.1) which approaches the inviscid rarefaction wave $R_i(B^i(x,t))$ time asymptotically. It follows from our analysis that a nonlinear stability result for the rarefaction wave can also be formulated and proved. We do not carry out this; it is our goal in the present paper to emphasize the strong coupling that a rarefaction wave induces. This is reflected in the different decay rates for the genuinely nonlinear transversal fields $j\in\Lambda_g$ and for the linearly degenerate fields $j\in\Lambda_l$ in (1.11). In fact, in our analysis we obtain explicit expression of the leading terms which leads to the lower bound estimate (1.12) for linearly degenerate transversal fields and (1.13) for the genuinely nonlinear transversal fields. We will simplify the situation by taking particular initial data $\boldsymbol{u}(x,0)=\boldsymbol{v}^a(x,\varepsilon^{-2})\equiv R_i(B^v(x,\varepsilon^{-2}))$. Such an initial data avoid the singularity near the Riemann data and the appearance of the initial layer. This allows us to highlight the fact that the above rates are mainly the consequence of the strong coupling, rather than the perturbation through the initial data.

2. Preliminaries

The elementary waves for the conservation laws (1.1) and (1.2) can be approximated by the simple scalar equations, the inviscid and viscous Burgers equations, [6], [7], [8]. Such approximations make it definite the strong couplings of the the rarefaction wave with the transversal fields.

2.1. Scalar rarefaction waves

Consider the inviscid and viscous Burgers equations

$$(B^i)_t + \left(\frac{(B^i)^2}{2}\right)_x = 0, \tag{2.1}$$

$$(B^v)_t + \left(\frac{(B^v)^2}{2}\right)_x = (B^v)_{xx}, \tag{2.2}$$

with the Riemann initial data corresponding to rarefaction waves:

$$B^i(x,0) = B^v(x,0) = \begin{cases} -\varepsilon & \text{for } x < 0, \\ \varepsilon & \text{for } x > 0, \end{cases} \tag{2.3}$$

where ε is a given positive constant. For the inviscid Burgers equation, the Riemann rarefaction wave is the well-known self-similar solution:

$$B^i(x,t) = \begin{cases} \varepsilon & \text{for } x > \varepsilon t, \\ x/t & \text{for } x \in (-\varepsilon t, \varepsilon t), \\ -\varepsilon & \text{for } x < -\varepsilon t. \end{cases} \tag{2.4}$$

The rarefaction wave for the inviscid Burgers equation can be used for the construction of the rarefaction waves for the general convex scalar hyperbolic conservation law

$$u_t + f(u)_x = 0, \;\; f''(u) > 0,$$

by the relation

$$B^i(x,t) \equiv f'(u(x,t)).$$

The solution of the viscous rarefaction wave for Burgers equation is constructed explicitly by the Hopf-Cole transformation, [2],

$$\begin{aligned}
B^v(x,t) &= -2\partial_x \log W(x,t),\\
W(x,t) &\equiv e^{-\varepsilon^2 t/4}\left(\int_{-\infty}^0 k(x-y,t)e^{\varepsilon y/2}dy + \int_0^\infty k(x-y,t)e^{-\varepsilon y/2}dy\right)\\
&= \frac{1}{2}e^{-\frac{\varepsilon x}{2}}\left(\mathrm{Erfc}\left(\frac{\varepsilon t - x}{2\sqrt{t}}\right) + e^{\varepsilon x}\mathrm{Erfc}\left(\frac{\varepsilon t + x}{2\sqrt{t}}\right)\right); \qquad (2.5)\\
B^v(x,t) &= \varepsilon\left(\frac{\mathrm{Erfc}\left(\frac{\varepsilon t-x}{2\sqrt{t}}\right) - e^{\varepsilon x}\mathrm{Erfc}\left(\frac{\varepsilon t+x}{2\sqrt{t}}\right)}{\mathrm{Erfc}\left(\frac{\varepsilon t-x}{2\sqrt{t}}\right) + e^{\varepsilon x}\mathrm{Erfc}\left(\frac{\varepsilon t+x}{2\sqrt{t}}\right)}\right), \qquad (2.6)
\end{aligned}$$

where $k(x,t)$ is the heat kernel and Erfc is the error function:

$$k(x,t) \equiv \frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}}, \ \ \mathrm{Erfc}(v) \equiv \frac{2}{\sqrt{\pi}}\int_v^\infty e^{-z^2}dz. \qquad (2.7)$$

From these, by direct computations, for $t \geq \varepsilon^{-2}$

$$\begin{aligned}
&|B^v(x,t) - B^i(x,t)|\\
&= O(1)\begin{cases}
\dfrac{e^{-\frac{(x+\varepsilon t)^2}{4t}}}{\sqrt{t}} & \text{for } x \leq -\varepsilon t + \sqrt{t},\\
\frac{1}{|x+\varepsilon t|} + \frac{1}{|x-\varepsilon t|} & \text{for } x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t}),\\
\dfrac{e^{-\frac{(-x+\varepsilon t)^2}{4t}}}{\sqrt{t}} & \text{for } x \geq \varepsilon t - \sqrt{t},
\end{cases} \qquad (2.8)\\
&|B_x^v(x,t) - B_x^i(x,t)|\\
&= O(1)\begin{cases}
\dfrac{e^{-\frac{(x+\varepsilon t)^2}{4t}}}{t} & \text{for } x < -\varepsilon t + \sqrt{t},\\
\dfrac{1}{t+(x+\varepsilon t)^2} + \dfrac{1}{t+(x-\varepsilon t)^2} & \text{for } x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t}),\\
\dfrac{e^{-\frac{(x-\varepsilon t)^2}{4t}}}{t} & \text{for } x > \varepsilon t - \sqrt{t},
\end{cases} \qquad (2.9)
\end{aligned}$$

and

$$|B^v_{xx}(x,t)| = O(1)\begin{cases} \dfrac{e^{-\frac{(x+\varepsilon t)^2}{4t}}}{\sqrt{t^3}} & \text{for } x < -\varepsilon t + \sqrt{t}, \\ \dfrac{1}{\sqrt{(t+(x+\varepsilon t)^2)^3}} + \dfrac{1}{\sqrt{(t+(x-\varepsilon t)^2)^3}} & \text{for } x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t}), \\ \dfrac{e^{-\frac{(x-\varepsilon t)^2}{4t}}}{\sqrt{t^3}} & \text{for } x > \varepsilon t - \sqrt{t}, \end{cases} \tag{2.10}$$

and for $t \in (0, \varepsilon^{-2})$

$$|B^v_x(x,t)| = O(1)\varepsilon \frac{e^{-\frac{(x+\varepsilon t)^2}{4t}} + e^{-\frac{(x-\varepsilon t)^2}{4t}}}{\sqrt{1+t}}, \tag{2.11}$$

$$|B^v_{xx}(x,t)| = O(1)\varepsilon \frac{e^{-\frac{(x+\varepsilon t)^2}{4t}} + e^{-\frac{(x-\varepsilon t)^2}{4t}}}{1+t}. \tag{2.12}$$

Remark 2.2. The estimate (2.8) is linear inside the rarefaction wave and degenerates to sublinear, $t^{-1/2}$, at the edge of the rarefaction wave. These rates can be generalized to the system in the characteristic region of the rarefaction wave, see the first term on the R.H.S. of (1.11). These rates are optimal. There is a continuum of parameters to translate a rarefaction wave, and unlike the shock wave, no single translation can accommodate the initial perturbation to obtain higher rate. The rate inside the rarefaction wave is higher due to the hyperbolic expansion at linear rate in t. The unattainable higher rate of $t^{-1}\log t$ has been claimed on the edge of the rarefaction wave in [12].

2.2. Approximate rarefaction waves for systems

For the hyperbolic system

$$\partial_t \boldsymbol{u} + \partial_x \boldsymbol{F}(\boldsymbol{u}) = 0, \tag{2.13}$$

the rarefaction waves exist for genuinely nonlinear fields, the first case of (1.5). The inviscid Burgers rarefaction wave $B^i(x,t)$ can be used to construct the exact rarefaction waves $(\boldsymbol{u}_-, \boldsymbol{u}_+)$ for the system by setting

$$\boldsymbol{u}(x,t) = R_i(B^i(x,t)),$$

where R_i is the integral curve of the i-th characteristic direction

$$\frac{dR_i(\lambda)}{d\lambda} \parallel \boldsymbol{r}_i(R_i(\lambda)).$$

The parameter, the i-th characteristic value $\lambda = \lambda_i(R_i(\boldsymbol{u}))$, is the inviscid Burgers rarefaction wave (2.4). Similarly, we use the Burgers rarefaction wave $B^v(x,t)$, (2.6), to construct, not exact, but an approximate, viscous rarefaction wave for the system:

$$\boldsymbol{v}^a(x,t) \equiv R_i(B^v(x,t)). \tag{2.14}$$

Direct calculations yield the truncation error:

$$\begin{cases} \partial_t \boldsymbol{v}^a + \partial_x \boldsymbol{F}(\boldsymbol{v}^a) - \partial_x^2 \boldsymbol{v}^a = -(B_x^v)^2 R_i''(B^v). \\ \lim\limits_{x\to-\infty} \boldsymbol{v}^a(x,t) = R_i(\varepsilon);\ \lim\limits_{x\to\infty} \boldsymbol{v}^a(x,t) = R_i(-\varepsilon). \end{cases} \tag{2.15}$$

Denote the perturbation

$$\vec{\zeta}(x,t) \equiv \boldsymbol{v}(x,t) - \boldsymbol{v}^a(x,t),$$

where $\boldsymbol{v}$ is a solution of

$$\boldsymbol{v}_t + \boldsymbol{F}(\boldsymbol{v})_x - \boldsymbol{v}_{xx} = 0.$$

We are interested in the interplay of the nonlinear hyperbolicity, the linear rate of expansion, (2.4), and the sublinear rate of dissipation of heat kernel type. For this we will avoid the singularity at $t = 0$ and start at time ε^{-2}, and, to simplify the presentation, we assume that the initial data of the perturbation $\vec{\zeta}$ is identically equal to zero at $t = \varepsilon^{-2}$:

$$\vec{\zeta}(x, \varepsilon^{-2}) \equiv 0. \tag{2.16}$$

Our analysis for this situation can be applied straightforwardly to more general perturbations. With (2.16), our perturbation $\vec{\zeta}$ satisfies

$$\begin{cases} \vec{\zeta}_t + (\boldsymbol{F}'(\boldsymbol{v}^a)\vec{\zeta})_x - \vec{\zeta}_{xx} + N_x = (B^v_x)^2 R''_i(B^v), \\ \vec{\zeta}(x, \varepsilon^{-2}) = 0, \end{cases} \tag{2.17}$$

where

$$N \equiv \boldsymbol{F}(\boldsymbol{v}^a + \vec{\zeta}) - \boldsymbol{F}(\boldsymbol{v}^a) - \boldsymbol{F}'(\boldsymbol{v}^a)\vec{\zeta}.$$

The nonlinear term $N(x,t)$ can be expanded component-wise as

$$\begin{cases} N(x,t) &= \sum_{j=1}^{n} N^j(x,t)\boldsymbol{r}_j(x,t), \\ N^j(x,t) &\equiv \sum_{1\le k,l\le n} C^j_{kl}\zeta^k\zeta^l + O(1)|\vec{\zeta}|^3, \\ C^j_{kl}(x,t) &\equiv \frac{1}{2}\boldsymbol{l}_j \boldsymbol{F}''(\boldsymbol{v}^a)(\boldsymbol{r}_k, \boldsymbol{r}_l), \\ \boldsymbol{l}_j(x,t) &\equiv \boldsymbol{l}_j(\vec{\boldsymbol{v}}^a(x,t)), \boldsymbol{r}_j(x,t) \equiv \boldsymbol{r}_j(\vec{\boldsymbol{v}}^a(x,t)), \\ \vec{\zeta}(x,t) &\equiv \sum_{l=1}^{n} \zeta^l(x,t)\boldsymbol{r}_l(x,t). \end{cases} \tag{2.18}$$

2.3. Primary nonlinear system

The main task is to understand the nonlinear coupling of waves pertaininig to distinct characteristic fields. For this, for the rest of this section and much of remaining of the present paper, we will make the assumption that the system (1.2) is 2×2 and genuinely nonlinear in both characteristic fields. Thus one has that $C^l_{ll} \neq 0$ for $l = 1, 2$. With normalization, we can assume C^l_{ll}:

$$C^l_{ll}(x,t) = \frac{1}{2} + O(1)(B^v(x,t) - \varepsilon) \text{ for } l = 1, 2. \tag{2.19}$$

By a suitable rescaling, we may assume that

$$\begin{cases} \lambda_1(\boldsymbol{v}^a(x,t)) = B^v(x,t), \\ \lambda_2(\boldsymbol{v}^a(x,t)) = 1 + O(1)(B^v(x,t) - \varepsilon). \end{cases} \tag{2.20}$$

We introduce a **diagonal nonlinear system** $\vec{\zeta}_D$ as an auxiliary system to (2.17)

$$\begin{cases}\vec{\zeta}_{Dt} + (\boldsymbol{F}'(\boldsymbol{v}^a)\vec{\zeta}_D)_x - \vec{\zeta}_{Dxx} + N^D_x = (B^v_x)^2 R''_i(\boldsymbol{v}^a),\\ \vec{\zeta}_D(x,\varepsilon^{-2}) = 0,\end{cases} \tag{2.21}$$

where

$$\begin{cases} N^D(x,t) \equiv \sum_{l=1}^{2} C^l_{ll}(x,t)(\zeta^l_D)^2 \boldsymbol{r}_l(x,t),\\ \vec{\zeta}_D(x,t) \equiv \sum_{l=1}^{2} \zeta^l_D(x,t)\boldsymbol{r}_l(x,t).\end{cases} \tag{2.22}$$

One can decompose the system (2.21) for $\vec{\zeta}_D(x,t)$ as follows

$$\begin{cases}\zeta^1_{Dt} + (B^v\zeta^1_D)_x - \zeta^1_{Dxx} + (C^1_{11}(\zeta^1_D)^2)_x\\ \quad = \boldsymbol{l}_1\left(\sum_{k=1}^{2}\boldsymbol{r}_{kx}(2\zeta^k_{Dx} - C^k_{kk}(\zeta^k_D)^2) - \boldsymbol{r}_{2x}(\lambda_2 - B^v)\zeta^2_D + \boldsymbol{r}_{kxx}\zeta^k_D + (B^v_x)^2 R''_1\right),\\ \zeta^2_{Dt} + (\lambda_2\zeta^2_D)_x - \zeta^2_{Dxx} + (C^2_{22}(\zeta^2_D)^2)_x\\ \quad = \boldsymbol{l}_2\left(\sum_{k=1}^{2}\boldsymbol{r}_{kx}(2\zeta^k_{Dx} - C^k_{kk}(\zeta^k_D)^2) - \boldsymbol{r}_{2x}(\lambda_2 - B^v)\zeta^2_D + \boldsymbol{r}_{kxx}\zeta^k_D + (B^v_x)^2 R''_1\right),\\ \vec{\zeta}_D(x,\varepsilon^{-2}) = 0.\end{cases} \tag{2.23}$$

It turns out that the RHS of (2.23) is dominated by $|B^v_x|^2$ and the primary system is a decoupled nonlinear system with the inhomogeneous terms $\boldsymbol{l}_k R''_1 (B^v_x)^2$. In fact, we will see later that the other terms on the RHS of (2.23) have the faster decay of

$$|\vec{\zeta}_D B^v_{xx}|, |\partial_x \vec{\zeta}_D B^v_x|, |\zeta^2_D B^v_x| \ll 1/t^2 \text{ for } |x| \le \varepsilon t. \tag{2.24}$$

This leads us to the consideration of the **primary nonlinear system**:

$$\begin{cases}\zeta^1_{pt} + (B^v\zeta^1_p)_x - \zeta^1_{pxx} + (C^1_{11}(\zeta^1_p)^2)_x = (\boldsymbol{l}_1 R''_1)(B^v_x)^2, \text{ (primary field)}\\ \zeta^2_{pt} + \lambda^*_2\zeta^2_{px} + d(x,t)\zeta^2_p - \zeta^2_{pxx} + (C^2_{22}(\zeta^2_p)^2)_x = (\boldsymbol{l}_2 R''_1)(B^v_x)^2, \text{ (transverse field)},\\ \qquad \zeta^l_p(x,\varepsilon^{-2}) \equiv 0 \text{ for } l = 1,2,\end{cases} \tag{2.25}$$

where

$$\begin{cases} \lambda_2^*(x,t) \equiv \lambda_2(R_1(B^i(x,t))), \\ d(x,t) \equiv \Big(\lambda_{2x}^* + (\lambda_2^* - B^i)\boldsymbol{l}_2(R_1(B^i))\boldsymbol{r}_{2x}(R_1(B^i))\Big), \end{cases} \tag{2.26}$$

are the 2-characteristics of the hyperbolic rarefaction wave, (2.4), (2.19) and (2.20). Corresponding to the second equation of (2.25) for the transverse field, we introduce an auxiliary linear hyperbolic equation

$$\begin{cases} z_t + \lambda_2^* z_x + dz = (\boldsymbol{l}_2^* R_1''(B^i))(B_x^i)^2, \ \boldsymbol{l}_2^*(x,t) \equiv \boldsymbol{l}_2(R_1(B^i(x,t))), \\ z(-\varepsilon t, t) = 0. \end{cases} \tag{2.27}$$

Remark 2.3. The system (2.25) contain Burgers nonlinearity in both characteristic fields. For the first characteristic field, the nonlinearity is encoded in the transport term $(B^v\zeta_p^1)_x$ over the Burgers rarefaction wave B^v, and the term $(C_{11}^1(\zeta_p^1)^2)_x$ can be viewed as the source. For the second characteristic field the term $(C_{22}^2(\zeta_p^2)^2)_x$ should be taken as Burgers nonlinearity and dealt with by Hopf-Cole technique. In both characteristic fields, there are the nonlinear sources $(\boldsymbol{l}_1 R_1'')(B_x^v)^2$ and $(\boldsymbol{l}_2 R_1'')(B_x^v)^2$. Thus in solving (2.25), the analysis has to be essentially nonlinear in both characteristic fields. In contrast to this, for the study of the shock stability, the method for handling the transversal fields is basically linear.

Let $\mathbb{G}(x,t;y,s)$ be the Green's function of the Burgers equation linearized around B^v, see (3.4) in Section 3. We have from the first equation in (2.25)

$$\begin{aligned} \zeta_p^1(x,t) \ = \ \int_{\varepsilon^{-2}}^t \int_{\mathbb{R}} \mathbb{G}(x,t;y,s)\Big(& -(C_{11}^1(y,s)\zeta_p^1(y,s)^2)_y \\ & + B_y^v(y,s)^2\boldsymbol{l}_1(y,s)R_1''(B^v(y,s))\Big)dyds. \end{aligned} \tag{2.28}$$

2.4. Secondary nonlinear system

The secondary waves, the solution minus the primary waves

$$\begin{cases} \vec{\xi}(x,t) \equiv \vec{\zeta} - \sum_{j=1}^2 \zeta_p^l(x,t)\boldsymbol{r}_l(x,t), \\ \xi^k(x,t) \equiv \boldsymbol{l}^k(x,t)\cdot\vec{\xi}(x,t) \text{ for } k=1,2, \end{cases} \tag{2.29}$$

satisfies the following **secondary nonlinear system**:

$$
\begin{cases}
\xi^1_t+(B^v\xi^1)_x-\xi^1_{xx}+\Big(C^1_{11}\xi^1(2\xi^1+\zeta^1_p)+\sum\limits_{(j,k,l)\neq(1,1,1)}C^l_{jk}(\xi^j+\zeta^j_p)(\xi^k+\zeta^k_p)\Big)_x \\
=\boldsymbol{l}_1\Big(\sum\limits_{k=1}^{2}\boldsymbol{r}_{kx}\Big(2(\zeta^k_p+\xi^k)_x-\sum\limits_{1\le l,m\le 2}C^k_{lm}(\zeta^l_p+\xi^l)(\zeta^m_p+\xi^m)\Big) \\
\qquad -\boldsymbol{r}_{2x}(\lambda_2-B^v)(\zeta^2_p+\xi^2)+\boldsymbol{r}_{kxx}(\zeta^k_p+\xi^k)\Big), \\
\xi^2_t+(\lambda_2\xi^2)_x+(\lambda_2-B^v)\xi^2\boldsymbol{l}_2\boldsymbol{r}_{2x}-\xi^2_{xx} \\
+\Big(C^2_{22}\xi^2(2\xi^2+\zeta^2_p)+\sum_{(j,k,l)\neq(2,2,2)}C^l_{jk}(\xi^j+\zeta^j_p)(\xi^k+\zeta^k_p)\Big)_x \\
=(\lambda_2-\lambda^*_2)\zeta^2_{px}+(\lambda_{2x}-\lambda^*_{2x})\zeta^2_p \\
+\boldsymbol{l}_2\Big(\sum\limits_{k=1}^{2}2\boldsymbol{r}_{kx}\Big((\xi^k+\zeta^k_p)_x-\sum\limits_{1\le l,m\le 2}C^k_{lm}(\zeta^l_p+\xi^l)(\zeta^m_p+\xi^m)\Big)+\boldsymbol{r}_{kxx}(\xi^k+\zeta^k_p)\Big), \\
\vec{\xi}(x,\varepsilon^{-2})=0.
\end{cases}
\tag{2.30}
$$

Remark 2.4. One can rewrite (2.30) in the form

$$
\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix}_t+\left(\begin{pmatrix}B^v & 0\\ 0 & \lambda_2\end{pmatrix}\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix}_t\right)_x-\begin{pmatrix}\xi^1\\ \xi^2\end{pmatrix}_{xx}=L\begin{pmatrix}\xi^1+\xi^1_p\\ \xi^2+\xi^2_p\end{pmatrix}B^v_x+H.O.T.
$$

Here, the matrix L is the form

$$
L=\begin{pmatrix}0 & O(1)\\ 0 & O(1)\end{pmatrix}.
$$

It is important that the first element in the matrix is zero, meaning that the source for the rarefaction field comes only from the transversal field. This is due to the fact that the approximate rarefaction wave $R_1(B^v(x,t))$ has been put into the construction of the vector $\boldsymbol{r}_1(x,t)=\boldsymbol{r}_1(R_1(B^v(x,t)))$. In other word, the structure of the hyperbolic rarefaction wave has already been used.

3. The Green's Functions

The Green's function for the system is approximated by two explicitly constructed scalar Green's functions. For the characteristic field associated with the rarefaction wave, we use the exact Burgers Green's function for the construction. For construction of the Green's function for the transversal characteristic field we use the basic characteristic method.

3.1. Burgers Green's functions

The Burgers equation linearized around the viscous rarefaction wave $B^v(x,t)$ is

$$v_t + (B^v(x,t)v)_x = v_{xx}, \tag{3.1}$$

and the equation for the anti-derivative variable $w_x(x,t) \equiv v(x,t)$ is

$$w_t + B^v(x,t)w_x = w_{xx}. \tag{3.2}$$

We consider the corresponding two types of Green's function:

$$\begin{cases} G_t(x,t;y,s) + B^v(x,t)G_x(x,t;y,s) = G_{xx}(x,t;y,s) \text{ for } x \in \mathbb{R}, t > s, \\ G(x,s;y,s) = \delta(x-y), \end{cases} \tag{3.3}$$

$$\begin{cases} \mathbb{G}_t(x,t;y,s) + (B^v(x,t)\mathbb{G})_x(x,t;y,s) = \mathbb{G}_{xx}(x,t;y,s) \text{ for } x \in \mathbb{R}, t > s, \\ \mathbb{G}(x,s;y,s) = \delta(x-y), \end{cases} \tag{3.4}$$

With the function $W(x,t)$ in (2.5) from the Hopf-Cole transformation, one can apply the linear version of the Hopf-Cole transformation to (3.2),

$$V(x,t) = W(x,t)w(x,t),$$

and the function $V(x,t)$ satisfies

$$V_t + \frac{\varepsilon^2}{4}V = V_{xx}.$$

This leads to the representation of the solution of (3.2):

$$w(x,t) = \int_{\mathbb{R}} \frac{W(y,0)}{W(x,t)} e^{-\frac{\varepsilon^2 t}{4}} k(x-y,t) w(y,0) dy. \tag{3.5}$$

The Green's function $G(x,t;y,s)$ is therefore given by

$$G(x,t;y,s) \equiv \frac{W(y,s)}{W(x,t)} e^{-\frac{\varepsilon^2 (t-s)}{4}} k(x-y,t-s). \tag{3.6}$$

By the property that $w_x(x,t) = v(x,t)$, the Green's function $\mathbb{G}(x,t;y,s)$ is obtained by the relation

$$\partial_y \mathbb{G}(x,t;y,s) = -\partial_x G(x,t;y,s). \tag{3.7}$$

By straightforward calculations, we have the following expansion of the error function

$$\frac{\sqrt{\pi}}{2}\mathrm{Erfc}(v) = \int_v^\infty e^{-z^2} dz = e^{-v^2}\left(\frac{1}{2v} - \frac{1}{4v^3} + \frac{3}{8v^5} + O(1)\frac{1}{v^7}\right) \text{ for } v \geq 1, \tag{3.8}$$

and use it to expand components for $W(x,t)$ in (2.5) to yield

$$W(x,t) = \begin{cases} O(1)e^{\frac{\varepsilon x}{2}} & \text{for } x \leq -\varepsilon t + \sqrt{t}, \\ \left(\frac{\sqrt{t}}{(\varepsilon t + x)} + \frac{\sqrt{t}}{(\varepsilon t - x)} + \frac{O(1)t}{(\varepsilon t - x)^2} + \frac{O(1)t}{(\varepsilon t + x)^2}\right) e^{-\frac{\varepsilon^2 t}{4} - \frac{x^2}{4t}} & \\ & \text{for } x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t}), \\ O(1)e^{-\frac{\varepsilon x}{2}} & \text{for } x \geq \varepsilon t - \sqrt{t}. \end{cases} \tag{3.9}$$

This and (3.6) yield the estimates for the Green's function for $t, s \geq \varepsilon^{-2}$

$$
\begin{aligned}
&G(x,t;y,s)\\
&= O(1)\begin{cases}
\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{4(t-s)}}}{\sqrt{t-s}} & \text{for } x \leq -\varepsilon t+\sqrt{t}, y \leq -\varepsilon s+\sqrt{s},\\
(\frac{\sqrt{s}}{y+\varepsilon s}+\frac{\sqrt{s}}{\varepsilon s-y})\frac{e^{-\frac{t(y-sx/t)^2}{4s(t-s)}}e^{-\frac{(x+\varepsilon t)^2}{4t}}}{\sqrt{(t-s)}} & \\
\qquad \text{for } x \leq -\varepsilon t+\sqrt{t}, y \in (-s\varepsilon+\sqrt{s}, \varepsilon s-\sqrt{s}),\\
\frac{e^{-\frac{(x+\varepsilon t)^2}{4t}}e^{-\frac{s(x-\varepsilon t)^2}{4t(t-s)}}e^{-\frac{(y-\varepsilon s)^2}{4(t-s)}}e^{\frac{(x-\varepsilon t)(y-\varepsilon s)}{2(t-s)}}}{\sqrt{(t-s)}} & \\
\qquad \text{for } x \leq -\varepsilon t+\sqrt{t},\ y \geq \varepsilon s-\sqrt{s},\\
\frac{1}{\frac{\sqrt{t}}{x+\varepsilon t}+\frac{\sqrt{t}}{\varepsilon t-x}}\frac{e^{\frac{(x+\varepsilon t)(y+\varepsilon s)}{2(t-s)}}e^{-\frac{(y+\varepsilon s)^2}{4(t-s)}}e^{\frac{-s(x+\varepsilon t)^2}{4t(t-s)}}}{\sqrt{t-s}} & \\
\qquad \text{for } x \in (-\varepsilon t+\sqrt{t}, \varepsilon t+\sqrt{t}),\ y \leq -\varepsilon s+\sqrt{s},\\
\frac{\frac{\sqrt{s}}{y+\varepsilon s}+\frac{\sqrt{s}}{\varepsilon s-y}}{\frac{\sqrt{t}}{x+\varepsilon t}+\frac{\sqrt{t}}{\varepsilon t-x}}\frac{e^{-\frac{t(y-sx/t)^2}{4s(t-s)}}}{\sqrt{(t-s)}} & \\
\qquad \text{for } x \in (-\varepsilon t+\sqrt{t}, \varepsilon t+\sqrt{t}), y \in (-\varepsilon s+\sqrt{s}, \varepsilon s+\sqrt{s}),\\
\frac{1}{\frac{\sqrt{t}}{x+\varepsilon t}+\frac{\sqrt{t}}{\varepsilon t-x}}\frac{e^{\frac{-s(x-\varepsilon t)^2}{4t(t-s)}}e^{-\frac{(y-\varepsilon s)^2}{4(t-s)}}e^{\frac{(x-\varepsilon t)(y-\varepsilon s)}{2(t-s)}}}{\sqrt{t-s}} & \\
\qquad \text{for } x \in (-\varepsilon t+\sqrt{t}, \varepsilon t+\sqrt{t}),\ y \geq \varepsilon s-\sqrt{s},\\
\frac{e^{\frac{(x+\varepsilon t)(y+\varepsilon s)}{2(t-s)}}e^{\frac{-(y+\varepsilon s)^2}{4s}}e^{\frac{-s(x+\varepsilon t)^2}{4t(t-s)}}e^{-\frac{(x-\varepsilon t)^2}{4t}}}{\sqrt{t-s}} & \\
\qquad \text{for } x \geq \varepsilon t-\sqrt{t}, y < -\varepsilon s+\sqrt{s},\\
(\frac{\sqrt{s}}{y+\varepsilon s}+\frac{\sqrt{s}}{\varepsilon s-y})\frac{e^{\frac{-t(y-sx/t)^2}{4s(t-s)}}e^{-\frac{(x-\varepsilon t)^2}{4t}}}{\sqrt{t-s}} & \\
\qquad \text{for } x \geq \varepsilon t+\sqrt{t},\ y \in (-\varepsilon s+\sqrt{s}, \varepsilon s-\sqrt{s}),\\
\frac{e^{-\frac{(x-y-\varepsilon(t-s))^2}{4(t-s)}}}{\sqrt{t-s}} & \text{for } x \geq \varepsilon t-\sqrt{t},\ y \geq \varepsilon s-\sqrt{s}.
\end{cases}
\end{aligned}
\tag{3.10}
$$

Lemma 3.5. *The Green's function $G(x,t;y,s)$ satisfies the following estimates for $t, s \geq \varepsilon^{-2}$:*

For $x \le -\varepsilon t - \sqrt{t}$,

$$
|G_x(x,t;y,s)| \le O(1) \begin{cases} \dfrac{1}{(t-s)} e^{-\frac{(x-y+\varepsilon(t-s))^2}{5(t-s)}} & \text{for } y \le -\varepsilon s + \sqrt{s}, \\ \dfrac{\left(\frac{\sqrt{s}}{y+\varepsilon s} + \frac{\sqrt{s}}{-y+\varepsilon s}\right)}{(t-s)} e^{-\frac{(x+\varepsilon t)^2}{5t}} e^{-\frac{t(y-\frac{x}{t}s)^2}{5s(t-s)}} & \\ & \text{for } y \in [-\varepsilon s + \sqrt{s}, \varepsilon s - \sqrt{s}] \\ \dfrac{1}{(t-s)} e^{-\frac{(x+\varepsilon t)^2}{5t}} e^{-\frac{s(x-\varepsilon t)^2}{5t(t-s)}} e^{-\frac{(y-\varepsilon s)^2}{5(t-s)}} e^{\frac{(x-\varepsilon t)(y-\varepsilon s)}{5(t-s)}} & \\ & \text{for } y \ge \varepsilon s - \sqrt{s}. \end{cases} \tag{3.11}
$$

For $x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t})$,

$$
\begin{aligned} &|G_x(x,t;y,s)| \\ &\le O(1) \begin{cases} \dfrac{\frac{|(x+\varepsilon t)|s}{t} + |y+\varepsilon s|}{\left(\frac{\sqrt{t}}{x+\varepsilon t} + \frac{\sqrt{t}}{-x+\varepsilon t}\right)(t-s)^{3/2}} e^{-\frac{s(x+\varepsilon t)^2}{5t(t-s)}} e^{-\frac{(y+\varepsilon s)^2}{5(t-s)}} e^{\frac{(x+\varepsilon t)(y+\varepsilon s)}{5(t-s)}} & \\ & \text{for } y \le -\varepsilon s + \sqrt{s}, \\ \dfrac{|\frac{xs}{t} - y| \left(\frac{\sqrt{s}}{y+\varepsilon s} + \frac{\sqrt{s}}{-y+\varepsilon s}\right)}{\left(\frac{\sqrt{t}}{x+\varepsilon t} + \frac{\sqrt{t}}{-x+\varepsilon t}\right)(t-s)^{3/2}} e^{-\frac{t(y-\frac{x}{t}s)^2}{5s(t-s)}} & \\ & \text{for } y \in [-\varepsilon s + \sqrt{s}, \varepsilon s - \sqrt{s}] \\ \dfrac{\frac{|(x-\varepsilon t)|s}{t} + |y-\varepsilon s|}{\left(\frac{\sqrt{t}}{x+\varepsilon t} + \frac{\sqrt{t}}{-x+\varepsilon t}\right)(t-s)^{3/2}} e^{-\frac{s(x-\varepsilon t)^2}{5t(t-s)}} e^{-\frac{(y-\varepsilon s)^2}{5(t-s)}} e^{\frac{(x-\varepsilon t)(y-\varepsilon s)}{5(t-s)}} & \\ & \text{for } y \ge \varepsilon s - \sqrt{s}, \end{cases} \end{aligned} \tag{3.12}
$$

For $x \ge \varepsilon t + \sqrt{t}$,

$$
\begin{aligned} &|G_x(x,t;y,s)| \\ &\le O(1) \begin{cases} \dfrac{1}{(t-s)} e^{-\frac{(x-\varepsilon t)^2}{5t}} e^{-\frac{s(x-\varepsilon t)^2}{5t(t-s)}} e^{-\frac{(y+\varepsilon s)^2}{5(t-s)}} e^{\frac{(x+\varepsilon t)(y+\varepsilon s)}{5(t-s)}} & \\ & \text{for } y \le -\varepsilon s + \sqrt{s}, \\ \dfrac{\left(\frac{\sqrt{s}}{y+\varepsilon s} + \frac{\sqrt{s}}{-y+\varepsilon s}\right)}{(t-s)} e^{-\frac{(x-\varepsilon t)^2}{5t}} e^{-\frac{t(y-\frac{x}{t}s)^2}{5s(t-s)}} & \\ & \text{for } y \in [-\varepsilon s + \sqrt{s}, \varepsilon s - \sqrt{s}] \\ \dfrac{1}{(t-s)} e^{-\frac{(x-y-\varepsilon(t-s))^2}{5t}} & \text{for } y \ge \varepsilon s - \sqrt{s}. \end{cases} \end{aligned} \tag{3.13}
$$

Proof. From (3.6) and (2.5) one has that

$$G_x(x,t;y,s)=\left(\frac{1}{2}B^v(x,t)-\frac{x-y}{2(t-s)}\right)\frac{W(x,t)}{W(y,s)}\frac{e^{-\frac{(x-y)^2}{4(t-s)}-\varepsilon^2\frac{(t-s)}{4}}}{\sqrt{4\pi(t-s)}}. \tag{3.14}$$

By (3.14), (3.9), and $B^v\sim-\varepsilon$ for $x\le\varepsilon t$, one has the following three cases.

For $x<-\varepsilon t-\sqrt{t}$ and $y<-\varepsilon s-\sqrt{s}$,

$$\begin{aligned}G_x(x,t;y,s) &= O(1)\left(-\frac{\varepsilon(t-s)+(x-y)}{2(t-s)}\right)\frac{W(y,s)}{W(x,t)}\frac{e^{-\frac{(x-y)^2}{4(t-s)}-\varepsilon^2\frac{(t-s)}{4}}}{\sqrt{4\pi(t-s)}}\\ &= O(1)\left(\frac{\varepsilon(t-s)+(x-y)}{-2(t-s)}\right)\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\\ &\le O(1)\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{5(t-s)}}}{(t-s)}. \end{aligned}\tag{3.15}$$

For $x<-\varepsilon t-\sqrt{t}$ and $y\in(-\varepsilon s+\sqrt{s},\varepsilon s-\sqrt{s})$,

$$\begin{aligned}&G_x(x,t;y,s)\\ &= O(1)\left(-\frac{\varepsilon(t-s)+(x-y)}{2(t-s)}\right)e^{-\frac{(y+\varepsilon s)^2}{4s}}\left(\frac{\sqrt{s}}{\varepsilon s+y}+\frac{\sqrt{s}}{\varepsilon s-y}\right)\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\\ &\le O(1)e^{-\frac{(y+\varepsilon s)^2}{5s}}\left(\frac{\sqrt{s}}{\varepsilon s+y}+\frac{\sqrt{s}}{\varepsilon s-y}\right)\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{5(t-s)}}}{(t-s)}\\ &\le O(1)\left(\frac{\sqrt{s}}{\varepsilon s+y}+\frac{\sqrt{s}}{\varepsilon s-y}\right)\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}e^{-t\frac{(y-\frac{x}{t}s)^2}{5(t-s)s}}}{(t-s)}. \end{aligned}\tag{3.16}$$

For $x<-\varepsilon t-\sqrt{t}$ and $y\ge\varepsilon s+\sqrt{s}$,

$$\begin{aligned}G_x(x,t;y,s) &= O(1)\left(-\frac{\varepsilon(t-s)+(x-y)}{2(t-s)}\right)e^{-\varepsilon y}\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{4(t-s)}}}{\sqrt{4\pi(t-s)}}\\ &\le O(1)e^{-\frac{\varepsilon y}{5/4}}\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{5(t-s)}}}{(t-s)}\\ &\le O(1)\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}e^{-t\frac{(y-\frac{x}{t}s)^2}{5(t-s)s}}e^{\frac{(y-\varepsilon s)^2}{5s}}}{(t-s)}\end{aligned}$$

$$\leq O(1) \frac{e^{-\frac{(x+\varepsilon t)^2}{5t}} e^{-s\frac{(x-\varepsilon t)^2}{5(t-s)t}} e^{-\frac{(y-\varepsilon s)^2}{5(t-s)}} e^{-\frac{(x-\varepsilon t)(y-\varepsilon s)}{5}}}{(t-s)}. \quad (3.17)$$

Similar to the above three cases, one uses $B^v \sim x/t$ for $x \in (-\varepsilon t, \varepsilon t)$ to yields the following two cases.

For $x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t})$ and $y \leq -\varepsilon s - \sqrt{s}$,

$$G_x(x,t;y,s) = O(1)\left(\frac{x}{2t} - \frac{x-y}{2(t-s)}\right) \frac{W(y,s)}{W(x,t)} \frac{e^{-\frac{(x-y)^2}{4(t-s)} - \varepsilon^2\frac{(t-s)}{4}}}{\sqrt{4\pi(t-s)}}$$

$$\leq O(1)\left(\frac{\frac{|x+\varepsilon t|s}{t} + |y+\varepsilon s|}{(t-s)}\right) \frac{1}{\frac{\sqrt{t}}{\varepsilon t - x} + \frac{\sqrt{t}}{\varepsilon t + x}} \frac{e^{-\frac{(y+\varepsilon s)^2}{4(t-s)} - s\frac{(x+\varepsilon t)^2}{4t(t-s)}} e^{\frac{(x+\varepsilon t)(y+\varepsilon s)}{5(t-s)}}}{\sqrt{(t-s)}}. \quad (3.18)$$

For $x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t})$ and $y \in (-\varepsilon s + \sqrt{s}, \varepsilon s - \sqrt{s})$,

$$G_x(x,t;y,s) = O(1)\left(-\frac{\frac{xs}{t} - y}{(t-s)}\right) \frac{\frac{\sqrt{s}}{\varepsilon s - y} + \frac{\sqrt{s}}{\varepsilon s + y}}{\frac{\sqrt{t}}{\varepsilon t - x} + \frac{\sqrt{t}}{\varepsilon t + x}} \frac{e^{-t\frac{(y - \frac{x}{t}s)^2}{4s(t-s)}}}{\sqrt{(t-s)}}. \quad (3.19)$$

The remaining cases are symmetric to the above cases and are dealt with the same way. This completes the proof of the lemma. □

Lemma 3.6. *For $t, s \geq \varepsilon^{-2}$, we have the following estimates:*

For $x < -\varepsilon t - \sqrt{t}$ and $y \leq -\varepsilon s - \sqrt{s}$,

$$\mathbb{G}(x,t;y,s) = \int_{-\infty}^{y} G_x(x,t;z,s)dz \leq O(1)\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{4(t-s)}}}{\sqrt{t-s}}. \quad (3.20)$$

For $x < -\varepsilon t - \sqrt{t}$ and $y \geq \varepsilon s + \sqrt{s}$,

$$\mathbb{G}(x,t;y,s) = \int_{y}^{\infty} G_x(x,t;z,s)dz$$

$$\leq O(1)\min\left(\frac{\sqrt{s}}{t-s}, \frac{1}{\varepsilon t - x}\right) e^{-\frac{(x+\varepsilon t)^2}{5t}} e^{-\frac{s(x-\varepsilon t)^2}{5t(t-s)}} e^{\frac{-(y-\varepsilon s)^2}{5(t-s)}} e^{\frac{(x-\varepsilon t)(y-\varepsilon s)}{5(t-s)}}. \quad (3.21)$$

For $x < -\varepsilon t - \sqrt{t}$ and $y \in (-\varepsilon s + \sqrt{s}, \varepsilon s - \sqrt{s})$,

$$\mathbb{G}(x,t;y,s) = \int_{y}^{\infty} G_x(x,t;z,s)dz$$

$$\leq O(1)\min(\frac{\sqrt{s}}{t-s},\frac{1}{\varepsilon t-x})e^{-\frac{(x+\varepsilon t)^2}{5t}}e^{-\frac{s(x-\varepsilon t)^2}{5t(t-s)}}+\frac{\log(s)\sqrt{s}}{t-s}e^{-\frac{(x+\varepsilon t)^2}{5t}-\frac{t(y-\frac{x}{t}s)^2}{5s(t-s)}}. \tag{3.22}$$

For $x\in(-\varepsilon t+\sqrt{t},0)$ *and* $y>\varepsilon s+\sqrt{s}$*,*

$$\begin{aligned}\mathbb{G}(x,t;y,s)&=\int_y^\infty G_x(x,t;z,s)dz\\ &\leq O(1)\left(\frac{(x+\varepsilon t)\sqrt{t-s}}{(x-\varepsilon t)^2\sqrt{t}}+\frac{(x+\varepsilon t)s}{\sqrt{(t-s)t^3}}\right)e^{-\frac{(x-\varepsilon t)^2 s}{8t(t-s)}}e^{-\frac{(y-\varepsilon s)^2}{5(t-s)}}e^{\frac{(x-\varepsilon t)(y-\varepsilon s)}{4(t-s)}}.\end{aligned} \tag{3.23}$$

For $x\in(-\varepsilon t+\sqrt{t},0)$*,* $s\in(\varepsilon^{-2},(t/(x+\varepsilon t))^2)$*, and* $y\in(-\varepsilon s+s(x+\varepsilon t)/t,\varepsilon s)$*,*

$$\begin{aligned}\mathbb{G}(x,t;y,s)&=\int_y^\infty G_x(x,t;z,s)dz\\ &\leq O(1)\left[\left(\frac{(x+\varepsilon t)\sqrt{t-s}}{(x-\varepsilon t)^2\sqrt{t}}+\frac{(x+\varepsilon t)s}{\sqrt{(t-s)t^3}}\right)e^{-\frac{(x-\varepsilon t)^2 s}{8t(t-s)}}+\frac{(x+\varepsilon t)s}{t(t-s)}\right]e^{-\frac{t(y-\frac{sx}{t})^2}{8s(t-s)}}.\end{aligned} \tag{3.24}$$

For $x\in(-\varepsilon t+\sqrt{t},0)$*,* $s\in(\varepsilon^{-2},(t/(x+\varepsilon t))^2)$*, and* $y\in(-\varepsilon s+\sqrt{s},-\varepsilon s+s(x+\varepsilon t)/t)$*,*

$$\begin{aligned}\mathbb{G}(x,t;y,s)&=-\int_{-\infty}^y G_x(x,t;z,s)dz\\ &\leq O(1)\left[\left(\frac{\sqrt{t-s}}{(x+\varepsilon t)\sqrt{t}}+\frac{(x+\varepsilon t)s}{\sqrt{(t-s)t^3}}\right)e^{-\frac{(x+\varepsilon t)^2 s}{8t(t-s)}}+\frac{(x+\varepsilon t)s}{t(t-s)}\right]e^{-\frac{t(y-\frac{sx}{t})^2}{8s(t-s)}}.\end{aligned} \tag{3.25}$$

For $x\in(-\varepsilon t+\sqrt{t},0)$*,* $s\in((t/(x+\varepsilon t))^2,t)$*, and* $s(x+\varepsilon t)/t<y+\varepsilon s<2\varepsilon s-\sqrt{s}$*,*

$$\begin{aligned}\mathbb{G}(x,t;y,s)&=\int_y^\infty G_x(x,t;z,s)dz\\ &\leq O(1)\left(\frac{(x+\varepsilon t)\sqrt{t-s}}{(x-\varepsilon t)^2\sqrt{t}}+\frac{(x+\varepsilon t)s}{\sqrt{(t-s)t^3}}\right)e^{-\frac{(x-\varepsilon t)^2}{8t(t-s)}s-\frac{(y-\frac{sx}{t})^2 t}{8s(t-s)}}\\ &\quad+O(1)e^{-\frac{(y-\frac{sx}{t})^2 t}{8s(t-s)}}\left(\frac{(x+\varepsilon t)s}{t(t-s)}e^{-\frac{(x-\varepsilon t)^2 s}{8(t-s)t}}+\frac{\sqrt{s}}{\sqrt{t(t-s)}}\right).\end{aligned} \tag{3.26}$$

For $x \in (-\varepsilon t + \sqrt{t}, 0)$, $s \in \left((t/(x+\varepsilon t))^2, t\right)$, *and* $\sqrt{s} < y + \varepsilon s < s(x+\varepsilon t)/t$,

$$\begin{aligned} \mathbb{G}(x,t;y,s) &= -\int_{-\infty}^{y} G_x(x,t;z,s)dz \\ &\leq O(1)\left(\frac{\sqrt{t-s}}{(x+\varepsilon t)\sqrt{t}} + \frac{(x+\varepsilon t)s}{\sqrt{(t-s)t^3}}\right) e^{-\frac{(x+\varepsilon t)^2}{8t(t-s)}s - \frac{(y-\frac{sx}{t})^2 t}{8s(t-s)}} \\ &\quad + O(1) e^{-\frac{(y-\frac{sx}{t})^2 t}{8s(t-s)}} \left(\frac{(x+\varepsilon t)s}{t(t-s)} e^{-\frac{(x+\varepsilon t)^2}{8t(t-s)}s} + \frac{\sqrt{s}}{\sqrt{t(t-s)}}\right). \end{aligned} \tag{3.27}$$

For $x \in (-\varepsilon t + \sqrt{t}, 0)$ *and* $y < -\varepsilon s - \sqrt{s}$,

$$\begin{aligned} \mathbb{G}(x,t;y,s) &= -\int_{-\infty}^{y} G_x(x,t;z,s)dz \\ &\leq O(1)\frac{1}{\sqrt{t(t-s)}} e^{-\frac{(y+\varepsilon s)^2}{5(t-s)}} e^{\frac{(x+\varepsilon t)(y+\varepsilon s)}{5(t-s)}} e^{-\frac{(x+\varepsilon t)^2 s}{8t(t-s)}}. \end{aligned} \tag{3.28}$$

Proof. (3.20) (respectively (3.21)) is a consequence of (3.11) in the case $y \leq -\varepsilon s + \sqrt{s}$ (respetively $y \geq \varepsilon s + \sqrt{t}$.) For (3.22), we apply (3.11) in the case $y \in (-\varepsilon s + \sqrt{s}, \varepsilon s - \sqrt{s})$ together with (3.21).

(3.23) is a consequence of (3.12) in the case $y > \varepsilon s + \sqrt{s}$. (3.26), (3.25) are consequences of (3.23) and (3.12) in the case $y \in [-\varepsilon s + \sqrt{s}, \varepsilon s - \sqrt{s}]$. The rationale for (3.26) and (3.27) is similar to that for (3.23). (3.28) follows from (3.12) in the case $y < -\varepsilon s - \sqrt{s}$. □

3.2. Green's function for the transverse field

For the transverse field, second equation in (2.25), we construct the approximate Green's function for the equation

$$v_t + \lambda_2^* v_x + d(x,t)v - v_{xx} = 0. \tag{3.29}$$

Thus we construct the Green's function

$$G_2(x,t;y,s) = g(x,t)$$

satisfying

$$\begin{cases}(G_2)_t+\lambda_2^*(x,t)(G_2)_x+d(x,t)(G_2)_x=(G_2)_{xx} \text{ for } x\in\mathbb{R}, t>s,\\ G_2(x,s)=\delta(x-y).\end{cases} \tag{3.30}$$

Unlike the case for linearized Burgers equation (3.1), for which there is an explicit formula; for G_2 we need to construct accurate approximation $\tilde{G}_2$. Consider the characteristic curve $z=\Xi(\sigma;x,t)$ in the (z,σ) domain for the inviscid characteristic field $\lambda_2(B_*^i(x,t))$,

$$\begin{cases}\partial_\sigma\Xi(\sigma;x,t)=\lambda_2^*(\Xi(\sigma;x,t),\sigma),\\ \Xi(t;x,t)=x.\end{cases} \tag{3.31}$$

The approximate Green's function $\tilde{G}_2(x,t;y,s)$ for (3.30) is defined by

$$\tilde{G}_2(x,t;z,\sigma)\equiv e^{-\int_\sigma^t d(\Xi(\tau;x,t),\tau)d\tau}\frac{e^{-\frac{(z-\Xi(\sigma;x,t))^2}{4(t-\sigma)}}}{\sqrt{4\pi(t-\sigma)}}. \tag{3.32}$$

One has the representation of ζ_p^2 as follows

$$\begin{aligned}\zeta_p^2(x,t) \;=\; &\int_{\varepsilon^{-2}}^t\int_{\mathbb{R}}\Big(\hat{G}_{2y}(x,t;y,s)(\lambda_2^*(y,s)-\lambda_2^*(\Xi(s;x,t),s))\\ &-\hat{G}_2(x,t;y,s)(d(y,s)-d(\Xi(s;x,t),s))\Big)\zeta_p^2(y,s)dyds\\ &+\int_0^t\int_{\mathbb{R}}\hat{G}_{2y}(x,t;y,s)C_{22}^2(y,s)\zeta_p^2(y,s)^2\\ &\qquad+\hat{G}_2(x,t;y,s)(\boldsymbol{l}_2R_1'')B_y^v(y,s)^2dyds.\end{aligned} \tag{3.33}$$

By considering the identity,

$$\begin{aligned}0 \;=\; &\int_s^t\int_{\mathbb{R}}\tilde{G}_2(x,t;z,\sigma)(\partial_\sigma G_2(z,\sigma;y,s)+\lambda_2^*(z,\sigma)\partial_z G_2(z,\sigma;y,s)\\ &+d(z,\sigma)G_2(z,\sigma;y,s)-\partial_z^2G_2(z,\sigma;y,s))dzd\sigma,\end{aligned} \tag{3.34}$$

we obtain the relation

$$\begin{aligned}&G_2(x,t;y,s)=\tilde{G}_2(x,t;y,s)\\ &\quad+\int_s^t\int_{\mathbb{R}}\Big((\lambda_2^*(z,\sigma)-\Xi_\sigma(\sigma;x,t))\tilde{G}_2(x,t;z,\sigma)\Big)_z G_2(z,\sigma;y,s)dzd\sigma\end{aligned}$$

$$-\int_s^t \int_{\mathbb{R}} (d(z,\sigma) - d(\Xi(\sigma;x,t),\sigma))\tilde{G}_2(x,t;z,\sigma)G_2(z,\sigma;y,s)dzd\sigma. \quad (3.35)$$

Lemma 3.7. *Suppose that ε is sufficiently small. Then there exists a uniform constant $C > 0$ such that for s, $\tau \geq 1$,*

$$1 - C\varepsilon \leq |\partial_\eta \Xi(s;\eta,\tau)| \leq 1 + C\varepsilon. \quad (3.36)$$

Proof. The equation for $\Xi_\eta(s;\eta,\tau)$ is

$$\begin{cases} \frac{d}{ds}\Xi_\eta(s;\eta,\tau) = \partial_x \lambda_2^*(\Xi(s;\eta,\tau),s)\Xi_\eta(s;\eta,\tau), \\ \Xi_\eta(\tau;\eta,\tau) = 1. \end{cases} \quad (3.37)$$

From (2.4), we know that $\partial_x \lambda_2^*(x,t)$ times the width of its support at time t is of the order of ε. Moreover, the characteristic curve $\Xi(s;x,t)$ crosses the fan region, the support of $\partial_x \lambda_2^*(x,t)$, transversally. Thus

$$\int_1^\infty |\partial_\eta \lambda_2^*(\Xi(s;\eta,\tau),s)|ds = O(1)\varepsilon. \quad (3.38)$$

This and (3.37) yield the lemma. □

The next lemma is proved by the direct usage of the explicit expression (2.4) of the inviscid rarefaction wave, and is omitted.

Lemma 3.8. *The characteristic curve $\Xi(s;x,t)$, (3.37), satisfies, for $s \in (0,t)$,*

$$\Xi(s;x,t) = \begin{cases} x - t + s \text{ for } x > \varepsilon t \text{ and } x - t + s > \varepsilon s + 1, \\ -\int_s^{\left(\frac{t-x}{1-\varepsilon+O(1)}\right)} \frac{\tau^{\kappa_0}}{s^{\kappa_0}} d\tau + \varepsilon \frac{\left(\frac{t-x}{1-\varepsilon+O(1)}\right)^{\kappa_0}}{s^{\kappa_0}} \\ \qquad\qquad \text{for } x > \varepsilon t \text{ and } \Xi(s;x,t) \in (-\varepsilon s, \varepsilon s), \\ -\varepsilon s_0 + (1-\kappa_0\varepsilon)(s-s_0) \qquad \text{for } x > \varepsilon t \text{ and } \Xi(s;x,t) < -\varepsilon s, \\ -\int_s^t \frac{\tau^{\kappa_0}}{s^{\kappa_0}} d\tau + \frac{t^{\kappa_0}}{s^{\kappa_0}} \quad \text{for } x \in (-\varepsilon t, \varepsilon t) \text{ and } \Xi(s;x,t) \in (-\varepsilon s, \varepsilon s), \\ -\varepsilon s_0 + (1-\kappa_0\varepsilon)(s-s_0) \text{ for } x \in (-\varepsilon t, \varepsilon t) \text{ and } \Xi(s;x,t) < -\varepsilon s, \\ x - \lambda_2(\vec{\boldsymbol{u}}_-)(t-s) \qquad \text{for } x < -\varepsilon t, \end{cases} \quad (3.39)$$

and

$$|\Xi_s(s;x,t)-1| = O(1)\varepsilon. \tag{3.40}$$

Here, $\kappa_0 \equiv t\partial_x\lambda_2^*(0,t)$ *and* $s_0\in(0,t)$ *is defined by* $\Xi(s_0;x,t)=-\varepsilon s_0$.

Lemma 3.9. *Suppose that* $\varepsilon \ll 1$. *Then, for* $t \geq s \gg 1$

$$|\tilde{G}_2(x,t;y,s)-G_2(x,t;y,s)| \leq O(1)\sqrt{\varepsilon}\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{10(t-s)}}}{\sqrt{t-s}}, \tag{3.41}$$

and

$$|\tilde{G}_{2y}(x,t;y,s)-G_{2y}(x,t;y,s)| = O(1)\sqrt{\varepsilon}\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{5(t-s)}}}{\sqrt{t-s}}\frac{1}{\sqrt{s}} \text{ for } t,s\geq 1. \tag{3.42}$$

Proof. We only consider the case that $y<0$; the case $y>0$ is simpler. Assume that we have a priori assumption

$$|\tilde{G}_2(x,t;y,s)-G_2(x,t;y,s)| \leq O(1)\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{10(t-s)}}}{\sqrt{t-s}} \text{ for all } t>s\geq 1. \tag{3.43}$$

Then, from (3.35) and (3.36),

$$\begin{aligned}
&|\tilde{G}_2(x,t;y,s)-G_2(x,t;y,s)| \\
&\leq \int_s^t\int_{\mathbb{R}}\Big((|\partial_z\lambda_2^*(z,\sigma)|+|d(z,\sigma)-d(\Xi(\sigma;x,t),\sigma)|)\tilde{G_2}(z,\sigma;x,t) \\
&\qquad +|(\lambda_2^*(z,\sigma)-\Xi_s(\sigma;x,t))|\tilde{G}_{2z}(z,\sigma;x,t)\Big)G_2(z,\sigma;y,s)dzd\sigma \\
&\leq O(1)\int_s^t\int_{\mathbb{R}}\left(|\partial_z\lambda_2^*(z,\sigma)|+\frac{|\lambda_2^*(z,\sigma)-\Xi_s(\sigma;x,t)|}{\sqrt{t-\sigma}}\right) \\
&\qquad \times\frac{e^{-\frac{(\Xi(s;z,\sigma)-\Xi(s;x,t))^2}{5(t-\sigma)}}}{\sqrt{(t-\sigma)}}\frac{e^{-\frac{(y-\Xi(s;z,\sigma))^2}{10(\sigma-s)}}}{\sqrt{(\sigma-s)}}dzd\sigma.
\end{aligned} \tag{3.44}$$

Use the property that $\partial_z\lambda_2^*(z,\sigma)=O(1)\partial_zB^i(z,\sigma)$ and $\lambda_2^*(z,\sigma)-\Xi_s(\sigma;x,t)= O(1)(B^i(z,\sigma)-B^i(\Xi(\sigma;x,t),\sigma))$, and the properties of B^i in (2.4) to yield that

$$\begin{aligned}&|\tilde{G}_2(x,t;y,s)-G_2(x,t;y,s)|\\&\le O(1)\int_s^t\int_{\mathbb{R}}\left(|B_z^i(z,\sigma)|+\frac{|B^i(z,\sigma)-B^i(\Xi(\sigma;x,t),s)|}{\sqrt{t-\sigma}}\right)\\&\times\frac{e^{-\frac{(\Xi(s;z,\sigma)-\Xi(s;x,t))^2}{5(t-\sigma)}-\frac{(y-\Xi(s;z,\sigma))^2}{10(\sigma-s)}}}{\sqrt{(t-\sigma)(\sigma-s)}}dzd\sigma. \qquad (3.45)\end{aligned}$$

Direct calculations from (2.4) give the estimate for the first term on the R.H.S. of (3.45):

$$\int_s^t\int_{\mathbb{R}}|B_z^i(z,\sigma)|\frac{e^{-\frac{(\Xi(s;z,\sigma)-\Xi(s;x,t))^2}{10(t-\sigma)}-\frac{(y-\Xi(s;z,\sigma))^2}{8(\sigma-s)}}}{\sqrt{(t-\sigma)(\sigma-s)}}dzd\sigma\le O(1)\varepsilon\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{10(t-s)}}}{\sqrt{(t-s)}}. \qquad (3.46)$$

The estimate for the second term on the R.H.S. of (3.45) will be done in the following two cases.

Case 1. $|x-\Xi(t;y,s)|\le(t-s)/4$ and $t>2|y|+\varepsilon^{-1}$.

The condition $t>2|y|+\varepsilon^{-1}$ assures $\Xi(t;y,s)>\varepsilon t$. The characteristic curve $z=\Xi(\sigma;x,t)$ crosses the rarefaction fan before $\sigma=t$, see Figure 1.

Define

$$\begin{aligned}I_1\equiv\Big\{&(z,\sigma)|\sigma\in(s,t),|\Xi(s;z,\sigma)-y-\tfrac{2(\sigma-s)}{t+\sigma-2s}(\Xi(s;x,t)-y)|\\&\le K|\log\left(\tfrac{(t-\sigma)(\sigma-s)}{t+\sigma-2s}\right)|\sqrt{\tfrac{(t-\sigma)(\sigma-s)}{t+\sigma-2s}}\Big\},\\I_2\equiv\Big\{&(z,\sigma)|\sigma\in(s,t),|\Xi(s;z,\sigma)-y-\tfrac{2(\sigma-s)}{t+\sigma-2s}(\Xi(s;x,t)-y)|\\&\ge K|\log\left(\tfrac{(t-\sigma)(\sigma-s)}{t-s}\right)|\sqrt{\tfrac{(t-\sigma)(\sigma-s)}{t-s}}\Big\}.\end{aligned} \qquad (3.47)$$

We have

$$\begin{aligned}&\iint\limits_{I_2}\frac{|B^i(z,\sigma)-B^i(\Xi(\sigma;x,t),s)|}{\sqrt{t-\sigma}}\frac{e^{-\frac{(\Xi(s;z,\sigma)-\Xi(s;x,t))^2}{5(t-\sigma)}-\frac{(y-\Xi(s;z,\sigma))^2}{10(\sigma-s)}}}{\sqrt{(t-\sigma)(\sigma-s)}}dzd\sigma\\&\le\iint\limits_{I_2}\frac{O(1)\varepsilon}{\sqrt{t-\sigma}}\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{5(t+\sigma-2s)}-\frac{(t+\sigma-2s)}{10(t-\sigma)(\sigma-s)}(\Xi(s;z,\sigma)-y-\frac{2(\sigma-s)}{t+\sigma-2s}(\Xi(s;x,t)-y))^2}}{\sqrt{(t-\sigma)(\sigma-s)}}dzd\sigma\end{aligned}$$

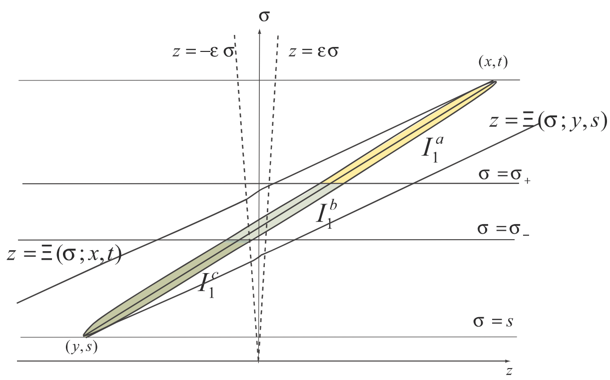

Figure 1.

$$
\begin{aligned}
&\le O(1)\varepsilon \frac{e^{-\frac{(y-\Xi(s;x,t))^2}{10(t-s)}}}{\sqrt{t-s}} \int_s^t \frac{1}{\sqrt{t-\sigma}} \left(\frac{(t-\sigma)(\sigma-s)}{(t-s)}\right)^{-K} d\sigma \\
&\le O(1)\varepsilon \frac{e^{-\frac{(y-\Xi(s;x,t))^2}{8(t-s)}}}{\sqrt{t-s}} \qquad (3.48)
\end{aligned}
$$

for some $K > 1$.

To analyze the integral over I_1, one needs to divide the region I_1 into components, Figure 1:

$$
\begin{cases}
I_1 = I_1^a \cup I_1^b \cup I_1^c, \\
I_1^a \equiv I_1 \cap \{\sigma \in (\sigma_+, t)\}, I_1^b \equiv I_1 \cap \{\sigma \in (\sigma_-, \sigma_+)\}, I_1^c \equiv I_1 \cap \{\sigma \in (s, \sigma_-)\}, \\
\sigma_+ \equiv \max\{s, \inf\{\tau \le t | ((I_1 \cup \{z = \Xi(\sigma; x, t)\}) \cap \{\sigma \in (\tau, t)\}) \\
\qquad\qquad\qquad\qquad \cap \{z \in (-\varepsilon\sigma, \varepsilon s)\} = \emptyset\}\}, \\
\sigma_- \equiv \max\{s, \sup\{\tau < \sigma_+ | ((I_1 \cup \{z = \Xi(\sigma; x, t)\}) \cap \{\sigma = \tau\}) \\
\qquad\qquad\qquad\qquad \cap \{z \in (-\varepsilon\sigma, \varepsilon s)\} = \emptyset\}\}.
\end{cases}
$$

Thus, for $(z, \sigma) \in I_1^a \cup I_1^c$ one has

$$
|B^i(z,\sigma) - B^i(\Xi(\sigma; x, t), \sigma)| = 0. \qquad (3.49)
$$

This yields that

$$\iint\limits_{I_1^a\cup I_1^c} \frac{|B^i(z,\sigma)-B^i(\Xi(\sigma;x,t),s)|}{\sqrt{t-\sigma}} \frac{e^{-\frac{(\Xi(s;z,\sigma)-\Xi(s;x,t))^2}{8(t-\sigma)}-\frac{(y-\Xi(s;z,\sigma))^2}{8(\sigma-s)}}}{\sqrt{(t-\sigma)(\sigma-s)}} dzd\sigma = 0. \tag{3.50}$$

For $(z,\sigma)\in I_1^b$, under the conditions $t>2(|y|+\varepsilon^{-1})$ and $|x-\Xi(t;y,s)|\le (t-s)/4$ it follows

$$\begin{cases} \sigma_+ \le t/2, \\ |\sigma_+-\sigma_-| \le O(1)|y-\Xi(s;x,t)|. \end{cases} \tag{3.51}$$

By (3.51) and $B^i(z,\sigma)-B^i(\Xi(\sigma;x,t),\sigma)\le O(1)\varepsilon$. This yields that

$$\begin{aligned} &\iint\limits_{I_1^b} \frac{|B^i(z,\sigma)-B^i(\Xi(\sigma;x,t),s)|}{\sqrt{t-\sigma}} \\ &\qquad \times \frac{e^{-\frac{(y-\Xi(s;x,t))^2}{5(t+\sigma-2s)}-\frac{(t+\sigma-2s)}{10(t-\sigma)(\sigma-s)}(\Xi(s;z,\sigma)-y-\frac{2(\sigma-s)}{t+\sigma-2s}(\Xi(s;x,t)-y))^2}}{\sqrt{(t-\sigma)(\sigma-s)}} dzd\sigma \\ &\le O(1)\varepsilon \frac{e^{-\frac{(y-\Xi(s;x,t))^2}{15(t-s)/2}}}{t-s}\int_{\sigma_-}^{\sigma_+} d\sigma \le O(1)\varepsilon \frac{e^{-\frac{(y-\Xi(s;x,t))^2}{15(t-s)/2}}|y-\Xi(s;x,t)|}{t-s} \\ &= O(1)\varepsilon \frac{e^{-\frac{(y-\Xi(s;x,t))^2}{10(t-s)}}}{\sqrt{t-s}}. \end{aligned} \tag{3.52}$$

Thus, (3.46), (3.48), (3.50), and (3.52) yield the proof for this case.

Case 2. $x\le t/2$.

Similar to the decomposition $I_1\cup I_2$, one introduces a decomposition:

$$\begin{aligned} J_1 &\equiv \{(z,\sigma)|\sigma\in(s,t), |\Xi(\sigma;x,t)-z| \le K\sqrt{t-\sigma}\log(t-\sigma+1), \\ J_2 &\equiv \{(z,\sigma)|\sigma\in(s,t), |\Xi(\sigma;x,t)-z| \ge K\sqrt{t-\sigma}\log(t-\sigma+1). \end{aligned}$$

And similar to (3.48) and (3.50), we have

$$\iint\limits_{J_2\cup J_1^a\cup J_1^c} \frac{|B^i(z,\sigma)-B^i(\Xi(\sigma;x,t),\sigma)|}{\sqrt{t-\sigma}} \frac{e^{-\frac{(\Xi(s;z,\sigma)-\Xi(s;x,t))^2}{5(t-\sigma)}-\frac{(y-\Xi(s;z,\sigma))^2}{10(\sigma-s)}}}{\sqrt{(t-\sigma)(\sigma-s)}} dzd\sigma$$

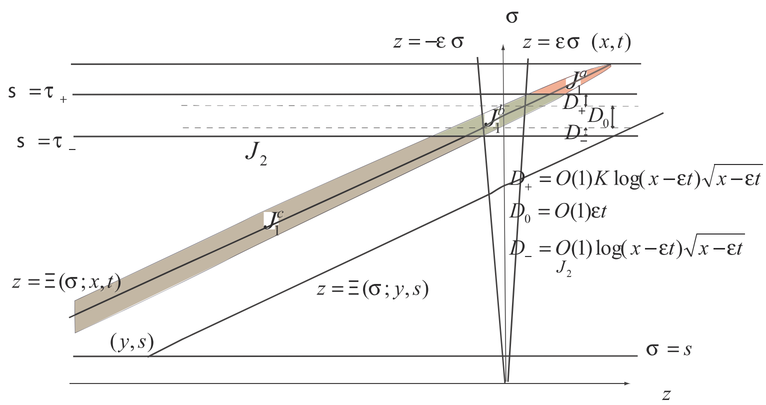

Figure 2.

$$\leq\ O(1)\varepsilon\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{8(t-s)}}}{\sqrt{t-s}}. \tag{3.53}$$

It is easy to see that the condition $x < t/2$ results in, see Figure 2,

$$\begin{cases}(t-s)/4 < \tau_- - s,\\ \tau_+ - \tau_- \leq O(1)\left(K\log(t)\sqrt{t}+\varepsilon t\right).\end{cases} \tag{3.54}$$

Then, one uses that $B^i(z,\sigma) - B^i(\Xi(\sigma;x,t),\sigma) \leq O(1)\frac{|z-\Xi(\sigma;x,t)|}{\sigma}$ to result in

$$\begin{aligned}&\iint\limits_{J_1^b}\frac{|B^i(z,\sigma)-B^i(\Xi(\sigma;x,t),s)|}{\sqrt{t-\sigma}}\frac{e^{-\frac{(\Xi(s;z,\sigma)-\Xi(s;x,t))^2}{8(t-\sigma)}-\frac{(y-\Xi(s;z,\sigma))^2}{8(\sigma-s)}}}{\sqrt{(t-\sigma)(\sigma-s)}}dzd\sigma\\
&\leq\ O(1)\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{8(t-s)}}}{\sqrt{t-s}}\int_{\tau_-}^{\tau_+}\frac{1}{s}d\sigma \leq O(1)\left(\varepsilon+\frac{\log(t)}{\sqrt{t}}\right)\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{8(t-s)}}}{\sqrt{t-s}}\\
&\leq\ O(1)\sqrt{\varepsilon}\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{8(t-s)}}}{\sqrt{t-s}}.\end{aligned} \tag{3.55}$$

Thus, (3.53) and (3.55) conclude this case.

The other cases for verifying a prior assumption (3.43) can be obtained easily. They are omitted. Thus, (3.43) is true. This completes the proof of the lemma. □

4. Primary Nonlinear System

The construction of the solutions of the primary nonlinear system (2.25) is divided into two subsections. The study of waves for the primary field of the rarefaction wave is based on the Burgers Green's function. The more interesting study of the transversal field highlights the role of two types of nonlinearity. The first one is the coupling term from the primary field, represented by the source $B_y^v(y,s)^2$. The second nonlinearity is the genuine nonlinearity of the second characteristic field that we are assuming. To study the total effect of these two nonlinearities, we design a intricate inner-outer estimates. Each subsections starts with some preparatory lemmas.

4.1. Primary field

We start with the convolution of the source with both types of Burgers Green's function G and $\mathbb{G}$.

Lemma 4.1 *For* $t \geq \varepsilon^{-2}$,

$$\int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} \mathbb{G}(x,t;y,s)B_y^v(y,s)^2 dyds$$
$$\leq\ O(1)\frac{\log t}{t}+O(1)\varepsilon|\log\varepsilon|\begin{cases}\dfrac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{\sqrt{1+t}} & \textit{for } x<-\varepsilon t+\sqrt{t},\\ \dfrac{1}{|x+\varepsilon t|}+\dfrac{1}{|x-\varepsilon t|} & \textit{for } x\in(-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}),\\ \dfrac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{\sqrt{1+t}} & \textit{for } x>\varepsilon t-\sqrt{t}.\end{cases} \tag{4.1}$$

Proof. By (3.20), (3.21), (3.22), and (2.9), for $x \leq -\varepsilon t - \sqrt{t}$,

$$\int_{\varepsilon^{-2}}^{t}\left(\int_{-\infty}^{-\varepsilon s-\sqrt{s}}+\int_{-\varepsilon s+\sqrt{s}}^{\varepsilon s-\sqrt{s}}+\int_{\varepsilon s+\sqrt{s}}^{\infty}\right)\mathbb{G}(x,t;y,s)B_y^v(y,s)^2 dyds$$
$$\leq\ \int_{\varepsilon^{-2}}^{t}\int_{-\infty}^{-\varepsilon s-\sqrt{s}}\frac{e^{-\frac{(x-y+\varepsilon(t-s))^2}{4(t-s)}}}{\sqrt{t-s}}\frac{e^{-\frac{(y+\varepsilon s)^2}{4s}}}{(s+1)^2}dyds$$

$$
\begin{aligned}
&+\int_{\varepsilon^{-2}}^{t}\int_{-\infty}^{-\varepsilon s-\sqrt{s}} \frac{e^{-\frac{(x+\varepsilon t)^2}{5t}-\frac{t(y-\frac{x}{t}s)^2}{5s(t-s)}}}{\sqrt{t-s}} \frac{1}{(s+1)^2}dyds \\
&+\int_{\varepsilon^{-2}}^{t}\int_{-\infty}^{-\varepsilon s-\sqrt{s}} \frac{e^{-\frac{(x+\varepsilon t)^2}{5t}} e^{-\frac{s(x-\varepsilon t)^2}{5t(t-s)}} e^{\frac{-(y-\varepsilon s)^2}{5(t-s)}} e^{\frac{(x-\varepsilon t)(y-\varepsilon s)}{5(t-s)}}}{\sqrt{t-s}} \frac{e^{-\frac{(y-\varepsilon s)^2}{4s}}}{(s+1)^2}dyds \\
\le\ & O(1)\varepsilon \frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{\sqrt{t}}. \qquad (4.2)
\end{aligned}
$$

By (3.28) and (2.9), for $x \in (-\varepsilon t+\sqrt{t}, 0)$,

$$
\begin{aligned}
&\int_{\varepsilon^{-2}}^{t}\int_{-\infty}^{-\varepsilon s-\sqrt{s}} \mathbb{G}(x,t;y,s)B_y^v(y,s)^2 dyds \\
&\quad \le\ O(1)\int_{\varepsilon^{-2}}^{t} \frac{1}{\sqrt{t(t-s)s^3}}ds \le O(1)\frac{\varepsilon}{t}. \qquad (4.3)
\end{aligned}
$$

By (3.24), (3.25), and (2.9), for $x \in (-\varepsilon t+\sqrt{t}, 0)$,

$$
\begin{aligned}
&\iint\limits_{\substack{\varepsilon^{-2}<s<t^2/(x+\varepsilon t)^2 \\ -\varepsilon s+\sqrt{s}<y<\varepsilon s}} \mathbb{G}(x,t;y,s)B_y^v(y,s)^2 dyds \\
&\le\ O(1)\int_{\varepsilon^{-2}}^{t^2/(x+\varepsilon t)^2} \left(\frac{1}{(\varepsilon t-x)s^{3/2}}+\frac{x+\varepsilon t}{t^2 s^{1/2}}\right)ds \\
&\le\ O(1)\left(\frac{\varepsilon}{x+\varepsilon t}+\frac{1}{t}\right) \qquad (4.4)
\end{aligned}
$$

By (3.26), (3.27), and (2.9), for $x \in (-\varepsilon t+\sqrt{t}, 0)$,

$$
\begin{aligned}
&\iint\limits_{\substack{t^2/(x+\varepsilon t)^2<s<t \\ -\varepsilon s+\sqrt{s}<y<\varepsilon s}} \mathbb{G}(x,t;y,s)B_y^v(y,s)^2 dyds \\
&\le\ O(1)\left(\frac{\varepsilon}{x+\varepsilon t}+\frac{1}{t}\right) \\
&\quad +O(1)\int_{t^2/(x+\varepsilon t)^2}^{t} \left(\frac{(x+\varepsilon t)e^{-\frac{(x-\varepsilon t)^2}{8t(t-s)}s}}{t(t-s)s^{1/2}}+\frac{1}{(t(t-s))^{1/2}s}\right)ds \\
&\le\ O(1)\left(\frac{\varepsilon}{x+\varepsilon t}+\frac{\log(t)}{t}\right). \qquad (4.5)
\end{aligned}
$$

By (3.23) and (2.9), for $x \in (-\varepsilon t + \sqrt{t}, 0)$,

$$\iint_{\substack{\varepsilon^{-2}<s<t \\ y>\varepsilon s+\sqrt{s}}} \mathbb{G}(x,t;y,s) B_y^v(y,s)^2 dyds$$

$$\leq O(1) \int_{\varepsilon^{-2}}^{t} \frac{1}{(\varepsilon t - x)s^{3/2}} ds$$

$$+O(1) \left(\int_{\varepsilon^{-2}}^{t/2} \frac{(x+\varepsilon t)e^{-\frac{(x-\varepsilon t)^2 s}{8t^2}}}{t^2\sqrt{s}} ds + \int_{t/2}^{t} \frac{(x+\varepsilon t)e^{-\frac{(x-\varepsilon t)^2}{8(t-s)}}}{t^{5/2}} \right)$$

$$\leq O(1) \left(\frac{\varepsilon}{\varepsilon t - x} + \frac{1}{t} \right). \tag{4.6}$$

From (4.3), (4.4), (4.5), and (4.6), one concludes that for $x \in (-\varepsilon t + \sqrt{t}, 0)$

$$\int_{\varepsilon^{-2}}^{t} \left(\int_{-\infty}^{-\varepsilon s - \sqrt{s}} + \int_{-\varepsilon s + \sqrt{s}}^{\varepsilon s - \sqrt{s}} + \int_{\varepsilon s + \sqrt{s}}^{\infty} \right) \mathbb{G}(x,t;y,s) B_y^v(y,s)^2 dyds$$

$$\leq O(1) \left(\frac{\log t}{t} + \frac{\varepsilon |\log \varepsilon|}{x + \varepsilon t} + \frac{\varepsilon |\log \varepsilon|}{-x + \varepsilon t} \right). \tag{4.7}$$

This proves the cases $x > 0$ by (4.7) and (4.2). The case, $x < 0$ is the analogous. This proves the lemma. □

The following lemma follows from Lemma 3.5 through straightforward computations.

Lemma 4.2.

$$\int_0^t \int_{-\varepsilon s + \sqrt{s}}^{\varepsilon s - \sqrt{s}} G_x(x,t;y,s) \left(\frac{1}{(y+\varepsilon s)^2} + \frac{1}{(y - \varepsilon s)^2} \right) dyds$$

$$\leq O(1) \begin{cases} \frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{\sqrt{t}} & \textit{for } x < -\varepsilon t, \\ \frac{1}{\sqrt{(x+\varepsilon t)^2 + t}} + \frac{1}{\sqrt{(x-\varepsilon t)^2+t}} & \textit{for } x \in (-\varepsilon t, \varepsilon t), \\ \frac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{\sqrt{t}} & \textit{for } x < -\varepsilon t. \end{cases} \tag{4.8}$$

$$\int_0^t \int_{\mathbb{R}} G_x(x,t;y,s)\frac{e^{-\frac{(y+\varepsilon s)^2}{5s}}+e^{-\frac{(y-\varepsilon s)^2}{5s}}}{s}$$
$$\leq O(1)\begin{cases} \frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{\sqrt{t}} & \textit{for } x\leq -\varepsilon t+\sqrt{t}, \\ \frac{1}{t}+\frac{1}{x+\varepsilon t}+\frac{1}{\varepsilon t-x} & \textit{for } x\in(-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}), \\ \frac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{\sqrt{t}} & \textit{for } x\geq \varepsilon t-\sqrt{t}. \end{cases} \tag{4.9}$$

For $\gamma\in(1/2,1]$

$$\int_0^t \int_{-\varepsilon s+\sqrt{s}}^{\varepsilon s-\sqrt{s}} G_x(x,t;y,s)\left(\frac{1}{(y+\varepsilon s)s^\gamma}+\frac{1}{(y-\varepsilon s)s^\gamma}\right)dyds$$
$$\leq O(1)\begin{cases} \frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^\gamma} & \textit{for } x<-\varepsilon t, \\ \frac{1}{t^\gamma} & \textit{for } x\in(-\varepsilon t,\varepsilon t), \\ \frac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{t^\gamma} & \textit{for } x<-\varepsilon t. \end{cases} \tag{4.10}$$

Theorem 4.3. *The solution* $\zeta_p^1(x,t)$ *of* (2.28) *satisfies, for* $t\geq\varepsilon^{-2}$,

$$|\zeta_p^1(x,t)| \leq O(1)\Bigg(\frac{\log t}{t}+\varepsilon|\log\varepsilon|\Bigg(\frac{\chi_{[-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)}{|x+\varepsilon t|}+\frac{\chi_{[-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)}{|x-\varepsilon t|}$$
$$+\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}+e^{-\frac{(x-\varepsilon t)^2}{5t}}}{\sqrt{t}}\Bigg)\Bigg). \tag{4.11}$$

Proof. By (3.7), one can rewrite (2.28) as

$$\zeta_p^1(x,t) = \int_{\varepsilon^{-2}}^t\int_{\mathbb{R}}\Big(G_x(x,t;y,s)C_{11}^1(y,s)\zeta_p^1(y,s)^2$$
$$+\mathbb{G}(x,t;y,s)B_y^v(y,s)^2\boldsymbol{l}_1(y,s)R_1''(B^v(y,s))\Big)dyds. \tag{4.12}$$

By (4.1), there exists C_0 such that

$$\zeta_p^1(x,t)\leq(1+O(1)\varepsilon)\int_{\varepsilon^{-2}}^t\int_{\mathbb{R}}G_x(x,t;y,s)(\zeta_p^1(y,s))^2dyds+C_0\Psi_1(x,t), \tag{4.13}$$

where

$$\Psi_1(x,t) \equiv \frac{\log t}{t} + \varepsilon|\log\varepsilon| \begin{cases} \dfrac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{\sqrt{1+t}} \text{ for } x < -\varepsilon t + \sqrt{t}, \\ \dfrac{1}{|x+\varepsilon t|} + \dfrac{1}{|x-\varepsilon t|} \text{ for } x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t}), \\ \dfrac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{\sqrt{1+t}} \text{ for } x > \varepsilon t - \sqrt{t}. \end{cases} \tag{4.14}$$

By making an ansatz assumption $\zeta_p^1(x,t) \le 2C_0\Psi_1(x,t)$ and substituting it into (4.13), we have, by Lemma 4.2, $\zeta_p^1(x,t) \le (1 + O(1)\varepsilon|\log\varepsilon|)C_0\Psi_1(x,t)$. Thus, the ansatz assumption holds for $\varepsilon \ll 1$ and the theorem follows. □

4.2. The transverse fields

For the problem (2.25), we will consider the following linear hyperbolic equation first.

$$\begin{cases} Z_t + \lambda_2^*(x,t)Z_x + d(x,t)Z = B_x^v(x,t)^2 \boldsymbol{l}_2(x,t)R_1''(B^v(x,t)), \\ Z(x,\varepsilon^{-2}) \equiv 0. \end{cases} \tag{4.15}$$

This is solved by the characteristic method:

$$Z(x,t) = \int_{\varepsilon^{-2}}^t e^{\int_\tau^t -d(\Xi(\sigma,x,t),\sigma)d\sigma} \left((B_y^v)^2 \boldsymbol{l}_2 R_1''(B^v)\right)(\Xi(\tau,x,t),\tau)d\tau. \tag{4.16}$$

Here $d(x,t)$ is given in (2.26). and $\partial_x^k B^v$, $\partial_x^j B^i$, $k = 0,1,2$, $j = 0,1$ in (2.8), (2.9), (2.10), and (2.4).

Lemma 4.4. *There exists D_0 such that for $t \ge 1$,*

$$\left| Z\left(\frac{t}{2}, t\right) - \varepsilon\frac{D_0}{t} \right| \le \frac{O(1)}{t^{3/2}}. \tag{4.17}$$

Furthermore,

$$|Z(x,t)| \le O(1)\frac{\varepsilon}{|x-t|+1} \text{ for } x \in \mathbb{R}, \tag{4.18}$$

and

$$|Z_x(x,t)| \le O(1)\frac{1}{|x-t|^2+1} \text{ for } x \in \mathbb{R}. \tag{4.19}$$

Proof. The problem (2.27) for $x \in (-\varepsilon t, \varepsilon t)$ admits a self-similar solution of the form

$$z(x,t) = t^{-1} m(x/t),$$

with $m(\xi)$ given by

$$\begin{cases} -\xi m_\xi + \lambda_2^* m_\xi + \bar{d}(\xi) m = \bar{B}^S \text{ for } \xi \in (-\varepsilon, \varepsilon), \\ m(-\varepsilon) = 0, \end{cases} \tag{4.20}$$

and

$$\begin{cases} \bar{d}(\xi) \equiv t d(x,t), \ \xi \equiv x/t, \\ \bar{B}^S(\xi) \equiv t^2 B_x^i(x,t)^2 \boldsymbol{l}_2^*(x,t) R_1''(B^i(x,t)), \ \xi \equiv x/t. \end{cases}$$

Thus, under the condition $\varepsilon \ll 1$ one has that

$$m(\varepsilon) = O(1)\varepsilon, \tag{4.21}$$

and

$$Z(t/2,t) = Z(\frac{\varepsilon}{2}(1+2\varepsilon)(1-\varepsilon)t, \frac{1}{2}(1+2\varepsilon)(1-\varepsilon)t).$$

This leads to

$$Z(t/2,t) \sim \frac{m(\varepsilon)}{\frac{1}{2}(1+2\varepsilon)(1-\varepsilon)t}, \tag{4.22}$$

and this determines the constant D_0:

$$D_0 \equiv \frac{2m(\varepsilon)}{\varepsilon(1+2\varepsilon)(1-\varepsilon)}. \tag{4.23}$$

By (2.8), (2.9), $|d(x,t)| \le |O(1)B_x^i(x,t)|$, and the representation (4.16), one has that

$$\begin{aligned} &|z(t/2,t) - Z(t/2,t)| \\ &= \Big| \int_{\varepsilon^{-2}}^t e^{\int_\tau^t -d(\Xi(\sigma,t/2,t),\sigma)d\sigma} \Big((B_y^v)^2 \boldsymbol{l}_2 R_1''(B^v) \end{aligned}$$

$$
\begin{aligned}
& -(B^i_y)^2\boldsymbol{l}^*_2R''_1(B^i)\Big)(\Xi(\tau,t/2,t),\tau)d\tau\Big| \\
\le\ & O(1)\int_{\varepsilon^{-2}}^{t}\left|\left((B^v_y)^2\boldsymbol{l}_2R''_1(B^v)-(B^i_y)^2\boldsymbol{l}^*_2R''_1(B^i)\right)\right|(\Xi(\tau,x,t),\tau)d\tau \\
=\ & O(1)t^{-3/2}. \qquad (4.24)
\end{aligned}
$$

This and (4.23) conclude (4.17). The estimate (4.18) follows by a similar analysis; we omit its proof. For the proof of the estimate (4.19), we differentiate (4.15) in x, and use the estimate (4.18) for the inhomogeneous term $\lambda^*_{2x}Z$ and (2.9) and (2.10) for the inhomogeneous term $B^v_xB^v_{xx}$. Finally, the estimate (4.19) is proved by the characteristic method. □

The self-similar solution $z(x,t)$ of (2.27) asserts the solution $Z(x,t)$ of the hyperbolic problem is rather regular in a domain close to $x=t/2$. Then, one considers a viscous nonlinear problem without the quadratic nonlinearity in the second equation of (2.25):

$$
\begin{cases}
\partial_tv^O+\lambda^*_2\partial_xv^O+dv^O-\partial_x^2v^O+\partial_x(\chi_{[-\infty,2t/3]}(x)C^2_{22}(v^O)^2)=(B^v_x)^2\boldsymbol{l}_2R''_1, \\
v^O(x,\varepsilon^{-2})\equiv 0.
\end{cases}
\qquad (4.25)
$$

Lemma 4.5. (Outer Nonlinearity) *The solution $v^O(x,t)$ of (4.25) satisfies*

$$
\left|v^O(t/2,t)-\frac{D_0\varepsilon}{t}\right|\le O(1)t^{-3/2}, \qquad (4.26)
$$

where D_0 is given by (4.23).

Proof. By substituting $Z(x,t)$ the solution of (4.15) into (4.25), the difference

$$
\hat v(x,t)\equiv v^O(x,t)-Z(x,t) \qquad (4.27)
$$

satisfies

$$
\begin{cases}
\partial_t\hat v+\lambda^*_2\partial_x\hat v+d\hat v-\partial_x^2\hat v+\partial_x(\chi_{[-\infty,2t/3]}(x)C^2_{22}\hat v(2\hat v+Z))=\partial_x^2Z \\
\hat v(x,\varepsilon^{-2})\equiv 0.
\end{cases}
\qquad (4.28)
$$

Then, one uses the Green's function $G_2(x,t;y,s)$ to represent $\hat{v}(x,t)$ as follows

$$\hat{v}(x,t)=\int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}}G_{2y}(x,t;y,s)(\chi_{[-\infty,2s/3]}(y)C_{22}^2(Z+2\hat{v})\hat{v}+Z_y)(y,s)dyds. \tag{4.29}$$

By the estimate of Z_x in (4.19), one has that

$$\int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}}G_{2y}(x,t;y,s)Z_y(y,s)dyds\le O(1)\left(\log te^{-\frac{(x-t)^2}{8t}}+\frac{\sqrt{t}}{|x-t|^2+t}\right). \tag{4.30}$$

This leads to the ansatz assumption

$$\hat{v}(x,t)\le O(1)\left(\log te^{-\frac{(x-t)^2}{8t}}+\frac{\sqrt{t}}{|x-t|^2+t}\right). \tag{4.31}$$

Under this ansatz assumption and together with the cut-off nonlinearity,

$$\chi_{[-\infty,2s/3]}C_{22}^2(Z+2\hat{v})\hat{v}\ll|Z_y|, \tag{4.32}$$

we have from (3.41), (3.42), and the representation (4.29) that

$$|\hat{v}(t/2,t)|\le O(1)t^{-3/2}. \tag{4.33}$$

This, (4.17), and (4.27) conclude the lemma. □

With the constant D_0 given in (4.23), we now concentrate on the effect of genuine nonlinearity to the solution around the characteristic $x=t$ by considering the following initial value problem for the translated Burgers equation

$$\begin{aligned}&v_t^I+v_x^I+v^Iv_x^I=v_{xx}^I,\\&v^I(x,0)=\begin{cases}0 \text{ for } x\ge 1,\\ \dfrac{\varepsilon D_0}{2(1-x)} \text{ for } x<0.\end{cases}\end{aligned} \tag{4.34}$$

Lemma 4.6.(Inner Nonlinearity) *The solution $v^I(x,t)$ of (4.34) satisfies*

$$\left|v^I(t/2,t)-\varepsilon\frac{D_0}{t}\right|\le\frac{O(1)\varepsilon}{t^{3/2}}. \tag{4.35}$$

Furthermore, the function $v^I(x,t)$ satisfies

$$|v^I(x,t)| \le O(1)\frac{\varepsilon}{\sqrt{t+(x-t)^2}}. \tag{4.36}$$

Proof. The function

$$u(x,t) \equiv 1 + v^I(x,t),$$

satisfies the Burgers equation $u_t + uu_x - u_{xx} = 0$. Then, one can use the Hopf-Cole transformation to express $u(x,t)$ as follows

$$\begin{aligned} u(x,t) &= -2\frac{\partial_x \left\{ e^{-\frac{x}{2}} \left(\int_{-\infty}^0 \frac{e^{-\frac{(x-y-t)^2}{4t}}(1-y)^{\frac{\varepsilon D_0}{4}}}{\sqrt{4\pi t}}dy + \int_0^\infty \frac{e^{-\frac{(x-y-t)^2}{4t}}}{\sqrt{4\pi t}}dy \right) \right\}}{e^{-\frac{x}{2}} \left(\int_{-\infty}^0 \frac{e^{-\frac{(x-y-t)^2}{4t}}(1-y)^{\frac{\varepsilon D_0}{4}}}{\sqrt{4\pi t}}dy + \int_0^\infty \frac{e^{-\frac{(x-y-t)^2}{4t}}}{\sqrt{4\pi t}}dy \right)} \\ &= 1 + \frac{\varepsilon D_0}{2} \frac{\int_{-\infty}^0 \frac{e^{-\frac{(x-y-t)^2}{4t}}(1-y)^{\frac{\varepsilon D_0}{4}-1}}{\sqrt{4\pi t}}dy}{\int_{-\infty}^0 \frac{e^{-\frac{(x-y-t)^2}{4t}}(1-y)^{\frac{\varepsilon D_0}{4}}}{\sqrt{4\pi t}}dy + \int_0^\infty \frac{e^{-\frac{(x-y-t)^2}{4t}}}{\sqrt{4\pi t}}dy}. \end{aligned} \tag{4.37}$$

Thus, for $t \gg 1$,

$$\begin{aligned} v^I(t/2,t) &= u(t/2,t) - 1 \\ &= \frac{\varepsilon D_0}{2} \frac{(1+t/2)^{\frac{\varepsilon D_0}{4}-1}\left(1 + \int_{-t/2}^\infty \frac{e^{-\frac{\eta^2}{4t}}}{\sqrt{4\pi t}}((1+\frac{\eta}{1+\frac{t}{2}})^{\frac{\varepsilon D_0}{4}-1} - 1)d\eta\right)}{(1+t/2)^{\frac{\varepsilon D_0}{4}}\left(1 + \int_{-t/2}^\infty \frac{e^{-\frac{\eta^2}{4t}}}{\sqrt{4\pi t}}((1+\frac{\eta}{1+\frac{t}{2}})^{\frac{\varepsilon D_0}{4}} - 1)d\eta\right)} \\ &= \frac{\varepsilon D_0}{t}\left(1 + O(1)\frac{1}{\sqrt{t}}\right). \end{aligned} \tag{4.38}$$

The same expansion also works for $x < t - \sqrt{t}$ and we have

$$\begin{aligned} v^I(x,t) &= \frac{\varepsilon D_0}{2} \frac{(t-x+1)^{\frac{\varepsilon D_0}{4}-1}\left(1 + \int_{x-t}^\infty \frac{e^{-\frac{\eta^2}{4t}}}{\sqrt{4\pi t}}((1+\frac{\eta}{1+t-x})^{\frac{\varepsilon D_0}{4}-1} - 1)d\eta\right)}{(t-x+1)^{\frac{\varepsilon D_0}{4}}\left(1 + \int_{x-t}^\infty \frac{e^{-\frac{\eta^2}{4t}}}{\sqrt{4\pi t}}((1+\frac{\eta}{1+t-x})^{\frac{\varepsilon D_0}{4}} - 1)d\eta\right)} \\ &= \frac{\varepsilon D_0}{t-x+1}\left(1 + O(1)\frac{\sqrt{t}}{t-x+1}\right). \end{aligned} \tag{4.39}$$

Around the characteristic direction, $|x-t| \le \sqrt{t}$, we consider the following cases:

Case 1. $D_0 < 0$

From

$$\begin{cases} \int_0^\infty \frac{e^{-\frac{(x-y-t)^2}{4t}}}{\sqrt{4\pi t}} dy \ge 1/4, \\ \int_{-\infty}^0 \frac{e^{-\frac{(x-y-t)^2}{4t}}}{\sqrt{4\pi t}} (1-y)^{\frac{\varepsilon D_0}{4}-1} dy \le O(1)/\sqrt{t}, \end{cases} \tag{4.40}$$

we have

$$|v^I(x,t)| \le O(1)\frac{\varepsilon}{\sqrt{t}} \text{ for } |x-t| \le \sqrt{t}. \tag{4.41}$$

Case 2. $D_0 \in (0,1)$.

For any fixed $\alpha \in (-1,\infty)$, one has the following scaling property

$$\int_0^\infty e^{-\frac{x^2}{t}} x^\alpha dx = O(1) t^{(\alpha+1)/2}, \tag{4.42}$$

whence one has, for $|x-t| \le O(1)\sqrt{t}$,

$$\begin{cases} \int_{-\infty}^0 \frac{e^{-\frac{(x-t-y)^2}{4t}}}{\sqrt{4\pi t}} (1-y)^{\frac{\varepsilon D_0}{4}} dy \sim (1+t)^{\frac{\varepsilon D_0}{8}}, \\ \int_{-\infty}^0 \frac{e^{-\frac{(x-y-t)^2}{4t}}}{\sqrt{4\pi t}} (1-y)^{\frac{\varepsilon D_0}{4}-1} dy \sim (1+t)^{\frac{\varepsilon D_0}{8}-\frac{1}{2}}. \end{cases} \tag{4.43}$$

From this one also concludes that

$$|v^I(x,t)| \le O(1)\frac{\varepsilon}{\sqrt{t}} \text{ for } |x-t| \le \sqrt{t}. \tag{4.44}$$

For $x > t+\sqrt{t}$, one has

$$\frac{\int_{-\infty}^0 \frac{e^{-\frac{(x-y-t)^2}{4t}}(1-y)^{\frac{\varepsilon D_0}{4}-1}}{\sqrt{4\pi t}} dy}{\int_{-\infty}^0 \frac{e^{-\frac{(x-y-t)^2}{4t}}(1-y)^{\frac{\varepsilon D_0}{4}}}{\sqrt{4\pi t}} dy + \int_0^\infty \frac{e^{-\frac{(x-y-t)^2}{4t}}}{\sqrt{4\pi t}} dy}$$

$$\leq O(1)\begin{cases} \dfrac{t^{\frac{\varepsilon D_0}{4}-\frac{1}{2}}e^{-\frac{(x-t)^2}{5t}}}{t^{\frac{\varepsilon D_0}{4}}e^{-\frac{(x-t)^2}{5t}}+1} \text{ for } D_0>0, \\ \\ \dfrac{t^{-\frac{1}{2}}e^{-\frac{(x-t)^2}{5t}}}{t^{\frac{\varepsilon D_0}{4}}e^{-\frac{(x-t)^2}{5t}}+1} \text{ for } D_0<0, \end{cases} \tag{4.45}$$

and again we conclude

$$|v^I(x,t)| \leq O(1)\varepsilon \frac{e^{-\frac{(x-t)^2}{5t}}}{\sqrt{t}}. \tag{4.46}$$

This completes the proof of the lemma. □

Remark 4.7. The series of reductions that leads to the problem (4.34) allows us to use the Hopf-Cole transformation for exact estimates of the effect of the rarefaction wave on the transversal genuinely nonlinear field. This exact analysis of the leading term yields, in particular, the lower bound estimate (1.13). Without these reductions, one would need to study problem of the Burgers-like equation with a source, a problem to which the Hopf-Cole transformation in general cannot be applied for definite estimates. Such an approach, however, has been used, c.f. (3.18) to (3.22) in [12].

One denotes by $v^M(x,t)$ the interpolation

$$v^M(x,t) \equiv \chi_+(x-\frac{1}{2}t)v^I(x,t) + \chi_-(x-\frac{1}{2}t)v^O(x,t),$$

where $\chi_\pm$ is a partition of unity with *supp*$(\chi_+') \subset (-1,1)$ and $\chi_+(1)=1$.

Lemma 4.8.(Matching Nonlinearity) *The truncation error of function* $v^M(x,t)$,

$$\mathscr{E}(x,t) \equiv -\left(\partial_t v^M + \lambda_2^* v_x^M + dv^M - v_{xx}^M + ((v^M)^2C_{22}^2)_x - (B_x^v)^2 \boldsymbol{l}_2 R_1''\right), \tag{4.47}$$

satisfies

$$|\mathscr{E}(x,t)| = O(1)\varepsilon \frac{-e^{|x-\frac{t}{2}|}}{t^{3/2}}. \tag{4.48}$$

This lemma is a consequence of (4.26), (4.35), (2.19) and (2.26).

Theorem 4.9 *The solution* $\zeta_p^2(x,t)$ *of* (2.25) *satisfies*

$$|\zeta_p^2(x,t)| \le O(1)\varepsilon \begin{cases} \dfrac{e^{-\frac{(x-t)^2}{5t}}}{\sqrt{t}} \text{ for } x \ge t, \\ \dfrac{1}{\sqrt{t+|x-t|^2}} \text{ for } x \in (-\varepsilon t, t), \\ \dfrac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^2} \text{ for } x < -\varepsilon t. \end{cases} \tag{4.49}$$

Proof. The function

$$w(x,t) \equiv \zeta_p^2(x,t) - v^M(x,t),$$

satisfies

$$\begin{cases} w_t + \lambda_2^* w_x + dw - w_{xx} + (w(2w+v^M)C_{xx}^2)_x = \mathscr{E}, \\ w(x,\varepsilon^{-2}) = 0. \end{cases} \tag{4.50}$$

Similar to (4.29), the representation of $w(x,t)$ is

$$\begin{aligned} w(x,t) \;=\; & \int_{\varepsilon^{-2}}^{t} \int_{\mathbb{R}} G_{2y}(x,t;y,s)(C_{22}^2 w(2w+v^M))(y,s)dyds \\ & + \int_{\varepsilon^{-2}}^{t} \int_{\mathbb{R}} G_2(x,t;y,s)\mathscr{E}(y,s)dyds. \end{aligned} \tag{4.51}$$

Under the property (4.48), one has

$$\int_{\varepsilon^{-2}}^{t} \int_{\mathbb{R}} G_2(x,t;y,s)\mathscr{E}(y,s)dyds \le O(1)\frac{\varepsilon}{\sqrt{t+|x-t|^2}}. \tag{4.52}$$

This leads us to the ansatz assumption for $w(x,t)$:

$$w(x,t) \le O(1)\frac{\varepsilon}{\sqrt{t+|x-t|^2}}. \tag{4.53}$$

By substituting this ansatz assumption into (4.51), one concludes that $\int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_2(x,t;y,s)\mathscr{E}(y,s)dyds$ is the dominant term in the representation (4.51), and that the ansatz assumption (4.53) is justified. This proves the theorem. □

5. Linear and Nonlinear Coupling

We now study the secondary nonlinear system (2.30). Its solution $\vec{\xi}$ satisfies

$$\begin{aligned}
\xi^1(x,t) \;=\; & \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_x(x,t;y,s)\Big(\sum_{(j,k,l)\neq(1,1,1)} C^l_{jk}\zeta^j_p\zeta^k_p + \sum_{k=1}^{2} 2\zeta^k_p \boldsymbol{l}_1\boldsymbol{r}_{ky}\Big) \\
& \qquad -\mathbb{G}(x,t;y,s)\sum_{k=1}^{2} 2\zeta^k_p(\boldsymbol{l}_1\boldsymbol{r}_{ky})_y dyds \\
& + \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} \mathbb{G}(x,t;y,s)\boldsymbol{l}_1\Big(\sum_{k=1}^{2}\Big(\sum_{1\le l,m\le 2}\zeta^l_p\zeta^m_p C^k_{lm}\boldsymbol{r}_{ky} - \boldsymbol{r}_{2y}(\lambda_2 - B^v)\zeta^2_p \\
& \qquad +\boldsymbol{r}_{kyy}\zeta^k_p\Big)\Big)dyds \\
& + \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_x(x,t;y,s)\Big(C^1_{11}\xi^1(2\xi^1+\zeta^1_p) + \sum_{(j,k,l)\neq(1,1,1)}\frac{\zeta^k_p\xi^l + 2\xi^k\xi^l}{2} \\
& \qquad +\sum_{k=1}^{2} 2\xi^k \boldsymbol{l}_1\boldsymbol{r}_{ky}\Big)dyds \\
& + \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} \mathbb{G}(x,t;y,s)\Big(-\boldsymbol{l}_1\boldsymbol{r}_{2x}(\lambda_2 - B^v)\xi^2 + \sum_{k=1}^{2}\Big(-2\xi^k(\boldsymbol{l}_1\boldsymbol{r}_{ky})_y \\
& \qquad - \sum_{1\le l,m\le 2}\frac{\zeta^l_p\xi^m + 2\xi^l\xi^m}{2} C^k_{lm}\boldsymbol{r}_{ky} + \boldsymbol{r}_{kyy}\xi^k\Big)\Big)dyds, \qquad (5.1)
\end{aligned}$$

$$\begin{aligned}
\xi^2(x,t) = & \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_{2y}(x,t;y,s)\Big(\sum_{(j,k,l)\neq(2,2,2)} C^l_{jk}\zeta^j_p\zeta^k_p - (\lambda_2 - \lambda_2^*)\Big)dyds \\
& + \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_2(x,t;y,s)\boldsymbol{l}_2\sum_{k=1}^{2}\Big(\sum_{1\le l,m\le 2} -C^k_{lm}\zeta^l_p\zeta^m_p 2\boldsymbol{r}_{ky} + \boldsymbol{r}_{kyy}\zeta^k_p\Big)dyds \\
& + \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} -G_{2y}(x,t;y,s)\boldsymbol{l}_2\sum_{k=1}^{2} 2\boldsymbol{r}_{ky}\zeta^k_p - G_2(x,t;y,s)\sum_{k=1}^{2} 2(\boldsymbol{l}_2\boldsymbol{r}_{ky})_y\zeta^k_p dyds \\
& + \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_{2y}(x,t;y,s)\Big(C^2_{22}\xi^2(2\xi^2+\zeta^2_p) + \sum_{(j,k,l)\neq(2,2,2)}\frac{C^l_{jk}\zeta^j_p\xi^k + 2\xi^j\xi^k}{2}\Big)dyds
\end{aligned}$$

$$+\int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_2(x,t;y,s)\boldsymbol{l}_2\sum_{k=1}^{2}\Big(\sum_{1\le l,m\le 2}\frac{C_{lm}^k(\zeta_p^l\xi^m+2\xi^l\xi^m)}{2}2\boldsymbol{r}_{ky}+\boldsymbol{r}_{kyy}\xi^k\Big)dyds$$

$$+\int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}}-G_{2y}(x,t;y,s)\boldsymbol{l}_2\sum_{k=1}^{2}2\boldsymbol{r}_{ky}\xi^k-\tilde{G}_2(x,t;y,s)\sum_{k=1}^{2}2(\boldsymbol{l}_2\boldsymbol{r}_{ky})_y\xi^k dyds. \quad (5.2)$$

The representations (5.1) and (5.2) can be expressed in the forms:

$$\xi^1(x,t) = \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_x(x,t;y,s)(\mathbb{S}_a^1(y,s)+\mathbb{U}_a^1[y,s;\vec{\xi}]) +\mathbb{G}(x,t;y,s)(\mathbb{S}_b^1(y,s)+\mathbb{U}_b^1[y,s;\vec{\xi}])dyds, \quad (5.3)$$

$$\xi^2(x,t) = \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_{2y}(x,t;y,s)(\mathbb{S}_a^2(y,s)+\mathbb{U}_a^2[y,s;\vec{\xi}]) +G_2(x,t;y,s)(\mathbb{S}_b^2(y,s)+\mathbb{U}_b^2[y,s;\vec{\xi}])dyds. \quad (5.4)$$

The functions $\mathbb{S}_a^i$, and $\mathbb{S}_b^i$, $i=1,2$ are determined by $\vec{\zeta}_p(x,t)$, estimated in (4.11) and (4.49). $\mathbb{U}_a^i$ and $\mathbb{U}_b^i$, $i=1,2$ are functional of $\vec{\xi}$. The functions $\mathbb{S}_a^i$ and $\mathbb{S}_b^i$ satisfy

$$\mathbb{S}_a^1(x,t)=O(1)((\zeta_p^2)^2+|\zeta_p^1\zeta_p^2|+|B_x^v|(|\zeta_p^1|+|\zeta_p^2|)), \quad (5.5a)$$

$$\mathbb{S}_b^1(x,t)=O(1)\left(|B_x^v\zeta_p^2|+|\vec{\zeta}_p|^2|B_x^v|+|\vec{\zeta}_p|(|B_{xx}^v|+|B_x^v|^2)\right), \quad (5.5b)$$

$$\mathbb{S}_a^2(x,t)=O(1)\left(|\zeta_p^1\zeta_p^2|+(\zeta_p^1)^2+|B^v-B^i||\zeta_p^2||\right), \quad (5.5c)$$

and

$$\mathbb{S}_b^2(x,t)=O(1)|B_{xx}^v\zeta_p^1|. \quad (5.5d)$$

5.1. Lemmas for linear and nonlinear coupling

We list in below the lemmas that will be used later for the study of coupling of waves. Their proofs are by direct calculations and omitted.

Lemma 5.1.(Nonlinear coupling to the primary field)

$$\begin{aligned}
&\int_{\varepsilon^{-2}}^{t}\int_{-\varepsilon s-\sqrt{s}}^{\varepsilon s+\sqrt{s}} G_x(x,t;y,s)\frac{|\log s|^\alpha}{s^2}dyds\\
&= O(1)\begin{cases} \frac{1}{t}e^{-\frac{(x+\varepsilon t)^2}{5t}} & \textit{for } x\le -\varepsilon t+\sqrt{t},\\ \frac{\varepsilon|\log\varepsilon|^{\alpha+1}}{t} & \textit{for } x\in(-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}),\\ \frac{1}{t}e^{-\frac{(x-\varepsilon t)^2}{5t}} & \textit{for } x\ge \varepsilon t-\sqrt{t},\end{cases}
\end{aligned}\tag{5.6}$$

$$\begin{aligned}
&\int_{\varepsilon^{-2}}^{t}\int_{-\varepsilon s-\sqrt{s}}^{\varepsilon s+\sqrt{s}} G_x(x,t;y,s)\frac{1}{s\sqrt{|y-\varepsilon s|^2+s}}dyds\\
&= O(1)\begin{cases} \frac{1}{t}e^{-\frac{(x+\varepsilon t)^2}{5t}} & \textit{for } x\le -\varepsilon t+\sqrt{t},\\ \frac{1}{t} & \textit{for } x\in(-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}),\\ \frac{1}{t}e^{-\frac{(x-\varepsilon t)^2}{5t}} & \textit{for } x\ge \varepsilon t-\sqrt{t},\end{cases}
\end{aligned}\tag{5.7}$$

$$\begin{aligned}
&\int_{\varepsilon^{-2}}^{t}\int_{\varepsilon s-\sqrt{s}}^{s+\sqrt{s}} G_x(x,t;y,s)\frac{1}{(y-s)^2+s}dyds\\
&= O(1)\begin{cases} \frac{1}{\sqrt{t}}e^{-\frac{(x+\varepsilon t)^2}{5t}} & \textit{for } x\le -\varepsilon t+\sqrt{t},\\ \frac{1}{t} & \textit{for } x\in(-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}),\\ \frac{1}{t^{3/4}} & \textit{for } x\in(\varepsilon t-\sqrt{t},\varepsilon t+\sqrt{t}),\\ \frac{1}{\sqrt{(t-x)(x-\varepsilon t)}} & \textit{for } x\in(\varepsilon t+\sqrt{t},t-\sqrt{t}),\\ \frac{1}{t^{3/4}}e^{-\frac{(x-t)^2}{5t}}, & \textit{for } x\ge t-\sqrt{t},\end{cases}
\end{aligned}\tag{5.8}$$

$$\int_{\varepsilon^{-2}}^{t}\int_{\varepsilon s-\sqrt{s}}^{s+\sqrt{s}} G_x(x,t;y,s)\frac{1}{s\sqrt{|y-\varepsilon s|^2+s}}dyds$$

$$= O(1)\begin{cases} \dfrac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t} & \textit{for } x \le -\varepsilon t+\sqrt{t}, \\ \dfrac{1}{t} & \textit{for } x \in (-\varepsilon t+\sqrt{t}, \varepsilon t-\sqrt{t}), \\ \dfrac{\log t}{t} & \textit{for } x \in (\varepsilon t-\sqrt{t}, \varepsilon t+\sqrt{t}), \\ \dfrac{\log t}{(x-\varepsilon t)\sqrt{(t-x)}} & \textit{for } x \in (\varepsilon t+\sqrt{t}, t-\sqrt{t}), \\ \dfrac{\log t}{t^{3/4}} e^{-\frac{(x-t)^2}{5t}} & \textit{for } x \ge t-\sqrt{t}, \end{cases} \tag{5.9}$$

$$\int_{\varepsilon^{-2}}^{t}\int_{\varepsilon s-\sqrt{s}}^{s+\sqrt{s}} G_x(x,t;y,s)\frac{1}{s\sqrt{|y-s|^2+s}}dyds$$

$$= O(1)\begin{cases} \dfrac{1}{t} e^{-\frac{(x+\varepsilon t)^2}{5t}} & \textit{for } x \le -\varepsilon t+\sqrt{t}, \\ \dfrac{1}{t} & \textit{for } x \in (-\varepsilon t+\sqrt{t}, \varepsilon t+\sqrt{t}), \\ \dfrac{\sqrt{t-x}}{t(x-\varepsilon t)} + \dfrac{\log t}{(t-x)\sqrt{(x-\varepsilon t)}} & \textit{for } x \in (\varepsilon t+\sqrt{t}, t-\sqrt{t}), \\ \dfrac{\log t}{t} e^{-\frac{(x-t)^2}{5t}} & \textit{for } x \ge t-\sqrt{t}. \end{cases} \tag{5.10}$$

$$\int_{\varepsilon^{-2}}^{t}\int_{\varepsilon s-\sqrt{s}}^{s+\sqrt{s}} G_x(x,t;y,s)\frac{1}{|y-\varepsilon s|^2+s}dyds$$

$$= O(1)\begin{cases} \dfrac{\log t}{t} e^{-\frac{(x+\varepsilon t)^2}{5t}} & \textit{for } x \le -\varepsilon t, \\ \dfrac{1}{\sqrt{t}t} & \textit{for } x \in (\varepsilon t-\sqrt{t}, \varepsilon t+\sqrt{t}), \\ \dfrac{\log t}{t} + \dfrac{1}{\sqrt{t(\varepsilon t-x)}} & \textit{for } x \in (-\varepsilon t+\sqrt{t}, \varepsilon t-\sqrt{t}), \\ \dfrac{1}{|x-\varepsilon t|} & \textit{for } x \in (\varepsilon t+\sqrt{t}, t-\sqrt{t}), \\ \dfrac{e^{-\frac{(x-t)^2}{5t}}}{t} & \textit{for } x \ge t-\sqrt{t}, \end{cases} \tag{5.11}$$

Lemma 5.2.(Linear coupling to the primary field)

$$\int_{\varepsilon^{-2}}^{t}\int_{-\varepsilon s+\sqrt{s}}^{\varepsilon s-\sqrt{s}}\mathbb{G}(x,t;y,s)\frac{1}{s^{3/2}}\left(\frac{1}{\sqrt{y+\varepsilon s}}+\frac{1}{\sqrt{-y+\varepsilon s}}\right)dyds$$

$$=O(1)\begin{cases}\dfrac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^{\frac{1}{2}}} & \textit{for } x\le -\varepsilon t+\sqrt{t},\\ \left(\dfrac{1}{x+\varepsilon t}+\dfrac{1}{\varepsilon t-x}\right) & \textit{for } x\in(-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}),\\ \dfrac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{t^{\frac{1}{2}}} & \textit{for } x\ge \varepsilon t-\sqrt{t}.\end{cases} \tag{5.12}$$

Lemma 5.3.(Nonlinear coupling to the transverse field, A)

$$\int_{\varepsilon^{-2}}^{t}\int_{-\varepsilon s+\sqrt{s}}^{s-\sqrt{s}}\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{5(t-s)}}}{t-s}\left(\frac{1}{(y-s)^2}+\frac{1}{(y-\varepsilon s)^2}+\frac{1}{(y+\varepsilon s)^2}\right)dyds$$

$$+\int_{\varepsilon^{-2}}^{t}\int_{-\varepsilon s+\sqrt{s}}^{s-\sqrt{s}}\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{5(t-s)}}}{t-s}\Bigg(\frac{1}{\sqrt{((y-s)^2+s)((y+\varepsilon s)^2+s)}}$$

$$+\frac{1}{\sqrt{((y-s)^2+s)((y-\varepsilon s)^2+s)}}\Bigg)dyds$$

$$=\ O(1)\begin{cases}\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t} & \textit{for } x\le -\varepsilon t+\sqrt{t}\\ \frac{1}{\sqrt{(\varepsilon t+x)t}}+\frac{1}{(\varepsilon t-x)^{3/2}} & \textit{for } x\in(-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}),\\ \frac{1}{\sqrt{(t-x)(x+\varepsilon t)}}+\frac{1}{\sqrt{(t-x)(x-\varepsilon t)}} & \textit{for } x\in(\varepsilon t+\sqrt{t},t-\sqrt{t}),\\ \frac{1}{t^{3/4}}e^{-\frac{(x-t)^2}{5t}} & \textit{for } x\ge t-\sqrt{t}.\end{cases} \tag{5.13}$$

Lemma 5.4.(Nonlinear coupling to the transverse field, B)

$$\int_{\varepsilon^{-2}}^{t}\int_{-\varepsilon s+\sqrt{s}}^{\varepsilon s-\sqrt{s}}\frac{e^{-\frac{(y-\Xi(s;x,t))^2}{5(t-s)}}}{\sqrt{t-s}}\left(\frac{(\log s)^2}{s^2}+\frac{1}{(y-\varepsilon s)^2}+\frac{1}{(y+\varepsilon s)^2}\right)\frac{1}{s}dyds$$

$$= O(1)\begin{cases} \dfrac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^2} & \textit{for } x \le -\varepsilon t + \sqrt{t} \\ \dfrac{1}{t} & \textit{for } x \in (-\varepsilon t + \sqrt{t}, \varepsilon t - \sqrt{t}), \\ \frac{1}{(t-x)^{3/2}} & \textit{for } x \in (\varepsilon t + \sqrt{t}, t - \sqrt{t}), \\ \frac{e^{-\frac{(x-t)^2}{5t}}}{\sqrt{t}} & \textit{for } x \ge t + \sqrt{t}. \end{cases} \tag{5.14}$$

5.2. The Global Pointwise Estimates of the Perturbations

Finally, we finish the estimates for the secondary waves in the perturbation of the rarefaction wave. From (5.5), one has

$$\int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_x(x,t;y,s)\mathbb{S}_a^1(y,s) + \mathbb{G}(x,t;y,s)\mathbb{S}_b^1(y,s)dyds$$

$$= O(1)\begin{cases} \varepsilon\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{\sqrt{t}} & \text{for } x < -\varepsilon t - \sqrt{t}, \\ \left(\frac{\varepsilon \log t}{t} + \frac{\varepsilon^2|\log\varepsilon|^2}{\sqrt{t+(x+\varepsilon t)^2}} + \frac{\varepsilon^2|\log\varepsilon|^2}{\sqrt{t+(x-\varepsilon t)^2}}\right) & \text{for } x \in (-\varepsilon t - \sqrt{t}, \varepsilon t + \sqrt{t}), \\ \frac{\varepsilon^2}{\sqrt{(x-\varepsilon t)(t-x)}} & \text{for } x \in (\varepsilon t + \sqrt{t}, t - \sqrt{t}), \\ \varepsilon^2\frac{e^{-\frac{(x-t)^2}{5t}}}{\sqrt{t}} & \text{for } x > t - \sqrt{t}, \end{cases} \tag{5.15}$$

$$\int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_{2y}(x,t;y,s)\mathbb{S}_a^2(y,s) + G_2(x,t;y,s)\mathbb{S}_b^2(y,s)dyds$$

$$\le O(1)\begin{cases} \dfrac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^{3/2}} & \text{for } x \le -\varepsilon t + \sqrt{t}, \\ \dfrac{\varepsilon}{(x-t)^{3/2}} & \text{for } x \in (-\varepsilon t + \sqrt{t}, \varepsilon t + \sqrt{t}), \\ \frac{\varepsilon}{\sqrt{(t-x)(x-\varepsilon t)}} & \text{for } x \in (\varepsilon t + \sqrt{t}, t - \sqrt{t}), \\ \varepsilon\dfrac{e^{-\frac{(x-t)^2}{5t}}}{t^{1/2}} & \text{for } x \ge t - \sqrt{t}. \end{cases} \tag{5.16}$$

With (5.15) and (5.16), one is led to the following ansatz assumption on $\vec{\xi}$:

$$\begin{cases} |\xi^1(x,t)| = O(1)\bar{\Psi}^1(x,t), \\ |\xi^2(x,t)| = O(1)\bar{\Psi}^2(x,t). \end{cases} \tag{5.17}$$

$$\begin{aligned} \bar{\Psi}^1(x,t) \;&\equiv\; \varepsilon\left(\frac{\chi_{[-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)}{\sqrt{t+(x+\varepsilon t)^2}} + \frac{\chi_{[\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)}{\sqrt{t+(x-\varepsilon t)^2}} + \frac{\chi_{[-\varepsilon t,t]}(x)}{\sqrt{(x-\varepsilon t)(t-x)}}\right) \\ &\quad + \frac{\sqrt{\varepsilon}\log t}{t}\chi_{[-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x) \\ &\quad + \varepsilon\left(\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^{1/2}} + \frac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{t^{1/2}} + \frac{e^{-\frac{(x-t)^2}{5t}}}{t^{1/2}}\right), \\ \bar{\Psi}^2(x,t) \;&\equiv\; \varepsilon\left(\frac{\chi_{[-\varepsilon t+\sqrt{t},t-\sqrt{t}]}(x)}{\sqrt{(x-t)(x+\varepsilon t)}} + \frac{\chi_{[\varepsilon t+\sqrt{t},t-\sqrt{t}]}(x)}{\sqrt{(x-t)(x-\varepsilon t)}}\right) \\ &\quad + \varepsilon\left(\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^{3/4}} + \frac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{t^{3/4}} + \frac{e^{-\frac{(x-t)^2}{5t}}}{t^{1/2}}\right) \\ &\quad + O(1)\varepsilon\chi_{[-\varepsilon t,t]}\frac{1}{\sqrt{(x-t)^2+t}} \end{aligned} \tag{5.18}$$

Under the ansatz assumption (5.17) and (5.18), the functions $\mathbb{U}_a^i$ and $\mathbb{U}_b^i$ satisfy

$$\mathbb{U}_a^1[y,s,\vec{\xi}],\ \mathbb{U}_a^2[y,s,\vec{\xi}] = O(1)\mathbb{S}(x,t), \tag{5.19}$$

where

$$\mathbb{S}(x,t) \equiv \begin{cases} \dfrac{\varepsilon^2 e^{-\frac{(y+\varepsilon s)^2}{5s}}}{s} & \text{for } y \le -\varepsilon s, \\ \frac{\varepsilon^2}{s+(y-\varepsilon s)^2} + \frac{\varepsilon^2}{s+(y+\varepsilon s)^2} + \frac{\varepsilon(\log s)^2}{s^2} & \text{for } y \in (-\varepsilon s-\sqrt{s}, \varepsilon s+\sqrt{s}), \\ \dfrac{\varepsilon^2}{(y-\varepsilon s)(s-y)} & \text{for } y \in (\varepsilon s+\sqrt{s}, s-\sqrt{s}), \\ \dfrac{\varepsilon^2 e^{-\frac{(y-s)^2}{5s}}}{s} & \text{for } y \ge s-\sqrt{s}. \end{cases} \tag{5.20}$$

(5.19), Lemma 5.1, (5.13), and (5.14) together yield

$$\begin{cases} \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_x(x,t;y,s)\mathbb{U}_a^1[y,s;\vec{\xi}]dyds \ll \bar{\Psi}^1(x,t), \\ \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_{2y}(x,t;y,s)\mathbb{U}_a^2[y,s;\vec{\xi}]dyds \ll \bar{\Psi}^2(x,t). \end{cases} \tag{5.21}$$

The function $\mathbb{U}_b^1$ is dominated by the linear coupling term in the expressions with a coefficient $\xi^2 \boldsymbol{l}_k \partial_x \boldsymbol{r}_j$:

$$\mathbb{U}_b^1[y,s;\vec{\xi}],\ \mathbb{U}_b^2[y,s;\vec{\xi}] \le O(1)|B_y^v|\bar{\Psi}^2(y,s). \tag{5.22}$$

This results in

$$\begin{cases} \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} \mathbb{G}(x,t;y,s)\mathbb{U}_b^1[y,s;\vec{\xi}]dyds \ll \bar{\Psi}^1(x,t), \\ \int_{\varepsilon^{-2}}^{t}\int_{\mathbb{R}} G_2(x,t;y,s)\mathbb{U}_b^2[y,s;\vec{\xi}]dyds \ll \bar{\Psi}^2(x,t). \end{cases} \tag{5.23}$$

Thus, (5.15), (5.16), (5.22), and (5.23) conclude the ansatz assumption (5.17). From the decomposition (2.29), and the estimates (4.11), (4.49) for the primary waves, and finally, the estimate (5.17) for the secondary waves, we have established the estimates for the perturbation of the rarefaction wave of the same form as our main Theorem 1.1 when the characteristic fields are genuinely nonlinear. We have carried out our analysis when there is only one transversal characteristic field besides the one associated with the rarefaction wave and that transversal field is genuinely nonlinear. In the next section we will consider the case when there is a transversal field that is linearly degenerate.

6. A System with Genuinely Nonlinear and Linearly Degenerate Fields

Finally, we indicate through the Euler equations in gas dynamics the difference between genuinely nonlinear and linear degenerate fields in term of the coupling with the rarefaction wave. For simplicity we consider the polytropic gases:

$$\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = 0, \tag{6.1}$$

$$\boldsymbol{u} \equiv \begin{pmatrix} \rho \\ \rho u \\ \frac{1}{2}\rho(u^2+3\theta) \end{pmatrix}, \ \boldsymbol{F}(\boldsymbol{u}) \equiv \begin{pmatrix} \rho u \\ \rho u^2+\rho\theta \\ \frac{1}{2}\rho u(u^2+5\theta) \end{pmatrix}.$$

The first and third characteristic fields

$$\lambda_1 = u - \sqrt{5\theta/3}, \ \lambda_3 = u + \sqrt{5\theta/3},$$

are genuinely nonlinear, and the 2nd characteristic field

$$\lambda_2 = u$$

is linearly degenerate. This gives rise to three different behavior for the perturbation of a rarefaction wave for the viscous system

$$\boldsymbol{u}_t + \boldsymbol{F}(\boldsymbol{u})_x = \boldsymbol{u}_{xx}. \tag{6.2}$$

Let $(\boldsymbol{u}_-, \boldsymbol{u}_+)$ be a weak 1-rarefaction wave with the property

$$-\lambda_1(\boldsymbol{u}_-) = \lambda_1(\boldsymbol{u}_+) = \varepsilon > 0. \tag{6.3}$$

As before, one uses (2.14) to define the approximate solution $\boldsymbol{v}^a(x,t)$ to approximate the hyperbolic rarefaction wave $R_1(\lambda(x,t))$:

$$\boldsymbol{v}^a(x,t) \equiv R_1(B^v(x,t)); \tag{6.4}$$

and denote

$$\vec{\zeta}(x,t) \equiv \vec{\boldsymbol{u}}(x,t) - \boldsymbol{v}^a(x,t). \tag{6.5}$$

This function $\vec{\zeta}$ satisfies

$$\vec{\zeta}_t + (\boldsymbol{F}'(\boldsymbol{v}^a)\vec{\zeta})_x - \vec{\zeta}_{xx} + N_x = -(B^v_x)^2 R_1''(B^v), \tag{6.6}$$

where

$$N \equiv \boldsymbol{F}(\boldsymbol{v}^a + \vec{\zeta}) - \boldsymbol{F}(\boldsymbol{v}^a) - \boldsymbol{F}'(\boldsymbol{v}^a)\vec{\zeta}.$$

The nonlinear term $N(x,t)$ can be expanded component-wise as

$$\begin{cases} N(x,t) = \sum_{j=1}^{3} N^j(x,t)\boldsymbol{r}_j(x,t), \\ N^j(x,t) \equiv \sum_{1\le k,l\le 3} C^j_{kl}\zeta^k\zeta^l + O(1)|\vec{\zeta}|^3, \\ C^j_{kl}(x,t) \equiv \frac{1}{2}\boldsymbol{l}_j\boldsymbol{F}''(\boldsymbol{v}^a)(\boldsymbol{r}_k,\boldsymbol{r}_l), \\ C^2_{22}(\pm\infty,t) = 0, \\ \boldsymbol{l}_j(x,t) \equiv \boldsymbol{l}_j(\vec{\boldsymbol{v}}^a(x,t)), \boldsymbol{r}_j(x,t) \equiv \boldsymbol{r}_j(\vec{\boldsymbol{v}}^a(x,t)), \\ \vec{\zeta}(x,t) \equiv \sum_{l=1}^{3} \zeta^l(x,t)\boldsymbol{r}_l(x,t). \end{cases} \tag{6.7}$$

The primary nonlinear system for (6.7) is

$$\begin{cases} \zeta^1_{pt} + (B^v\zeta^1_p)_x - \zeta^1_{pxx} + (C^1_{11}(\zeta^1_p)^2)_x = (\boldsymbol{l}_1R''_1)(B^v_x)^2, \\ \qquad\qquad\qquad \text{(primary field)} \\ \zeta^2_{pt} + \lambda^*_2\zeta^2_{px} + d_2(x,t)\zeta^2_p - \zeta^2_{pxx} = (\boldsymbol{l}_2R''_1)(B^v_x)^2, \\ \qquad\qquad\qquad \text{(degenerated transverse field)}, \\ \zeta^3_{pt} + \lambda^*_3\zeta^3_{px} + d_3(x,t)\zeta^3_p - \zeta^3_{pxx} + \left(C^3_{33}(\zeta^3_p)^2\right)_x = (\boldsymbol{l}_3R''_1)(B^v_x)^2, \\ \qquad\qquad\qquad \text{(nonlinear transverse field)}, \end{cases} \tag{6.8}$$
$$\zeta^l_p(x,\varepsilon^{-2}) \equiv 0 \text{ for } l = 1,2.$$

$$\begin{cases} \lambda^*_2(x,t) \equiv \lambda_2(R_1(B^i(x,t))), \\ \lambda^*_3(x,t) \equiv \lambda_3(R_1(B^i(x,t))), \\ d_j(x,t) \equiv \left(\lambda^*_{jx} + (\lambda^*_j - B^i)\boldsymbol{l}_j(R_1(B^i))\boldsymbol{r}_{jx}(R_1(B^i))\right) \text{ for } j = 2,3. \end{cases} \tag{6.9}$$

The second equation in (6.8) is a linear equation with almost constant coefficient transport-diffusion equation, as (3.29) plus a source. The source

$$(\boldsymbol{l}_2R''_1)(B^v_x)^2$$

is non-zero as the coupling coefficient $\boldsymbol{l}_2R''_1$ is non-zero for general constitutive relations, and in particular for the polytropic gases. The main part of the source therefore is of the form

$$C_1\frac{1}{t^2}\le|(\boldsymbol{l}_2R_1'')(B_x^v)^2(x,t)|\le C_2\frac{1}{t^2},\ |x|\le\varepsilon t.$$

Thus the main contribution to ζ_p^2, the solution of the second equation of (6.8), is an integral involving the Green's function $\tilde{G}_2$, like that of (3.32):

$$\int_{\varepsilon^{-2}}^{t}\int_{-\varepsilon s}^{\varepsilon s}\tilde{G}_2(x,t;y,s)\frac{1}{s^2}dyds. \tag{6.10}$$

Direct calculations yield the following upper and lower bounds for the integral and thus for the solution $\zeta_p^2(x,t),\ t>\varepsilon^{-2}$:

$$\begin{aligned}&\frac{1}{C}\varepsilon\log(1+t)\frac{e^{-\frac{(x-\lambda_2^+t)^2}{D_0(1+t)}}}{\sqrt{1+t}}\\ &\qquad\le\ |\zeta_p^2(x,t)|\le C\varepsilon\log(1+t)\frac{e^{-\frac{(x-\lambda_2^+t)^2}{D_0(1+t)}}}{\sqrt{1+t}}\ \text{ for } |x-\lambda_2^+t|\le\sqrt{t},\end{aligned} \tag{6.11}$$

$$|\zeta_p^2(x,t)|\ \le\ C\varepsilon\left(\frac{1}{|x-\lambda_2^+t|}+\log(1+t)\frac{e^{-\frac{(x-\lambda_2^+t)^2}{D_0(1+t)}}}{\sqrt{1+t}}\right)\ \text{ for } x\in(-\varepsilon t,\lambda_2^+t-\sqrt{t}), \tag{6.12}$$

$$|\zeta_p^2(x,t)|\le C\frac{e^{-\frac{(x+\varepsilon t)^2}{D_0(1+t)}}}{(1+t)^2}\qquad\text{for } x\le-\varepsilon t, \tag{6.13}$$

and

$$|\zeta_p^2(x,t)|\le C\frac{e^{-\frac{(x-\lambda_2^+t)^2}{D_0(1+t)}}}{(1+t)^{1/2}}\qquad\text{for } x\ge\lambda_2^-t+\sqrt{t}. \tag{6.14}$$

The component ζ_p^1 is estimated by similar arguments for the proof of Theorem 4.3: for $t\ge\varepsilon^{-2}$

$$\begin{aligned}|\zeta_p^1(x,t)|\ \le\ &O(1)\Bigg(\frac{\log t}{t}+\varepsilon|\log\varepsilon|\Bigg(\frac{\chi_{[-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)}{|x+\varepsilon t|}+\frac{\chi_{[-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)}{|x-\varepsilon t|}\\ &+\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}+e^{-\frac{(x-\varepsilon t)^2}{5t}}}{\sqrt{t}}\Bigg)\Bigg);\end{aligned} \tag{6.15}$$

and one can apply the idea for the proof of Theorem 4.9 to yield the estimate for ζ_p^3:

$$|\zeta_p^3(x,t)| \le O(1)\varepsilon \begin{cases} \dfrac{e^{-\frac{(x-\lambda_3^+ t)^2}{5t}}}{\sqrt{t}} & \text{for } x \ge \lambda_3^+ t, \\ \dfrac{1}{\sqrt{t+|x-\lambda_3^+ t|^2}} & \text{for } x \in (-\varepsilon t, \lambda_3^+ t), \\ \dfrac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^2} & \text{for } x < -\varepsilon t. \end{cases} \tag{6.16}$$

The system for

$$\vec{\xi} \equiv \vec{\zeta} - \sum_{j=1}^{3} \zeta_p^j(x,t)\boldsymbol{r}_j(x,t)$$

satisfies

$$\begin{cases} \xi_t^1 + (B^v\xi^1)_x - \xi_{xx}^1 + \Big(C_{11}^1\xi^1(2\xi^1+\zeta_p^1) + \sum_{(j,k,l)\neq(1,1,1)} C_{jk}^l(\xi^j+\zeta_p^j)(\xi^k+\zeta_p^k)\Big)_x \\ = \boldsymbol{l}_1\Big(\sum_{k=1}^{2}\boldsymbol{r}_{kx}\Big(2(\zeta_p^k+\xi^k)_x - \sum_{1\le l,m\le 2} C_{lm}^k(\zeta_p^l+\xi^l)(\zeta_p^m+\xi^m)\Big) \\ \qquad + \boldsymbol{r}_{kxx}(\zeta_p^k+\xi^k) - \sum_{l=2}^{3}\boldsymbol{r}_{lx}(\lambda_l - B^v)\zeta_p^2\Big), \\ \xi_t^2 + (\lambda_2\xi^2)_x + (\lambda_2 - B^v)\xi^2\boldsymbol{l}_2\boldsymbol{r}_{2x} - \xi_{xx}^2 + \Big(\sum_{(j,k,l)\neq(2,2,2)} C_{jk}^l(\xi^j+\zeta_p^j)(\xi^k+\zeta_p^k)\Big)_x \\ = (\lambda_2-\lambda_2^*)\zeta_{px}^2 + (\lambda_{2x}-\lambda_{2x}^*)\zeta_p^2 \\ + \boldsymbol{l}_2\Big(\sum_{k=1}^{2} 2\boldsymbol{r}_{kx}\Big((\xi^k+\zeta_p^k)_x - \sum_{1\le l,m\le 2} C_{lm}^k(\zeta_p^l+\xi^l)(\zeta_p^m+\xi^m)\Big) + \boldsymbol{r}_{kxx}(\xi^k+\zeta_p^k)\Big), \\ \xi_t^3 + (\lambda_3\xi^3)_x + (\lambda_3 - B^v)\xi^3\boldsymbol{l}_3\boldsymbol{r}_{3x} - \xi_{xx}^3 \\ + \Big(C_{33}^3\xi^3(2\xi^3+\zeta_p^3) + \sum_{(j,k,l)\neq(3,3,3)} C_{jk}^l(\xi^j+\zeta_p^j)(\xi^k+\zeta_p^k)\Big)_x \\ = (\lambda_3-\lambda_3^*)\zeta_{px}^3 + (\lambda_{3x}-\lambda_{3x}^*)\zeta_p^3 \\ + \boldsymbol{l}_3\Big(\sum_{k=1}^{3} 2\boldsymbol{r}_{kx}\Big((\xi^k+\zeta_p^k)_x - \sum_{1\le l,m\le 3} C_{lm}^k(\zeta_p^l+\xi^l)(\zeta_p^m+\xi^m)\Big) + \boldsymbol{r}_{kxx}(\xi^k+\zeta_p^k)\Big), \\ \vec{\xi}(x,\varepsilon^{-2}) = 0. \end{cases} \tag{6.17}$$

Similar to the representation in (5.3) and (5.4), one has that

$$\begin{aligned}\xi^1(x,t) &= \int_{\varepsilon^{-2}}^t \int_{\mathbb{R}} G_x(x,t;y,s)(\mathbb{S}_a^1(y,s)+\mathbb{U}_a^1[y,s;\vec{\xi}]) + \mathbb{G}(x,t;y,s)(\mathbb{S}_b^1(y,s)\\ &\qquad +\mathbb{U}_b^1[y,s;\vec{\xi}])dyds, \qquad (6.18)\\ \xi^2(x,t) &= \int_{\varepsilon^{-2}}^t \int_{\mathbb{R}} G_{2y}(x,t;y,s)(\mathbb{S}_a^2(y,s)+\mathbb{U}_a^2[y,s;\vec{\xi}]) + G_2(x,t;y,s)(\mathbb{S}_b^2(y,s)\\ &\qquad +\mathbb{U}_b^2[y,s;\vec{\xi}])dyds, \qquad (6.19)\\ \xi^3(x,t) &= \int_{\varepsilon^{-2}}^t \int_{\mathbb{R}} G_{3y}(x,t;y,s)(\mathbb{S}_a^3(y,s)+\mathbb{U}_a^3[y,s;\vec{\xi}]) + G_3(x,t;y,s)(\mathbb{S}_b^3(y,s)\\ &\qquad +\mathbb{U}_b^3[y,s;\vec{\xi}])dyds, \qquad (6.20)\end{aligned}$$

where the functions satisfy

$$\begin{cases}\mathbb{S}_a^1(x,t) \le O(1)((\zeta_p^2)^2 + |\zeta_p^1\zeta_p^2| + |B_x^v\zeta_p^2|),\\ \mathbb{S}_b^1(x,t) \le O(1)|B_x^v\zeta_p^2|,\\ \mathbb{S}_a^2(x,t) \le O(1)\left(|\zeta_p^1\zeta_p^2| + (\zeta_p^1)^2\right),\\ \mathbb{S}_b^2(x,t) \le O(1)|B_{xx}^v\zeta_p^1|,\\ \mathbb{S}_a^3(x,t) \le O(1)\left(|\zeta_p^1\zeta_p^2| + (\zeta_p^1)^2\right),\\ \mathbb{S}_b^3(x,t) \le O(1)|B_{xx}^v\zeta_p^1|,\end{cases} \qquad (6.21)$$

and $\tilde{\mathbb{G}}_2$ and $\tilde{\mathbb{G}}_3$ are similar to the $\tilde{\mathbb{G}}_2$ constructed in (3.32).

One makes an ansatz assumption on $\vec{\xi}$,

$$|\xi^i(x,t)| \le O(1)\bar{\Psi}^i(x,t) \text{ for } i=1,2,3 \text{ for } t \ge \varepsilon^{-2}, \qquad (6.22)$$

where

$$\begin{aligned}\bar{\Psi}^1(x,t) \equiv\ & \varepsilon\Bigg(\frac{\chi_{[-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)}{\sqrt{t+(x+\varepsilon t)^2}} + \frac{\chi_{[\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)}{\sqrt{t+(x-\varepsilon t)^2}} + \frac{\chi_{[-\varepsilon t,\lambda_2^+ t-\sqrt{t}]}(x)}{\sqrt{(x-\varepsilon t)(\lambda_2^+ t-x)}}\\ &+\frac{\chi_{[-\varepsilon t,\lambda_3^- t-\sqrt{t}]}(x)}{\sqrt{(x-\varepsilon t)(\lambda_3^+ t-x)}}\Bigg) + \frac{\sqrt{\varepsilon}\log t}{t}\chi_{[-\varepsilon t+\sqrt{t},\varepsilon t-\sqrt{t}]}(x)\end{aligned}$$

$$+\varepsilon\Big(\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^{1/2}}+\frac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{t^{1/2}}+\frac{e^{-\frac{(x-\lambda_2^+t)^2}{5t}}}{t^{1/2}}+\frac{e^{-\frac{(x-\lambda_3^-t)^2}{5t}}}{t^{1/2}}\Big),$$

$$\begin{aligned}\bar\Psi^2(x,t) \;&\equiv\; \varepsilon\Big(\frac{\chi_{[-\varepsilon t+\sqrt{t},\lambda_2^+t-\sqrt{t}]}(x)}{\sqrt{(\lambda_2^+t-x)(x+\varepsilon t)}}+\frac{\chi_{[\varepsilon t+\sqrt{t},\lambda_2^+t-\sqrt{t}]}(x)}{\sqrt{(\lambda_2^+t-x)(x-\varepsilon t)}}\\ &\quad+\frac{\chi_{[-\varepsilon t+\sqrt{t},\lambda_2^+t-\sqrt{t}]}(x)}{\sqrt{(x-\lambda_2^+t)^2+t}}+\frac{\chi_{[\varepsilon t+\sqrt{t},\lambda_3^+t-\sqrt{t}]}(x)}{(\lambda_3^+t-x)^{3/2}}\Big)\\ &\quad+\varepsilon\Big(\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^{3/4}}+\frac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{t^{3/4}}+\frac{e^{-\frac{(x-\lambda_2^+t)^2}{5t}}}{t^{1/2}}+\frac{e^{-\frac{(x-\lambda_3^+t)^2}{5t}}}{t}\Big),\\ \bar\Psi^3(x,t) \;&\equiv\; \varepsilon\Big(\frac{\chi_{[-\varepsilon t+\sqrt{t},\lambda_3^+t-\sqrt{t}]}(x)}{\sqrt{(\lambda_3^+t-x)(x+\varepsilon t)}}+\frac{\chi_{[\varepsilon t+\sqrt{t},\lambda_3^+t-\sqrt{t}]}(x)}{\sqrt{(\lambda_3^+t-x)(x-\varepsilon t)}}\\ &\quad+\frac{\chi_{[-\varepsilon t+\sqrt{t},\lambda_3^+t-\sqrt{t}]}(x)}{\sqrt{(x-\lambda_3^+t)(x-\lambda_2^+t)}}+\frac{\chi_{[\varepsilon t+\sqrt{t},\lambda_3^+t-\sqrt{t}]}(x)}{(\lambda_3^+t-x)}\Big)\\ &\quad+\varepsilon\Big(\frac{e^{-\frac{(x+\varepsilon t)^2}{5t}}}{t^{3/4}}+\frac{e^{-\frac{(x-\varepsilon t)^2}{5t}}}{t^{3/4}}+\frac{e^{-\frac{(x-\lambda_2^+t)^2}{5t}}}{t}+\frac{e^{-\frac{(x-\lambda_3^+t)^2}{5t}}}{t^{1/2}}\Big).\end{aligned}\tag{6.23}$$

Then, by the procedure in Subsection 5.2 the ansatz assumption can be justified. It yields the global estimate (6.22). Finally, by (6.22) and (6.11) have the lower bounded estimates

$$\zeta^2(x,t)\ge\Big(\frac{1}{C}\frac{\log t}{\sqrt{t}}-C\frac{1}{\sqrt{t}}\Big)e^{-\frac{(x-\lambda_2^+t)^2}{5t}}\text{ for }|x-\lambda_2^+|\le\sqrt{t}.\tag{6.24}$$

This proves the main theorem Theorem 1.1 in this case.

Remark 6.5. There are important differences between genuinely nonlinear 3-characteristic field and linear degenerate 2-characteristic field: The perturbation decays at the rate $t^{-1/2}$ for 3-waves; and $t^{-1/2}\log t$ for 2-waves. The Burgers nonlinearity for the 3-characteristic field induces this faster decay rate as we have seen in Section 4 and Section 5. For the linearly degenerate 2-characteristic field, there is no quadratic Burgers term. Thus the main contribution is the integral (6.10) with the Green's function $\tilde G_2$ similar to the heat kernel with speed λ_2. This gives the $t^{-1/2}\log t$ rate. The fact that there is no quadratic nonlinear term as the source also allows ansatz (6.22) to be closed such a lower decay rate.

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Institute of Mathematics, Academia Sinica, Taipei and Department of Mathematics, Stanford University.

E-mail: liu@math.stanford.edu

Department of Mathematics, National University of Singapore.

E-mail: matysh@nus.edu.sg