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EIGENVALUE INTERVALS FOR TWO-POINT GENERAL THIRD ORDER DIFFERENTIAL EQUATION

BY

K. R. PRASAD, A. KAMESWARA RAO AND P. MURALI

Abstract

Values of the parameter λ are determined for which there exist positive solutions to the third order eigenvalue problem satisfying general two-point boundary conditions. We establish the results by applying cone theory and the Krasnosel'skii fixed point theorem.

1. Introduction

We are concerned with determining eigenvalues, λ, for which there exist positive solutions with respect to the cone, of the general nonlinear two-point boundary value problem

$$y'''(t) + \lambda f(t,\ y,\ y',\ y'') = 0, \quad t \in [a,b], \tag{1.1}$$

$$\begin{aligned} \alpha_{11}y(a) + \alpha_{12}y(b) &= 0 \\ \alpha_{21}y'(a) + \alpha_{22}y'(b) &= 0 \\ \alpha_{31}y''(a) + \alpha_{32}y''(b) &= 0 \end{aligned} \tag{1.2}$$

where the coefficients α_{11}, α_{12}, α_{21}, α_{22}, α_{31}, α_{32} are real constants. The BVPs of this form arise in the modeling of nonlinear diffusions generated by nonlinear sources, in thermal ignition of gases, and in concentration in

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chemical or biological problems. In these applied settings, only positive solutions are meaningful.

The study of determining the values of the parameter, λ, for which there exist positive solutions was first employed by Erbe and Wang [7] when they worked to establish the existence of positive solutions in a cone for boundary value problems (BVPs) for second order ordinary differential equations. Continuing in a similar manner, Erbe, Hu and Wang [6] along with Eloe and Henderson [4], Henderson and Wang [5] obtained further results. In addition, the work done by Erbe and Wang [7], the extensions can be viewed in the following studies [2, 3, 10, 12, 15]. Sun and Wen [16] extended the results to third order eigenvalue problem,

$$u'''(t) = \lambda a(t) f(u(t)), \quad 0 < t < 1$$

subject to the two-point boundary conditions

$$\begin{aligned} \alpha u'(0) - \beta u''(0) &= 0, \\ u(1) = u'(1) &= 0 \end{aligned}$$

and establish multiple positive solutions by utilizing Krasnosel'skii fixed point theorem. We extend these results to general two-point boundary value problems in the interval $[a, b]$, where $b > a \geq 0$, and also involving the derivatives of y in f. Some of the previous results will be subcases of our problem. We use the following notation for simplicity,

$$\gamma_i = \alpha_{i1} + \alpha_{i2}, \quad i = 1, 2, 3 \text{ and } \beta_i = a\alpha_{i1} + b\alpha_{i2}, \quad i = 1, 2.$$

Throughout this paper we assume the following:

(A1) $f : [a, b] \times {\mathbb{R}^+}^3 \to \mathbb{R}^+$ is continuous, where $\mathbb{R}^+$ is the set of nonnegative real numbers.

(A2) $\alpha_{11} > 0, \quad \alpha_{12} < 0, \quad \alpha_{21} > 0, \quad \alpha_{22} < 0, \quad \alpha_{31} < 0, \quad \alpha_{32} > 0, \quad \gamma_1 > 0, \quad \gamma_2 > 0, \quad \gamma_3 > 0.$

(A3) $\frac{\beta_2}{\gamma_2} - \frac{\alpha_{22}\gamma_3}{\alpha_{32}\gamma_2}(b - a) \leq a$ and $\frac{\beta_2}{\gamma_2} + \frac{\alpha_{21}\gamma_3}{\alpha_{31}\gamma_2}(b - a) \leq a$.

(A4) $\frac{-\alpha_{11}}{2\gamma_1} + \frac{\alpha_{21}}{\gamma_2} - \frac{\alpha_{31}}{2\gamma_3} < 0, \quad \frac{\alpha_{12}}{2\gamma_1} - \frac{\alpha_{22}}{\gamma_2} + \frac{\alpha_{32}}{2\gamma_3} < 0.$

We define the nonnegative extended real numbers f_0, f^0, f_∞ and f^∞ by

$$f_0 = \lim_{(y,y',y'')\to(0^+,0^+,0^+)} \min_{t\in[a,b]} \frac{f(t,y,y',y'')}{y},$$
$$f^0 = \lim_{(y,y',y'')\to(0^+,0^+,0^+)} \max_{t\in[a,b]} \frac{f(t,y,y',y'')}{y},$$
$$f_\infty = \lim_{(y,y',y'')\to(\infty,\infty,\infty)} \min_{t\in[a,b]} \frac{f(t,y,y',y'')}{y},$$

$$f^\infty = \lim_{(y,y',y'')\to(\infty,\infty,\infty)} \max_{t\in[a,b]} \frac{f(t,y,y',y'')}{y},$$

and assume that they will exist.

The rest of the paper is organized as follows. In Section 2, as a fundamental importance, we estimate the bounds for the Green's function. In Section 3, we present some fundamental lemmas which are needed in the main result as well as establish the existence of eigenvalue intervals for which the two point BVP (1.1)−(1.2) has a positive solution, by using Krasnosel'skii fixed point Theorem. Finally, we give an example to demonstrate our result as an application.

2. Green's Function and Bounds

In this section, we estimate the bounds of the Green's function for the homogeneous two-point BVP corresponding to (1.1)−(1.2).

The Green's function for the homogeneous problem $-y''' = 0$, satisfying the boundary conditions (1.2) can be constructed after computation and is given by

$$G(t,s) = \begin{cases} \frac{\alpha_{12}\gamma_2\gamma_3(b-s)^2+2\alpha_{22}\gamma_3(-\beta_1+t\gamma_1)(b-s)+\alpha_{32}(A-2t\gamma_1\beta_2+t^2\gamma_1\gamma_2)}{2\gamma_1\gamma_2\gamma_3} & \mathrm{a\le t\le s\le b} \\ \frac{-\alpha_{11}\gamma_2\gamma_3(s-a)^2+2\alpha_{21}\gamma_3(-\beta_1+t\gamma_1)(s-a)-\alpha_{31}(A-2t\gamma_1\beta_2+t^2\gamma_1\gamma_2)}{2\gamma_1\gamma_2\gamma_3} & \mathrm{a\le s\le t\le b.} \end{cases} \tag{2.1}$$

where $A = 2\beta_1\beta_2 - \gamma_2(a^2\alpha_{11} + b^2\alpha_{12})$. We now state two Lemmas for minimum and maximum values of Green's function.

Lemma 2.1. *For $t < s$, $G(t,s)$ attains minimum value at*

$$t = \frac{\alpha_{12}\alpha_{32}\gamma_2\beta_2 - b\alpha_{12}\alpha_{22}\gamma_2\gamma_3 - \alpha_{22}^2\gamma_3\beta_1 + b\alpha_{12}\alpha_{22}\gamma_2\gamma_3}{\alpha_{12}\alpha_{32}\gamma_2^2 - \alpha_{22}^2\gamma_1\gamma_3}, \quad and$$
$$s = \frac{\alpha_{22}\alpha_{32}\gamma_1\beta_2 - b\alpha_{22}^2\gamma_1\gamma_3 - \alpha_{22}\alpha_{32}\gamma_2\beta_1 + b\alpha_{12}\alpha_{32}\gamma_2^2}{\alpha_{12}\alpha_{32}\gamma_2^2 - \alpha_{22}^2\gamma_1\gamma_3}.$$

And also, for $s < t$, $G(t,s)$ attains minimum value at

$$t = \frac{\alpha_{11}\alpha_{31}\gamma_2\beta_2 - a\alpha_{11}\alpha_{21}\gamma_2\gamma_3 - \alpha_{21}^2\gamma_3\beta_1 + a\alpha_{11}\alpha_{21}\gamma_2\gamma_3}{\alpha_{11}\alpha_{31}\gamma_2^2 - \alpha_{21}^2\gamma_1\gamma_3}, \quad and$$
$$s = \frac{\alpha_{21}\alpha_{31}\gamma_1\beta_2 - a\alpha_{21}^2\gamma_1\gamma_3 - \alpha_{21}\alpha_{31}\gamma_2\beta_1 + a\alpha_{11}\alpha_{31}\gamma_2^2}{\alpha_{11}\alpha_{31}\gamma_2^2 - \alpha_{21}^2\gamma_1\gamma_3}.$$

Lemma 2.2. *Assume that the condition $(A4)$ holds, then $G(s,s)$ has a maximum value at*

$$s = \frac{b\alpha_{12}\gamma_2\gamma_3 - b\alpha_{22}\gamma_1\gamma_3 - \alpha_{22}\gamma_3\beta_1 + \alpha_{32}\gamma_1\beta_2}{\alpha_{12}\gamma_2\gamma_3 - 2\alpha_{22}\gamma_1\gamma_3 + \alpha_{32}\gamma_1\gamma_2}.$$

Theorem 2.1. *Let $G(t,s)$ be the Green's function for the homogeneous BVP corresponding to* (1.1)−(1.2), *then*

$$mG(s,s) \le G(t,s) \le G(s,s), \quad for\ all\ (t,s) \in [a,b]\times[a,b] \tag{2.2}$$

where $0 < m = \min\{m_1, m_2\} \le 1$.

Proof. The Green's function $G(t,s)$ for the homogeneous problem of the BVP (1.1)−(1.2) is given in (2.1). Clearly

$$G(t,s) > 0 \quad on\ [a,b]\times[a,b]. \tag{2.3}$$

First we establish the right side inequality by assuming the conditions given by (A2)−(A3). For $t < s$,

$$\begin{aligned} G(t,s) &\le \frac{\alpha_{12}}{2\gamma_1}(b-s)^2 + \frac{\alpha_{22}}{\gamma_2}\left(\frac{-\beta_1}{\gamma_1} + s\right)(b-s) + \frac{\alpha_{32}}{2\gamma_3}\left(\frac{A}{\gamma_1\gamma_2} - 2s\frac{\beta_2}{\gamma_2} + s^2\right) \\ &= G(s,s), \end{aligned}$$

and for $s < t$,

$$\begin{aligned} G(t,s) &\le -\frac{\alpha_{11}}{2\gamma_1}(s-a)^2+\frac{\alpha_{21}}{\gamma_2}\left(\frac{-\beta_1}{\gamma_1}+s\right)(s-a)-\frac{\alpha_{31}}{2\gamma_3}\left(\frac{A}{\gamma_1\gamma_2}-2s\frac{\beta_2}{\gamma_2}+s^2\right) \\ &= G(s,s). \end{aligned}$$

Thus we have established the right hand side inequality of (2.2). By assuming the conditions given by (A2)$-$(A4), we establish the other inequality.

For $t < s$, from Lemma 2.1 and Lemma 2.2, we have

$$\frac{G(t,s)}{G(s,s)} \ge \frac{\min G(t,s)}{\max G(s,s)} = m_1$$

and for $s < t$, we have

$$\frac{G(t,s)}{G(s,s)} \ge \frac{\min G(t,s)}{\max G(s,s)} = m_2.$$

Therefore,

$$mG(s,s) \le G(t,s), \quad \text{for all} \quad (t,s) \in [a,b]\times[a,b],$$

where $0 < m = \min\{m_1, m_2\} \le 1$. □

3. Existence of Positive Solutions

In this section, first we prove some fundamental lemmas which are needed in main result and then, establish a criteria to determine the value of λ for the existence of at least one positive solution of the BVP given by (1.1)$-$(1.2).

Let $y(t)$ be the solution of a two-point BVP (1.1)$-$(1.2), and is given by

$$y(t) = \lambda\int_a^b G(t,s)f(s,y(s),y'(s),y''(s))ds, \quad \text{for all } t\in[a,b]. \tag{3.1}$$

Define

$$X = \left\{\, u \mid u \in C^3[a,b] \,\right\},$$

with

$$\| u \| = \max_{t\in[a,b]} | u(t) | .$$

Now $(X, \| . \|)$ is a Banach space. Define a set κ by

$$\kappa = \left\{ u \in X : u(t) \geq 0 \text{ on } [a,b] \text{ and } \min_{t\in[a,b]} u(t) \geq m \| u \| \right\}. \tag{3.2}$$

Definition 3.1. Let X be a Banach space. A nonempty closed convex set κ is called **cone** of X, if it satisfies the following conditions:

(1) $\alpha_1 u + \alpha_2 v \in \kappa$ for all $u, v \in \kappa$ and $\alpha_1, \alpha_2 \geq 0$;

(2) If $u \in \kappa$ and $-u \in \kappa$, then $u = 0$.

Definition 3.2. Let X and Y be Banach spaces and $T : X \to Y$. T is said to be completely continuous, if T is continuous, and for each bounded sequence $\{x_n\} \subset X$, $\{Tx_n\}$ has a convergent subsequence.

Lemma 3.1. *κ is a cone in X, where κ is defined by the equation* (3.2).

Proof. Let $\{u_n\} \in \kappa$ be such that $\| u_n - u_0 \| \to 0$ as $n \to \infty$, where $u_0 \in X$. Then $u_n(t) \geq 0$ on $[a,b]$, and $\min\{u_n(t)\} \geq m \| u_n \|$, for all n. Thus, given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $-\epsilon < u_n(t) - u_0(t) < \epsilon$, $t \in [a,b]$, $n \geq N$ and so $0 \leq u_n(t) \leq u_0(t) + \epsilon$, $t \in [a,b]$, $n \geq N$. Hence $u_0(t) \geq 0$ on $[a,b]$, then $\lim_{n\to\infty} \min\{u_n(t)\} \geq m \lim_{n\to\infty} \| u_n \|$ and $\min u_0(t) \geq m \| u_0 \|$, $t \in [a,b]$, implies $u_0 \in \kappa$ and κ is closed.

Now let $u, v \in \kappa$ and $\alpha_1, \alpha_2 \geq 0$. Then $\alpha_1 u(t) + \alpha_2 v(t) \geq 0$, $t \in [a,b]$, and

$$\begin{aligned} \min\{\alpha_1 u(t) + \alpha_2 v(t)\} &\geq \alpha_1 \min\{u(t)\} + \alpha_2 \min\{v(t)\} \\ &\geq \alpha_1 m \| u \| + \alpha_2 m \| v \| \\ &\geq m \| \alpha_1 u + \alpha_2 v \| . \end{aligned}$$

Therefore, $\alpha_1 u + \alpha_2 v \in \kappa$. Finally, if $u \in \kappa$ and $-u \in \kappa$ then $u(t) = 0$ for all $t \in [a,b]$. Hence the proof. □

Define the operator $T : \kappa \to X$ by

$$(Ty)(t) = \lambda \int_a^b G(t,s) f(s, y(s), y'(s), y''(s)) ds, \quad \text{for all } t \in [a,b]. \tag{3.3}$$

If $y \in \kappa$ is a fixed point of T, then y satisfies (3.1) and hence y is a positive solution of the BVP (1.1)−(1.2). We seek the fixed point of the operator T in the cone κ.

Lemma 3.2. *The operator T is defined in* (3.3) *is self map on κ.*

Proof. Let $y \in \kappa$. From (2.3), we have $(Ty)(t) \geq 0$ for all $t \in [a, b]$. Then

$$\begin{aligned}(Ty)(t) &= \lambda \int_a^b G(t,s) f(s, y(s), y'(s), y''(s))ds \\ &\geq \lambda \int_a^b mG(s,s) f(s, y(s), y'(s), y''(s))ds \\ &\geq \lambda m \int_a^b \max_{t\in[a,b]} G(t,s) f(s, y(s), y'(s), y''(s))ds \\ &\geq m \max_{t\in[a,b]} \lambda \int_a^b G(t,s) f(s, y(s), y'(s), y''(s))ds \\ &= m \parallel Ty \parallel .\end{aligned}$$

Therefore,

$$\min_{t\in[a,b]} (Ty)(t) \geq m \parallel Ty \parallel .$$

Hence the proof. □

Lemma 3.3. *The operator T is completely continuous, where T is defined in* (3.3).

Proof. Let $y \in \kappa$ and $\epsilon > 0$ be given. By the continuity of f, there exists $\delta > 0$ such that

$$\mid f(t, y, y', y'') - f(t, w, w', w'') \mid < \epsilon,$$

whenever $\mid y - w \mid < \delta$, $\mid y' - w' \mid < \delta$, and $\mid y'' - w'' \mid < \delta$.

$$\begin{aligned}\mid (Ty)(t) - (Tw)(t) \mid \; &= \; \lambda \int_a^b G(t,s) \mid f(s, y, y', y'') - f(s, w, w', w'') \mid ds \\ &\leq \; \epsilon\lambda \int_a^b G(t,s) ds, \quad t \in [a,b].\end{aligned}$$

Thus,

$$\| (Ty)(t) - (Tw)(t) \| \leq \epsilon\lambda \int_a^b G(t,s)ds$$

and T is continuous. Now, let $\{y_n\}$ be a bounded sequence in κ. Since f is continuous, there exists $N > 0$, such that $| f(t, y(t), y'(t), y''(t)) | \leq N$ for all y with $0 \leq y < \infty$ then, for each $t \in [a, b]$ and for each n,

$$\begin{aligned} | (Ty_n)(t) | &= | \lambda \int_a^b G(t,s) f(s, y_n, y_n', y_n'')ds | \\ &\leq \lambda \int_a^b | G(s,s) || f(s, y_n, y_n', y_n'') | \, ds \\ &\leq N\lambda \int_a^b G(s,s)ds \end{aligned}$$

By choosing successive subsequences, there exists a subsequence $\{Ty_{n_j}\}$ which converges uniformly on $[a, b]$. Hence T is completely continuous. □

Theorem 3.1. (**Krasnosel'skii**) [13] *Let X be a Banach space, $K \subseteq X$ be a cone, and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$ is completely continuous operator such that either*

(i) $\| Tu \| \leq \| u \|$, $u \in K \cap \partial\Omega_1$ *and* $\| Tu \| \geq \| u \|$, $u \in K \cap \partial\Omega_2$, *or*
(ii) $\| Tu \| \geq \| u \|$, $u \in K \cap \partial\Omega_1$ *and* $\| Tu \| \leq \| u \|$, $u \in K \cap \partial\Omega_2$

holds. Then T has a fixed point in $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

Theorem 3.2. *Assuming the conditions* (A1)−(A4) *hold and if*

$$\frac{1}{[m^2 \int_a^b G(s,s)ds] f_\infty} < \lambda < \frac{1}{[\int_a^b G(s,s)ds] f^0}, \tag{3.4}$$

then the two-point BVP (1.1)−(1.2) *has at least one positive solution in κ.*

Proof. Let λ be given as in (3.4) and let $\epsilon > 0$ be such that

$$\frac{1}{[m^2 \int_a^b G(s,s)ds](f_\infty - \epsilon)} \leq \lambda \leq \frac{1}{[\int_a^b G(s,s)ds](f^0 + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined as in

(3.3). By the definition of f^0, there exists $H_i^1 > 0$ such that

$$\max_{t\in[a,b]} \frac{f(t,y,y',y'')}{y} \leq (f^0+\epsilon), \quad \text{for} \ \ 0 < y^{(i)} \leq H_i^1, \ i = 0,1,2.$$

Let $H^1 = \min\{H_i^1 : i = 0,1,2\}$. It follows that

$$f(t,y,y',y'') \leq (f^0+\epsilon)y, \quad \text{for} \ \ 0 < y^{(i)} \leq H^1, i = 0,1,2.$$

Let us choose $y \in \kappa$ with $\| y \|= H^1$. Then, from (2.2),

$$\begin{aligned}(Ty)(t) &= \lambda \int_a^b G(t,s) f(s,y,y',y'')ds \\ &\leq \lambda \int_a^b G(s,s) f(s,y,y',y'')ds \\ &\leq \lambda \int_a^b G(s,s)(f^0+\epsilon)y(s)ds \\ &\leq \lambda \int_a^b G(s,s)(f^0+\epsilon) \parallel y \parallel ds \\ &\leq \parallel y \parallel, \quad t \in [a,b].\end{aligned}$$

Therefore, $\| Ty \| \leq \| y \|$. Hence, if we set

$$\Omega_1 = \{u \in X : \| u \| < H^1\},$$

then

$$\| Ty \| \leq \| y \|, \ \text{for} \ \ y \in \kappa \cap \partial\Omega_1. \tag{3.5}$$

By the definition of f_∞, there exists $\overline{H}_i^2 > 0$ such that

$$\min_{t\in[a,b]} \frac{f(t,y,y',y'')}{y} \geq (f_\infty - \epsilon), \quad \text{for} \ \ y^{(i)} \geq \overline{H}_i^2, i = 0,1,2.$$

Let $\overline{H}^2 = \max\{\overline{H}_i^2 : i = 0,1,2\}$. It follows that

$$f(t,y,y',y'') \geq (f_\infty - \epsilon)y, \quad \text{for} \ \ y^{(i)} \geq \overline{H}^2, i = 0,1,2.$$

Let

$$H^2 = \max\left\{2H^1, \frac{1}{m}\overline{H}^2\right\},$$

and define

$$\Omega_2 = \{u \in X : \| u \| < H^2\}.$$

If $y \in \kappa \cap \partial\Omega_2$, so that $\| y \| = H^2$, then

$$\min_{t\in[a,b]} y(t) \geq m \| y \| \geq \overline{H}^2.$$

Consider,

$$\begin{aligned}
(Ty)(t) &= \lambda \int_a^b G(t,s) f(s,y,y',y'')ds \\
&\geq \lambda \int_a^b mG(s,s) f(s,y,y',y'')ds \\
&\geq m\lambda \int_a^b G(s,s)(f_\infty - \epsilon) y(s)ds \\
&\geq m^2\lambda \int_a^b G(s,s)(f_\infty - \epsilon) \| y \| \, ds \\
&\geq \| y \| .
\end{aligned}$$

Thus, $\| Ty \| \geq \| y \|$, and so

$$\| Ty \| \geq \| y \|, \text{for } \; y \in \kappa \cap \partial\Omega_2. \tag{3.6}$$

An application of Theorem (3.1) to (3.5) and (3.6) yields a fixed point of T that lies in $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is the positive solution of the two-point BVP (1.1)−(1.2). □

Theorem 3.3. *Assuming the conditions* (A1)−(A4) *hold and if*

$$\frac{1}{[m^2 \int_a^b G(s,s)ds] f_0} < \lambda < \frac{1}{[\int_a^b G(s,s)ds] f^\infty}, \tag{3.7}$$

then the two-point BVP (1.1)−(1.2) *has at least one positive solution in* κ.

Proof. Let λ be given as in (3.7) and let $\epsilon > 0$ be such that

$$\frac{1}{[m^2 \int_a^b G(s,s)ds](f_0 - \epsilon)} \leq \lambda \leq \frac{1}{[\int_a^b G(s,s)ds](f^\infty + \epsilon)}.$$

Let T be the cone preserving, completely continuous operator defined as in

(3.3). By the definition of f_0, there exists $J_i^1 > 0$ such that

$$\min_{t\in[a,b]} \frac{f(t,y,y',y'')}{y} \geq f_0 - \epsilon, \quad \text{for} \quad 0 < y^{(i)} \leq J_i^1, i = 0,1,2.$$

Let $J^1 = \min\{J_i^1 : i = 0,1,2\}$. It follows that

$$f(t,y,y',y'') \geq (f_0 - \epsilon)y, \quad \text{for} \quad 0 < y^{(i)} \leq J^1, i = 0,1,2.$$

In this case, define $\Omega_1 = \{u \in X : \parallel u \parallel < J^1\}$. Then, for $y \in \kappa \cap \partial\Omega_1$, we have $f(s,y,y',y'') \geq (f_0 - \epsilon)y, \ \ s \in [a,b]$, and moreover, $y(t) \geq m \parallel y \parallel, \ \ t \in [a,b]$, and we have

$$\begin{aligned}
(Ty)(t) &= \lambda \int_a^b G(t,s) f(s,y,y',y'')ds \\
&\geq \lambda \int_a^b mG(s,s) f(s,y,y',y'')ds \\
&\geq m\lambda \int_a^b G(s,s)(f_0 - \epsilon)y(s)ds \\
&\geq m^2\lambda \int_a^b G(s,s)(f_0 - \epsilon) \parallel y \parallel ds \\
&\geq \parallel y \parallel .
\end{aligned}$$

Thus, $\parallel Ty \parallel \geq \parallel y \parallel$, and so

$$\parallel Ty \parallel \geq \parallel y \parallel, \quad \text{for} \quad y \in \kappa \cap \partial\Omega_1. \tag{3.8}$$

It remains for us to consider f^∞. By the definition of f^∞, there exists $\overline{J}_i^2 > 0$ such that

$$\max_{t\in[a,b]} \frac{f(t,y,y',y'')}{y} \leq (f^\infty + \epsilon), \quad \text{for} \quad y^{(i)} \geq \overline{J}_i^2, i = 0,1,2.$$

Let $\overline{J}^2 = \max\{\overline{J}_i^2 : i = 0,1,2\}$. It follows that

$$f(t,y,y',y'') \leq (f^\infty + \epsilon)y, \ \text{for} \quad y^{(i)} \geq \overline{J}^2, i = 0,1,2.$$

There are two cases.

Case(i). f is bounded. Suppose $L > 0$ is such that $\max_{t\in[a,b]} f(t,y,y',y'') \leq$

L, for all $0 < y < \infty$, $0 < y' < \infty$, $0 < y'' < \infty$. Let

$$J^2 = \max\left\{2J^1, L\lambda\int_a^b G(s,s)ds\right\},$$

and let

$$\Omega_2 = \{u \in X :\| u \|< J^2\}.$$

Then, for $y \in \kappa \cap \partial\Omega_2$, we have

$$\begin{aligned}(Ty)(t) &= \lambda\int_a^b G(t,s)f(s,y,y',y'')ds\\ &\leq \lambda\int_a^b G(s,s)f(s,y,y',y'')ds\\ &\leq \lambda L\int_a^b G(s,s)ds\\ &\leq \| y \|, \quad t \in [a,b].\end{aligned}$$

and so

$$\| Ty \| \leq \| y \|, \text{ for } \ y \in \kappa \cap \partial\Omega_2. \tag{3.9}$$

Case(ii). f is unbounded. Let $J_i^2 > \max\{2J_i^1, \overline{J}_i^2\}$ be such that $f(t,u,u',u'') \leq f(t,J_0^2,J_1^2,J_2^2)$, for $0 < u^{(i)} \leq J_i^2$, $i = 0,1,2$. Let $J^2 = \max\{J_i^2 : i = 0,1,2\}$, and let

$$\Omega_2 = \{u \in X :\| u \|< J^2\}.$$

Choosing $y \in \kappa \cap \partial\Omega_2$,

$$\begin{aligned}(Ty)(t) \ &= \ \lambda\int_a^b G(t,s)f(s,y,y',y'')ds\\ &\leq \ \lambda\int_a^b G(s,s)f(s,y,y',y'')ds\\ &\leq \ \lambda\int_a^b G(s,s)f(s,J_0^2,J_1^2,J_2^2)ds\\ &\leq \ \lambda\int_a^b G(s,s)(f^\infty+\epsilon)y(s)ds\\ &\leq \ \lambda\int_a^b G(s,s)(f^\infty+\epsilon) \| y \| \, ds\end{aligned}$$

$$\leq \| y \|, \quad t \in [a, b].$$

And so

$$\| Ty \| \leq \| y \|, \text{ for } \; y \in \kappa \cap \partial\Omega_2. \tag{3.10}$$

An application of Theorem (3.1) to (3.8), (3.9) and (3.10) yields a fixed point of T that lies in $\kappa \cap (\overline{\Omega}_2 \backslash \Omega_1)$. This fixed point is the positive solution of the two-point BVP (1.1)−(1.2). □

4. Example

Now, we give an example to illustrate the above results. Consider the following two-point eigenvalue problem

$$y''' + \lambda y(20 - 19.5e^{-7y})(25 - 24e^{-5y'})(72 - 71e^{-3y''}) = 0, t \in [0, 1] \tag{4.1}$$

with the boundary conditions

$$\begin{aligned} 5y(0) - \frac{9}{2}y(1) &= 0 \\ 3y'(0) - 2y'(1) &= 0 \\ y''(0) - 2y''(1) &= 0. \end{aligned} \tag{4.2}$$

We found that $m = 0.1757$, $f_\infty = 18000$, and $f^0 = 1$. Employing Theorem 3.2, we get the optimal eigenvalue interval $0.0000493 < \lambda < 0.02739$, for which (4.1)−(4.2) has a positive solution.

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Department of Applied Mathematics, Andhra University, Visakhapatnam 530003, India.
E-mail: rajendra92@rediffmail.com

E-mail: kamesh_1724@yahoo.com

E-mail: murali_uoh@yahoo.co.in