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EXISTENCE OF COMMON RANDOM FIXED POINT AND RANDOM BEST APPROXIMATION FOR NON-COMMUTING RANDOM OPERATORS

BY

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Abstract

We present existence of common random fixed point as random best approximation results for non-commuting random operators. These results improve, extend and generalize some existing known results in the literature.

1. Introduction

Probabilistic functional analysis is an important mathematical discipline because of its applications to probabilistic models in applied problems. Random operator theory is needed for the study of various classes of random equations. The theory of random fixed point theorems was initiated by the Prague school of probabilistic in the 1950s. The interest in this subject enhanced after publication of the survey paper by Bharucha Reid [9]. Random fixed point theory has received much attention in recent years (see [4, 15, 16, 18]).

Random fixed point theorems and random approximations are stochastic generalization of classical fixed point and approximation theorems, and have application in probability theory and nonlinear analysis. The random fixed

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point theory for self-maps and nonself-maps has been developed during the last decade by various author (see e.g. [3, 4, 10]). Recently, this theory has been further extended for 1-set contractive, nonexpansive, semi-contractive and completely continuous random maps, etc.

Random fixed point theorems have been applied in many instances in the field of random best approximation theory and several interesting and meaningful results have been studied. The theory of approximation has become so vast that it intersects with every other branch of analysis and plays an important role in the applied sciences and engineering. Approximation theory is concerned with the approximation of functions of a certain kind by other functions. In this point of view, in the year 1963, Meinardus [13] was first to observe the general principle and to use a Schauder Fixed Point Theorem. Later on, number of results were developed in this direction under different conditions following the line made by Meinardus (see [3, 4, 6, 8, 14]).

The purpose of this paper is to first find existence results on common random fixed point as random best approximation for $\mathcal{R}$-subweakly commuting random operators satisfying $\mathcal{S}$-nonexpansive condition and affinity of random operator $\mathcal{S}$ in the setup of compact and weakly compact subset of Banach space. Secondly, an existence result on common random fixed point for uniformly $\mathcal{R}$-subweakly commuting satisfying asymptotically $\mathcal{S}$-nonexpansive condition and affinity of random operator $\mathcal{S}$ has been established in the same frame work, which is further applied to derived random best approximation results. In this way, results of Nashine [14] are improved and generalized with the aid of more general class of noncommuting random operators and weakening the condition of linearity of random operators by affinity. Incidently, results of Beg and Shahzad [6, Theorem 2] and Beg and Shahzad [8, Theorem B] have also been generalized.

2. Preliminary

For the sake of convenience, we gather some basic definitions and set out our terminology needed in the sequel.

Definition 2.1.([8]) Let $(\Omega, \mathcal{A})$ be a measurable space and $\mathcal{X}$ be a metric space. Let $2^{\mathcal{X}}$ be the family of all nonempty subsets of $\mathcal{X}$ and $\mathcal{C}(\mathcal{X})$ denote the family of all nonempty compact subsets of $\mathcal{X}$. A mapping $\mathcal{F} : \Omega \rightarrow$

$2^{\mathcal{X}}$ is called measurable (respectively, weakly measurable) if, for any closed (respectively, open) subset $\mathcal{B}$ of $\mathcal{X}$, $\mathcal{F}^{-1}(\mathcal{B}) = \{\omega \in \Omega : \mathcal{F}(\omega) \cap \mathcal{B} \neq \phi\} \in \mathcal{A}$. Note that, if $\mathcal{F}(\omega) \in \mathcal{C}(\mathcal{X})$ for every $\omega \in \Omega$, then $\mathcal{F}$ is weakly measurable if and only if measurable.

A mapping $\xi : \Omega \to \mathcal{X}$ is said to be measurable selector of a measurable mapping $\mathcal{F} : \Omega \to 2^{\mathcal{X}}$, if ξ is measurable and, for any $\omega \in \Omega, \xi(\omega) \in \mathcal{F}(\omega)$. A mapping $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$ is called a random operator if, for any $x \in \mathcal{X}$, $\mathcal{T}(., x)$ is measurable. A measurable mapping $\xi : \Omega \to \mathcal{X}$ is called a random fixed point of a random operator $\mathcal{T} : \Omega \times \mathcal{X} \to \mathcal{X}$, if for every $\omega \in \Omega$, $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega))$.

Let $\mathcal{X}$ be a normed space. A map $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is said to be

(1) a uniformly asymptotically regular on $\mathcal{X}$ if, for each $\eta > 0$, there exists $N(\eta) = N$ such that $\|\mathcal{T}_n x - \mathcal{T}_{n+1} x\| < \eta$ for all $\eta \geq 0$ and all $x \in \mathcal{X}$.

(2) $\mathcal{S}$-nonexpansive, if there exists a self-map $\mathcal{S}$ on $\mathcal{X}$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \|\mathcal{S}x - \mathcal{S}y\| \text{ for all } x, y \in \mathcal{X}.$$

(3) asymptotically $\mathcal{S}$-nonexpansive, if there exists a sequence $\{k_n\}$ of real numbers with $k_n \geq 1$ and $lim_{n\to\infty} k_n = 1$ such that $\|\mathcal{T}^n x - \mathcal{T}^n y\| \leq k_n \|\mathcal{S}x - \mathcal{S}y\|$ for all $x, y \in \mathcal{X}$ and $n = 1, 2, 3, \ldots \infty$.

Two maps $\mathcal{T}, \mathcal{S} : \mathcal{X} \to \mathcal{X}$ are called

(4) $\mathcal{R}$-weakly commuting, if there exists some $\mathcal{R} > 0$ such that

$$\|\mathcal{T}\mathcal{S}(x) - \mathcal{S}\mathcal{T}(x)\| \leq \mathcal{R}\|\mathcal{T}(x) - \mathcal{S}(x)\| \ \ for \ all \ \ x \in \mathcal{X}.$$

Suppose $p \in Fix(\mathcal{S})$, $\mathcal{M} \subset \mathcal{X}$ is p-starshaped and both $\mathcal{T}$ and $\mathcal{S}$ are invariant. Then $\mathcal{T}$ and $\mathcal{S}$ are said to be

(5) $\mathcal{R}$-subweakly commutating on $\mathcal{M}$, if there exists a real number $\mathcal{R} > 0$ such that

$$\|\mathcal{T}\mathcal{S}x - \mathcal{S}\mathcal{T}x\| \leq \mathcal{R} dist(\mathcal{S}x, [\mathcal{T}x, p])$$

for all $x \in \mathcal{M}$ where $dist(y, \mathcal{K}) = \inf\{dist(y, z) : z \in \mathcal{K}\}$ for $\mathcal{K} \in \mathcal{M}$ and $y \in \mathcal{M}$. Obviously, commutativity implies $\mathcal{R}$-subweak commutativity but the converse is not true in general [17].

(6) uniformly $\mathcal{R}$-subweakly commuting on $\mathcal{M}-\{p\}$ if there exists a real number $\mathcal{R}>0$ such that

$$\|\mathcal{T}^n\mathcal{S}x-\mathcal{S}\mathcal{T}^nx\|\leq\mathcal{R}dist(\mathcal{S}x,[\mathcal{T}^nx,p])$$

for all $x\in\mathcal{M}-\{p\}$, where $[\mathcal{T}^nx,p]=k\mathcal{T}^nx+(1-k)p$ for some $k\in(0,1]$ and $n\in N$ [2].

It is clear from (6) that uniformly $\mathcal{R}$-subweakly commuting mappings on $\mathcal{M}-\{p\}$ are $\mathcal{R}$-subweakly commuting on $\mathcal{M}-\{p\}$, but $\mathcal{R}$-subweakly commuting mappings on $\mathcal{M}-\{p\}$ need not be uniformly $\mathcal{R}$-subweakly commuting on $\mathcal{M}-\{p\}$. To see, this we consider the following example.

Example 2.2. Let $\mathcal{X}=\mathcal{R}$ with norm $\|x\|=|x|$, $\mathcal{M}=[1,0)$ and let $\mathcal{S}$ and $\mathcal{T}$ be self-mappings on $\mathcal{M}$ defined by $\mathcal{T}x=2x-1$, $\mathcal{S}x=x^2$. Then $\mathcal{S}$ and $\mathcal{T}$ are $\mathcal{R}$-subweakly commuting on $\mathcal{M}-\{p\}$. In fact, $\|\mathcal{T}\mathcal{S}x-\mathcal{S}\mathcal{T}x\|\leq\mathcal{R}dist(\mathcal{S}x,[\mathcal{T}x,p])$ for all $x\in\mathcal{M}-\{p\}$, where $\mathcal{R}=12$ and $p=1$ is the fixed point of $\mathcal{S}$. But $\mathcal{S}$ and $\mathcal{T}$ are not uniformly $\mathcal{R}$-subweakly commuting on $\mathcal{M}-\{p\}$ because if we take $n=2$, $x>1$ then

$$\|\mathcal{T}^2\mathcal{S}x-\mathcal{S}\mathcal{T}^2x\|\leq\mathcal{R}dist(\mathcal{S}x,[\mathcal{T}^2x,p])$$

with $\mathcal{R}=12$ and $p=1\in Fix(\mathcal{S})$.

A random operator $\mathcal{T}:\Omega\times\mathcal{X}\to\mathcal{X}$ is continuous (respectively, nonexpansive, $\mathcal{S}$-nonexpansive) if, for each $\omega\in\Omega$, $\mathcal{T}(\omega,\cdot)$ is continuous(respectively, nonexpansive, $\mathcal{S}$-nonexpansive). Random operators $\mathcal{T},\mathcal{S}:\Omega\times\mathcal{X}\to\mathcal{X}$ are $\mathcal{R}$-weakly commuting (respectively $\mathcal{R}$-subweakly commuting, uniformly $\mathcal{R}$-subweakly commuting), if $\mathcal{T}(\omega,\cdot)$ and $\mathcal{S}(\omega,\cdot)$ are $\mathcal{R}$-weakly commuting (respectively $\mathcal{R}$-subweakly commuting, uniformly $\mathcal{R}$-subweakly commuting) for each $\omega\in\Omega$.

Definition 2.3. A Banach space $\mathcal{X}$ satisfies Opial's condition if for every sequence $\{x_n\}$ in $\mathcal{X}$ weakly convergent to $x\in\mathcal{X}$, the inequality $liminf_{n\to\infty}\|x_n-x\|<liminf_{n\to\infty}\|x_n-y\|$ holds for all $y\neq x$. Every Hilbert space and the space $l_q(1\leq q<\infty)$ satisfy Opial's condition. The map $\mathcal{T}:\mathcal{M}\to\mathcal{X}$ is said to be demiclosed at 0 if for every sequence $\{x_n\}$ in $\mathcal{M}$ such that $\{x_n\}$ converges weakly to x and $\{\mathcal{T}x_n\}$ converges strongly to $0\in\mathcal{X}$, then $0=\mathcal{T}x$.

Definition 2.4.([8]) Let $\mathcal{M}$ be a nonempty subset of a normed space $\mathcal{X}$. For $x_0 \in \mathcal{X}$, let us define

$$dist(x_0, \mathcal{M}) = \inf_{y \in \mathcal{M}} \|x_0 - y\|$$

and

$$\mathcal{P}_{\mathcal{M}}(x_0) = \{y \in \mathcal{M} : \|x_0 - y\| = dist(x_0, \mathcal{M})\}.$$

An element $y \in \mathcal{P}_{\mathcal{M}}(x_0)$ is called a *best approximant* of x_0 out of $\mathcal{M}$. The set $\mathcal{P}_{\mathcal{M}}(x_0)$ is the set of all best approximation of x_0 out of $\mathcal{M}$.

3. Main results

The following result is needed in the sequel:

Theorem 3.1.([7]) *Let $\mathcal{X}$ be a Polish space and $\mathcal{T}, \mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ be two random operators such that, for each $\omega \in \Omega$, $\mathcal{T}(\omega, \mathcal{X}) \subseteq \mathcal{S}(\omega, \mathcal{X})$. If $\mathcal{T}$ and $\mathcal{S}$ are $\mathcal{R}$-weakly commutative, $\mathcal{T}$ is continuous and $d(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \leq hd(\mathcal{S}(\omega, x), \mathcal{S}(\omega, y))$ for all $x, y \in \mathcal{X}$, $\omega \in \Omega$ and $h \in (0, 1)$ such that $\mathcal{S}(\omega, x) \neq \mathcal{S}(\omega, y)$, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common random fixed point.*

Following is a common random fixed point theorem as random best approximation for $\mathcal{R}$-subweakly commuting random operators in the setting of compact subset.

Theorem 3.2. *Let $\mathcal{X}$ be a normed space. Let $\mathcal{T}, \mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ be $\mathcal{R}$-subweakly commuting random operators and $\mathcal{M} \subseteq \mathcal{X}$ such that $\mathcal{T}(\omega, \cdot) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$ and $x_0 = \mathcal{T}(\omega, x_0) = \mathcal{S}(\omega, x_0)$ for each $\omega \in \Omega$. Suppose $\mathcal{T}$ is $\mathcal{S}$-nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, and $\mathcal{S}(\omega, \cdot)$ be affine and nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0)$. Suppose $\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty compact, p-starshaped and $\mathcal{S}(\omega, \mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$ for each $\omega \in \Omega$, then there exists a measurable map $\xi : \Omega \to \mathcal{P}_{\mathcal{M}}(x_0)$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \xi(\omega))$ for each $\omega \in \Omega$.*

Proof. Let $y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Then $\mathcal{S}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$, since $\mathcal{S}(\omega, \mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$ for each $\omega \in \Omega$. Also, if $y \in \partial\mathcal{M}$ and so $\mathcal{T}(\omega, y) \in \mathcal{M}$, since

$\mathcal{T}(\omega,\partial\mathcal{M}) \subseteq \mathcal{M}$ for each $\omega \in \Omega$. Now, since $x_0 = \mathcal{T}(\omega, x_0)$ and $\mathcal{T}$ is $\mathcal{S}$-nonexpansive map, we have

$$\|\mathcal{T}(\omega,y)-x_0\| = \|\mathcal{T}(\omega,y)-\mathcal{T}(\omega,x_0)\| \leq \|\mathcal{S}(\omega,y)-\mathcal{S}(\omega,x_0)\|.$$

As $\mathcal{S}(\omega,x_0) = x_0$, we therefore have,

$$\|\mathcal{T}(\omega,y)-x_0\| \leq \|\mathcal{S}(\omega,x_0)-x_0\| = d(x_0,\mathcal{M}),$$

since $\mathcal{S}(\omega,y) \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}(\omega,y)$ is also closest to x_0, so $\mathcal{T}(\omega,y) \in \mathcal{P}_{\mathcal{M}}(x_0)$; consequently $\mathcal{P}_{\mathcal{M}}(x_0)$ is $\mathcal{T}(\omega,\cdot)$-invariant, that is, $\mathcal{T}(\omega,\mathcal{P}_{\mathcal{M}}(x_0)) \subseteq \mathcal{P}_{\mathcal{M}}(x_0)$.

Choose a fixed sequence of measurable mappings $k_n : \Omega \to (0,1)$ such that $k_n(\omega) \to 1$ as $n \to \infty$. For $n \geq 1$, define a sequence of random operators $\mathcal{T}_n : \Omega \times \mathcal{P}_{\mathcal{M}}(x_0) \to \mathcal{P}_{\mathcal{M}}(x_0)$ as

$$\mathcal{T}_n(\omega,x) = k_n(\omega)\mathcal{T}(\omega,x) + (1-k_n(\omega))p. \tag{3.1}$$

It is clear that $\mathcal{T}_n$ is a well-defined map from $\mathcal{P}_{\mathcal{M}}(x_0)$ into $\mathcal{P}_{\mathcal{M}}(x_0)$ for each n and $\omega \in \Omega$, since $\mathcal{P}_{\mathcal{M}}(x_0)$ is p-starshaped. It follows from (3.1) and $\mathcal{S}$-nonexpansiveness of $\mathcal{T}$ that

$$\begin{aligned}\|\mathcal{T}_n(\omega,x)-\mathcal{T}_n(\omega,y)\| &= k_n(\omega)\|\mathcal{T}(\omega,x)-\mathcal{T}(\omega,y)\| \\ &\leq k_n(\omega)\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\|\end{aligned}$$

i.e.,

$$\|\mathcal{T}_n(\omega,x)-\mathcal{T}_n(\omega,y)\| \leq k_n(\omega)\|\mathcal{S}(\omega,x)-\mathcal{S}(\omega,y)\| \tag{3.2}$$

whenever $\mathcal{S}(\omega,x) \neq \mathcal{S}(\omega,y)$, for all $x,y \in \mathcal{P}_{\mathcal{M}}(x_0)$ and $\omega \in \Omega$.

Also, from the affinity of $\mathcal{S}$ and $\mathcal{R}$-subweakly commutativity of $\mathcal{T}$ and $\mathcal{S}$

$$\begin{aligned}&\|\mathcal{T}_n(\omega,\mathcal{S}(\omega,x))-\mathcal{S}(\omega,\mathcal{T}_n(\omega,x))\| \\ &= k_n(\omega)\|\mathcal{T}(\omega,\mathcal{S}(\omega,x))-\mathcal{S}(\omega,\mathcal{T}(\omega,x))\| \\ &= k_n(\omega)\mathcal{R}\|(k_n(\omega)\mathcal{T}(\omega,x)+(1-k_n(\omega))p)-\mathcal{S}(\omega,x)\| \\ &\leq k_n(\omega)\mathcal{R}\|\mathcal{T}_n(\omega,x)-\mathcal{S}(\omega,x)\|\end{aligned}$$

i.e.,

$$\|\mathcal{T}_n(\omega, \mathcal{S}(\omega, x)) - \mathcal{S}(\omega, \mathcal{T}_n(\omega, x))\| \leq k_n(\omega)\mathcal{R}\|\mathcal{T}_n(\omega, x) - \mathcal{S}(\omega, x)\| \tag{3.3}$$

for all $x, y \in \mathcal{P}_{\mathcal{M}}(x_0)$ and $\omega \in \Omega$ which implies that $\mathcal{T}_n$ and $\mathcal{S}$ are $k_n(\omega)\mathcal{R}$ -weakly commuting on $\mathcal{P}_{\mathcal{M}}(x_0)$ for each n and $\mathcal{T}_n(\omega, \mathcal{P}_{\mathcal{M}}(x_0)) \subseteq \mathcal{P}_{\mathcal{M}}(x_0) = \mathcal{S}(\omega, \mathcal{P}_{\mathcal{M}}(x_0))$ for each $\omega \in \Omega$. Moreover, $\mathcal{S}(\omega, \cdot)$ is nonexpansive and so continuous on $\mathcal{P}_{\mathcal{M}}(x_0)$. Thus, all the condition of the Theorem 3.1 are satisfied on $\mathcal{P}_{\mathcal{M}}(x_0)$ and so, there exists a common random fixed point ξ_n of $\mathcal{T}_n$ and $\mathcal{S}$ such that

$$\xi_n(\omega) = \mathcal{T}_n(\omega, \xi_n(\omega)) = \mathcal{S}(\omega, \xi_n(\omega)). \tag{3.4}$$

For each n, define $\mathcal{G}_n : \Omega \to \mathcal{C}(\mathcal{P}_{\mathcal{M}}(x_0))$ by $\mathcal{G}_n = cl\{\xi_i(\omega) : i \geq n\}$ where $\mathcal{C}(\mathcal{P}_{\mathcal{M}}(x_0))$ is the set of all nonempty compact subset of $\mathcal{P}_{\mathcal{M}}(x_0)$.

Let $\mathcal{G} : \Omega \to \mathcal{C}(\mathcal{P}_{\mathcal{M}}(x_0))$ be a mapping defined as $\mathcal{G}(\omega) = \cap_{n=1}^{\infty}\mathcal{G}_n(\omega)$. By Himmelberg [10, Theorem 4.1] implies that $\mathcal{G}$ is measurable. The Kuratowski and Ryll-Nardzewski selection Theorem [11] further implies that $\mathcal{G}$ has a measurable selector $\xi : \Omega \to \mathcal{P}_{\mathcal{M}}(x_0)$. We show that ξ is the random fixed point of $\mathcal{T}$ and $\mathcal{S}$. Fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, therefore there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges to $\xi(\omega)$; that is $\xi_m(\omega) \to \xi(\omega)$. Since $\mathcal{T}_m(\omega, \xi_m(\omega)) = \xi_m(\omega)$, we have $\mathcal{T}_m(\omega, \xi_m(\omega)) \to \xi(\omega)$.
On the other hand, we have

$$\mathcal{T}_m(\omega, \xi_m(\omega)) = k_m(\omega)\mathcal{T}(\omega, \xi_m(\omega)) + (1 - k_m(\omega))p.$$

Proceeding to the limit as $m \to \infty$, $k_m(\omega) \to 1$ and continuity of $\mathcal{T}$, we have

$$\mathcal{T}(\omega, \xi(\omega)) = \xi(\omega).$$

Also from the continuity of $\mathcal{S}$, we have

$$\mathcal{S}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \lim_{m\to\infty} \xi_m(\omega)) = \lim_{m\to\infty} \mathcal{S}(\omega, \xi_m(\omega)) = \lim_{m\to\infty} \xi_m(\omega) = \xi(\omega).$$

Following is the result for weakly compact subset.

Theorem 3.3. *Let $\mathcal{X}$ be a Banach space. Let $\mathcal{T}, \mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ be $\mathcal{R}$-subweakly commutative weakly random operators and $\mathcal{M} \subseteq \mathcal{X}$ such that $\mathcal{T}(\omega, \cdot) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 \in \mathcal{X}$*

and $x_0 = \mathcal{T}(\omega, x_0) = \mathcal{S}(\omega, x_0)$ *for each* $\omega \in \Omega$. *Suppose* $\mathcal{T}$ *is* $\mathcal{S}$*-nonexpansive on* $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, *and* $\mathcal{S}(\omega, \cdot)$ *be affine and weakly continuous on* $\mathcal{P}_{\mathcal{M}}(x_0)$. *Suppose* $\mathcal{P}_{\mathcal{M}}(x_0)$ *is nonempty separable weakly compact, p-starshaped and* $\mathcal{S}(\omega, \mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$ *for each* $\omega \in \Omega$, *then there exists a measurable map* $\xi : \Omega \to \mathcal{P}_{\mathcal{M}}(x_0)$ *such that* $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \xi(\omega))$ *for each* $\omega \in \Omega$, *provided* $(\mathcal{S} - \mathcal{T})(\omega, \cdot)$ *is demiclosed at zero for each* $\omega \in \Omega$.

Proof. For each $n \in N$, define $\{k_n(\omega)\}, \{\mathcal{T}_n\}$ as in the proof of the Theorem 3.2. Also, we have

$$\|\mathcal{T}_n(\omega, x) - \mathcal{T}_n(\omega, y)\| \leq k_n(\omega)\|\mathcal{S}(\omega, x) - \mathcal{S}(\omega, y)\|$$

and

$$\|\mathcal{T}_n(\omega, \mathcal{S}(\omega, x)) - \mathcal{S}(\omega, \mathcal{T}_n(\omega, x))\| \leq k_n(\omega)\mathcal{R}\|\mathcal{T}_n(\omega, x) - \mathcal{S}(\omega, x)\|$$

for all $x, y \in \mathcal{P}_{\mathcal{M}}(x_0), \omega \in \Omega$. Since weak topology is Hausdorff and $\mathcal{P}_{\mathcal{M}}(x_0)$ is weakly compact, it follows that $\mathcal{P}_{\mathcal{M}}(x_0)$ is strongly closed and is a complete metric space. Thus by weakly continuity of $\mathcal{S}$ and Theorem 3.1, there exists a random fixed point ξ of $\mathcal{T}_n$ such that $\xi_n(\omega) = \mathcal{S}(\omega, \xi_n(\omega)) = \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$.
For each n, define $\mathcal{G}_n : \Omega \to \mathcal{WC}(\mathcal{P}_{\mathcal{M}}(x_0))$ by $\mathcal{G}_n = w - cl\{\xi_i(\omega) : i \geq n\}$, where $\mathcal{WC}(\mathcal{P}_{\mathcal{M}}(x_0))$ is the set of all nonempty weakly compact subset of $\mathcal{P}_{\mathcal{M}}(x_0)$ and $w - cl$ denotes the weak closure. Defined a mapping $\mathcal{G} : \Omega \to \mathcal{WC}(\mathcal{P}_{\mathcal{M}}(x_0))$ by $\mathcal{G}(\omega) = \cap_{n=1}^{\infty}\mathcal{G}_n(\omega)$. Because $\mathcal{P}_{\mathcal{M}}(x_0)$ is weakly compact and separable, the weak topology on $\mathcal{P}_{\mathcal{M}}(x_0)$ is a metric topology. Then by Himmelberg [10, Theorem 4.1] implies that $\mathcal{G}$ is w-measurable. The Kuratowski and Ryll-Nardzewski selection Theorem [11] further implies that $\mathcal{G}$ has a measurable selector $\xi : \Omega \to \mathcal{P}_{\mathcal{M}}(x_0)$. We show that ξ is the random fixed point of $\mathcal{T}$. Fix $\omega \in \Omega$. Since $\xi(\omega) \in \mathcal{G}(\omega)$, therefore there exists a subsequence $\{\xi_m(\omega)\}$ of $\{\xi_n(\omega)\}$ that converges weakly to $\xi(\omega)$; that is $\xi_m(\omega) \to \xi(\omega)$.
Now, from weakly continuity of $\mathcal{S}$, we have

$$\mathcal{S}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \lim_{m\to\infty} \xi_m(\omega)) = \lim_{m\to\infty} \mathcal{S}(\omega, \xi_m(\omega)) = \lim_{m\to\infty} \xi_m(\omega) = \xi(\omega).$$

Now,

$$\mathcal{S}(\omega, \xi_m(\omega)) - \mathcal{T}(\omega, \xi_m(\omega)) = \xi_m(\omega) - \mathcal{T}(\omega, \xi_m(\omega))$$

$$= \mathcal{T}_m(\omega, \xi_m(\omega)) - \mathcal{T}(\omega, \xi_m(\omega))$$
$$= (1 - k_m(\omega))(p - \mathcal{T}(\omega, \xi_m(\omega)).$$

Since $\mathcal{P}_{\mathcal{M}}(x_0)$ is bounded and $k_m(\omega) \to 1$, it follows that

$$\mathcal{S}(\omega, \xi_m(\omega)) - \mathcal{T}(\omega, \xi_m(\omega)) \to 0.$$

Now, $y_m = \mathcal{S}(\omega, \xi_m(\omega)) - \mathcal{T}(\omega, \xi_m(\omega)) = (\mathcal{S} - \mathcal{T})(\omega, \xi_m(\omega))$ and $y_m \to 0$. Since $(\mathcal{S} - \mathcal{T})(\omega, \cdot)$ is demiclosed at 0, so $0 \in (\mathcal{S} - \mathcal{T})(\omega, \xi(\omega))$. This implies that $\mathcal{S}(\omega, \xi(\omega)) = \mathcal{T}(\omega, \xi(\omega))$ and so, $\mathcal{S}(\omega, \xi(\omega)) = \mathcal{T}(\omega, \xi(\omega)) = \xi(\omega)$.

Theorem 3.4. *Let $\mathcal{M}$ be a subset of a Banach space $\mathcal{X}$ and $\mathcal{T}, \mathcal{S} : \Omega \times \mathcal{M} \to \mathcal{M}$ be two random operators such that, for each $\omega \in \Omega$, $\mathcal{T}(\omega, \mathcal{M} - \{p\}) \subseteq \mathcal{S}(\omega, \mathcal{M} - \{p\})$ where $p \in Fix(\mathcal{S})$. Suppose $\mathcal{T}$ is continuous and*

$$d(\mathcal{T}(\omega, x), \mathcal{T}(\omega, y)) \leq k(\omega) d(\mathcal{S}(\omega, x), \mathcal{S}(\omega, y))$$

for all $x, y \in \mathcal{M}$, $\omega \in \Omega$ and $k(\omega) \in (0, 1)$ such that $\mathcal{S}(\omega, x) \neq \mathcal{S}(\omega, y)$. If $\mathcal{T}$ and $\mathcal{S}$ are $\mathcal{R}$-weakly commutative on $\mathcal{M} - \{p\}$, then $\mathcal{T}$ and $\mathcal{S}$ have a unique common random fixed point.

Proof. It can be proved following the similar arguments of those given in the proof of [7].

Theorem 3.5. *Let $\mathcal{M}$ be a nonempty complete p-starshaped subset of a normed space $\mathcal{X}$ and let $\mathcal{T}, \mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ be uniformly $\mathcal{R}$-subweakly commutative random operators on $\mathcal{M} - \{p\}$ such that for each $\omega \in \Omega$, $\mathcal{S}(\omega, \mathcal{M}) = \mathcal{M}$ and $\mathcal{T}(\omega, \mathcal{M} - \{p\}) \subseteq \mathcal{S}(\omega, \mathcal{M} - \{p\})$ where $p \in Fix(\mathcal{S})$. Suppose $\mathcal{T}$ is continuous, asymptotically $\mathcal{S}$-nonexpansive with sequence $\{k_n(\omega)\}$ and $\mathcal{S}(\omega, \cdot)$ be affine. For each $n \geq 1$, define a random operator $\mathcal{T}_n(\omega, \cdot)$ by $\mathcal{T}_n(\omega, x) = \mu_n(\omega)\mathcal{T}^n(\omega, x) + (1 - \mu_n(\omega))p$, $x \in \mathcal{M}$, where $\mu_n(\omega) = \frac{\lambda_n(\omega)}{k_n(\omega)}$ and $\lambda_n(\omega)$ is a sequence in $(0, 1)$ such that $lim_{n\to\infty}\lambda_n(\omega) = 1$. Then for each $n \geq 1$, $\mathcal{T}_n$ and $\mathcal{S}$ have exactly one common random fixed point.*

Proof. For all $x, y \in \mathcal{M}$, we have

$$\|\mathcal{T}_n(\omega, x) - \mathcal{T}_n(\omega, y)\| = \mu_n(\omega)\|\mathcal{T}^n(\omega, x) - \mathcal{T}^n(\omega, y)\| \leq \lambda_n(\omega)\|\mathcal{S}(\omega, x) - \mathcal{S}(\omega, y)\|.$$

Also, $\mathcal{T}_n$ is a self-mapping of $\mathcal{M}$ such that $\mathcal{T}_n(\mathcal{M} - \{p\}) \subseteq \mathcal{S}(\mathcal{M}) - \{p\}$ for each n. From the uniformly $\mathcal{R}$-subweakly commutativity of $\mathcal{S}$ and $\mathcal{T}$ on $\mathcal{M} - \{p\}$ and affinity of $\mathcal{S}$, it follows that

$$\begin{aligned}
&\|\mathcal{T}_n(\omega, \mathcal{S}(\omega, x)) - \mathcal{S}(\omega, \mathcal{T}_n(\omega, x))\| \\
&= \|\mu_n(\omega)\mathcal{T}^n(\omega, \mathcal{S}(\omega, x)) + (1 - \mu_n(\omega))p - \mathcal{S}(\omega, \mu_n(\omega)\mathcal{T}^n x + (1 - \mu_n(\omega)p))\| \\
&= \mu_n(\omega)\|\mathcal{T}^n(\omega, \mathcal{S}(\omega, x)) - \mathcal{S}(\omega, \mathcal{T}^n(\omega, x))\| \\
&\leq \mu_n(\omega)\mathcal{R}dist(\mathcal{S}(\omega, x), [\mathcal{T}^n(\omega, x), p]) \\
&\leq \mu_n(\omega)\mathcal{R}\|\mu_n(\omega)\mathcal{T}^n(\omega, x) + (1 - \mu_n(\omega))p - \mathcal{S}(\omega, x)\| \\
&\leq \mu_n(\omega)\mathcal{R}\|\mathcal{T}_n(\omega, x) - \mathcal{S}(\omega, x)\|
\end{aligned}$$

for all $x \in \mathcal{M} - \{p\}$. Thus $\mathcal{T}_n$ and $\mathcal{S}$ are $\mu_n(\omega)\mathcal{R}$-weakly commuting. Therefore, Theorem 3.4 implies that there exists a random fixed point $\xi_n(\omega)$ of $\mathcal{T}_n$ such that $\xi_n(\omega) = \mathcal{S}(\omega, \xi_n(\omega)) = \mathcal{T}_n(\omega, \xi_n(\omega))$ for each $\omega \in \Omega$.

Following is the common random fixed point results for uniformly $\mathcal{R}$-weakly commuting random operators.

Theorem 3.6. *Let $\mathcal{M}$ be a nonempty p-starshaped subset of a normed space $\mathcal{X}$ and let $\mathcal{T}, \mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ be continuous random operator such that for each $\omega \in \Omega$, $\mathcal{S}(\omega, \mathcal{M}) = \mathcal{M}$ and $\mathcal{T}(\omega, \mathcal{M} - \{p\}) \subseteq \mathcal{S}(\omega, \mathcal{M} - \{p\})$ where $p \in Fix(\mathcal{S})$. Suppose $\mathcal{T}$ is uniformly asymptotically regular, asymptotically $\mathcal{S}$-nonexpansive with sequence $\{k_n(\omega)\}$ and $\mathcal{S}(\omega, \cdot)$ be affine on $\mathcal{M}$. If $\mathcal{T}, \mathcal{S}$ be uniformly $\mathcal{R}$-subweakly commutative random operators on $\mathcal{M}$, then there exists a measurable map $\xi : \Omega \to \mathcal{M}$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \xi(\omega))$ for each $\omega \in \Omega$, if one of the following conditions is satisfied:*

(1) *$\mathcal{M}$ is compact and $\mathcal{S}$ is continuous;*

(2) *$\mathcal{X}$ is Banach space, $\mathcal{M}$ is weakly compact, $\mathcal{S}$ is weakly continuous and $(\mathcal{S} - \mathcal{T}^n)(\omega, \cdot)$ is demiclosed at 0;*

(3) *$\mathcal{S}$ is weakly continuous, $\mathcal{M}$ is weakly compact and $\mathcal{X}$ is Banach space satisfying Opial's condition.*

Proof. From Theorem 3.5, for each $n \geq 1$, there exists exactly one point in $\mathcal{M}$ such that

$$\mathcal{S}(\omega, \xi_n(\omega)) = \xi_n(\omega) = \mu_n(\omega)\mathcal{T}^n(\omega, \xi_n(\omega)) + (1 - \mu_n(\omega))p.$$

Also

$$\|\xi_n(\omega) - \mathcal{T}^n(\omega, \xi_n(\omega))\| = (1 - \mu_n(\omega))\|\mathcal{T}^n(\omega, \xi_n(\omega)) - p\|.$$

Since $\mathcal{T}(\mathcal{M} - \{p\})$ is bounded and $k_n(\omega) \to 1$ as $n \to \infty$, it follows that $\|\xi_n(\omega) - \mathcal{T}^n(\omega, \xi_n(\omega))\| \to 0$. Now

$$\begin{aligned}
&\|\xi_n(\omega) - \mathcal{T}(\omega, \xi_n(\omega))\| \\
&\quad \leq \|\xi_n(\omega) - \mathcal{T}^n(\omega, \xi_n(\omega))\| + \|\mathcal{T}^n(\omega, \xi_n(\omega)) - \mathcal{T}^{n+1}(\omega, \xi_n(\omega))\| \\
&\qquad + \|\mathcal{T}^{n+1}(\omega, \xi_n(\omega)) - \mathcal{T}(\omega, \xi_n(\omega))\| \\
&\quad \leq \|\xi_n(\omega) - \mathcal{T}^n(\omega, \xi_n(\omega))\| + \|\mathcal{T}^n(\omega, \xi_n(\omega)) - \mathcal{T}^{n+1}(\omega, \xi_n(\omega))\| \\
&\qquad + k_1(\omega)\|\mathcal{S}(\omega, \mathcal{T}^n(\omega, \xi_n(\omega))) - \mathcal{S}(\omega, \xi_n(\omega))\|.
\end{aligned}$$

Since $\mathcal{S}$ is continuous, affine and $\mathcal{T}$ is uniformly asymptotically regular, we have

$$\begin{aligned}
&\|\xi_n(\omega) - \mathcal{T}(\omega, \xi_n(\omega))\| \\
&\quad \leq \|\xi_n(\omega) - \mathcal{T}^n(\omega, \xi_n(\omega))\| + \|\mathcal{T}^n(\omega, \xi_n(\omega)) - \mathcal{T}^{n+1}(\omega, \xi_n(\omega))\| \\
&\qquad + k_1(\omega)\|\mathcal{S}(\omega, \mathcal{T}^n(\omega, \xi_n(\omega))) - \xi_n(\omega)\| \;\; as \;\; n \to \infty.
\end{aligned}$$

Thus $\mathcal{T}(\omega, \xi_n(\omega)) - \xi_n(\omega) \to 0$ as $n \to \infty$.

(1) Since $\mathcal{M}$ is compact, therefore, in the line of Theorem 3.2, there exists a subsequence $\{\xi_m\}$ of $\{\xi_n\}$ such that $\xi_m(\omega) \to \xi(\omega) \in \mathcal{M}$ as $m \to \infty$. By the continuity of $\mathcal{T}$, we have $\mathcal{T}(\omega, \xi(\omega)) = \xi(\omega)$. Since $\mathcal{T}(\mathcal{M} - \{p\}) \subset \mathcal{S}(\mathcal{M} - \{p\})$, it follows that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \zeta(\omega))$ for some $\zeta \in \mathcal{M}$. Moreover,

$$\begin{aligned}
\|\mathcal{T}(\omega, \xi_m(\omega)) - \mathcal{T}(\omega, \zeta(\omega))\| &\leq k_1(\omega)\|\mathcal{S}(\omega, \xi_m(\omega)) - \mathcal{S}(\omega, \zeta(\omega))\| \\
&= k_1(\omega)\|\xi_m - \zeta(\omega)\|.
\end{aligned}$$

Taking the limit as $m \to \infty$, we get $\mathcal{T}(\omega, \xi(\omega)) = \mathcal{T}(\omega, \zeta(\omega))$. Thus, $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{T}(\omega, \zeta(\omega)) = \mathcal{S}(\omega, \zeta(\omega))$. Since $\mathcal{S}$ and $\mathcal{T}$ are uniformly $\mathcal{R}$-subweakly commuting on $\mathcal{M} - \{p\}$, it follows that

$$\begin{aligned}
\|\mathcal{T}(\omega, \xi(\omega)) - \mathcal{S}(\omega, \xi(\omega))\| &= \|\mathcal{T}(\omega, \mathcal{S}(\omega, \zeta(\omega))) - \mathcal{S}(\omega, \mathcal{T}(\omega, \zeta))\| \\
&\leq \mathcal{R}\|\mathcal{T}(\omega, \zeta(\omega)) - \mathcal{S}(\omega, \zeta(\omega))\| = 0.
\end{aligned}$$

Hence, we have $\mathcal{T}(\omega,\xi(\omega)) = \mathcal{S}(\omega,\xi(\omega)) = \xi(\omega)$.

(2) Since $\mathcal{M}$ is weakly compact, therefore, in the line of Theorem 3.2, there exists a subsequence $\{\xi_m\}$ of $\{\xi_n\}$ such that $\xi_m(\omega) \to \xi(\omega) \in \mathcal{M}$ as $m \to \infty$. Now, from weakly continuity of $\mathcal{S}$, we have

$$\mathcal{S}(\omega,\xi(\omega)) = \mathcal{S}(\omega, \lim_{m\to\infty} \xi_m(\omega)) = \lim_{m\to\infty} \mathcal{S}(\omega,\xi_m(\omega)) = \lim_{m\to\infty} \xi_m(\omega) = \xi(\omega).$$

Now,

$$\begin{aligned}\mathcal{S}(\omega,\xi_m(\omega)) - \mathcal{T}^m(\omega,\xi_m(\omega)) &= \xi_m(\omega) - \mathcal{T}^m(\omega,\xi_m(\omega))\\ &= \mathcal{T}_m(\omega,\xi_m(\omega)) - \mathcal{T}^m(\omega,\xi_m(\omega))\\ &= (1-\mu_m(\omega))(p - \mathcal{T}^m(\omega,\xi_m(\omega)).\end{aligned}$$

Since $\mathcal{M}$ is bounded and $\mu_m(\omega) \to 1$, it follows that

$$\|\mathcal{S}(\omega,\xi_m(\omega)) - \mathcal{T}^m(\omega,\xi_m(\omega))\| \to 0.$$

Since $(\mathcal{S} - \mathcal{T}^m)(\omega,\cdot)$ is demiclosed at 0, so $\mathcal{S}(\omega,\xi(\omega)) = \mathcal{T}^m(\omega,\xi(\omega))$ and so, $\mathcal{S}(\omega,\xi(\omega)) = \mathcal{T}^m(\omega,\xi(\omega)) = \xi(\omega)$. It is remaining to show that $\mathcal{T}(\omega,\xi(\omega)) = \xi(\omega)$.

$$\begin{aligned}\|\mathcal{T}(\omega,\xi(\omega)) - \mathcal{T}^m(\omega,\xi(\omega))\| &= \|\mathcal{T}(\omega,\xi(\omega)) - \mathcal{T}(\omega,\mathcal{T}^{m-1}(\omega,\xi(\omega)))\|\\ &\le k_1(\omega)\|\mathcal{S}(\omega,\xi(\omega)) - \mathcal{S}(\omega,\mathcal{T}^{m-1}(\omega,\xi(\omega)))\|\\ \|\mathcal{T}(\omega,\xi(\omega)) - \xi(\omega)\| &\le k_1(\omega)\|\xi(\omega) - \mathcal{S}(\omega,\xi(\omega))\|\\ &= k_1(\omega)\|\xi(\omega) - \xi(\omega)\| = 0.\end{aligned}$$

a contradiction. Hence $\mathcal{T}(\omega,\xi(\omega)) = \xi(\omega)$ which implies $\mathcal{T}(\omega,\xi(\omega)) = \mathcal{S}(\omega,\xi(\omega)) = \xi(\omega)$.

(3) As in (2), $\mathcal{S}(\omega,\xi(\omega)) = \xi(\omega)$ and $\|(\mathcal{S} - \mathcal{T}^m)(\omega,\xi_m(\omega))\| \to 0$ as $m \to \infty$. If $\mathcal{S}(\omega,\xi(\omega)) \neq \mathcal{T}^m(\omega,\xi(\omega))$, then by Opial's condition of $\mathcal{X}$ and asymptotically $\mathcal{S}$-nonexpansiveness of $\mathcal{T}$, it follows that

$$\begin{aligned}&\liminf_{m\to\infty} \|\mathcal{S}(\omega,\xi_m(\omega)) - \mathcal{S}(\omega,\xi(\omega))\|\\ &\quad < \liminf_{m\to\infty} \|\mathcal{S}(\omega,\xi_m(\omega)) - \mathcal{T}^m(\omega,\xi(\omega))\|\\ &\quad < \liminf_{m\to\infty} \|\mathcal{S}(\omega,\xi_m(\omega)) - \mathcal{T}^m(\omega,\xi_m(\omega))\|\\ &\qquad + \liminf_{m\to\infty} \|\mathcal{T}^m(\omega,\xi_m(\omega)) - \mathcal{T}^m(\omega,\xi(\omega))\|\end{aligned}$$

$$
\begin{aligned}
&< \liminf_{m\to\infty} \|\mathcal{T}^m(\omega,\xi_m(\omega)) - \mathcal{T}^m(\omega,\xi(\omega))\| \\
&\leq k_m(\omega)\|\mathcal{S}(\omega,\xi_m(\omega)) - \mathcal{S}(\omega,\xi(\omega))\|
\end{aligned}
$$

a contradiction. Hence $\mathcal{S}(\omega,\xi(\omega)) = \mathcal{T}^m(\omega,\xi(\omega)) = \xi(\omega)$. We can show that $\mathcal{T}(\omega,\xi(\omega)) = \mathcal{S}(\omega,\xi(\omega))$ as in (2).

An analogue of Theorem 3.2 is presented in the following in the frame work of uniformly $\mathcal{R}$-subweakly commuting random operators.

Theorem 3.7. *Let $\mathcal{X}$ be a normed space. Let $\mathcal{T},\mathcal{S} : \Omega\times\mathcal{X} \to \mathcal{X}$ be continuous random operators and $\mathcal{M} \subseteq \mathcal{X}$ such that $\mathcal{T}(\omega,\cdot) : \partial\mathcal{M}\cap\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 = \mathcal{T}(\omega,x_0) = \mathcal{S}(\omega,x_0)$ for each $x_0 \in \mathcal{X}$ and $\omega \in \Omega$. Suppose $\mathcal{T}$ is uniformly asymptotically regular, asymptotically $\mathcal{S}$-nonexpansive and $\mathcal{S}(\omega,\cdot)$ be affine on $\mathcal{P}_{\mathcal{M}}(x_0)$ with $\mathcal{S}(\omega,\mathcal{P}_{\mathcal{M}}(x_0)) = \mathcal{P}_{\mathcal{M}}(x_0)$. If $\mathcal{P}_{\mathcal{M}}(x_0)$ is nonempty, p-starshaped and $\mathcal{T}$ and $\mathcal{S}$ are uniformly $\mathcal{R}$-subweakly commuting mappings on $\mathcal{P}_{\mathcal{M}}(x_0)\cup\{x_0\}$ satisfying $\|\mathcal{T}(\omega,x) - \mathcal{T}(\omega,x_0)\| \leq \|\mathcal{S}(\omega,x) - \mathcal{S}(\omega,x_0)\|$, then there exists a measurable map $\xi : \Omega \to \mathcal{P}_{\mathcal{M}}(x_0)$ such that $\xi(\omega) = \mathcal{T}(\omega,\xi(\omega)) = \mathcal{S}(\omega,\xi(\omega))$ for each $\omega \in \Omega$, if one of the following conditions is satisfied:*

(1) *$\mathcal{P}_{\mathcal{M}}(x_0)$ is compact and $\mathcal{S}$ is continuous;*
(2) *$\mathcal{X}$ is Banach space, $\mathcal{P}_{\mathcal{M}}(x_0)$ is weakly compact, $\mathcal{S}$ is weakly continuous and $(\mathcal{S}-\mathcal{T}^n)(\omega,\cdot)$ is demiclosed at 0;*
(3) *$\mathcal{S}$ is weakly continuous, $\mathcal{P}_{\mathcal{M}}(x_0)$ is weakly compact and $\mathcal{X}$ is Banach space satisfying Opial's condition.*

Proof. Let $y \in \mathcal{P}_{\mathcal{M}}(x_0)$. Then $\|y - x_0\| = dist(x,\mathcal{M})$. Note that for any $t(\omega) \in (0,1)$,

$$\|t(\omega)x_0 + (1-t(\omega))y - x_0\| = (1-t(\omega))\|y - x_0\| < dist(x_0,\mathcal{M}).$$

It follows that the line segment $\{t(\omega)x_0 + (1-t(\omega))y : 0 < t(\omega) < 1\}$ and the set $\mathcal{M}$ are disjoint. Thus y is not in the interior of $\mathcal{M}$ and so $y \in \partial\mathcal{M}\cap\mathcal{M}$. Since $\mathcal{T}(\partial\mathcal{M}\cap\mathcal{M}) \subset \mathcal{M}$, $\mathcal{T}x$ must be in $\mathcal{M}$. Also since $\mathcal{S}(\omega,y) \in \mathcal{P}_{\mathcal{M}}(x_0)$, $x_0 = \mathcal{T}(\omega,x_0) = \mathcal{S}(\omega,x_0)$ and therefore by the given contractive condition, we have

$$\|\mathcal{T}(\omega,y) - x_0\| \;=\; \|\mathcal{T}(\omega,y) - \mathcal{T}(\omega,x_0)\| \leq \|\mathcal{S}(\omega,x) - \mathcal{S}(\omega,x_0)\|$$

$$= \|\mathcal{S}(\omega, y) - x_0\| = dist(x_0, \mathcal{M}).$$

Consequently $\mathcal{P}_{\mathcal{M}}(x_0)$ is $\mathcal{T}(\omega, \cdot)$-invariant. Hence,

$$\mathcal{T}(\omega, \mathcal{P}_{\mathcal{M}}(x_0)) \subseteq \mathcal{P}_{\mathcal{M}}(x_0) = \mathcal{S}(\omega, \mathcal{P}_{\mathcal{M}}(x_0)).$$

Thus, the result follows from Theorem 3.6.

Define $\mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0) = \{x \in \mathcal{M} : \mathcal{S}x \in \mathcal{P}_{\mathcal{M}}(x_0)\}$ and $\mathcal{D}^{\mathcal{S}}_{\mathcal{M}}(x_0) = \mathcal{P}_{\mathcal{M}}(x_0) \cap \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0)$ [1].

Theorem 3.8. *Let $\mathcal{X}$ be a normed space. Let $\mathcal{T}, \mathcal{S} : \Omega \times \mathcal{X} \to \mathcal{X}$ be random operators and $\mathcal{M} \subseteq \mathcal{X}$ such that $\mathcal{T}(\omega, \cdot) : \partial\mathcal{M} \to \mathcal{M}$, where $\partial\mathcal{M}$ stands for the boundary of $\mathcal{M}$. Let $x_0 = \mathcal{T}(\omega, x_0) = \mathcal{S}(\omega, x_0)$ for each $x_0 \in \mathcal{X}$ and $\omega \in \Omega$. Suppose $\mathcal{T}$ is continuous, uniformly asymptotically regular, asymptotically $\mathcal{S}$-nonexpansive and $\mathcal{S}(\omega, \cdot)$ be nonexpansive $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$ and affine on $\mathcal{D} = \mathcal{D}^{\mathcal{S}}_{\mathcal{M}}(x_0)$ with $\mathcal{S}(\omega, \mathcal{D}) = \mathcal{D}$. If $\mathcal{D}$ is nonempty, p-starshaped and $\mathcal{T}$ and $\mathcal{S}$ are uniformly $\mathcal{R}$-subweakly commuting mappings on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$ satisfying $\|\mathcal{T}(\omega, x) - \mathcal{T}(\omega, x_0)\| \leq \|\mathcal{S}(\omega, x) - \mathcal{S}(\omega, x_0)\|$, then there exists a measurable map $\xi : \Omega \to \mathcal{P}_{\mathcal{M}}(x_0)$ such that $\xi(\omega) = \mathcal{T}(\omega, \xi(\omega)) = \mathcal{S}(\omega, \xi(\omega))$ for each $\omega \in \Omega$ if one of the following conditions is satisfied:*

(1) *$\mathcal{D}$ is compact and $\mathcal{S}$ is continuous;*

(2) *$\mathcal{X}$ is Banach space, $\mathcal{D}$ is weakly compact, $\mathcal{S}$ is weakly continuous and $(\mathcal{S} - \mathcal{T}^n)(\omega, \cdot)$ is demiclosed at 0;*

(3) *$\mathcal{S}$ is weakly continuous, $\mathcal{D}$ is weakly compact and $\mathcal{X}$ is Banach space satisfying Opial's condition.*

Proof. Let $y \in \mathcal{D}$, then $\mathcal{S}(\omega, y) \in \mathcal{D}$, since $\mathcal{S}(\omega, \mathcal{D}) = \mathcal{D}$ for each $\omega \in \Omega$. Also, if $y \in \partial\mathcal{M}$ and so $\mathcal{T}(\omega, y) \in \mathcal{M}$, since $\mathcal{T}(\omega, \partial\mathcal{M}) \subseteq \mathcal{M}$ for each $\omega \in \Omega$. Now since $x_0 = \mathcal{T}(\omega, x_0)$ and $\mathcal{T}$ is $\mathcal{S}$-nonexpansive map, we have

$$\|\mathcal{T}(\omega, y) - x_0\| = \|\mathcal{T}(\omega, y) - \mathcal{T}(\omega, x_0)\| \leq \|\mathcal{S}(\omega, y) - \mathcal{S}(\omega, x_0)\|.$$

As $\mathcal{S}(\omega, x_0) = x_0$, we therefore have,

$$\|\mathcal{T}(\omega, y) - x_0\| \leq \|\mathcal{S}(\omega, x_0) - x_0\| = dist(x_0, \mathcal{M}),$$

since $\mathcal{S}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}(\omega, y)$ is also closest to x_0, so, $\mathcal{T}(\omega, y) \in \mathcal{P}_{\mathcal{M}}(x_0)$; consequently $\mathcal{P}_{\mathcal{M}}(x_0)$ is $\mathcal{T}(\omega, \cdot)$-invariant, that is,

$\mathcal{T}(\omega,\cdot) \subseteq \mathcal{P}_{\mathcal{M}}(x_0)$. As $\mathcal{S}$ is nonexpansive on $\mathcal{P}_{\mathcal{M}}(x_0) \cup \{x_0\}$, so for each $\omega \in \Omega$, we have

$$\begin{aligned}\|\mathcal{S}(\omega,\mathcal{T}(\omega,y)) - x_0\| &= \|\mathcal{S}(\omega,\mathcal{T}(\omega,y)) - \mathcal{S}(\omega,x_0)\| \leq \|\mathcal{T}(\omega,y) - x_0\| \\ &= \|\mathcal{T}(\omega,y) - \mathcal{T}(\omega,x_0)\| \leq \|\mathcal{S}(\omega,y) - \mathcal{S}(\omega,x_0)\| \\ &= \|\mathcal{S}(\omega,y) - x_0\|.\end{aligned}$$

Thus, $\mathcal{S}(\omega,\mathcal{T}(\omega,y)) \in \mathcal{P}_{\mathcal{M}}(x_0)$. This implies that $\mathcal{T}(\omega,y) \in \mathcal{C}^{\mathcal{S}}_{\mathcal{M}}(x_0)$ and hence $\mathcal{T}(\omega,y) \in \mathcal{D}$. So, $\mathcal{T}(\omega,\cdot)$ and $\mathcal{S}(\omega,\cdot)$ are self-maps on $\mathcal{D}$. Hence, all the condition of the Theorem 3.6 are satisfied. Thus, there exists a measurable map $\xi : \Omega \to \mathcal{D}$ such that $\xi(\omega) = \mathcal{T}(\omega,\xi(\omega)) = \mathcal{S}(\omega,\xi(\omega))$ for each $\omega \in \Omega$.

Remark 3.9. With the remark given by Beg et al. [2] that uniformly $\mathcal{R}$-subweakly commuting random operators are $\mathcal{R}$-subweakly commuting but not conversely and weakening the condition of linearity of $\mathcal{S}$, Theorem 3.7 and Theorem 3.8 are generalization of the results of Nashine [14].

Remark 3.10. With the Remark 3.9 and remark given by Shahzad [17] that $\mathcal{R}$-subweakly random operators includes the class of commutative random operators, Theorem 3.2 to Theorem 3.8 are generalization of the results due to Beg and Shahzad [6, 8].

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