Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **4** (2009), No. 3, pp. 235-297

VISCOUS SHOCK WAVE TRACING, LOCAL CONSERVATION LAWS, AND POINTWISE ESTIMATES

BY

TAI-PING LIU AND SHIH-HSIEN YU

Abstract

We introduce a new approach to decompose a system of viscous conservation laws with respect to each characteristic wave structures. Under this new decomposition, the global wave interactions of the system are reduced to coupling of nonlinear waves around constant states outside shock region and a scalar conservation law in the shock region to determine the behavior of local internal shock layers. The behavior is characterized by the motion of the viscous shock fronts. It is analyzed by the local conservation laws. We also introduce generalized diffusion waves to localize waves in initial data.

We prove stability of a viscous shock layer of 2×2 system; and obtain the optimal rate of convergence.

1. Introduction

We are interested in the stability problem of a viscous shock profile $\phi(x-st)$ connecting a shock wave (u_-,u_+) for a system of viscous conservation laws

$$u_t+f(u)_x=u_{xx},\ u\in\mathbf{R}^n. \tag{1.1}$$

Received and accepted on August 26, 2009.

The research of the first author is supported by NSC Grant 96-2628-M-001-011 and NSF Grant DMS-0709248. The research of the second author is supported by the NUS startup grant R-146-000-108-133.

Here, the system $u_t + f(u)_x = 0$ is a strictly hyperbolic system:

$$\begin{cases} \lambda_1(u) < \cdots < \lambda_n(u), \\ f'(u)\ r_j(u) = \lambda_j(u)\ r_j(u), \\ l_j(u)\ f'(u) = \lambda_j(u)\ l_j(u). \end{cases}$$

where $\lambda_j(u)$, $r_j(u)$, and $l_j(u)$ are the j-th eigenvalues, right eigenvector, and left eigenvector of $f'(u)$. A shock wave (u_-, u_+) pertaining a genuine nonlinear characteristic field λ_i is a two-valued weak solution of $u_t + f(u)_x = 0$ satisfying both the Rankine-Hugoniot Condition,

$$f(u_-) - f(u_+) = s\ (u_- - u_+), \tag{R-H}$$

and Lax's entropy condition,

$$\begin{cases} \lambda_i(u_-) > s > \lambda_i(u_+), \\ \lambda_{i+1}(u_+) > s > \lambda_{i-1}(u_-). \end{cases} \tag{Lax}$$

Here, s is the speed of the shock wave (u_-, u_+). Due to the Lax's entropy condition, we call (u_-, u_+) an i-th shock wave. A shock profile $\phi(x - st)$ is a travelling wave solution $u(x,t) = \phi(x - st)$ of (1.1) connecting (u_-, u_+):

$$\begin{cases} -s\ \phi' + f'(\phi)\ \phi' = \phi'', \\ \lim_{x\to\pm\infty} \phi(x) = u_\pm. \end{cases}$$

The stability problem of $\phi(x-st)$ has been studied by energy estimates, [1], [3], [4], [5], [6], [18]. Particularly in [7] and [14], the stability problem is analyzed through detailed wave interactions; and the nonlinearity is effectively analyzed.

In this paper, we introduce local conservation laws and wave front tracing to decompose the solution of (1.1) into the form $u(x,t) = \mathbb{V}(x,t) + \phi(x - st - \gamma(t))$:

$$\int_{-L+st}^{L+st} \Big[u(x,t) - \phi(x - st - \gamma(t))\Big]\ dx = \sum_{j<i} C_j(t)\ r_j(u_-) + \sum_{j>j} C_j(t)\ r_j(u_+), \tag{1.2}$$

where L is sufficiently large to essentially contain the shock layer, and (u_-, u_+) is an i-th shock; and $C_j(t)$ are real-valued functions.

The function $\gamma(t)$ is the wave front of $u(x,t)$; and it represents the motion of viscous shock fronts due to the nonlinear wave interactions. Indeed, $\gamma(t)$ essentially depends on the local structure of $\mathbb{V}(x,t)$. With this concept of wave fronts, one can efficiently obtain an optimal convergence rate of the solution to the shock profile. This wave front tracing approach requires the presence of a local structure. However, we do not need to require the initial data $u(x,0) - \phi(x)$ to have such a local structure such as the condition $\|u(x,0) - \phi(x)| \le O(1)(|x|+1)^{-3/2}$. We need to introduce a generalized nonlinear diffusion wave $\Theta(x,t)$ to localize the initial data of $\mathbb{V}(x,0)$:

$$\partial_t \Theta + f(u_- + \Theta)_x - \Theta_{xx} = E(x,t)(u_+ - u_-), \tag{1.3}$$

$$\|\Theta(x,0) - [u(x,0) - \phi(x)]\| \le \delta\, e^{-\alpha|x|},\ \alpha > 0. \tag{1.4}$$

Here, $E(x,t)$ is a real-valued function with a compact support in $[-1,1]$ and with the decaying structure in time

$$\sup_{x\in\mathbf{R}} |E(x,t)| \le O(1)\frac{\delta}{\epsilon}(1+t)^{-1/2}(1+\epsilon^2 t)^{-1},$$
$$\epsilon \equiv \|u_- - u_+\|;$$

δ is the magnitude of the perturbation:

$$\delta \equiv \sup_{x\in\mathbf{R}} \|u(x,0) - \phi(x)\|;$$

and α is a positive number independent of ϵ and δ. So, for the decomposition

$$u(x,t) = \mathbb{V}(x,t) + \phi(x - st - \gamma(t)) + \Theta(x,t)$$

the perturbation $\mathbb{V}(x,t)$ has a local structure in the initial data, (1.4).

Wave front tracing have been introduced to study time asymptotic behaviors of viscous conservation laws, [2], [5], [9], [12], [13], [15], [17], and [20]. The coupling of wave front and wave interactions could result in a sub-linear phenomenon. Such coupling was studied in a singular behavior of Burgers equation in the quarter plane (see [9]) and in a coupling of viscous shock profile and viscous rarefaction wave (see [13]).

The approach of pointwise estimates was originated from [7]. Such approach gives a rather detailed structure of solution. With the detailed structure one can analyze the nonlinear couple effectively. Thus, various types of nonlinear viscous problem can be studied by the pointwise estimates, see [8], [10], [11].

For the purpose to illustrate the role of localization of initial data, local conservation laws, and wave front tracing we consider a 2x2 system with the following setting:

Settings of the problem

$$u_t + f(u)_x - u_{xx} = 0,\ u \in \mathbf{R}^2, \tag{1.5}$$

$$(u_-, u_+):\ \text{a stationary 2-shock shock wave: } s = 0, \tag{1.6}$$

$$\|u_- - u_+\| \equiv \epsilon, \tag{1.7}$$

$$\phi(x):\ \text{a stationary profile of (1.5) connecting } (u_-, u_+), \tag{1.8}$$

$$|u(x,0) - \phi(x)| \le \delta \begin{cases} e^{-|x|} \text{ for } x \ge 0, \\ (1 + |x|)^{-3/2} \text{ for } x \end{cases} \tag{1.9}$$

$$|\partial_x(u(x,0) - \phi(x))| \le \delta \begin{cases} e^{-|x|} \text{ for } x \ge 0, \\ (1 + |x|)^{-3/2} \text{ for } x \end{cases} \tag{1.10}$$

Here, the parameters ϵ and δ are assumed sufficiently small

$$\epsilon,\ \delta/\epsilon^6 \ll 1.$$

The wave front $\gamma(t)$ will satisfy

$$\gamma(t) = \gamma(\infty) + O(1)\ \frac{\delta}{\epsilon^2}(1+t)^{-1/2}.$$

Remark 1.1. The assumptions $\epsilon \ll 1$ and $u \in \mathbf{R}^2$ are not necessary for the wave front tracing. It is interesting to relax those conditions in the future. Indeed, under this frame work many problems related to interactions of different structures such as shock layers to boundary layers, shock layers to rarefaction lagers are possible to study, e.g. [13].

Main Theorem. *Suppose that $\delta\epsilon^{-6}$ and ϵ are sufficiently small. Then, there exist an uniformly bounded function γ and constants $\mathscr{C}$, $J > 1$ such* *that*

$$\lim_{t\to\infty}\sup_{x\in\mathbf{R}}\|u(x,t)-\phi(x-\gamma(t))\|=0,$$
$$\lim_{t\to\infty}\gamma(t) \text{ exists},$$
$$\|u(x,t)-\phi(x-\gamma(t))\|\le\ \frac{\mathscr{C}\delta}{(1+\epsilon^2 t)\sqrt{(1+t)}} \text{ for } |x|\le\frac{J|\log\epsilon|}{\epsilon}.$$

Remark 1.2. Before this research, the optimal rate of convergence is $O(1)t^{-1/2}$. This may appears to be inconsistent with our present result. However, in the previous approaches the wave front is fixed to be the time asymptotic wave front $\gamma(\infty)$. The perturbation satisfies

$$\|u(x,t)-\phi(x-\gamma(\infty))\|\le\|u(x,t)-\phi(x-\gamma(t))\|+\|\phi(x-\gamma(t))-\phi(x-\gamma(\infty)\|.$$

This results in

$$\|u(x,t)-\phi(x-\gamma(\infty))\|\le O(1)\frac{\delta}{\epsilon^2}(1+t)^{-1/2} \text{ for } |x|\le J\epsilon^{-1}|\log\epsilon|.$$

Thus, our result is consistent with the old approaches.

In our analysis for obtaining the Main Theorem, we introduce two sets of variables $\mathbb{V}(x,t)$ and $\mathbb{W}(x,t)$:

$$\begin{cases}\mathbb{V}(x,t)\equiv\sum_{j=1}^{2}\mathbb{V}^j(x,t)\ r_j(u_-),\\ \mathbb{W}_x(x,t)\equiv\mathbb{V}(x,t),\\ \mathbb{W}(x,t)\equiv\sum_{j=1}^{2}\mathbb{W}^j(x,t)\ r_j(\phi(x-\gamma(t))),\\ \mathbb{V}^j(x,t)=\sum_{k=1}^{2}\mathbb{W}^k_x(x,t)\ l_j(u_-)\cdot r_k(\phi(x-\gamma(t)))\\ \qquad\qquad+\sum_{k=1}^{2}\mathbb{W}^k(x,t)\ l_j(u_-)\cdot\partial_x r_k(\phi(x-\gamma(t))).\end{cases} \tag{1.11}$$

Here, the variable $\mathbb{W}(x,t)$ is considered in the region of shock wave only.

We will introduce Theorems A and B before proving the Main Theorem.

In **Theorem A**, we make a weak assumptions on both $\mathbb{V}^2(x,t)$ in the region of shock wave $x \geq st - J_0$ and $\gamma(t)$. Under this weak assumption, one can obtain a global estimate of $\mathbb{V}^1(x,t)$, a semi-global estimate of $\mathbb{V}^2(x,t)$ for $x \leq st - J_0$, and a weak estimate of $\|\mathbb{W}(x,t)\|$ in the region of shock. In the analysis, we also introduce a special decomposition for the $\mathbb{V}$. It separates the function $\mathbb{V}$ into two parts $\mathbb{V}_\pi$ and $\mathbb{V}_{\pi^\perp}$. The part $\mathbb{V}_\pi$ decays slower and contains smaller norm. In contrast to this, $\mathbb{V}_{\pi^\perp}$ decays faster and contains larger norm.

In **Theorem B**, under the necessary conditions of Theorem A we can obtain sharper estimates about both $\mathbb{V}^2(x,t)$ in the region of shock wave and $\gamma(t)$.

Thus, the necessary conditions of Theorems A and B yield the optimal rates of $\|\mathbb{V}(x,t)\|$.

In Section 3 and 4, we will outline the constructions of the generalize diffusion waves and the sketch of the proof of Theorems A and B. In Section 5, we will provide the technical lemmas needed for proving the theorems. Finally, in the last sections we will give the detailed proof of the theorems.

2. Preliminaries

Notation 2.1.

$$\begin{cases} \lambda_j^\pm \equiv \lambda_j(u_\pm),\ j = 1,\dots,n \\ \Lambda_\pm(\xi) \equiv -\lambda_i^\pm \tanh\left(\frac{\lambda_i^\pm \xi}{2}\right). \end{cases}$$

Suppose that the strength, ϵ, of the shock (u_-, u_+) is sufficiently small. The eigen vectors are normalized

$$l_i(u_-)f''(u_-)(r_i(u_-), r_i(u_-)) = 1.$$

Then, one has the following analytic structure of the shock profile $\phi(\xi)$:

$$|\lambda_i^\pm| = \frac{\epsilon(1+O(\epsilon))}{2}, \tag{2.1}$$

$$\|\phi'(\xi)\| \leq O(1)\ \epsilon^2\ e^{-\lambda_2^-|\xi|(1+O(\epsilon))}, \tag{2.2}$$

for $k = 0, 1$

$$\left|\partial_x^k\Big[\lambda_i(\phi(\xi)) - \Lambda_-(\xi)\Big]\right| \le O(1)\begin{cases}\epsilon^{2+k}\ e^{-\lambda_i^-(1+O(\epsilon))|\xi|} \text{ for } \xi < 0,\\ \epsilon^{2+k} \text{ for } \xi \ge 0,\end{cases} \tag{2.3}$$

$$\left|\partial_x^k\Big[\lambda_i(\phi(\xi)) - \Lambda_+(\xi)\Big]\right| \le O(1)\begin{cases}\epsilon^{2+k}\ e^{-\lambda_i^-(1+O(\epsilon))|\xi|} \text{ for } \xi > 0,\\ \epsilon^{2+k} \text{ for } \xi \le 0.\end{cases} \tag{2.4}$$

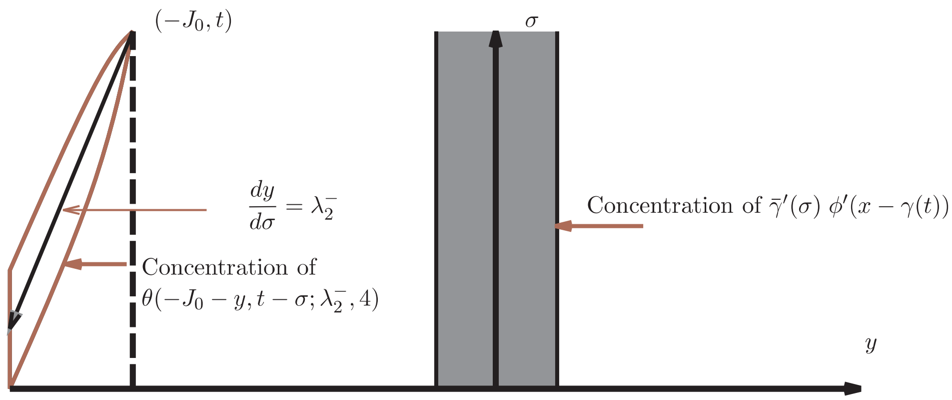
Structure due to the stationary 2-shock (u_-, u_+)

Since (u_-, u_+) is taken to be a stationary 2-shock of the 2×2 system (1.5), the Lax's entropy condition becomes

$$\begin{cases} s = 0,\\ \lambda_1^- < 0,\\ \lambda_1^+ < \lambda_2^+ < 0 < \lambda_2^-,\\ \lambda_2^- = -\lambda_2^+(1+O(\epsilon)) = \dfrac{\epsilon}{2}(1+O(1)\epsilon).\end{cases}$$

There are two characteristic curves entering the shock from the right of the shock; one characteristic curve catching up with the shock from the left of the shock; and one characteristic curve leaving to the left of the shock. Combine this property with our initial condition that there essentially no waves in front of the shock in the initial data. We can conclude that time-asymptotically there are no waves in front of the shock. So, we only need to take care the waves behind the shock; and decompose the vector $r_2(u_-)$ as follows

$$r_2(u_-) = c^+\ (u_+ - u_-) + c^-\ r_1(u_-),$$



where $c^{\pm}$ are real-valued constants satisfying

$$c^{+} = O(1)\,\epsilon^{-1},\ c^{-} = O(1)\,\epsilon. \tag{2.5}$$

The coefficients $c^{\pm}$ actually are the scattering data of a vector $r_2(u_-)$ into the shock wave (u_-, u_+). c^- is the reflection coefficient; and c^+ is the absorbing coefficient into the shock.

3. Generalized Diffusion Waves

Due to (1.9) and (1.10), the initial data $\|u(x,0)-\phi(x)\|$ does not decay fast enough outside the shock wave region. We will take out a nonlinear wave $\Theta(x,t)$ from $u(x,t)-\phi(x)$ such that the resulted wave has a localized structure in the initial data.

Before we outline the construction of $\Theta(x,t)$, let's introduce the following notations:

Notation 3.1.

$$\theta^j(\xi,\tau;\lambda,D) \equiv \frac{e^{-\frac{(\xi-\lambda\tau)^2}{D\tau}}}{\sqrt{\pi D\tau}^{\,j}} \text{ for } j \ge 0,$$

$$\psi^i(x,t;\lambda) \equiv \left(|x-\lambda\, t|^2 + t\right)^{-i/2},$$

$$\eta^\alpha(x,t) \equiv \begin{cases} \left\{\sqrt{x-\lambda_1^- t}\cdot\left[1+\epsilon^2(x-\lambda_1^- t)\right]\right\}^{-\alpha} & \text{for } x \in [\lambda_1^- t + \sqrt{t}, 0] \\ 0 \text{ else,} \end{cases}$$

$$\rho^\alpha(t) \equiv \left[\sqrt{1+t}\,(1+\epsilon^2 t)\right]^{-\alpha}.$$

$$\varrho(x) \equiv \begin{cases} e^{-\frac{1}{1-|x|}} \text{ for } |x|<1, \\ 0 \text{ for } |x| \ge 1, \end{cases}$$

$$\chi_+(x) \equiv \frac{\displaystyle\int_{-\infty}^{x} \varrho(y)\,dy}{\displaystyle\int_{-\infty}^{\infty} \varrho(y)\,dy},\ \chi_-(x) = 1-\chi_+(x),$$

$$\boldsymbol{v}(x,t) \equiv u(x,t)-\phi(x),\ \boldsymbol{v}(x,t) \equiv \sum_{j=1}^{2} \boldsymbol{v}^j(x,t)\, r_j(u_-).$$

Due to (1.9) and (1.10), one needs only to localize the initial data for $x \le -1$. We will take out the waves in region $x \le -1$, that is, $\boldsymbol{v}^1(x,0)\, r_1(u_-) + \chi_-(x)\boldsymbol{v}^2(x,0)\, r_2(u_-)$. In the same time, we also need to maintain the total masses of the residue waves parallel to $(u_- - u_+)$. This can be done by finding a function $\Psi(x)$ with *supp* $\Psi \subset [-1,1]$ and

$$\int_R \chi_+(x)\ \boldsymbol{v}^2(x,0)\ r_2(u_-) - \Psi(x)\ r_1(u_-)\ dx \parallel (u_+ - u_-). \tag{3.1}$$

Thus, we consider the vector-valued function $\Theta_1(x,t) \equiv \sum_{j=1}^{2} \Theta_1^j(x,t)\ r_j(u_-)$:

$$\begin{cases} (\bar\Theta_1^2)_t + (\lambda_2^-\ \bar\Theta_1^2)_x - (\bar\Theta_1^2)_{xx} = 0, \\ \bar\Theta_1^2(x,0) = \boldsymbol{v}^2(x,0). \end{cases} \tag{3.2a}$$

$$\begin{cases} \Theta_1^2(x,t) \equiv \chi_-(x)\ \bar\Theta_1^2(x,t), \\ \mathscr{E}_1(x,t) \equiv (\Theta_1^2)_t + (\lambda_2^-\ \Theta_1^2)_x - (\Theta_1^2)_{xx}, \\ \mathscr{E}_1^+(x,t) \equiv c^+ \cdot \mathscr{E}_1(x,t),\ \ \mathscr{E}_1^-(x,t) \equiv c^- \cdot \mathscr{E}_1(x,t). \end{cases} \tag{3.2b}$$

$$\begin{cases} (\Theta_1^1)_t + (\lambda_1^-\ \Theta_1^1)_x - (\Theta_-^1)_{xx} = -\mathscr{E}_1^-(x,t), \\ \Theta_1^1(x,0) = \boldsymbol{v}^1(x) + \Psi(x). \end{cases} \tag{3.2c}$$

$$\Theta_1(x,t) \equiv \Theta_1^1(x,t)\ r_1(u_-) + \Theta_1^2(x,t)\ r_2(u_-). \tag{3.2d}$$

The function $\bar\Theta_1^2(x,t)$ is devised to absorb the initial data $\boldsymbol{v}^2(x,0)$. The behavior of this function is not proper when $x \ge -1$, because the transportation term in (3.2a) does not contain the structure of shock profile. We need to cut off the part of $\bar\Theta_1^2$ for $x \ge -1$ by multiplying the cut-off function $\chi_-(x)$. It results in an extra truncation error $\mathscr{E}_1(x,t)r_2(u_-)$ when one substitutes $\chi_-\ \bar\Theta_1^2$ into (3.2a). One can decompose the truncation error $\mathscr{E}_1 r_2(u_-)$ into two parts

$$\mathscr{E}_1 r_2(u_-) = \mathscr{E}_1^+\ (u_+ - u_-) + \mathscr{E}_1^-\ r_1(u_-).$$

The component $\mathscr{E}_1^-\ r_1(u_-)$ will not interact with the viscous shock profile, and it will travel away the region of shock wave according to the behavior of (3.2c). This results in the following property of $\Theta_1(x,t)$:

$$\begin{cases} \partial_t\Theta_1 + \partial_x f'(u_-)\Theta_1 - \partial_x^2\Theta_1 = \mathscr{E}_1^+ \,(u_+ - u_-), \\ supp(\mathscr{E}_1^{\pm}(\cdot,t)) \subset [-1,1], \\ \|\boldsymbol{v}(x,0) - \Theta_1(x,0)\| \equiv 0 \text{ for } x \le -1, \\ \|\boldsymbol{v}(x,0) - \Theta_1(x,0)\| \le \delta\, e^{-|x|} \text{ for } x \ge -1, \\ \int_R (\boldsymbol{v} - \Theta_1)(x,0)\, dx \parallel (u_+ - u_-). \end{cases}$$

We will need to use Lemma 5.12 to estimate $\Theta_1^1(x,t)$.

The function $\Theta_1(x,t)$ localizes the initial data of $\boldsymbol{v}(x,0)$ linearly. In order to effectively localize the initial data, we need a nonlinear localization. One needs to consider the following sequence:

$$\begin{cases} (\bar{\mathbb{D}}_i^2)_t + (\lambda_2^- \, \bar{\mathbb{D}}_i^2)_x - (\bar{\mathbb{D}}_i^2)_{xx} = (\mathscr{N}^2[\Theta_i] - \mathscr{N}^2[\Theta_{i-1}])_x, \\ \bar{\mathbb{D}}_i^2(x,0) \equiv 0, \end{cases} \tag{3.3a}$$

where $\mathscr{N}(\mathbb{U}) \equiv f(u_- + \mathbb{U}) - f(u_-) - f'(u_-)\,\mathbb{U}$

$$= \int_0^1 \int_0^1 \xi f''(u_- + \xi\bar{\xi}\mathbb{U}) \cdot (\mathbb{U},\mathbb{U})\, d\xi d\bar{\xi},$$

$$\mathscr{N}[\mathbb{U}] \equiv \sum_{j=1}^{2} \mathscr{N}^j[\mathbb{U}]\, r_j(u_-),$$

$$\Theta_0 \equiv 0,$$

$$\Theta_{i+1} \equiv \mathbb{D}_i + \Theta_i \text{ for } i \ge 1,$$

$$\mathbb{D}_i^2(x,t) \equiv \chi_-(x)\, \bar{\mathbb{D}}_i^2(x,t),$$

$$\mathscr{E}_{i+1}(x,t) \equiv (\mathbb{D}_i^1)_t + (\lambda_2^- \mathbb{D}_i^1)_x - (\mathbb{D}_i^1)_{xx} + (\mathscr{N}^2[\Theta_i] - \mathscr{N}^2[\Theta_{i-1}])_x,$$

$$\mathscr{E}_{i+1}^+ \equiv c^+ \cdot \mathscr{E}_{i+1}, \quad \mathscr{E}_{i+1}^- \equiv c^- \cdot \mathscr{E}_{i+1},$$

$$\begin{cases} (\mathbb{D}_i^1)_t + (\lambda_1^- \mathbb{D}_i^1)_x - (\mathbb{D}_i^1)_{xx} + (\mathscr{N}^1[\Theta_i] - \mathscr{N}^1[\Theta_{i-1}])_x = -\mathscr{E}_{i+1}^-, \\ \mathbb{D}_i^1(x,0) \equiv 0, \end{cases} \tag{3.3b}$$

$$\mathbb{D}_i(x,t) \equiv \mathbb{D}_i^1(x,t)\, r_1(u_-) + \mathbb{D}_i^2\, r_2(u_-).$$

In the above iteration, the updated waves Θ_i eventually nonlinearly localize the initial data; and leave a sequence of waves $\mathscr{E}_i\ (u_+ - u_-)$ to interact with the shock profile.

In the updating procedure, we need to solve a sequence of initial value problems in (3.3a) and (3.3b). We will need to use Lemmas 5.2, 5.4, 5.5, 5.8, 5.12, and 5.13 to analyze the convergence of this sequence. In particular, we need a special cancellation from equations itself in order to obtain a proper decaying rate in time for $\mathbb{D}_i^2(x,t)$, see (6.14).

Finally, we define the diffusion wave $\Theta(x,t)$ as follows

$$\begin{cases} \Theta(x,t) \equiv \lim_{i\to\infty} \Theta_i(x,t); \\ \Theta(x,t) \equiv \sum_{j=1}^{2} \Theta^j(x,t)\ r_j(u_-) \end{cases} \tag{3.4}$$

and it satisfies

$$\Theta_t + f(\Theta + u_-)_x - \Theta_{xx} = E(x,t)\ (u_+ - u_-), \tag{3.5}$$

$$\begin{cases} |\Theta(x,0) - \boldsymbol{v}(x,0)| \equiv 0 \text{ for } x < -1, \\ l_1(u_-)\cdot(\boldsymbol{v}(x,0) - \Theta(x,0)) = 0 \text{ for } x \geq 1, \\ \int_R \boldsymbol{v}(x,0) - \Theta(x,0)\ dx \parallel (u_+ - u_-), \\ E(x,t) \equiv \sum_{i=1}^{\infty} \mathscr{E}_i^+(x,t). \end{cases} \tag{3.6}$$

Theorem DW (Diffusion Wave). *Suppose that both δ/ϵ^4 and ϵ are sufficiently small. Then, the sequence $\{\mathbb{D}_i\}_{i\geq 1}$ converges and there exists a constant $\bar{c}$ satisfying*

for $j = 0, 1, 2$

$|\partial_x^j \Theta^1(x,t)|,\ |\partial_x^j \Theta_1^1(x,t)|$

$$\leq O(1)\ \delta \begin{cases} \theta^{j+1}(x,t;\lambda_1^-,D) + \epsilon\ \eta(x,t)\ \ (|x|+1)^{-j/2} & \textit{for } x \leq 1, \\ \epsilon\rho^1(t)\ e^{-\frac{|(\lambda_1^- - \lambda_2^-)x|}{2}} & \textit{for } x \geq 1, \end{cases}$$

$$|\partial_x^j \mathbb{D}_i^1(x,t)| \le O(1)\ \delta\ (\bar{c}\bar{\epsilon})^i \begin{cases} \theta^{j+1}(x,t;\lambda_1^-,D)+\epsilon\eta(x,t)(|x|+1)^{-j/2} & \textit{for } x\le 1, \\ \epsilon\ \rho^1(t)\ e^{-\frac{|(\lambda_1^- - \lambda_2^-)x|}{2}} & \textit{for } x\ge 1; \end{cases}$$

for $j = 0, 1$

$$|\partial_x^j \Theta^2(x,t)|,\ |\partial_x^j \Theta_1^2(x,t)|$$

$$\le O(1)\ \delta \begin{cases} 0 \ \textit{for } x \ge 1, \\ \left(\frac{1}{\sqrt{t^j}} + \varrho(x)\right) \left[\theta^1(x,t;\lambda_2^-,D) + \psi^{3/2}(x,t;\lambda_2^-)\right] \ \textit{for } x \le 1, \end{cases}$$

$$|\partial_x^j \mathbb{D}_i^2(x,t)|$$

$$\le O(1)\ \delta\ (\bar{c}\bar{\epsilon})^i \begin{cases} 0 \ \textit{for } x \ge 1, \\ \left(\frac{1}{\sqrt{t^j}} + \varrho(x)\right) \left[\theta^1(x,t;\lambda_2^-,D) + \psi^{3/2}(x,t;\lambda_2^-)\right] \ \textit{for } x \le 1, \end{cases}$$

$$supp(\mathscr{D}_i(\cdot,t)) \subset [-1,1], \quad \sup_{x\in\mathbf{R}} |\mathscr{E}_i(x,t)| \le O(1)\ \delta\ \bar{\epsilon}^{\ i-1}\ \rho^1(t),$$

where $\bar{\epsilon} \equiv \delta\epsilon^{-4} \ll 1$ *and* $D > 4$.

Corollary 3.2. *The function* $E(x,t)$ *satisfies*

$$|E(x,t)| \le O(1)\ \frac{\delta\ \varrho(x/2)}{\epsilon}\ \rho^1(t).$$

By adjusting the wave front properly, we may assume that

$$\int_R \Theta(x,0) - \boldsymbol{v}(x,0)\ dx = 0. \tag{3.7}$$

4. Local Conservation Laws and Wave Front Tracing

When we take out the generalized diffusion wave $\Theta(x,t)$, from (3.5) there is a wave $E\ (u_+ - u_-)$ left in the region of shock. This wave will residue and result in $H(t)$ phase shift in the shock profile. The function $H(t)$ satisfies

$$H'(t) = \int_R E(x,t)\ dx.$$

We denote the total phase shift

$$\gamma(t) \equiv H(t) + \bar{\gamma}(t).$$

4.1. System of equations with wave fronts

We set

$$\begin{aligned}
&H(t) \equiv \int_0^t \left(\int_R E(x,\tau)\ dx \right)\ d\tau, \\
&\begin{cases} H(t) = O(1)\ \dfrac{\delta}{\epsilon} \displaystyle\int_0^t \rho^1(\sigma)\ d\sigma, \\ H'(t) = O(1)\ \dfrac{\delta}{\epsilon}\ \rho^1(t), \end{cases} \\
&\mathbb{A}(x,t) \equiv \phi(x - H(t) - \bar{\gamma}(t)), \\
&\mathscr{A}(x,t) \equiv \mathbb{A}(x,t) + \Theta(x,t), \\
&\mathbb{V}(x,t) \equiv u(x,t) - \mathscr{A}(x,t).
\end{aligned} \tag{4.1}$$

The function $\mathbb{V}(x,t)$ satisfies

$$\mathbb{V}_t + (f'(\mathbb{A})\mathbb{V})_x - \mathbb{V}_{xx} = -\left[\left((\boldsymbol{N} + \boldsymbol{I}_0)\,[\mathbb{V}] + \boldsymbol{I}_1[\Theta] + \boldsymbol{I}_2[\bar{\gamma}] \right)_x + \boldsymbol{I}_3[\bar{\gamma}] \right] \tag{4.2}$$

where

$$\begin{aligned}
&\boldsymbol{N}[\mathbb{V}] \equiv \int_0^1 \int_0^1 \xi f''(\mathscr{A} + \bar{\xi}\xi\mathbb{V})(\mathbb{V},\mathbb{V})\ d\xi d\bar{\xi}, \\
&\bar{\boldsymbol{N}}[\mathbb{V}] \equiv \int_0^1 \int_0^1 \xi f''(\mathscr{A})(\mathbb{V},\mathbb{V})\ d\xi d\bar{\xi}, \\
&\boldsymbol{I}_0[\mathbb{V}] \equiv [f'(\mathscr{A}) - f'(\mathbb{A})]\ \mathbb{V}, \\
&\boldsymbol{I}_1[\Theta] \equiv \int_0^1 \int_0^1 \bar{\xi}[f''(u_- + \xi\bar{\xi}\Theta) - f''(\phi + \xi\bar{\xi}\Theta)](\Theta,\Theta) d\xi d\bar{\xi} + (f'(u_-) - f'(\mathbb{A}))\Theta
\end{aligned} \tag{4.3}$$

$$\boldsymbol{I}_2[\bar{\gamma}] \equiv \int_{-\infty}^x -E(y,t)\ (u_+ - u_-) + H'(t)\ \phi'(y - H(t) - \bar{\gamma}(t))\ dy, \tag{4.4}$$

$$\begin{aligned}
&\boldsymbol{I}_3[\bar{\gamma}] \equiv \bar{\gamma}'(t)\ \phi'(x - H(t) - \bar{\gamma}(t)), \\
&\boldsymbol{I}_4[\mathbb{V}] \equiv \left(f'(u_-) - f'(\mathbb{A}) \right)\ \mathbb{V}.
\end{aligned} \tag{4.5}$$

The term $\bar{\boldsymbol{N}}[\mathbb{V}]$ is a quadratic nonlinear term. Here, $\boldsymbol{N}[\mathbb{V}]$, $\boldsymbol{I}_0[\mathbb{V}]$, $\boldsymbol{I}_1[\Theta]$, and $\boldsymbol{I}_2[\bar{\gamma}]$ are all functions of (x,t); and satisfy

$$\begin{cases} \|\boldsymbol{N}[\mathbb{V}](x,t)\| \le O(1)\ \|\mathbb{V}(x,t)\|^2, \\ \|\boldsymbol{I}_0[\mathbb{V}](x,t)\| \le O(1)\ \|\Theta(x,t)\| \cdot \|\mathbb{V}(x,t)\|, \\ \|\boldsymbol{I}_1[\Theta](x,t)\| \le O(1)\ \|\Theta(x,t)\| \cdot \|\phi(x-H(t)-\bar{\gamma}(t)) - u_-\|, \\ \|\boldsymbol{I}_2[\bar{\gamma}](x,t)\| \le O(1)\ \epsilon\ |H'(t)|\ e^{-\lambda_2^-|x-H(t)-\bar{\gamma}(t)|(1+O(1)\epsilon)}. \end{cases} \tag{4.6}$$

If $\|\bar{\gamma}\|_\infty \le 1$, then by Corollary 3.2, (4.1), and (4.4)

$$\begin{cases} |l_1(u_-)\cdot \boldsymbol{I}_2[\bar{\gamma}](x,t)| \le O(1)\ \delta\ \epsilon\ \rho^1(t)\big[\epsilon\ e^{-\lambda_2^-|x|(1+O(1)\epsilon)} + \varrho(\frac{x}{2})\big], \\ |l_2(u_-)\cdot \boldsymbol{I}_2[\bar{\gamma}](x,t)| \le O(1)\ \delta\ \rho^1(t)\big[\epsilon\ e^{-\lambda_2^-|x|(1+O(1)\epsilon)} + \varrho(\frac{x}{2})\big], \end{cases} \tag{4.7}$$

where $\lambda_2^- = (1+O(1)\epsilon)\frac{\epsilon}{2}$.

4.2. Local conservation laws

Since the shock wave (u_-,u_+) is a stationary 2-shock, the local conservation defined in (1.2) becomes

$$\int_{-L}^{L} \mathbb{V}(x,t)\ dx = C_1(t)\ r_1(u_-) \parallel r_1(u_-).$$

We set $L = J_0 = 2\epsilon^{-1}|\log\epsilon|\ J$ with $J \gg 1$. On the other hand, the wave structure $\|\mathbb{V}(x,0)\|$ decays exponential fast. Eventually, there is almost no wave contained in the region $x \ge J_0$. We can relax the upper limit in the above integral sign to redefine the local conservation laws for a 2-shock wave (u_-,u_+) of a 2x2 system as follows:

Definition 4.1.(Viscous Wave Front due to Local Conservation Laws) The function $\bar{\gamma}$ is chosen to satisfy

$$\int_{-J_0}^{\infty} \mathbb{V}(y,t)\ dy \parallel r_1(u_-) \tag{4.8}$$

where $J_0 \equiv 2J\epsilon^{-1}|\log\epsilon|$.

Since the wave structures of $\mathbb{V}$ essentially is contained in the region of left to the shock wave, we consider the following decomposition regarding the background field u_-

$$\begin{aligned}\mathbb{V}(x,t) &\equiv \sum_{j=1}^{2} \mathbb{V}^j(x,t)\ r_j(u_-) \text{ for } x \in \mathbf{R}, t \geq 0,\\ |\mathbb{V}|(x,t) &\equiv \sum_{j=1}^{j} |\mathbb{V}^j(x,t)|.\end{aligned}$$

Integrate (4.2) over $[-J_0,\infty)$ with respect to x and multiply it with $l_2(u_-)$ from the left, and use

$$\|\mathbb{A}(-J_0) - u_-\| = O(1)\ \epsilon^{1+J}$$

to yield that

$$\begin{aligned}&\big(-\lambda_2^-\ \mathbb{V}^2 + |\mathbb{V}_x^2| + O(1)\big(|\mathbb{V}|\cdot|\mathbb{A}_x| + |\mathbb{V}_x|\cdot|\mathbb{A}-u_-|\big)\big)\Big|_{x=-J_0}\\ &\qquad + \bar{\gamma}'(t)[l_2(u_-)(u_+-u_-) + O(1)\ \epsilon^{1+J}e^{O(1)\ \epsilon\ |\bar{\gamma}(t)|}]\\ &= O(1)\ \epsilon^{1+J}|\mathbb{V}|(-J_0,t) + \Big[|\boldsymbol{N}[\mathbb{V}]| + |\boldsymbol{I}_0[\mathbb{V}]| + |\boldsymbol{I}_1[\Theta]| + |\boldsymbol{I}_2[\bar{\gamma}]|\Big]\Big|_{x=-J_0}.\end{aligned} \tag{4.9}$$

This results in the equation of $\bar{\gamma}'(t)$:

$$\begin{aligned}\bar{\gamma}'(t) &= \frac{\lambda_2^-\ \mathbb{V}^2(-J_0,t) + \mathbb{V}_x^2(-J_0,t)}{2\epsilon}(1+O(\epsilon))\\ &\quad + O(1)\ \frac{\epsilon^{J+1}\|\mathbb{V}(-J_0,t)\| + \|\mathbb{V}(-J_0,t)\|^2}{\epsilon}.\end{aligned} \tag{4.10}$$

From this expression, the dominant term for determining the behavior of $\bar{\gamma}'(t)$ is due to $\mathbb{V}^2(-J_0,t)$. It illustrates the interactions between shock layer and its far fields.

We also need to consider the following variables $\mathbb{W}^i$ to analyze $\mathbb{V}$ in the region of shock.

$$\mathbb{W}(x,t) \equiv -\int_x^\infty \mathbb{V}(y,t)\ dy \text{ for } x \geq -J\epsilon^{-1}|\log\epsilon|,\ t \geq 0$$

$$\mathbb{W}(x,t) \equiv \sum_{j=1}^{2} \mathbb{W}^j(x,t)\ r_j(\mathbb{A}), \tag{4.11}$$

$$|\mathbb{W}|(x,t) \equiv \sum_{j=1}^{2} |\mathbb{W}^j(x,t)|.$$

The relationship between $\mathbb{W}^j$ and $\mathbb{V}^j$ is

$$\sum_{j=1}^{2} \mathbb{V}^j(x,t) r_j(u_-) = \sum_{j=1}^{2} \mathbb{W}_x^j(x,t) r_j(\mathbb{A}(x,t)) + \mathbb{W}^j(x,t)\partial_x r_j(\mathbb{A}(x,t)).$$

With the conditions $r_j(u_) = r_j(\mathbb{A}) + O(1)\|\mathbb{A} - u_-\|$ and $\|u_- - u_+\| = 2\epsilon$, we have that

$$\mathbb{V}^j(x,t) = \mathbb{W}_x^j(x,t) + O(1)\ \epsilon\|\mathbb{V}(x,t)\| + O(1)\ \epsilon^2\ \|\mathbb{W}(x,t)\|. \tag{4.12}$$

The system for $\mathbb{W}(x,t)$ is

$$\mathbb{W}_t + f'(\mathbb{A})\ \mathbb{W}_x - \mathbb{W}_{xx} = -((\boldsymbol{N} + \boldsymbol{I}_0)[\mathbb{V}] + \boldsymbol{I}_1[\Theta] + \boldsymbol{I}_2[\bar{\gamma}]) - \bar{\gamma}'(t)\ (\mathbb{A}(x,t) - u_-). \tag{4.13}$$

The equations for $\mathbb{V}^i$ and $\mathbb{W}^i$ are

$$\begin{aligned}
&\mathbb{W}_t^i + \lambda_i(\mathbb{A})\mathbb{W}_x^i - \mathbb{W}_{xx}^i \\
= \ & l_i(\mathbb{A}) \cdot \left(\sum_{j=1}^{2} \mathbb{W}^j\ \big(H'(t) + \bar{\gamma}'(t) - \lambda_i(\mathbb{A})\big) r_j(\mathbb{A})_x \right) \\
&+ l_i(\mathbb{A}) \cdot \left[\left(\sum_{j=1}^{2} 2\mathbb{W}_x^j\ r_j(\mathbb{A})_x + \mathbb{W}^j r_j(\mathbb{A})_{xx} \right) - \big((\boldsymbol{N} + \boldsymbol{I}_0)\,[\mathbb{V}] + \boldsymbol{I}_1[\Theta] + \boldsymbol{I}_2[\bar{\gamma}]\big) \right] \\
&- \bar{\gamma}'(t)\ l_i(u_-) \cdot \big(u_+ - \phi(x - H(t) - \bar{\gamma}(t))\big).
\end{aligned} \tag{4.14}$$

The condition (4.8) becomes a condition for $\mathbb{W}(x,t)$ at $x = -J_0$. We apply the expansion $\mathbb{A}(-J_0,t) = u_- + O(1)\ \epsilon\ e^{-\lambda_2^-|J_0 + H(t) + \bar{\gamma}(t)|}$ to

$$\begin{aligned}
\mathbb{W}^2(-J_0,t) \ &= \ -l_2(\mathbb{A}(-J_0,t)) \int_{-J_0}^{\infty} \mathbb{V}(x,t)dx \\
&= \ \Big(-l_2(u_-) + O(1)\epsilon\ e^{-\lambda_2^-|J_0 + H(t) + \bar{\gamma}(t)|}\Big) \int_{-J_0}^{\infty} \mathbb{V}(x,t)dx
\end{aligned}$$

$$= \epsilon\, e^{-\lambda_2^- |J_0+H(t)+\bar{\gamma}(t)|} \|\mathbb{W}(-J_0,t)\|$$

to obtain that

$$\mathbb{W}^2(-J_0,t) = O(1)\, \epsilon^{1+J}\, |\mathbb{W}|(-J_0,t). \tag{4.15}$$

For the variable $\mathbb{V}(x,t)$, we need to break $(f'(\mathbb{A})\mathbb{V})_x$ in (4.2) into $(f'(u_-)\mathbb{V}+[f'(\mathbb{A})-f'(u_-)]\mathbb{V})_x$; and then diagonalize it as follows

$$\begin{aligned}&(\mathbb{V}^i)_t + (\lambda_i^-\ \mathbb{V}^i)_x - (\mathbb{V}^i)_{xx}\\ &= -l_i(u_-)\cdot \big(\big\{\boldsymbol{I}_4[\mathbb{V}]+(\boldsymbol{N}+\boldsymbol{I}_0)\,[\mathbb{V}]+\boldsymbol{I}_1[\Theta]+\boldsymbol{I}_2[\bar{\gamma}]\big\}_x+\boldsymbol{I}_3[\bar{\gamma}]\big),\ i=1,2, \end{aligned} \tag{4.16}$$

with the initial data

$$\int_R \mathbb{V}(x,0)\, dx = 0,$$

$$|\mathbb{V}^1(x,0)| \le \delta \begin{cases} 0 \text{ for } |x|\ge 1,\\ 1 \text{ for } |x|\le 1.\end{cases}$$

$$|\mathbb{V}^2(x,0)| \le \delta \begin{cases} 0 \text{ for } x\le -1,\\ e^{-|x|} \text{ for } x\ge -1.\end{cases}$$

The integral representations of the variables $\mathbb{V}^i(x,t)$:

$$\begin{aligned}\mathbb{V}^i(x,t) &= \int_R \theta^1(x-y;t;\lambda_i^-,4)\mathbb{V}^i(y,0)\, dy\\ &\quad + \left[\mathbb{I}_1^i[\Theta] + \mathbb{I}_2^i[\bar{\gamma}] + \mathbb{I}_3^i[\bar{\gamma}] + \sum_{j=4}^{6} \mathbb{I}_j^i[\mathbb{V}]\right](x,t),\end{aligned} \tag{4.17}$$

where

$$\mathbb{I}_1^i[\Theta](x,t) \equiv \int_0^t \int_R \theta_y^1(x-y,t-\sigma;\lambda_i^-,4)\cdot \big(l_i(u_-)\cdot \boldsymbol{I}_1[\Theta]\big)(y,\sigma)\, dy\, d\sigma,$$

$$\mathbb{I}_2^i[\bar{\gamma}](x,t) \equiv \int_0^t \int_R \theta_y^1(x-y,t-\sigma;\lambda_i^-,4)\cdot \big(l_i(u_-)\cdot \boldsymbol{I}_2[\bar{\gamma}]\big)\,(y,\sigma)\, dy\, d\sigma,$$

$$\mathbb{I}_3^i[\bar{\gamma}](x,t) \equiv \int_0^t \int_R \theta^1(x-y,t-\sigma;\lambda_i^-,4)\cdot \big(l_i(u_-)\cdot \boldsymbol{I}_3[\bar{\gamma}]\big)(y,\sigma)\, dy\, d\sigma,$$

$$\mathbb{I}_4^i[\mathbb{V}](x,t) \equiv \int_0^t \int_R \theta_y^1(x-y,t-\sigma;\lambda_i^-,4)\cdot(l_i(u_-)\cdot \boldsymbol{I}_4[\mathbb{V}])\,(y,\sigma)\,dy\,d\sigma,$$

$$\mathbb{I}_5^i[\mathbb{V}](x,t) \equiv \int_0^t \int_R \theta_y^1(x-y,t-\sigma;\lambda_i^-,4)\cdot\big(l_i(u_-)\cdot \boldsymbol{I}_0[\mathbb{V}]\big)(y,\sigma)\,dy\,d\sigma,$$

$$\mathbb{I}_6^i[\mathbb{V}](x,t) \equiv \int_0^t \int_R \theta_y^1(x-y,t-\sigma;\lambda_i^-,4)\cdot\big(l_i(u_-)\cdot \boldsymbol{N}[\mathbb{V}]\big)(y,\sigma)\,dy\,d\sigma,$$

$$\bar{\mathbb{I}}_6^i[\mathbb{V}](x,t) \equiv \int_0^t \int_R \theta_y^1(x-y,t-\sigma;\lambda_i^-,4)\cdot\big(l_i(u_-)\cdot \bar{\boldsymbol{N}}[\mathbb{V}]\big)(y,\sigma)\,dy\,d\sigma.$$

In order to obtain a proper decaying rate of $\mathbb{V}^2(-J_0,t)$, one needs carefully analyze the representation in (4.17) with $i=2$. To be precise, we want to have some cancellation in $\mathbb{I}_1^2[\Theta]$, $\mathbb{I}_6^2[\mathbb{V}]$, and $\bar{\mathbb{I}}_6^2[\mathbb{V}]$. This cancellation is particularly due to the nonlinear coupling from the $\mathbb{V}^1$ field to $\mathbb{V}^2$. We need a proper decomposition of $\mathbb{V}^1$ as the following in order to handle the cancellation properly.

$$\begin{aligned} \mathbb{V}_\pi(x,t) &\equiv \sum_{i=1}^{2} \mathbb{V}_\pi^i(x,t)\, r_i(u_-), \quad \mathbb{V}_\pi^2(x,t)\equiv 0,\\ \mathbb{V}_{\pi^\perp}(x,t) &\equiv \mathbb{V}(x,t)-\mathbb{V}_\pi(x,t),\\ \mathbb{V}_\pi^1 &\equiv \mathbb{I}_3^1[\bar\gamma]+\mathbb{I}_5^1[\mathbb{V}_\pi]+\bar{\mathbb{I}}_6^1[\mathbb{V}_\pi]. \end{aligned} \tag{4.18}$$

In this decomposition, we have separated the quadratic nonlinearity $\bar{\boldsymbol{N}}[\mathbb{V}]$ from the nonlinear term $\boldsymbol{N}[\mathbb{V}]$. So that, we can have a higher order estimates on $\mathbb{V}_\pi^1$. This function has a lower decaying rate in time and a smaller L^1 norm in time compared to $\mathbb{V}_{\pi^\perp}^1$. The functions $\mathbb{V}_\pi^1$ satisfies

$$\begin{cases} \partial_t\mathbb{V}_\pi^1+\lambda_1^-\partial_x\mathbb{V}_\pi^1-\partial_x^2\mathbb{V}_\pi^1 = -l_1(u_-)\cdot\big(\boldsymbol{I}_3[\bar\gamma]+\big\{\bar{\boldsymbol{N}}[\mathbb{V}_\pi]+\boldsymbol{I}_0[\mathbb{V}_\pi]\big\}_x\big),\\ \mathbb{V}_\pi^1(x,0)\equiv 0. \end{cases} \tag{4.19}$$

In the following we impose a general theorem for the function $\mathbb{V}_\pi^1$.

Let $\epsilon_0>0$ and $w(x,t)$ be the solution of

$$\begin{cases} w_t+\lambda_1^- w_x - w_{xx} = -l_1(u_-)\cdot\big(f''(\mathbb{A})(\Theta, w\, r_1(u_-)) + \bar{\boldsymbol{N}}[w\, r_1(u_-)]\big)_x + S(x,t),\\ w(x,0)\equiv 0,\\ |S(x,t)|\le \epsilon_0\,\rho^1(t)\,e^{-\lambda_2^-|x|}. \end{cases} \tag{4.20}$$

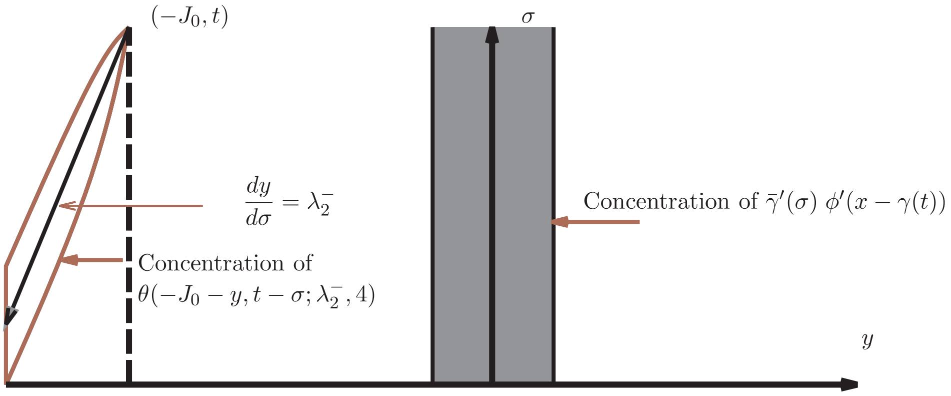
Lemma P. *Suppose that both ϵ_0 and δ are sufficiently small. Then, the function $w(x,t)$ satisfies*

$$|\partial_x^i w(x,t)| \le O(1)\,\frac{\epsilon_0}{|\lambda_2^-|}\begin{cases}\rho^1(t)\,e^{-\lambda_2^-|x|(1+O(\epsilon))}\ \textit{for } x\ge 0,\\ \min\left(|\lambda_2^-|, t^{-(i+1)/2}\right)\ \bar{\Gamma}^1(t)\ \theta^0(x,t;\lambda_1^-,D)\\ +\min\left(\sqrt{|\lambda_2^-|^{\min(2,i-1)}}, (|x|+1)^{-\frac{|i-1|}{2}}\right)\ \eta^1(x,t)\ \textit{for } x\le 0,\end{cases}$$

where $i=0,1,2$.

4.3. Scheme for constructing the ansatz of $\mathbb{V}(x,t)$

The system (4.16) contains an unknown variable $\bar{\gamma}'$; and from (4.10) the behavior of $\bar{\gamma}'$ essentially is dominated by the variable $\mathbb{V}^2(-J_0,t)$. The variables $\mathbb{V}^2(-J_0,t)$ and $\bar{\gamma}'$ are coupled each other. Indeed, these two variables are weakly coupled when both J_0 and $1/\delta$ are sufficiently large.



From the above figure, the essential domain of dependence for $\mathbb{V}^2(-J_0,t)$ is separated from the concentration of $\bar{\gamma}'(\sigma)\mathbb{A}_y(y,\sigma)$. Thus, the influence of from $\bar{\gamma}'(\sigma)\mathbb{A}_y(y,\sigma)$ to $\mathbb{V}^2(-J_0,t)$ is proportional to $e^{-O(1)\epsilon L}$. This influence can be neglected when ϵL is sufficiently large.

In this scheme we will construct the dominant behavior of $\mathbb{V}$ through the following procedures.

1. The leading ansatz of $\mathbb{V}^2(x,t)$ in $x\ge 0$.

We drop the terms $\Big((\boldsymbol{N}+\boldsymbol{I}_0)[\mathbb{V}]+\boldsymbol{I}_1[\Theta]\Big)_x+\boldsymbol{I}_3[\bar{\gamma}]$ from (4.2). Then, the system (4.2) becomes conservative system. The resulted system is sufficient to give the behavior of $\mathbb{V}^2(x,t)$ for $x\geq -O(1)\epsilon^{-1}$. So, we consider the following scalar equation:

$$\begin{cases}\partial_t U_0^2+\partial_x(\bar{\lambda}_2(x,t)U_0^2)-\partial_x^2 U_0^2=-l_2(u_-)\cdot \boldsymbol{I}_2[\bar{\gamma}]_x,\\ U_0^2(x,0)=\mathbb{V}^2(x,0),\end{cases}\tag{4.21}$$

where $\bar{\lambda}_2(x,t)\equiv l_2(u_-)f'(\mathbb{A})r_2(u_-)$. The solution $U_0^2(x,t)$ decays exponentially fast in space but algebraically in time. This function $U_0^2(x,t)$ is evaluated by Lemma 5.17.

We will show that

$$|\mathbb{V}^2(x,t)|\leq O(1)U_0^2(x,t) \text{ for } x\geq 0.$$

2. $\mathbb{V}^1(x,t)$.

Since the function $U_0^2(x,t)$ decays exponentially in space, the main contribution to $\mathbb{V}^2(-J_0,t)$ is due to nonlinear coupling from $\mathbb{V}^1(x,t)$. The function $\mathbb{V}^1(x,t)$ can decomposed into two parts. The dominant behavior is similar to the solution of the initial value problem

$$\begin{cases}(U_f^1)_t+(\lambda_1^- U_f^1)_x-(U_f^1)_{xx}=-l_1(u_-)\cdot\Big(\boldsymbol{I}_4[U_0^2\ r_2(u_-)]+\boldsymbol{I}_2[\bar{\gamma}]\Big)_x,\\ U_f^1(x,0)=\mathbb{V}^1(x,0).\end{cases}\tag{4.22}$$

This is (4.16) with $i=1$ and with all the source terms replaced by $-l_1(u_-)\cdot(\boldsymbol{I}_4[U_0^2\ r_2(u_-)]+\boldsymbol{I}_2[\bar{\gamma}])_x$. The function $U_f^1(x,t)$ is evaluated by Lemma 5.13.

Next, we evaluate the solution $U_1^2(-J_0,t)$ of the following equation at $x=-J_0$:

$$\begin{cases}(U_f^2)_t+(\lambda_2^- U_f^2)_x-(U_f^2)_{xx}=-l_2(u_-)\cdot \boldsymbol{N}[U_f^1\ r_1(u_-)]_x,\\ U_1^2(x,0)\equiv 0.\end{cases}\tag{4.23}$$

The function U_f^2 is evaluated by Lemma 5.6.

The function $\displaystyle\int_0^\infty |U_f^2(-J_0,t)|\ dt$ is almost equal to $\|\bar{\gamma}'\|_{L^1}$. By (4.10), we define

$$\bar{\gamma}_f'(t) \equiv \frac{\lambda_2^- \ U_f^2(-J_0,t) - \partial_x U_f^2(-J_0,t)}{2\epsilon}.$$

Next, we use this function to construct the slower decaying term of $\mathbb{V}^1$:

$$\begin{cases} (U_s^1)_t + (\lambda_1^- U_s^1) - (U_s^1)_{xx} = l_1(u_-)\cdot \boldsymbol{I}_3[\bar{\gamma}_f], \\ U_s(x,0) = 0. \end{cases}$$

U_s^1 is evaluated by Lemma 5.13.

Finally, we will show that

$$|\mathbb{V}^1(x,t)| \le O(1)\Big[|U_s^1(x,t)| + |U_f^1(x,t)|\Big]. \tag{4.24}$$

3. $\mathbb{V}^2(x,t)$ for $x \le -J_0$ & $\bar{\gamma}'(t)$.

We consider the problem

$$\begin{cases} (U_1^2)_t + (\lambda_2^- \ U_1^2)_x - (U_1^2)_{xx} = -\partial_x \bar{\boldsymbol{N}}[(U_s^1 + U_f^2)\ r_1(u_-)], \\ U_1^2(x,0) \le |\mathbb{V}^2(x,0)|. \end{cases} \tag{4.25}$$

In the evaluation of $U_1^2(-J_0,t)$, we need the cancellation similar to that derived in (6.14) in order to handle the effect due to $\partial_x \bar{\boldsymbol{N}}[U_s^1 \ r_1(u_-)]$ and to obtain the decaying rate $t^{-3/2}$.

From the function U_1^2, we define

$$\bar{\gamma}_d'(t) \equiv \frac{\lambda_2^- \ U_1^2(-J_0,t) - \partial_x U_1^2(-J_0,t)}{2\epsilon}$$

We will show that

$$\begin{cases} |\mathbb{V}^2(x,t)| \le O(1)|U_1^2(x,t)| \text{ for } x \le -J_0, \\ |\bar{\gamma}'(t)| \le O(1)|\bar{\gamma}_d'(t)|. \end{cases} \tag{4.26}$$

Remark 4.2. For the variable $U_1^2(-J_0,t)$, one needs an extra cancellation on $\partial_x \bar{\boldsymbol{N}}[U_s^1 \ r_1(u_-)]$ in order to obtain the rate of decaying $t^{-3/2}$ in time.

4. $\mathbb{V}^2(x,t)$ for $x \in [-J_0+1,0]$.

The construction of this ansatz is a little different from the other. We need to consider the following equation which is modified from (4.14) with $i=2$:

$$\begin{cases} \mathscr{W}_t^2 + \lambda_2(\mathbb{A})\,\mathscr{W}_x^2 - \mathscr{W}_{xx}^2 \\ \qquad = \lambda_2(u_-)\cdot\Big[\boldsymbol{I}_1[\Theta]+\boldsymbol{I}_2[\bar\gamma_d]-\bar\gamma_d'\big(u_+-\phi(x-H(t)-\bar\gamma_f(t))\big)\Big] \\ \mathscr{W}^2(x,0) = l_2(u_-)\cdot\displaystyle\int_x^\infty \mathbb{V}(r,0)\,dr, \\ |\mathscr{W}^2(-J_0,t)| \le O(1)\epsilon^J\,|U_0^2(0,t)|, \\ |\partial_x\mathscr{W}^2(-J_r0,t)| \le |U_1^2(-J_0,t)|. \end{cases} \tag{4.27}$$

The function $\mathscr{W}_x^2(x,t)$ will give the ansatz of $\mathbb{V}^2(x,t)$ for $x\in[-J_0+1,0]$:

$$|\mathbb{V}^2(x,t)| \le O(1)|\mathscr{W}_x^2(x,t)| \text{ for } x\in[-J_0+1,0].$$

The evaluation of $\mathscr{W}_x^2(x,t)$ is a combination of the lemmas in Subsection 5.2.

Remark 4.3. The equation (4.27) itself is an ill-posed boundary value problem. However, our purpose is not the existence of the solution. In stead, we are performing a priori estimate of such solutions.

4.4. Global stability of viscous shock layer

In the previous subsection, we have mentioned the scheme for constructing the ansatz $\mathbb{V}$. With the ansatz, we proceed to establish the Theorems A and B. Here, we omit the calculation of the ansatz and just state the necessary conditions of Theorems A and B.

In the construction, we have ignored some nonlinear terms. Now, we will show that the ansatz is still valid for the full nonlinear problem. In the following Theorem A, both the sufficient conditions and the necessary conditions are the ansatz, but we relax the coefficient of the ansatz in the sufficient condition to a rather large constant in order to handle the nonlinear term.

Theorem A. *Let $C \le |\log\epsilon|$. Suppose that $\delta \le \epsilon^6$, that the solution* $\mathbb{V}$

of (4.2) *satisfies*

$$\mathbb{V}^2(x,t) \le 2K_0\delta\ \rho^1(t)e^{-\frac{\lambda_2^-|x|}{2}(1+O(1)\epsilon)} + C\ \frac{\delta^2|\log\epsilon|^3}{\epsilon^5}\begin{cases} t^{-1}\left[1+\epsilon\sqrt{t}\right]^{-1} & \textit{for } x\in[-J_0,0],\\ t^{-1}\left[1+\epsilon\sqrt{t}\right]^{-1}\ e^{-\frac{\lambda_2^-|x|}{2}(1+O(\epsilon))} & \textit{for } x\ge 0,\end{cases} \tag{4.28a}$$

for $x\in[-J_0,\infty)$, *and that the function* $\bar\gamma'(t)$ *satisfies that*

$$|\bar\gamma'(t)| \le C\left(\frac{\delta^2|\log\epsilon|^2}{\epsilon^5}\ t^{-1}\left[1+\epsilon\sqrt{t}\right]^{-1} + \delta\ \epsilon^{\frac{J}{2}}\rho^1(t)\right), \tag{4.28b}$$

where $J_0 = 2\epsilon^{-1}J$ *with* $J\gg 1$ *and* K_0 *is a constant independent of* ϵ.

Then, there exist $\mathscr{C}$, $\mathscr{C}_0 = O(1)$ *such that*

$$|\mathbb{V}^1_{\pi^\perp}(x,t)| \tag{4.29a}$$
$$\le \mathscr{C}\begin{cases} \delta\sqrt{\epsilon}\ \rho^1(t)\ e^{-\frac{\lambda_2^-|x|(1+O(\epsilon))}{2}} \quad \textit{for } x\ge 0,\\ \delta\bar\Gamma^1(t)\min\left(\sqrt{\epsilon},\frac{1}{\sqrt{t}}\right)\theta^1(x,t;\lambda_1^-,D)+\delta\min\left(\sqrt{\epsilon},\frac{1}{\sqrt{|x|+1}}\right)\eta(x,t)\\ +\frac{\delta^2|\log\epsilon|^2}{\epsilon^3}\left\{|x-\lambda_1^-t|\ \left(1+\epsilon|x-\lambda_1^-t|^{\frac12}\right)\right\}^{-1} \quad \textit{for } \lambda_1^-t+\sqrt{t}\le x\le 0,\\ \delta\ \bar\Gamma^1(t)\ \min\left(\sqrt{\epsilon},\frac{1}{\sqrt{t}}\right)\theta^1(x,t;\lambda_1^-,D)+\frac{\delta^2|\log\epsilon|^3}{\epsilon^3}\theta^{3/2}(x,t;\lambda_1^-,D')\\ \ \textit{for } x\le\lambda_1^-t+\sqrt{t},\end{cases}$$

for $i=0,1,2$

$$|\partial_x^i\mathbb{V}^1_\pi|(x,t) \le \mathscr{C}_0\frac{\delta^2|\log\epsilon|^3}{\epsilon^3}\begin{cases}\left(\theta^{i+1}(x,t;\lambda_1^-,D)+\frac{\epsilon\ \eta^1(x,t)}{\sqrt{(|x|+1)^i}}\right) & \textit{for } x\le 0,\\ \epsilon\ \rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))} & \textit{for } x\ge 0,\end{cases} \tag{4.29b}$$

$$|\partial_x\mathbb{V}^2|(x,t),\ |\mathbb{V}^2|(x,t) \le \mathscr{C}\ \delta\ \epsilon^{J/2}\ \rho^1(t)\ e^{-\lambda_2^-|x|(1+O(\epsilon))/2} + \mathscr{C}\frac{\delta^2|\log\epsilon|^2}{\epsilon^4}\begin{cases}\psi^{3/2}(x,t;\lambda_2^-) & \textit{for } x\in[\lambda_1^-t,-J_0],\\ \theta^3(x,t;\lambda_1^-,D) & \textit{for } x\le\lambda_1^-t,\end{cases} \tag{4.29c}$$

$$|\mathbb{W}|(x,t) \le \mathscr{C}\delta\,\rho^1(t) \begin{cases} \left[\epsilon^{-1} + \frac{\delta\,|\log\epsilon|^3}{\epsilon^6}\right] \ \textit{for } x \in [-J_0, 0], \\ \left[\epsilon^{-1} + \frac{\delta\,|\log\epsilon|^3}{\epsilon^6}\right] \ e^{-\lambda_2^-|x|(1+O(\epsilon))} \ \textit{for } x \ge 0, \end{cases} \tag{4.29d}$$

for any fixed constants $D' > D > 4$.

Remark 4.4. In the evaluation of the ansatz $\mathbb{V}^1(x,t)$ in (4.24). we have ignored the effects of $\boldsymbol{I}_1[\Theta]$ (a small given source term), $\boldsymbol{I}_0[\mathbb{V}]$ (a weakly linear source term)[1], $\boldsymbol{N}[\mathbb{V}]$ (a the nonlinear term), and $\boldsymbol{I}_4[\mathbb{V} - U_0^2\, r_1(u_-)]$. Since all those terms contain small parameters so that we can relax the coefficient C in the sufficient condition of Theorem A and the coefficients $\mathscr{C}$ and $\mathscr{C}_0 = O(1)$ in the necessary condition are still valid.

About the domain of the ansatz $\mathbb{V}^2(x,t)$, in Theorem A we use the system (4.16) to approximate (4.2). So, we have ignored the structure of shock wave in the compressive field. Thus, a sharper estimate of $\mathbb{V}^2(x,t)$ is only valid in regions far away from the shock layer.

About the decomposition $\mathbb{V}_\pi^1$ and $\mathbb{V}_{\pi^\perp}$, when one apply a cancellation to obtain a better decaying rate of $\mathbb{V}^2(-J_0,t)$, one needs to handle either $\mathbb{V}_{\pi xx}^1\mathbb{V}_{\pi^\perp}^1$ or $\mathbb{V}_\pi^1\mathbb{V}_{\pi^\perp xx}^1$. We choose the former one. It is a solution of nonlinear scalar equation without coupling so that we can analyze it precisely.

Theorem B. *Suppose that the solutions* $\mathbb{V}$ *and* $\mathbb{W}$ *satisfy* (4.28a), (4.29c), *and* (4.29d). *Then, there exists a constant* $C_0 > 0$ *such that*

$$|\bar{\gamma}'(t)| \le C_0 \frac{\delta^2|\log\epsilon|^2}{\epsilon^5} t^{-1}(1+\epsilon\sqrt{t})^{-1}, \tag{4.30}$$

$$\begin{aligned} |\mathbb{W}^2|(x,t) \le {} & C_0 \frac{\delta}{\epsilon}\,\rho^1(t) e^{-\frac{\lambda_2^-|x|(1+O(\epsilon))}{2}} \\ & + C_0 \frac{\delta^2|\log\epsilon|^3}{\epsilon^6} t^{-1}(1+\epsilon\sqrt{t})^{-1} \begin{cases} e^{-\frac{\lambda_2^-|x|(1+O(\epsilon))}{2}} \text{ for } x \ge 0, \\ 1 \text{ for } x \in [-J_0, 0], \end{cases} \end{aligned} \tag{4.31}$$

$$\begin{aligned} |\mathbb{V}^2(x,t)| \le {} & 2K_0\delta\,\rho^1(t) e^{-\frac{\lambda_2^-|x|(1+O(\epsilon))}{2}} \\ & + C_0 \frac{\delta^2|\log\epsilon|^3}{\epsilon^5} t^{-1}(1+\epsilon\sqrt{t})^{-1} \begin{cases} e^{-\frac{\lambda_2^-|x|(1+O(\epsilon))}{2}} \text{ for } x \ge 0, \\ 1 \text{ for } x \in [-J_0, 0]. \end{cases} \end{aligned} \tag{4.32}$$

---

[1]The ansatz for $\boldsymbol{I}_0[\mathbb{V}]$ contains a factor δ^2. So, we classify the effect from this term as a weak term.

Remark 4.5. This theorem gives the ansatz $\mathbb{V}^2(x,t)$ in the shock region with information $\mathbb{V}^2(-J_0,t)$ obtained from the necessary condition of Theorem A. This theorem uses the variable $\mathbb{W}(x,t)$ in order to encounter the structure of the compressive field in the shock region effectively in the linear stability level. Finally, we obtain that this necessary condition is stronger than the sufficient condition of Theorem A.

Theorem C.(Local Existence Theorem) *Suppose that δ is sufficient small. Then, there existence $\tau > 0$ independent of t such that whenever* (4.29b), (4.29c), *and* (4.32) *are true for $t \in [0,T]$, then the condition* (4.28a) *is true for $t \in [0,T+\tau]$.*

The proof of this theorem is a standard procedure. We omit it.

Proof of the Main Theorem.

Proof. We will prove the Main Theorem by inductions. By Theorem C, there exists $\tau > 0$ such that (4.28a) and (4.28b) is true for $t \in [0,\tau]$. Thus, Theorem A and Theorem B show that the Main Theorem is true for $t \in [0,\tau]$. We suppose that for some $k \in \mathbb{N}$ the Main Theorem is true for $t \in [0,k\tau]$. Then, from Theorem C we can conclude that (4.28a) and (4.28b) are true for $t \in [0,(k+1)\tau]$. Combine this conclusion, Theorem A, and Theorem B. It follows that the Main Theorem is true for $t \in [0,(k+1)\tau]$. So, for all $k \in \mathbb{N}$ the Main Theorem is true for $t \in [0,k\tau]$. □

5. Technical Lemma

5.1. Dissipation of waves

$$\theta^\alpha(x,t;\mu,D) \equiv (t+1)^{-\alpha/2}\, e^{-\frac{[x-\mu(t+1)]^2}{D(t+1)}}, \tag{5.1}$$

$$\psi^\alpha(x,t;\mu) \equiv [(x-\mu(t+1))^2 + t + 1]^{-\alpha/2}, \tag{5.2}$$

$$\bar{\psi}^\alpha(x,t;\mu) \equiv [(x-\mu(x+1))^3 + (t+1)^2]^{-\alpha/3}, \tag{5.3}$$

$$\begin{aligned}&\rho^\alpha(t) \equiv \left[\sqrt{1+t}\,(1+\epsilon^2 t)\right]^{-\alpha},\\ &\text{for } 0 \le t_1 \le t_2 \le t,\\ &I^{\alpha,\beta,\gamma}(x,t;t_1,t_2;\lambda,\mu,D)\end{aligned} \tag{5.4}$$

$$\equiv \int_{t_1}^{t_2}\int_R (t-\sigma)^{(-\beta+\gamma)/2}(t-\sigma+1)^{-\frac{\gamma}{2}} e^{-\frac{[y-x-\lambda(t-\sigma)]^2}{D(t-\sigma)}} \cdot \theta^\alpha(y,s;\mu,D)dyd\sigma,$$

$$J^{\alpha,\beta,\gamma}(x,t;t_1,t_2;\lambda,\mu,D) \tag{5.5}$$

$$\equiv \int_{t_1}^{t_2}\int_R (t-\sigma)^{(-\beta+\gamma)/2}(t-\sigma+1)^{-\frac{\gamma}{2}} e^{-\frac{[y-x-\lambda(t-\sigma)]^2}{D(t-\sigma)}} \cdot \psi^\alpha(y,s;\mu,D)dyd\sigma,$$

$$\Gamma^\alpha(t) \equiv \int_0^t (\sigma+1)^{-\frac{\alpha}{2}}\, d\sigma = O(1)\begin{cases} 1 & \text{for } \alpha > 2 \\ \log(t+1) & \text{for } \alpha = 2 \\ (t+1)^{(2-\alpha)/2} & \text{for } \alpha < 2, \end{cases} \tag{5.6}$$

$$\bar{\Gamma}^\alpha(t) \equiv \int_0^t \rho^\alpha(\sigma)\, d\sigma = O(1)\begin{cases} \min(\Gamma^\alpha(t), \Gamma^\alpha(\epsilon^{-2})) & \text{for } \alpha < 2, \\ \min(\Gamma^2(t), |\log\epsilon|) & \text{for } \alpha = 2, \\ 1 & \text{for } \alpha > 2, \end{cases}$$

$$\mathbb{J}^{\alpha_1,\alpha_2;\beta}(x,t;\lambda,\mu;E) \equiv \int_0^t\int_R \theta^\beta(x-y,t-\sigma;\lambda,E)\psi^{\alpha_1}(y,\sigma;\mu)(1+\sigma)^{-\alpha_2}dyd\sigma; \tag{5.7}$$

$$\Psi_1^\alpha(x,t) \equiv \begin{cases} 0 \text{ for } x < \lambda_1^-\, t \text{ or } x > 0, \\ (|x-\lambda_1^- t|^2 + t)^{-\alpha/2} \text{ for } x \in [\lambda_1^-\, t, 0], \end{cases} \tag{5.8}$$

$$\Psi_2^\alpha(x,t) \equiv \begin{cases} 0 \text{ for } x < \lambda_1^-\, t \text{ or } x > 0, \\ \left(|x| + \epsilon t + \sqrt{t}\right)^{-\alpha} \text{ for } x \in [\lambda_1^-\, t, 0], \end{cases} \tag{5.9}$$

$$\mathfrak{J}_{j,i}^{\alpha,\beta}(x,t;t_1,t_2;D) \equiv \int_{t_1}^{t_2}\int_R (t-\sigma)^{-\beta/2} e^{-\frac{(x-y-\lambda_i^-(t-\sigma))^2}{D(t-\sigma)}}\ \Psi_j^\alpha(y,\sigma)\, dy\, d\sigma \tag{5.10}$$

for $0 \le t_1 \le t_2 < t$,

$$K^{\alpha,\beta,\gamma}(x,t;t_1,t_2,\lambda,D) \tag{5.11}$$

$$\equiv \int_{t_1}^{t_2}\int_R (t-\sigma+1)^{-\frac{\gamma}{2}}(t-\sigma)^{(-\beta+\gamma)/2} e^{-\frac{[y-x+\lambda(t-\sigma)]^2}{D(t-\sigma)}}(\sigma+1)^{-\frac{\alpha}{2}} e^{-\frac{4\epsilon|y|}{D}} dyd\sigma.$$

$$\mathbb{H}_j^{\alpha_1,\alpha_2;\beta}(x,t;D) \equiv \int_0^t\int_{\frac{\lambda_1^-\sigma}{2}}^1 \theta^\beta(x-y,t-\sigma;\lambda_j^-,D)\frac{\rho^{\alpha_2}(\sigma)}{\sqrt{|y|^{\alpha_1}}}\, dy\, d\sigma,$$

$$\widehat{K}^{\alpha,\beta}(x,t;\lambda,D;\nu) \equiv \int_0^t\int_R \theta^\beta(x-y,t-\sigma,\lambda,D)\ \rho^\alpha(\sigma)\ e^{-\nu|y|}dy\, d\sigma \tag{5.12}$$

$$\mathscr{K}_{j,i}^{\alpha,\beta}(x,t;t_1,t_2;D) \equiv \int_{t_1}^{t_2}\int_{\lambda_1^-\sigma}^0 \epsilon\, \frac{e^{-\frac{(x-y-\lambda_i^-(t-\sigma))^2}{D(t-\sigma)}}}{(t-\sigma)^{\beta/2}}\ \psi^\alpha(y,\sigma;\lambda_j^-)\ e^{-\frac{\epsilon|y|}{4D}}dyd\sigma, \tag{5.13}$$

$$\mathbb{K}^{\alpha,\beta}_{j,i}(x,t;t_1,t_2;D) \equiv \int_{t_1}^{t_2}\int_R \epsilon\, \frac{e^{-\frac{(x-y-\lambda_i^-(t-\sigma))^2}{D(t-\sigma)}}}{(t-\sigma)^{\beta/2}} \theta^\alpha(y,\sigma;\lambda_j^-,D)\, e^{-\frac{\epsilon|y|}{4D}} dyd\sigma, \tag{5.14}$$

Most of the following lemmas are due to [7]:

Lemma 5.1. *Suppose that $\alpha > 0$, $\beta \geq \gamma \geq 0$, and $\beta - \gamma < 3$. Then,*

$$\begin{aligned} I^{\alpha,\beta}(x,t;0,t;\lambda,\lambda;D) &= O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(t+1) \\ &\quad +(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(t+1)]\theta^1(x,t;\lambda,D). \end{aligned} \tag{5.15}$$

In particular,

$$I^{\alpha,\beta}(x,t;0,t;\lambda,\lambda;D) = O(1) \begin{cases} \theta(x,t;\lambda,D) \text{ for } \alpha \geq 3, \beta = 1, \\ \theta^{3/2}(x,t;\lambda,D) \text{ for } \alpha \geq 2.5, \beta = 2. \end{cases} \tag{5.16}$$

Lemma 5.2. *Suppose that $\alpha \geq 1$, $\beta \geq \gamma \geq 0$, $\beta - \gamma < 3$, and $\lambda < \mu$. Then for any given constant $E > D$,*

$$\begin{aligned} &I^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\mu;D) \\ &= O(1)(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1})\theta(x,t;\lambda,D) \\ &+O(1)(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\theta(x,t;\mu,D) \\ &+ \begin{cases} 0 \textit{ for } x < \lambda(t+1)+\sqrt{t+1} \textit{ or } x > \mu(t+1)-\sqrt{t+1} \\ O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(x-\lambda t)\theta(x,t;\lambda,E) \\ +(\mu t - x)^{(-\beta+1)/2}(x-\lambda t)^{(-\alpha+1)/2} \\ +(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\mu t - x)\theta(x,t;\mu,E)] \\ \textit{ for } \lambda(t+1)+\sqrt{t+1} < x < \mu(t+1)-\sqrt{t+1}. \end{cases} \end{aligned} \tag{5.17}$$

In particular,

$$I^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\mu,D) = O(1) \begin{cases} \psi^{\frac{1}{2}}(x,t;\lambda) & \textit{for } \alpha = 2, \beta = 1, \\ \psi^{\frac{3}{2}}(x,t;\lambda)+\bar{\psi}^{\frac{3}{2}}(x,t;\mu) & \textit{for } \alpha = 3, \beta = 2. \end{cases} \tag{5.18}$$

Lemma 5.3. *Suppose that* $\beta < 3$. *Then for any positive constant* $E > D$

$$\begin{aligned} & J^{\alpha,\beta,\gamma}(x,t;0,0;\lambda,\lambda,D) \\ = \ & O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(t+1)\theta(x,t;\lambda,E)+\Gamma^{\beta-1}(t+1)\psi^{\alpha}(x,t;\lambda)]. \end{aligned} \tag{5.19}$$

In particular,

$$J^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,D) = O(1)\begin{cases} \psi^{\frac{1}{2}}(x,t;\lambda) & \text{for } \alpha \geq 2.5,\ \beta = 1, \\ \psi(x,t;\lambda) & \text{for } \alpha > 1,\ \beta = 1, \\ \psi^{\frac{3}{2}}(x,t;\lambda) & \text{for } \alpha \geq 2.5,\ \beta = 2. \end{cases} \tag{5.20}$$

Lemma 5.4. *Suppose that* $\alpha > 1$, $3 > \beta \geq 1$, *and* $\lambda < \mu$. *Then, for any given constant* $E > D$,

$$\begin{aligned} & J^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\mu,D) \\ = \ & O(1)((t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1}) + (t+1)^{-\alpha}\Gamma^{\beta-1}(t+1))\theta(x,t;\lambda,E) \\ & +(t+1)^{(-\alpha+1)/2}\Gamma^{\beta-1}(\sqrt{t+1})\theta(x,t;\mu,E) \\ & + \begin{cases} O(1)[(t+1)^{(-\beta+1)/2}(x-\lambda t)^{(-2\alpha+2)/2} \\ \cdot(1+(1+\frac{t}{|x-\lambda t|})^{-1/2}\Gamma^{2\alpha-2}(\frac{t}{|x-\lambda t|}+1)) \\ +\Gamma^{\beta-1}(t+1)(|x-\lambda t|+|\lambda-\mu|t)^{-\alpha} + ((t+1)^{(-\beta+1)/2}\Gamma^{2\alpha}(t+1) \\ +(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(\sqrt{t+1}) \\ +(t+1)^{(-2\alpha+1)/2}\Gamma^{\beta-1}(t+1))\theta(x,t;\lambda,E)] \\ \quad \text{for } x \leq \lambda t - \sqrt{t+1}, \\ O(1)[(t+1)^{(-\beta+1)/2}\Gamma^{\alpha-1}(x-\lambda t)\theta(x,t;\lambda,E) \\ +(\mu t - x)^{(-\beta+1)/2}(x-\lambda t)^{(-\alpha+1)/2} + (x-\lambda t)^{-\alpha}\Gamma^{\beta-1}(t+1)] \\ \quad \text{for } \lambda t + \sqrt{t+1} < x < \mu t - \sqrt{t+1}, \\ O(1)(x-\mu t)^{-\alpha}\Gamma^{\beta-1}(\frac{(t+1)(x-\mu t)}{x-\lambda t}) \\ \quad \text{for } x > \mu t + \sqrt{t+1}. \end{cases} \end{aligned} \tag{5.21}$$

In particular,

$$J^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,\mu,D)=O(1)\begin{cases}\psi^{\frac{1}{2}}(x,t;\lambda) \text{ for } \alpha\geq 2,\beta=1,\\ \theta(x,t;\lambda,D)+\psi^{\frac{3}{2}}(x,t;\lambda) \text{ for } \alpha\geq 4,\beta=1,\\ \psi^{\frac{3}{2}}(x,t;\lambda)+\bar{\psi}^{\frac{3}{2}}(x,t;\mu) \text{ for } \alpha\geq 3,\beta=2.\end{cases} \tag{5.22}$$

Lemma 5.5. *Suppose that* $\lambda<\mu$, α_1, α_2, $\beta\geq 1$. *Then,*

$$\begin{aligned}&\mathbb{J}^{\alpha_1,\alpha_2;\beta}(x,t;\lambda,\mu,D)\\ &\leq O(1)\begin{cases}|x-\lambda t|^{-\alpha_1}\left[t^{-\frac{\beta-1}{2}}\Gamma^{2\alpha_2}(t)+\Gamma^{\beta-1}(t)t^{-\alpha_2}\right]+\Gamma^{\beta}(t)\theta^{2\alpha_2+\alpha_1}(x,t;\lambda,D')\\ +\Gamma^{2\alpha_2+\alpha_1}(t)\,\theta^{\beta}(x,t;\lambda,D') \text{ for } x-\lambda t\leq-\sqrt{t},\\ t^{-\frac{\beta}{2}}\Gamma^{\alpha_1+2\alpha_2-1}(t)+t^{-\frac{2\alpha_2+\alpha_1}{2}}\Gamma^{\beta-1}(t) \text{ for } |x-\lambda t|\leq\sqrt{t},\\ \theta^{\beta}(x,t;\lambda,D')\Gamma^{2\alpha_2+\alpha_1-1}(t)+\theta^{2\alpha_2+\alpha_1-1}(x,t;\lambda,D')\Gamma^{\beta}(t)\\ +|x-\lambda t|^{-\alpha_1}\left\{t^{-\frac{\beta-1}{2}}\Gamma^{2\alpha_2}(t)+t^{-\alpha_2}\Gamma^{\beta-1}(t)\right\}\\ +|x-\lambda t|^{-\alpha_2}J^{\alpha_1,\beta}(x,t;1,t;\lambda,\mu,D) \text{ for } x\in[\lambda t+\sqrt{t},\mu t-\lambda t],\end{cases}\end{aligned}$$

where $D'>D$.

Proof. Let $E'>1$.
Case. $x-\lambda t<-\sqrt{t}$

$$\begin{aligned}&\int_1^{t-1}\left(\int_{y-\lambda\sigma<\frac{x-\lambda t}{E'}}+\int_{y-\lambda\sigma\geq\frac{x-\lambda t}{E'}}\right)\theta^{\beta}(x-y,t-\sigma,\lambda;D)\sigma^{-\alpha_2}\psi^{\alpha_1}(y,\sigma,\mu)dyd\sigma\\ &\leq\ O(1)\left[\int_1^{t-1}\frac{(t-\sigma)^{-\frac{\beta-1}{2}}\sigma^{-\alpha_2}}{|x-\lambda t|^{\alpha_1}}d\sigma+\int_1^{t-1}\frac{\sigma^{-\alpha_2-\frac{(\alpha_1-1)}{2}}\ e^{-\frac{(x-\lambda t)^2}{D't}}}{(t-\sigma)^{\frac{\beta}{2}}}d\sigma\right]\\ &\leq\ O(1)\left[\frac{t^{-\frac{\beta-1}{2}}\Gamma^{2\alpha_2}(t)+\Gamma^{\beta-1}(t)t^{-\alpha_2}}{|x-\lambda t|^{\alpha_1}}+\Gamma^{\beta}(t)\theta^{2\alpha_2+\alpha_1}(x,t;\lambda,D')\right.\\ &\qquad\left.+\Gamma^{2\alpha_2+\alpha_1}(t)\ \theta^{\beta}(x,t;\lambda,D')\right].\end{aligned}$$

Case. $|x-\lambda t|\le\sqrt{t}$.

$$\left(\int_1^{\frac{t}{2}}+\int_{\frac{t}{2}}^{t-1}\right)\int_R \theta^\beta(x-y,t-\sigma,\lambda;D)\sigma^{-\alpha_2}\psi^{\alpha_1}(y,\sigma,\mu)\,dy\,d\sigma$$

$$\le\ O(1)\ \left[\int_1^{\frac{t}{2}}\frac{\sigma^{-\alpha_2-\frac{\alpha_1-1}{2}}}{t^{\frac{\beta}{2}}}d\sigma+\int_{\frac{t}{2}}^{t-1}\frac{\sigma^{-\alpha_2-\frac{\alpha_1}{2}}}{(t-\sigma)^{\frac{\beta-1}{2}}}\,d\sigma\right]$$

$$\le\ O(1)\ \left[\frac{\Gamma^{\alpha_1+2\alpha_2-1}(t)}{t^{\frac{\beta}{2}}}+\frac{\Gamma^{\beta-1}(t)}{t^{\frac{2\alpha_2+\alpha_1}{2}}}\right].$$

Case. $x-\lambda t\ge\sqrt{t}$.

$$\left[\int_1^{t-1}\int_{|y-\lambda s|\le\frac{|x-\lambda t|}{E'}}+\left(\int_1^{\frac{|x-\lambda t|}{4\mu}}+\int_{\frac{|x-\lambda t|}{4\mu}}^{t-1}\right)\int_{|y-\lambda\sigma|\ge\frac{|x-\lambda t|}{E'}}\right]$$

$$\times\theta^\beta(x-y,t-\sigma,\lambda;D)\sigma^{-\alpha_2}\psi^{\alpha_1}(y,\sigma,\mu)\,dy\,d\sigma$$

$$\le\ O(1)\left[\int_1^{t-1}\frac{\sigma^{-\alpha_2-\frac{\alpha_1-1}{2}}\ e^{-\frac{(x-\lambda t)^2}{D't}}}{(t-\sigma)^{\frac{\beta}{2}}}d\sigma\right.$$

$$\left.+\int_1^{\frac{|x-\lambda t|}{4\mu}}\frac{\sigma^{-\alpha_2}|x-\lambda t|^{-\alpha_1}}{(t-\sigma)^{\frac{\beta-1}{2}}}d\sigma+\int_{\frac{|x-\lambda t|}{4\mu}}^{t-1}\int_R\frac{|x-\lambda t|^{-\alpha_2}\psi^{\alpha_1}(y,\sigma;\mu)}{(t-\sigma)^{\frac{\beta}{2}}}\,dyd\sigma\right]$$

$$\le\ O(1)\Big[\theta^\beta(x,t;\lambda,D')\Gamma^{2\alpha_2+\alpha_1-1}(t)+\theta^{2\alpha_2+\alpha_1-}(t)\Gamma^\beta(t)$$

$$+|x-\lambda t|^{-\alpha_1}J^{\alpha_1,\beta}(x,t;0,t;\lambda,\mu)\Big]$$

$$+O(1)|x-\lambda t|^{-\alpha_2}\big[t^{-\frac{\beta}{2}}\Gamma^{\alpha_1-1}(t)+t^{-\frac{\alpha_1-1}{2}}\Gamma^\beta(t)\big].\qquad\square$$

Lemma 5.6. *Suppose that $\alpha>1$ and $1\le\beta<3$. Then, for any $E>D$,*

$$\mathfrak{I}_{1,1}^{\alpha,\beta}(x,t;0,t;D)$$
$$=\ O(1)\begin{cases}\left(t^{-\frac{\beta-1}{2}}\Gamma^{\alpha-1}(t)+t^{-\frac{\alpha-1}{2}}\Gamma^{\beta-1}(t)\right)\theta^1(x,t;\lambda_1^-,D)\text{ for }x\le\lambda_1^-\ t,\\ J^{\alpha,\beta}(x,t;1,t-1;\lambda_1^-,\lambda_1^-,D)\text{ for }x\in[\lambda_1^-\ t,0]\\ \theta^{2\alpha}(x,t;0,D)\ e^{-\frac{|(\lambda_1^--\lambda_2^-)x|}{2}}\text{ for }x>0,\end{cases}\tag{5.23}$$

$$\mathfrak{J}_{1,2}^{\alpha,\beta}(x,t;0,t;D)$$

$$= O(1)\begin{cases} e^{-\frac{|(\lambda_1^- - \lambda_2^-)(x-\lambda_1^- t)|}{2}}\ \theta^{\alpha}(x,t;\lambda_1^-,D) \text{ for } x \le \lambda_1^-\ t, \\ J^{\alpha,\beta}(x,t;1,t-1;\lambda_2^-,\lambda_1^-,D) \text{ for } x \in [\lambda_1^-\ t,\infty) \end{cases} \tag{5.24}$$

$$\mathfrak{J}_{2,1}^{\alpha,\beta}(x,t;0,t;D)$$

$$= O(1)\begin{cases} e^{-\frac{|(\lambda_1^- - \lambda_2^-)(x-\lambda_1^- t)|}{2}}\ \theta^{2\alpha}(x,t;\lambda_1^-,D) \text{ for } x \le \lambda_1^-\ t, \\ J^{\alpha,\beta}(x,t;1,t-1;\lambda_1^-,\lambda_2^-,D) \text{ for } x \in [\lambda_1^-\ t,0], \\ e^{-\frac{|(\lambda_2^- - \lambda_1^-)x|}{2}}\ \dfrac{e^{-\frac{x^2}{4Dt}}}{(1+\epsilon t)^{\alpha}} \text{ for } x \ge 0, \end{cases} \tag{5.25}$$

$$\mathfrak{J}_{2,2}^{\alpha,\beta}(x,t;0,t;D)$$

$$= O(1)\begin{cases} \theta^{2\alpha}(x,t;\lambda_1^-,D)\ e^{-\frac{|(\lambda_1^- - \lambda_2^-)(x-\lambda_1^- t)|}{2}} \text{ for } x \le \lambda_1^-\ t, \\ J^{\alpha,\beta}(x,t;1,t-1;\lambda_2^-,\lambda_2^-,D) \text{ for } x \in [\lambda_1^-\ t,\infty). \end{cases} \tag{5.26}$$

Remark 5.7. This Lemma is a consequence of straightforward calculations. We omit it.

Lemma 5.8. *Suppose* α_1, α_2, $\beta \ge 1$. *Then,*

$$\mathbb{H}_2^{\alpha_1,\alpha_2;\beta}(x,t;D) \le O(1)\begin{cases} \theta^{\beta}(x,t;\lambda_2^-,D')\displaystyle\int_0^t \Gamma^{\alpha_1}(\sigma)\rho^{\alpha_2}(\sigma)\ d\sigma \\ +\int_{\frac{|x-\lambda_2^- t|}{|\lambda_1^-|}}^t (t-\sigma)^{-\frac{\beta-1}{2}}\rho^{\alpha_2}(\sigma)d\sigma \\ \textit{for } (\frac{\lambda_1}{2}-\lambda_2^-)t \le x \le 0, \\ e^{-\frac{(x-\lambda_2^- t)^2}{Dt}}\displaystyle\int_1^{t-1}\frac{1}{\sqrt{t-\sigma}^{\beta}}\ \rho^{\alpha_2}(\sigma)\Gamma^{\alpha_1}(\sigma)\ d\sigma \\ \textit{for } x \le (\frac{\lambda_1}{2}-\lambda_2^-)\ t. \end{cases}$$

Lemma 5.9. *Suppose that* $\alpha \ge 0$, $3 > \beta-\gamma \ge 0$, $\gamma > 0$, *and* λ *a positive constant. Then, there exists a positive constant* C *such that for any* *fixed positive constant $E > D + O(1)\epsilon$ and a bound $O(1)$ independent of ϵ,*

$$K^{\alpha,\beta,\gamma}(x,t;0,t;D)$$
$$= O(1)\begin{cases} [(t+1)^{\frac{-\beta+1}{2}}\Gamma^\alpha(t+1)e^{-\epsilon t}+(t+1)^{-\frac{\alpha}{2}}\Gamma^{\beta-1}(\epsilon^{-1})]e^{-\frac{4\epsilon|x|}{D}} \text{ for } x < D, \\ (x+1)^{\frac{-\beta+2}{2}}(\lambda t-x)^{-\frac{\alpha}{2}}(1+\epsilon\sqrt{x+1})^{-1}+\Gamma^{\beta-1}(\epsilon^{-1})(t+1)^{-\frac{\alpha}{2}}e^{-C\epsilon x} \\ +\Gamma^\alpha(\epsilon^{-1})(t+1)^{\frac{-\beta+1}{2}}e^{-C\epsilon|x-\lambda t|} \text{ for } 0 < x < \lambda(t+1)-\sqrt{t}, \\ (t+1)^{\frac{-\beta+1}{2}}[\Gamma^\alpha(t+1)e^{-C\epsilon t}+\Gamma^\alpha(\sqrt{t+1})(1+\epsilon\sqrt{t})^{-1}] \\ +(t+1)^{-\frac{\alpha}{2}}\Gamma^{\beta-1}(t+1)e^{-C\epsilon t} \text{ for } |x-\lambda t| \le \sqrt{t+1}, \\ (t+1)^{\frac{-\beta+1}{2}}[\Gamma^\alpha(\sqrt{t+1})(1+\epsilon\sqrt{t})^{-1}e^{-\frac{(x-\lambda t)^2}{D(t+1)}} \\ +(t+1)^{\frac{1}{2}}(x-\lambda t)^{-\frac{\alpha}{2}}e^{-C\epsilon|x-\lambda t|} \\ +\Gamma^\alpha(\epsilon^{-1})e^{-C\epsilon|x-\lambda t|}e^{-C\epsilon^2 t}+\epsilon^{\frac{\alpha-2}{2}}e^{-\frac{4\epsilon(x-\lambda t)}{D}}] \\ +(x-\lambda t)^{\frac{-\alpha-2\beta+3}{2}}e^{-C\epsilon(x-\lambda t)}+(t+1)^{-\frac{\alpha}{2}}(\Gamma^{\beta-1}(\epsilon^{-1}) \\ +\Gamma^{\beta-1}(t+1))e^{-C\epsilon t}e^{-\frac{4\epsilon(x-\lambda t)}{D}} \\ \text{ for } x > \lambda(t+1)+\sqrt{t+1}. \end{cases} \tag{5.27}$$

In (5.27), *the term $\Gamma^\alpha(\sqrt{t+1})$ and $\Gamma^{\beta-1}(t+1)$, respectively, can be interchanged with $\Gamma^\alpha(\epsilon^{-1})$ and $\Gamma^{\beta-1}(\epsilon^{-1})$. The estimates* (5.27) *immediately yield the following:*

$$\epsilon K^{\alpha,\beta,\gamma}(x,t;0,t;\lambda,D) = O(1)\begin{cases} \psi^{\frac{1}{2}}(x,t;\lambda) \text{ for } \alpha \ge 1, \beta = 1, \\ \epsilon^{-\frac{1}{2}}\psi^{\frac{3}{2}}(x,t;\lambda)+\theta(x,t;\lambda,D) \text{ for } \alpha \ge 3, \beta = 1, \\ \psi(x,t;\lambda) \text{ for } \alpha \ge 1, \beta = 2, \\ \ln\epsilon\,\psi^{\frac{3}{2}}(x,t;\lambda) \text{ for } \alpha \ge 2, \beta = 2. \end{cases} \tag{5.28}$$

Lemma 5.10. *For $\alpha > 1$ and $\beta \ge 1$, the function $\mathscr{K}_{1,1}^{\alpha,\beta}(x,t;0,t;D)$ satisfies*

$$\mathscr{K}_{1,1}^{\alpha,\beta}(x,t;0,t;D)$$
$$\le O(1)\left[t^{-\frac{\beta-1}{2}}\psi^\alpha(x,t;\lambda_1^-)+\theta^\beta(x,t;\lambda_1^-,D')+\epsilon\,K^{2\alpha,\beta}(x,t;0,t;D)\right].$$

By the decomposition

$$\psi^\alpha(y,\sigma;\lambda_1^-)\, e^{-\frac{\epsilon|y|}{D}} \le O(1) \begin{cases} \sigma^{-\alpha}\, e^{-\frac{\epsilon|y|}{D}} & \text{for } y \ge \frac{\lambda_1^-\sigma}{2}, \\ \psi^\alpha(y,\sigma;\lambda_1^-) e^{-\frac{\epsilon|\lambda|\sigma}{2D}} & \text{for } y \le \frac{\lambda_1^-\sigma}{2}, \end{cases}$$

it follows

$$\begin{aligned} &\int_1^{t-1} \left(\int_{y\le \frac{\lambda_1^-\sigma}{2}} + \int_{y\ge \frac{\lambda_1^-\sigma}{2}} \right) \theta^\beta(x-y,t-\sigma;\lambda_1^-,D)\, \psi^\alpha(y,\sigma;\lambda_1^-)\, \epsilon\, e^{-\frac{\epsilon|y|}{D}} dyd\sigma \\ \le\ & O(1) \int_1^{t-1} \int_{y\le \frac{\lambda_1^-\sigma}{2}} \theta^\beta(x-y,t-\sigma;\lambda_1^-,D)\, \psi^\alpha(y,\sigma;\lambda_1^-)\, \epsilon\, e^{-\frac{\epsilon|\lambda_1^- y|}{D}} dyd\sigma \\ & +O(1) \int_1^{t-1} \int_{y\ge \frac{\lambda_1^-\sigma}{2}} \theta^\beta(x-y,t-\sigma;\lambda_1^-,D)\, \epsilon\, \sigma^{-\alpha}\, e^{-\frac{\epsilon|y|}{D}} dyd\sigma \\ \le\ & O(1) \left[t^{-\frac{\beta-1}{2}}\, \psi^\alpha(x,t;\lambda_1^-) + \theta^\beta(x,t;\lambda_1^-,D') + \epsilon\, K^{2\alpha,\beta}(x,t;0,t;\lambda_1^-,D) \right]. \end{aligned}$$

Lemma 5.11. *For $\alpha,\beta \ge 1$,*

$$\begin{aligned} &\mathbb{K}_{1,1}^{\alpha,\beta}(x,t;0,t;D) \le O(1)\, \epsilon\, K^{2\alpha,\beta}(x,t;0,t;D) \\ &\quad + O(1) \left(\epsilon^{\frac{\alpha-1}{2}}\, \theta^\beta(x,t;\lambda_1^-,D') + \epsilon\, e^{-\frac{\epsilon\, |\lambda_1^-|t}{4D}}\, t^{-\frac{\alpha}{2}}\, \Gamma^{\beta-1}(t) e^{-\frac{(x-\lambda_1^- t)^2}{D't}} \right). \end{aligned}$$

Proof. By the decomposition

$$\theta^\alpha(y,\sigma;\lambda_1^-,D)\, e^{-\frac{\epsilon|y|}{D}} \le O(1) \begin{cases} \sigma^{-\alpha}\, e^{-\frac{\epsilon|y|}{D}} & \text{for } y \ge \frac{\lambda_1^-\sigma}{2}, \\ \theta^\alpha(y,\sigma;\lambda_1^-,D) e^{-\frac{\epsilon|\lambda|\sigma}{2D}} & \text{for } y \le \frac{\lambda_1^-\sigma}{2}, \end{cases}$$

it follows

$$\begin{aligned} &\int_1^{t-1} \left(\int_{y\le \frac{\lambda_1^-\sigma}{2}} + \int_{y\ge \frac{\lambda_1^-\sigma}{2}} \right) \theta^\beta(x-y,t-\sigma;\lambda_1^-,D) \theta^\alpha(y,\sigma;\lambda_1^-,D) \epsilon e^{-\frac{\epsilon|y|}{D}} dyd\sigma \\ \le\ & O(1) \int_1^{t-1} \int_{y\le \frac{\lambda_1^-\sigma}{2}} \theta^\beta(x-y,t-\sigma;\lambda_1^-,D) \theta^\alpha(y,\sigma;\lambda_1^-,D) \epsilon e^{-\frac{\epsilon|\lambda_1^- y|}{D}} dyd\sigma \\ & +O(1) \int_1^{t-1} \int_{y\ge \frac{\lambda_1^-\sigma}{2}} \theta^\beta(x-y,t-\sigma;\lambda_1^-,D)\, \epsilon\, \sigma^{-\alpha}\, e^{-\frac{\epsilon|y|}{D}}\, dy\, d\sigma \end{aligned}$$

$$\leq O(1) \Big[t^{-1/2} \int_1^{t-1} (t-\sigma)^{-\frac{\beta-1}{2}} \sigma^{-\frac{\alpha-1}{2}} \epsilon e^{-\frac{\epsilon|\lambda|\sigma}{D2}} \, d\sigma \, e^{-\frac{(x-\lambda t)^2}{Dt}}$$
$$+\epsilon \, K^{2\alpha,\beta}(x,t;0,t;\lambda_1^-,D) \Big]. \qquad \square$$

Lemma 5.12. *Suppose that* α, $\beta \geq 1$, *and* $\lambda < 0$. *Then,*

$$\int_0^{t-1} \theta^\beta(x,t-\sigma,\lambda,D) \, \rho^\alpha(\sigma) \, d\sigma$$
$$\leq O(1) \begin{cases} \left(\dfrac{\rho^\alpha(t)}{|\lambda|} + e^{-\frac{\lambda^2 t}{D2}} \, \bar\Gamma^\alpha(t) t^{-1/2} \right) (1+|x|)^{-(\beta-1)} \, e^{-\frac{2\lambda x}{D'}} \text{ for } x > 0, \\ \frac{\rho^\alpha(x-\lambda t)}{|\lambda| \, (|x|+1)^{\beta-1}} + \frac{\rho^\alpha(x-\lambda t)}{|\lambda|^{\frac{3-\beta}{2}} \, (|x|+1)^{\frac{\beta-1}{2}}} + \theta^\beta(x,t;\lambda,D')\bar\Gamma^\alpha(x-\lambda t) \\ \text{for } \lambda t + \sqrt{t} < x < 0, \\ \dfrac{\rho^\alpha(t)}{|\lambda| \, (|\lambda t|+1)^{\beta-1}} + t^{-\beta/2} \, \bar\Gamma^\alpha(t) \text{ for } |x-\lambda t| \leq \sqrt{t}, \\ \theta^\beta(x,t;\lambda,D') \, \bar\Gamma^\alpha(t) \text{ for } x \leq \lambda t - \sqrt{t}, \end{cases} \tag{5.29}$$

where $D' > D$.

Proof. By (3.15) in the Lemma 3.3 of [10], we have that for $\lambda \neq 0$ and $D = O(1)$

$$\int_0^\infty \frac{e^{-\frac{x^2}{D\sigma} - \frac{\lambda^2\sigma}{4}}}{\sqrt{4\pi\sigma}} \, d\sigma = \frac{1}{|\lambda|} \, e^{-\frac{|\lambda x|}{\sqrt{D}}}. \tag{5.30}$$

Through this identity, we can have that for $i = 0, 1$

$$\int_0^\infty \left[\frac{|x|}{t} \right]^i \frac{e^{-\frac{x^2}{D\sigma} - \frac{\lambda^2\sigma}{4}}}{\sqrt{4\pi\sigma}} \, d\sigma = \frac{1}{|\lambda|^{1-i}} \, e^{-\frac{|\lambda x|}{\sqrt{D}}}. \tag{5.31}$$

This yields that for all $t \geq 0$

$$\int_0^t \theta^1(x,\sigma;\lambda,D) \, d\sigma \leq \frac{O(1)}{|\lambda|}. \tag{5.32}$$

Case. $x > 0$

$$\begin{aligned}
&\int_1^{t-1} \theta^\beta(x, t-\sigma, \lambda, D)\rho^\alpha(\sigma)\, d\sigma \\
&= O(1)\left(\int_{t/2}^{t-1} + \int_1^{t/2}\right) \frac{\theta^1(x, t-\sigma, 0, D')}{(x-\lambda(t-\sigma))^{\beta-1}} e^{-\frac{2\lambda|x|}{D} - \frac{\lambda^2}{D}\,(t-\sigma)}\, \rho^\alpha(\sigma) d\sigma \\
&\le \frac{O(1)\, e^{-\frac{2|\lambda x|}{D'}}}{(|x|+1)^{\beta-1}}\Big(\int_{t/2}^{t-1} \rho^\alpha(t/2)\, \theta^1(x, t-\sigma, 0, D')\, e^{-\frac{\lambda^2(t-\sigma)}{D}} d\sigma \\
&\qquad + \int_1^{t/2} e^{-\frac{\lambda^2 t}{2D}} \theta^1(x, t, 0, 2D')\, \rho^\alpha(\sigma)\, d\sigma\Big) \\
&= O(1)\frac{e^{-\frac{2|\lambda x|}{D'}}}{(|x|+1)^{\beta-1}}\left(\frac{\rho^\alpha(t)}{|\lambda|} + e^{-\frac{\lambda^2 t}{2D}}\, t^{-1/2}\bar{\Gamma}^\alpha(t)\right).
\end{aligned}$$

Case. $\lambda t + \sqrt{t} \le x \le 0$.
Let $E > 1$.

$$\begin{aligned}
&\left(\int_{t+\frac{x}{2\lambda}}^{t-1} + \int_{(x-\lambda t)/E}^{t+\frac{x}{2\lambda}} + \int_1^{(x-\lambda t)/E}\right) \theta^\beta(x, t-\sigma, \lambda, D)\, \rho^\alpha(\sigma)\, d\sigma \\
&\le \int_{t+\frac{x}{2\lambda}}^{t-1} \frac{\theta^1(x, t-\sigma, \lambda, D')}{|x-\lambda(t-\sigma)|^{\beta-1}} \rho^\alpha(x-\lambda t)\, d\sigma \\
&\quad + \int_{(x-\lambda t)/E}^{t+\frac{x}{2\lambda}} \frac{\theta^1(x, t-\sigma, \lambda, D)}{(|x|+1)^{(\beta-1)/2}} \rho^\alpha\left(\frac{x-\lambda t}{E}\right)\, d\sigma \\
&\quad + \int_1^{(x-\lambda t)/E} \frac{\theta^1(x, t, \lambda, \bar{D})}{\left(\frac{|x|+1}{\lambda}\right)^{\frac{\beta-1}{2}}} \rho^\alpha(\sigma)\, d\sigma \\
&= O(1)\left(\frac{\rho^\alpha(x-\lambda t)}{|\lambda|\, (|x|+1)^{\beta-1}} + \frac{\rho^\alpha\left(\frac{x-\lambda t}{E}\right)}{|\lambda|\, \left(\frac{|x|+1}{\lambda}\right)^{\frac{\beta-1}{2}}} + \frac{\theta^1(x, t; \lambda, \bar{D})\, \bar{\Gamma}^\alpha\left(\frac{x-\lambda t}{E}\right)}{\left(\frac{|x|+1}{\lambda}\right)^{\frac{\beta-1}{2}}}\right) \\
&= O(1)\left[\frac{\rho^\alpha(x-\lambda t)}{|\lambda|(|x|+1)^{\beta-1}} + \frac{\rho^\alpha\left(\frac{x-\lambda t}{E}\right)}{|\lambda|^{\frac{3-\beta}{2}}(|x|+1)^{\frac{\beta-1}{2}}} + \theta^\beta(x, t; \lambda, D')\bar{\Gamma}^\alpha(x-\lambda t)\right].
\end{aligned} \tag{5.33}$$

where $\bar{D} = (1-E^{-1})^2 D$.

Case. $|x-\lambda t| \le \sqrt{t}$.

$$\begin{aligned}&\left(\int_{t/2}^{t-1} + \int_1^{t/2}\right) \theta^\beta(x,t-\sigma;\lambda,D)\ \rho^\alpha(\sigma)\ d\sigma\\ &= O(1)\ \left[\frac{\rho^\alpha(t/2)}{|\lambda|\ (|\lambda t|+1)^{\beta-1}} + t^{-\beta/2}\ \bar\Gamma^\alpha(t/2)\right]\\ &\le O(1)\left[\frac{\rho^\alpha(t)}{|\lambda|\ (|\lambda t|+1)^{\beta-1}} + t^{-\beta/2}\ \bar\Gamma^\alpha(t)\right].\end{aligned}$$

Case. $x-\lambda t \le -\sqrt{t}$

$$\begin{aligned}&\int_1^{t-1} \theta^\beta(x,t-\sigma;\lambda,D)\rho^\alpha(\sigma)d\sigma \le \int_1^{t-1} \frac{e^{-\frac{(x-\lambda t)^2}{Dt}}}{|(x-\lambda t)+\lambda\sigma|^{\beta-1}\sqrt{t-\sigma}}\rho^\alpha(\sigma)d\sigma\\ &\le\ \frac{\theta^1(x,t;\lambda,D')}{|\sqrt{t}\ |^{\beta-1}}\ \int_1^t \rho^\alpha(\sigma)\ d\sigma \le O(1)\ \theta^\beta(x,t;\lambda,D')\ \bar\Gamma^\alpha(t). \qquad \square\end{aligned}$$

Lemma 5.13. *Suppose that $|\lambda| \ne 0$, $\nu \ge \epsilon$, α, $\beta \ge 1$. Then, the function $\widehat{K}^{\alpha,\beta}(x,t;\lambda,D;\nu)$ satisfies*

CaseA. $|\lambda| = O(1),\ \lambda < 0,$

When $0 \le t \le \nu^{-1}$,

$$\begin{aligned}&\widehat{K}^{\alpha,\beta}(x,t;\lambda,D;\nu) \le O(1)\ \nu\rho^\alpha(t)\ e^{-\frac{\nu|x|}{2}}\\ &+ O(1)\bar\Gamma^\alpha(t)\big\{\min\left(\frac{1}{\sqrt{t}},\nu\right)\theta^{\beta-1}(x,t;\lambda,\bar D) + \nu t^{-\frac{\beta-1}{2}} e^{-\frac{\nu|x-\lambda t|}{2}}\big\}\\ &+ O(1)\begin{cases}\nu e^{-\frac{\nu|x|}{2}}\left[\rho^\alpha(t)\Gamma^{\beta-1}(t) + \bar\Gamma^\alpha(t)t^{-\frac{\beta-1}{2}}e^{-\frac{\nu t}{2}}\right] & \textit{for } x \ge 0,\\ \nu\left[\rho^\alpha(t)\Gamma^{\beta-1}(t) + \bar\Gamma^\alpha(t)t^{-\frac{\beta-1}{2}}e^{-\frac{\nu t}{2}}\right] & \textit{for } x \in [\frac{\lambda t}{2},0],\\ \nu\left[t^{-\frac{\beta-1}{2}}\bar\Gamma^\alpha(t) + \Gamma^{\beta-1}(t)e^{-\frac{\nu t}{2}}\rho^\alpha(t)\right] & \textit{for } x \in [\lambda t,\frac{\lambda t}{2}],\\ \nu e^{-\frac{\nu|x-\lambda t|}{2}}\left[t^{-\frac{\beta-1}{2}}\bar\Gamma^\alpha(t) + \Gamma^{\beta-1}(t)e^{-\frac{\nu t}{2}}\rho^\alpha(t)\right] & \textit{for } x \le \lambda t;\end{cases}\end{aligned}$$

When $t \ge \nu^{-1}$,

$$\begin{aligned}&\widehat{K}^{\alpha,\beta}(x,t;\lambda,D;\nu) \le O(1)\ \nu\rho^\alpha(t)e^{-\frac{\nu|x|}{2}}\\ &+ O(1)\bar\Gamma^\alpha(t)\big\{\min\left(\frac{1}{\sqrt{t}},\nu\right)\theta^{\beta-1}(x,t;\lambda,\bar D) + t^{-\frac{\beta-1}{2}}e^{-\frac{\nu|x-\lambda t|}{2}}\big\}\end{aligned}$$

$$+\,O(1)\begin{cases}\nu e^{-\frac{\nu|x|}{2}}\left[\rho^\alpha(t)\Gamma^{\beta-1}(t)+\bar\Gamma^\alpha(t)t^{-\frac{\beta-1}{2}}e^{-\frac{\nu t}{2}}\right] \textit{ for } x\ge 0,\\ \rho^\alpha(t)\ \min\big((|x|+1)^{-\frac{\beta-1}{2}},\nu^{-\frac{\beta-1}{2}}\big) \textit{ for } x\in[\dfrac{\lambda t}{2},0],\\ t^{-\frac{\beta-1}{2}}\rho^\alpha(x-\lambda t) \textit{ for } x\in[\lambda t,\dfrac{\lambda t}{2}],\\ \nu e^{-\frac{\nu|x-\lambda t|}{2}}\ \left[t^{-\frac{\beta-1}{2}}\bar\Gamma^\alpha(t)+\Gamma^{\beta-1}(t)e^{-\frac{\nu t}{2}}\rho^\alpha(t)\right] \textit{ for } x\le\lambda t;\end{cases}$$

Case B. $\lambda=O(1)\,\epsilon,\ \nu\ge\epsilon,\ \lambda>0,\ x\le 0$

$$\widehat{K}^{\alpha,\beta}(x,t;\lambda,D;\nu)$$
$$\le O(1)\left(\frac{\rho^\alpha(t)}{|\lambda|}+e^{-\frac{\lambda^2 t}{D2}}\,\bar\Gamma^\alpha(t)t^{-1/2}\right)\,\nu\max(1,|\lambda|^{\beta-2})\,e^{-\nu|x|/2}.$$

Proof. We can treat $\widehat{K}^{\alpha,\beta}(x,t;\lambda,D;\nu)$ as

$$\widehat{K}^{\alpha,\beta}(x,t;\lambda,D;\nu)=\int_{\mathbf{R}}\left(\int_1^{t-1}\theta^\beta(x-y,t-\sigma,\lambda;D)\,\rho^\alpha(\sigma)\,d\sigma\right)\nu\,e^{-\nu|y|}\,dy.$$

This Lemma is consequence of the convolution of $\nu e^{-\nu|x|}$ with $\int_1^{t-1}\theta^\beta(x,t-\sigma,\lambda,D)\,d\sigma$. Consider the convolution of $\nu e^{-\nu|x|}$ and the upper bound of $\int_1^{t-1}\theta^\beta(x,t-\sigma,\lambda,D)\,d\sigma$ by Lemma 5.12. Then, this lemma follows. □

5.2. Technical lemmas for compressive fields

Denote $G_\pm(x,t;y,\sigma)$ the Green functions of the following two linearized Burgers equations:

$$u_t+\Lambda_\pm(x)\,u_x-u_{xx}=0. \tag{LB}$$

By Cole-Hopf transformation, we can have the Green functions $G_\pm(x,t;y,\sigma)$:

$$G_+(x,t;y,\sigma)\equiv\ \frac{\cosh\left(\frac{\lambda_2^+y}{2}\right)}{\cosh\left(\frac{\lambda_2^+x}{2}\right)}\ \frac{e^{-\frac{(x-y)^2}{4(t-\sigma)}}}{\sqrt{4\pi(t-\sigma)}}\ e^{-\frac{[\lambda_2^+\ (t-\sigma)]^2}{4}}$$
$$G_-(x,t;y,\sigma)\equiv\ \frac{\cosh\left(\frac{\lambda_2^-y}{2}\right)}{\cosh\left(\frac{\lambda_2^-x}{2}\right)}\ \frac{e^{-\frac{(x-y)^2}{4(t-\sigma)}}}{\sqrt{4\pi(t-\sigma)}}\ e^{-\frac{[\lambda_2^-\ (t-\sigma)]^2}{4}}. \tag{5.34}$$

We use $G_-(x,t;y,\sigma)$ to approximate the Green function of the linear problem:

$$u_t + (\lambda_2(\phi(x) - s)\, u_x - u_{xx} = 0.$$

The truncation error of $G_-(x,t;y,\sigma)$ for the backward equation:

$$\begin{aligned}
T(x,t;y,\sigma) &\equiv -\big(\partial_\sigma + \partial_y(\lambda_2(\phi) - s) + \partial_y^2\big) G_-(x,t;y,\sigma) \\
&= -\partial_y \left[(\lambda_2(\phi(y)) - s - \Lambda_-(y))\, G_-(x,t;y,\sigma) \right] \\
&= O(1)\, \epsilon^3\, e^{-\frac{\lambda_2^- |y|(1+O(\epsilon))}{4}}\, G_-(x,t;y,\sigma) \\
&\quad + O(1)\epsilon^2 \partial_y G_-(x,t;y,\sigma) \cdot \begin{cases} e^{-\lambda_2^- |y|(1+O(\epsilon))} & \text{for } y \le 0, \\ 1 & \text{for } y \ge 0. \end{cases} \qquad (5.35) \\
T_x(x,t;y,\sigma) &= O(1)\, \epsilon^3\, e^{-\lambda_2^- |y|(1+O(\epsilon))}\, \partial_x G_-(x,t;y,\sigma) \\
&\quad + \partial_y O(1)\, \epsilon^2\, \partial_y G_-(x,t;y,\sigma) \cdot \begin{cases} e^{-\lambda_2^- |y|(1+O(\epsilon))} & \text{for } y \le 0, \\ 1 & \text{for } y \ge 0. \end{cases} \qquad (5.36)
\end{aligned}$$

From the definition of $G_-(x,t;y,\sigma)$, we have that

$$\begin{aligned}
&\partial_x G_-(x,t;y,\sigma) \\
&= -\frac{\lambda_2^- \sinh \frac{\lambda_2^-(x-y)}{2}\, e^{-\frac{(x-y)^2}{4(t-\sigma)} - \frac{(\lambda_2^-(t-\sigma))^2}{4}}}{4\sqrt{\pi(t-\sigma)}\, \cosh^2 \frac{\lambda_2^- x}{2}} - \partial_y G_-(x,t;y,\sigma) \\
&= O(1)\, \lambda_2^- e^{-\lambda_2^- |x|}\, \theta^1(|x-y|, t-\sigma; \lambda_2^-, 4) - \partial_y G_-(x,t;y,\sigma). \qquad (5.37)
\end{aligned}$$

The expansion of the Green function $G_-(x,t;y,\sigma)$:

Case $x < 0, y < 0$. (5.38a)

$$\begin{cases}
G_- = O(1)\, \theta^1(x-y, t-\sigma; \lambda_2^-, 4), \\
\partial_y G_- = O(1)\, \left[\lambda_2^- e^{-|\lambda_2^- y|}\, G_- + \partial_y \theta^1(x-y, t-\sigma; \lambda_2^-, 4) \right], \\
\partial_y^2 G- = O(1)\big[(\lambda_2^- e^{-|\lambda_2^- y|})^2 G_- + \lambda_2^- e^{-|\lambda_2^- y|} \partial_y \theta^1(x-y, t-\sigma; \lambda_2^-, 4) \\
\qquad\qquad + \partial_y^2 \theta^1(x-y, t-\sigma; \lambda_2^-, 4) \big],
\end{cases}$$

Case $x > 0, y < 0$. (5.38b)

$$\begin{cases} G_- = O(1)\, e^{-|\lambda_2^- x|}\, \theta^1(x-y, t-\sigma; \lambda_2^-, 4), \\ \partial_y G_- = O(1)\, \big[\lambda_2^- e^{-|\lambda_2^- y|}\, G_- + e^{-|\lambda_2^- x|}\, \partial_y \theta^1(x-y, t-\sigma; \lambda_2^-, 4)\big], \\ \partial_y^2 G- = O(1)\, \big[(\lambda_2^- e^{-|\lambda_2^- y|})^2\, G_- + \lambda_2^- e^{-|\lambda_2^- x| - |\lambda_2^- y|}\, \partial_y \theta^1(x-y, t-\sigma; \lambda_2^-, 4) \\ \qquad + e^{-|\lambda_2^- x|}\, \partial_y^2 \theta^1(x-y, t-\sigma; \lambda_2^-, 4)\big], \end{cases}$$

Case $x > 0, y > 0$. (5.38c)

$$\begin{cases} G_- = O(1)\, \theta^1(x-y, t-\sigma; -\lambda_2^-, 4), \\ \partial_y G_- = O(1)\, \big[\lambda_2^- e^{-|\lambda_2^- y|}\, G_- + \partial_y \theta(x-y, t-\sigma; -\lambda_2^-, 4)\big], \\ \partial_y^2 G- = O(1)\, \big[(\lambda_2^- e^{-|\lambda_2^- y|})^2\, G_- + \lambda_2^- e^{-|\lambda_2^- y|} \partial_y \theta^1(x-y, t-\sigma; -\lambda_2^-, 4) \\ \qquad + \partial_y^2 \theta^1(x-y, t-\sigma; -\lambda_2^-, 4)\big], \end{cases}$$

Case $x < 0, y > 0$. (5.38d)

$$\begin{cases} G_- = O(1)\, e^{-|\lambda_2^- x|}\, \theta^1(x-y, t-\sigma; -\lambda_2^-, 4), \\ \partial_y G_- = O(1)\, \big[\lambda_2^- e^{-|\lambda_2^- y|}\, G_- + e^{-|\lambda_2^- x|}\, \partial_y \theta^1(x-y, t-\sigma; -\lambda_2^-, 4)\big], \\ \partial_y^2 G- = O(1) \big[(\lambda_2^- e^{-|\lambda_2^- y|})^2\, G_- + \lambda_2^- e^{-|\lambda_2^- x| - |\lambda_2^- y|} \partial_y \theta^1(x-y, t-\sigma; -\lambda_2^-, 4) \\ \qquad + e^{-|\lambda_2^- x|}\, \partial_y^2 \theta^1(x-y, t-\sigma; -\lambda_2^-, 4)\big]. \end{cases}$$

Lemma 5.14. *For $y \in [-J_0, 0]$ the function $G_-(x,t;y,\sigma)$ satisfies*

Case. $-J_0 \le x,\ y \le 0$,

$$\left| \int_0^t \partial_x G_-(x,t;y,\sigma) \big[(1+\sigma)(1+\epsilon\,\sqrt{\sigma+1})\big]^{-1}\, d\sigma \right| \tag{5.39}$$

$$\le\ O(1) \begin{cases} |\log\epsilon|\ \big[(1+t)(1+\epsilon\sqrt{1+t})\big]^{-1}\ e^{-\frac{\lambda_2^-|x-y|}{2}}\ \textit{for } x \le y, \\ |\log\epsilon| \big[(1+t)(1+\epsilon\sqrt{1+t})\big]^{-1}\ \textit{for } x \in [y, y + \frac{\lambda_2^- t}{3}], \\ |\log\epsilon|\ \Big[\frac{1}{t} + \frac{\epsilon}{\sqrt{t}}\Big]\ \textit{for } x \in [y + \frac{\lambda_2^- t}{3}, y + \lambda_2^- t], \\ \big[(1+t)(1+\epsilon\sqrt{1+t})\big]^{-1} + |\log\epsilon| \Big[\frac{1}{t} + \frac{\epsilon}{\sqrt{t}}\Big]\ e^{-\frac{(x-\lambda_2^- t)^2}{4t}}\ \textit{for } x \ge y + \lambda_2^- t, \end{cases}$$

Case. $x > 0;\ -J_0 \le y \le 0$

$$\left|\int_0^t \partial_x G_-(x,t;y,\sigma)\big[(1+\sigma)(1+\epsilon\,\sqrt{\sigma+1})\big]^{-1}\,d\sigma\right|$$
$$\le\ O(1)\ \frac{|\log\epsilon|\ e^{-\lambda_2^-|x|}\ e^{-\frac{\lambda_2^-|x-y|}{2}}}{(1+t)(1+\epsilon\sqrt{t})}. \tag{5.40}$$

Proof. By (5.31), we have that

$$\int_0^t |\partial_x\theta^1(x-y,t-\sigma;\lambda_2^-,4)|\ d\sigma$$
$$\le\ O(1)\ \int_0^t \left[\frac{|x|}{t-\sigma}+\lambda_2^-\right]\ \frac{e^{-\frac{(x-y)^2}{4(t-\sigma)}-\frac{(\lambda_2^-)^2(t-\sigma)}{4}+\frac{\lambda_2^-(x-y)}{2}}}{\sqrt{t-\sigma}}\ d\sigma$$
$$\le\ O(1)\ e^{-\frac{\lambda_2^-|x-y|}{2}+\frac{\lambda_2^-(x-y)}{2}}. \tag{5.41}$$

Case. $x \le y$.

By (5.41) and (5.38a), we have that

$$\left|\left(\int_0^{t/2}+\int_{t/2}^t\right)\partial_x G_-(x,t;y,\sigma)\ [(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\ d\sigma\right|$$
$$\le\ O(1)\max_{\sigma\in[0,\frac{t}{2}]}|\partial_x G_-(x,t;y,\sigma)|\int_0^{\frac{t}{2}}[(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\ d\sigma$$
$$+O(1)\int_{\frac{t}{2}}^t|\partial_x G_-(x,t;y,\sigma)|\ d\sigma\cdot[(1+t)(1+\epsilon\sqrt{1+t})]^{-1}$$
$$\le\ \frac{O(1)\ |\log\epsilon|\ e^{-\frac{\lambda_2^-|x-y|}{2}}}{(1+t)(1+\epsilon\sqrt{1+t})}.$$

Case. $y \le x \le y+\frac{\lambda_2^- t}{3}$ & $x \le 0$.

$$\left|\left(\int_0^{t/2}+\int_{t/2}^t\right)\partial_x G_-(x,t;y,\sigma)\ [(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\ d\sigma\right|$$
$$\le\ O(1)\max_{\sigma\in[0,\frac{t}{2}]}|\partial_x G_-(x,t;y,\sigma)|\int_0^{\frac{t}{2}}[(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\ d\sigma$$
$$+O(1)\int_{\frac{t}{2}}^t|\partial_x G_-(x,t;y,\sigma)|\ d\sigma\cdot[(1+t)(1+\epsilon\sqrt{1+t})]^{-1}$$

$$\leq \frac{O(1)\,|\log\epsilon|}{(1+t)(1+\epsilon\sqrt{1+t})}.$$

Case. $x\in[y+\frac{\lambda_2^- t}{3}, y+\lambda_2^- t]$ & $x\leq 0$.

$$\begin{aligned}
&\left|\left(\int_0^{t/2}+\int_{t/2}^t\right)\partial_x G_-(x,t;y,\sigma)\;[(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\,d\sigma\right|\\
&\leq O(1)\max_{\sigma\in[0,\frac{t}{2}]}|\partial_x G_-(x,t;y,\sigma)|\int_0^{\frac{t}{2}}[(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\,d\sigma\\
&\quad+O(1)\int_{\frac{t}{2}}^t|\partial_x G_-(x,t;y,\sigma)|\,d\sigma\cdot[(1+t)(1+\epsilon\sqrt{1+t})]^{-1}\\
&\leq O(1)\,|\log\epsilon|\,\left(\frac{1}{t+1}+\frac{\epsilon}{\sqrt{t+1}}\right)+O(1)\,\left[(t+1)(1+\epsilon\sqrt{t+1})\right]^{-1}.
\end{aligned}$$

Case. $x\geq y+\lambda_2^- t$ & $x\leq 0$.

$$\begin{aligned}
&\left|\left(\int_0^{t/2}+\int_{t/2}^t\right)\partial_x G_-(x,t;y,\sigma)\;[(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\,d\sigma\right|\\
&\leq O(1)\max_{\sigma\in[0,\frac{t}{2}]}|\partial_x G_-(x,t;y,\sigma)|\int_0^{\frac{t}{2}}[(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\,d\sigma\\
&\quad+O(1)\int_{\frac{t}{2}}^t|\partial_x G_-(x,t;y,\sigma)|\,d\sigma\cdot[(1+t)(1+\epsilon\sqrt{1+t})]^{-1}\\
&\leq O(1)\,|\log\epsilon|\,\left(\frac{1}{t+1}+\frac{\epsilon}{\sqrt{t+1}}\right)\,e^{-\frac{(x-\lambda_2^- t)^2}{4t}}\\
&\quad+O(1)\,\left[(t+1)(1+\epsilon\sqrt{t+1})\right]^{-1}.
\end{aligned}$$

Case. $x\geq 0$, $y\in[-J_0,0]$.
From (5.38b) and (5.41),

$$\begin{aligned}
&\left|\left(\int_0^{t/2}+\int_{t/2}^t\right)\partial_x G_-(x,t;y,\sigma)\;[(1+\sigma)(1+\epsilon\sqrt{1+\sigma})]^{-1}\,d\sigma\right|\\
&\leq O(1)\,e^{-\lambda_2^-|x|}\left(\int_0^{t/2}+\int_{t/2}^t\right)\Big[\lambda_2^-\;\theta^1(x-y,t-\sigma,-\lambda_2^-,4)\\
&\quad+\partial_y\theta^1(x-y,t-\sigma;-\lambda_2^-,4)\Big]\cdot\left[(1+\sigma)(1+\epsilon\sqrt{\sigma})\right]^{-1}\,d\sigma\\
&\leq O(1)\,|\log\epsilon|\,e^{-\lambda_2^-|x|}\,e^{-\frac{\lambda_2^-|x-y|}{2}}\,\left[(1+t)(1+\epsilon\sqrt{t})\right]^{-1}. \qquad\square
\end{aligned}$$

The following two lemmas are consequences of Lemma 5.14.

Lemma 5.15. *The function* $G_-(x,t;y,\sigma)$ *satisfies*

$$\lambda_2^- \left|\int_0^t \int_{-J_0}^0 |\partial_x G_-(x,t;y,\sigma)| \left[(\sigma+1)(1+\epsilon\sqrt{\sigma})\right]^{-1} dyd\sigma\right|$$
$$\le O(1) \begin{cases} \left(|\log\epsilon| + |\lambda_2^-|\,|x+J_0|\right) \left[(t+1)(1+\epsilon\sqrt{t})\right]^{-1} \text{ for } x\le 0, \\ |\log\epsilon|\; e^{-\lambda_2^-|x|} \left[(1+t)(1+\epsilon\sqrt{t})\right]^{-1} \text{ for } x\ge 0. \end{cases}$$

Lemma 5.16. *For* $x\ge -J_0+1$, *the function* $G_-(x,t;y,\sigma)$ *satisfies*

$$|\int_0^t |\partial_x\partial_y G_-(x,t;-J_0,\sigma)\,| \left[(\sigma+1)(1+\epsilon\sqrt{\sigma})\right]^{-1} d\sigma$$
$$\le O(1) \begin{cases} J|\log\epsilon|^2 \left[(1+t)(1+\epsilon\sqrt{1+t})\right]^{-1} \text{ for } x\in[-J_0,0], \\ \dfrac{J|\log\epsilon|^2\, e^{-\lambda_2^-|x|}\, e^{-\frac{\lambda_2^-|x-y|}{2}}}{(1+t)(1+\epsilon\sqrt{t})} \text{ for } x\ge 0. \end{cases}$$

Lemma 5.17. *There exists* $K_0\ge 2$ *which is independent of* ϵ *such that*

$$\int_0^t\int_R |\partial_x G_-(x,t;y,\sigma)|\; e^{-\lambda_2^-|y|(1+O(\epsilon))}\rho^1(\sigma)dyd\sigma \le \frac{K_0}{\epsilon} e^{-\frac{\lambda_2^-|x|}{2}}\rho^1(t),$$
$$\int_0^t\int_R |\partial_x G_-(x,t;y,\sigma)|\left[l_2(u_-)\cdot \boldsymbol{I}_2[\bar\gamma](y,\sigma)\right]dyd\sigma \le K_0\;\delta e^{-\frac{\lambda_2^-|x|}{2}(1+O(1)\epsilon)}\rho^1(t)$$

for all $t\ge 0$, *where* $\boldsymbol{I}_2[\bar\gamma]$ *is defined in* (4.2).

Proof. We use the formula (5.34) for the Green function $G_-(x,t;y,\sigma)$. So, we have that

$$\partial_x G_-(x,t;y,\sigma) \le O(1)\; e^{-\frac{|\lambda_2^- x|}{2}(|x|-|y|)}\Big(\frac{\epsilon}{\sqrt{t-\sigma}} + \frac{1}{t-\sigma}\Big)\; e^{-\frac{(x-y)^2}{8(t-\sigma)} - \frac{(\lambda_2^-)^2\,(t-\sigma)}{4}}$$

This yields

$$\int_0^{t-1}\int_R \partial_x G_-(x,t;y,\sigma)\;\rho^1(\sigma)\; e^{-\lambda_2^-|y|/2}dyd\sigma$$

$$\leq O(1)\, e^{-\frac{\lambda_2^-|x|}{2}} \int_0^{t-1} \Big[\epsilon + \frac{1}{\sqrt{t-\sigma}}\Big] \rho^1(\sigma)\, e^{-O(1)(\lambda_2^-)^2(t-\sigma)}\, d\sigma$$
$$\leq \frac{O(1)}{\epsilon} \rho^1(\sigma)\, e^{-\frac{\lambda_2^-|x|}{2}}. \tag{5.42}$$

The function $\boldsymbol{I}_2[\bar\gamma](y,\sigma)$ satisfies

$$\|\boldsymbol{I}_2[\bar\gamma](y,\sigma)\| \leq O(1)\, \frac{e^{-|y|} + |\lambda_2^-|\, e^{-|\lambda_2^-\, y|}}{\sqrt{\sigma}(1+\epsilon^2\sigma)}; \text{ where } |\lambda_2^-| = \frac{\epsilon}{2}(1+O(1)\epsilon).$$

From the above two, it follows

$$\int_0^{t-1} \int_R \partial_x G_-(x,t;y,\sigma)\, \boldsymbol{I}_2(y,\sigma)\, dy\, d\sigma \leq \frac{O(1)}{\sqrt{t}(1+\epsilon^2 t)}\, e^{-\frac{|\lambda_2^- x|}{2}}.$$

This concludes the lemma. □

6. Proofs Theorems DW, A and B

Proof of Lemma P

Proof. The integral representation of the solution $w(x,t)$ is

$$w(x,t) = \int_0^t \int_R \theta^1(x-y,t-\sigma;\lambda_1^-,4)\cdots\Big(l_1(u_-)\cdot\Big[f''(\mathbb{A})(\Theta, wr_1(u_-)) + \bar{\boldsymbol{N}}[w]\Big]_y(y,\sigma) + S(y,\sigma)\Big) dy d\sigma. \tag{6.1}$$

By Lemma 5.13 with $(\alpha,\beta)=(1,i+1)$ and $(\lambda,\nu)=(\lambda_1^-,\lambda_2^-)$ we have that

$$\left|\partial_x^i \int_0^t \int_R \theta^1(x-y,t-\sigma;\lambda_1^-,4) S(y,\sigma) dy d\sigma\right|$$
$$\leq O(1)\frac{\epsilon_0}{|\lambda_2^-|} \begin{cases} \rho^1(t) e^{-\lambda_2^-|x|(1+O(\epsilon))} \quad \text{for} \quad x \geq 0, \\ \min\Big(|\lambda_2^-|, t^{-(i-1)/2}\Big)\bar\Gamma^1(t)\theta^1(x,t;\lambda_1^-,D) \\ \quad + \min\Big(\sqrt{|\lambda_2^-|^{\min(2,i-1)}}, (|x|+1)^{-\frac{|i-1|}{2}}\Big) \\ \quad \times \eta^1(x,t) \quad \text{for} \quad x \leq 0. \end{cases} \tag{6.2}$$

When both δ and $\epsilon_0 \ll \lambda_2^-$, the effect from the nonlinear term $\bar{\boldsymbol{N}}[w]_y$ and

$f''(\mathbb{A})(\Theta, w)$ are much less than the source term $S(x,t)$. Then, a standard Picard's iteration,

$$\begin{aligned} w_k(x,t) &= \int_0^t \int_R \theta^1(x-y,t-\sigma;\lambda_1^-,4)\cdot\Big(l_1(u_-)\cdot\Big[f''(\mathbb{A})(\Theta, w_{k-1}r_1(u_-)) \\ &\quad +\bar{\boldsymbol{N}}[w_{k-1}]\Big]_y(y,\sigma)+S(y,\sigma)\Big)dyd\sigma, \end{aligned} \tag{6.3}$$

will prove this Lemma. We omit the details of this proof. □

Proof of Theorem DW

Proof. Let $E'\in(0,1)$ and $E=4/(1-(E')^2)$.

From the definition of $\bar{\Theta}_1^2(x,t)$,

$$\begin{aligned} \bar{\Theta}_1^2(x,t) &= \Big(\int_{|y|>E'|x-\lambda_2^- t|}+\int_{|y|<E'|x-\lambda_2^-|}\Big)\theta^1(x-y,t;\lambda_2^-,4)\boldsymbol{v}^2(y,0)dy \\ &= O(1)\delta\Big[\psi^{3/2}(x,t;\lambda_2^-)+\theta^1(x,t;\lambda_2^-,E)\Big]. \end{aligned}$$

Therefore,

$$\Theta_1^2(x,t) = \begin{cases} 0 \text{ for } x\geq 1, \\ O(1)\delta\Big[\theta^1(x,t;\lambda_2^-,E)+\psi^{3/2}(x,t;\lambda_2^-)\Big], \end{cases} \tag{6.4}$$

$$|\partial_x\Theta_1^2(x,t)| = \begin{cases} 0 \text{ for } x\geq 1, \\ O(1)\delta\Big[\theta^2(x,t;\lambda_2^-,2E)+t^{-1/2}\psi^{3/2}(x,t;\lambda_2^-) \\ \quad +\rho^1(t)\varrho(x)\Big], \end{cases} \tag{6.5}$$

$$\begin{aligned} \mathscr{E}_1(x,t) &= O(1)\delta\varrho(x,t)\rho^1(t), \\ \mathscr{E}_1^-(x,t) &= O(1)\delta\epsilon\varrho(x,t)\rho^1(t). \end{aligned}$$

By Lemma 5.12 with $(\alpha,\beta)=(1,1)$, we have that

$$\begin{aligned} \Theta_1^1(x,t) &= \int_R \theta^1(x-y,t;\lambda_1^-,4)\Big[\boldsymbol{v}^1(y,0)+\Psi(y)\Big]dy \\ &\quad +\int_0^t\int_R \theta^1(x-y,t-\sigma;\lambda_1^-,4)\mathscr{E}_1^-(y,\sigma)dyd\sigma \end{aligned}$$

$$
\begin{aligned}
&= O(1)\delta\Big[\theta^1(x,t;\lambda_1^-,E)+\psi^{3/2}(x,t;\lambda_1^-)\Big] \\
&\quad +O(1)\delta\int_0^t \theta^1(x;t-\sigma;\lambda_1^-,4)\epsilon\rho^1(\sigma)d\sigma \\
&\le O(1)\delta\begin{cases}\theta^1(x,t;\lambda_1^-,E)+\psi^{3/2}(x,t;\lambda_1^-) & \text{for } x<\lambda_1^-t,\\ \theta^1(x,t;\lambda_1^-,E)+\epsilon\eta^1(x,t) & \text{for } \lambda_1^-t\le x<1,\\ \epsilon\rho^1(t)e^{-|\lambda_1^-|x/2}+\theta^1(x,t;\lambda_1^-,E) & \text{for } x\ge 1.\end{cases}
\end{aligned} \tag{6.6}
$$

Differentiate the above with respect to x, we obtain that

$$
\begin{aligned}
&|\partial_x\Theta_1^1(x,t)| \\
&= O(1)\delta\begin{cases}\theta^2(x,t;\lambda_1^-,E)+t^{-\frac{1}{2}}\psi^{3/2}(x,t;\lambda_1^-) & \text{for } x<\lambda_1^-t,\\ \theta^2(x,t;\lambda_1^-,E)+\epsilon\eta^1(x,t)(|x|+1)^{-\frac{1}{2}} & \text{for } \lambda_1^-t\le x<1,\\ \epsilon\rho^1(t)e^{-|\lambda_1^-|x/2}(|x|+1)^{-1}+\theta^2(x,t;\lambda_1^-,E) & \text{for } x\ge 1.\end{cases}
\end{aligned} \tag{6.7}
$$

In order to evaluate $\bar{\mathbb{D}}_1^2$, we need some extra cancellations from the equations. First, we need to handle the nonlinear terms $\mathscr{N}[\Theta_1]$ as follows

$$
\mathscr{N}[\Theta_1]=\sum_{1\le i,j,k\le 2}\int_0^1\int_0^1 \xi\mathscr{C}_{ij}^k(u_-+\xi\bar{\xi}\Theta_1)\Theta_1^i\Theta_1^j d\xi d\bar{\xi} r_k(u_-)
$$

where $\mathscr{C}_{ij}^k(u)\equiv l_k(u_-)f''(u)(r_i(u_-),r_j(u_-))$. We define

$$
\begin{aligned}
\boldsymbol{i}_{ij}^k &\equiv \int_0^t\int_R\Big(\int_0^1\int_0^1 \theta^1(x-y,t-\sigma;\lambda_k^-,4) \\
&\quad \times\xi\Big[\mathscr{C}_{ij}^k(u_-+\bar{\xi}\xi\Theta_1)(\Theta_1^i\Theta_1^j)(y,\sigma)\Big]_y d\xi d\bar{\xi}\Big)dyd\sigma.
\end{aligned}
$$

The function $\bar{\mathbb{D}}_1^2(x,t)$ can be represented as

$$
\bar{\mathbb{D}}_1^2(x,t)=\sum_{1\le i,j\le 2}\boldsymbol{i}_{ij}^2(x,t). \tag{6.8}
$$

From the above estimates of Θ_1^1 and Θ_1^2, we have that

$$
\Theta_1^1(x,t)\Theta_1^2(x,t)
$$

$$\leq O(1)\delta^2 \begin{cases} \left[\theta^1(x,t;\lambda_1^-,E)+\psi^{3/2}(x,t;\lambda_1^-)\right]\left[\frac{e^{-\frac{(\lambda_1^-)^2 t}{4E}}}{\sqrt{t}}+(1+t)^{-3/2}\right] \\ \quad \text{for } x\leq \lambda_1^- t, \\ \left[\theta^1(x,t;\lambda_1^-,E)+\epsilon\eta^1(x,t)\right]\left[\frac{e^{-\frac{(\lambda_1^-)^2 t}{4E}}}{\sqrt{t}}+(1+t)^{-3/2}\right] \\ \quad \text{for } x\in[\lambda_1^- t,\lambda_1^- t/2], \\ \left[\frac{e^{-\frac{(\lambda_1^-)^2 t}{4E}}}{\sqrt{t}}+\epsilon\rho^1(t)\right]\left[\theta^1(x,t;\lambda_2^-,E)+\psi^{3/2}(x,t;\lambda_2^-)\right] \\ \quad \text{for } x\in[\lambda_1^- t/2,1], \\ 0 \text{ for } x\geq 1. \end{cases}$$

This, $\rho^\alpha(t)\leq\epsilon^{-2\alpha}(1+t)^{-3\alpha/2}$, and $\eta^\alpha(x,t)\leq\epsilon^{-2\alpha}\psi^{3\alpha}(x,t;\lambda_1^-)$ yield that

$$\Theta_1^1(x,t)\Theta_1^2(x,t)$$
$$\leq O(1)\begin{cases} \delta^2\left[\theta^4(x,t;\lambda_1^-,E)+\psi^{3/2}(x,t;\lambda_1^-)t^{-3/2}\right] \text{ for } x\leq\lambda_1^- t, \\ \delta^2\left[\theta^4(x,t;\lambda_1^-,E)+\epsilon^{-1}\psi^{3/2}(x,t;\lambda_1^-)t^{-3/2}\right] \\ \quad \text{for } x\in[\lambda_1^- t,\lambda_1^- t/2], \\ \delta^2\epsilon^{-1}\psi^3(x,t;\lambda_2^-) \text{ for } x\in[\lambda_1^- t/2,1], \\ 0 \text{ for } x\geq 1. \end{cases}$$

Substitute $(\lambda,\mu)=(\lambda_2^-,\lambda_1^-)$ into Lemma 5.2 with $(\alpha,\beta)=(4,2)$ to yield that

$$\int_0^t\int_R \theta^2(x-y,t-\sigma;\lambda_2^-)\theta^4(y,\sigma;\lambda_1^-,E)dyd\sigma = O(1)\psi^{3/2}(x,t;\lambda_2^-), \quad (6.9)$$

and substitute $(\lambda,\mu)=(\lambda_2^-,\lambda_2^-)$ into Lemma 5.4. with $(\alpha,\beta)=(3,2)$; and Lemma 5.5. with $(\beta,\alpha_1,\alpha_2)=(2,3/2,3/2)$ to yield that

$$\int_0^t\int_{y>\frac{\lambda_1^-\sigma}{2}} \theta^2(x-y,t-\sigma;\lambda_2^-)\psi^3(y,\sigma;\lambda_2^-)dyd\sigma$$
$$= O(1)\psi^{3/2}(x,t;\lambda_2^-), \quad (6.10)$$
$$\int_0^t\int_{y<\frac{\lambda_1^-\sigma}{2}} \theta^2(x-y,t-\sigma;\lambda_2^-)\psi^{3/2}(y,\sigma;\lambda_2^-)(\sigma+1)^{-3/2}dyd\sigma$$
$$\leq O(1)t^{-1/4}\psi^{3/2}(x,t;\lambda_2^-). \quad (6.11)$$

By combining (6.9), (6.10), and (6.11), we have that

$$\boldsymbol{i}_{21}^2(x,t), \boldsymbol{i}_{12}^2(x,t) \le O(1)\frac{\delta^2}{\epsilon}\psi^{3/2}(x,t;\lambda_2^-). \tag{6.12}$$

By Lemma 5.3 with $(\lambda,\mu)=(\lambda_2^-,\lambda_2^-)$ and $(\alpha,\beta)=(3,2)$,

$$\boldsymbol{i}_{22}^2(x,t) = O(1)\delta^2\psi^{3/2}(x,t;\lambda_2^-). \tag{6.13}$$

For the term $\boldsymbol{i}_{11}^2$, we need to consider that

$$\begin{aligned}
\boldsymbol{i}_{11}^2(x,t) &= \int_0^t\int_R \theta^1(x-y,t-\sigma;\lambda_2^-,4)\Big(\mathscr{C}_{11}^2\Theta_1^1\Theta_1^1\Big)_y dyd\sigma \\
&= O(1)\int_0^t\int_R \theta^1(x-y,t-\sigma;\lambda_2^-,4)\|\Theta_{1y}(y,\sigma)\|\cdot\Theta_1^1(y,\sigma)^2;dyd\sigma \\
&\quad -\int_0^t\int_R \theta^1(x-y,t-\sigma;\lambda_2^-,4)\mathscr{C}_{11}^2\frac{\{\partial_\sigma+\lambda_2^-\partial_y-\partial_y^2\}\Theta_1^1(y,\sigma)^2}{(\lambda_1^- - \lambda_2^-)}dyd\sigma \\
&\quad +\int_0^t\int_R \theta^1(x-y,t-\sigma;\lambda_2^-,4)\mathscr{C}_{11}^2\frac{\{\partial_\sigma+\lambda_1^-\partial_y-\partial_y^2\}\Theta_1^1(y,\sigma)^2}{(\lambda_1^- - \lambda_2^-)}dyd\sigma \\
&\equiv \boldsymbol{j}_1+\boldsymbol{j}_2+\boldsymbol{j}_3.
\end{aligned} \tag{6.14}$$

By the estimates for $\Theta_1(x,t)$,

$$\begin{aligned}
&\|\Theta_{1x}(x,t)\|\Theta_1^1(x,t)^2 \\
&\le O(1)\delta^3\begin{cases}
[1+\epsilon^{-2}t^{-\frac12}+\epsilon^{-1}]\theta^4(x,t;\lambda_1^-,2E)+\epsilon^{-3}t^{-\frac12}\psi^{\frac92}(x,t;\lambda_1^-) \\
\quad \text{for } x\le \lambda_1^- t/2, \\
\epsilon^{-3}t^{-\frac92}(|x|+1)^{-\frac12}+\epsilon^{-2}\Big[\theta^8(x,t;\lambda_2^-,2E)+t^{-\frac72}\psi^{\frac32}(x,t;\lambda_2^-)\Big] \\
\quad \text{for } x\le[\lambda_1^- t/2,-1], \\
\Big(\epsilon^3\rho^3(t)+e^{-\frac{(\lambda_1^-)^2t}{4E}}\Big)e^{-\frac{-|\lambda_1^- x|}{2E}} \text{ for } x\ge -1.
\end{cases}
\end{aligned}$$

By Lemma 5.2 with $(\alpha,\beta)=(4,1)$, $(\lambda,\mu)=(\lambda_2^-,\lambda_1^-)$; and by (5.22) with $(\alpha,\beta)=(9/2,1)$, $(\lambda,\mu)=(\lambda_2^-,\lambda_1^-)$, we have that

$$\begin{aligned}
&\int_0^t\int_{y\le\lambda_1^-\sigma/2}\theta^1(x-y,t-\sigma;\lambda_2^-,4)\|\Theta_{1y}(y,\sigma)\|\Theta_1^2(y,\sigma)^2dyd\sigma \\
&\quad\le O(1)\frac{\delta^3}{\epsilon^3}\Big[\theta^1(x,t;\lambda_2^-,E)+\psi^{3/2}(x,t;\lambda_2^-)\Big].
\end{aligned} \tag{6.15}$$

When $y \in [\lambda_1^- t/2, 0]$ and $\iota, \tau > 0$, we may treat $t^{-\iota}$ and $\psi^\tau(x,t;\lambda_2^-)$ as $\psi^\iota(x,t;\lambda_1^-)$ and $\psi^{\tau/2}(x,t;\lambda_1^-)$ respectively. Then, by Lemma 5.2, with $(\alpha,\beta) = (8,1)$, $(\lambda,\mu) = (\lambda_2^-,\lambda_1^-)$; and by (5.22) with $(\alpha,\beta) = (17/4,1)$, $(\lambda,\mu) = (\lambda_2^-,\lambda_1^-)$,

$$\begin{aligned} &\int_0^t \int_{\lambda_1^- \sigma/2}^{-1} \theta^1(x-y,t-\sigma;\lambda_2^-,4)\|\Theta_{1y}(y,\sigma)\|\Theta_1^2(y,\sigma)^2 dy d\sigma \\ &\quad \le O(1)\frac{\delta^3}{\epsilon^3}\Big[\theta^1(x,t;\lambda_2^-,E) + \psi^{3/2}(x,t;\lambda_2^-)\Big]. \end{aligned} \tag{6.16}$$

By Lemma 5.13 with $\nu = O(1)$ and $\lambda = \lambda_2^-$

$$\begin{aligned} &\int_0^t \int_{y\ge -1} \theta^1(x-y,t-\sigma;\lambda_2^-,4)\|\Theta_{1y}(y,\sigma)\|\Theta_1^2(y,\sigma)^2 dy d\sigma \\ &\quad \le O(1)\int_0^t \int_{y\ge -1} \theta^1(x-y,t-\sigma;\lambda_2^-,4)\Big[\epsilon^3\rho^3(\sigma) + e^{-O(1)\sigma}\Big]e^{-O(1)|y|} dy d\sigma \\ &\quad \le O(1)\delta^3 \begin{cases} \Big[\epsilon^2\rho^3(t) + \epsilon^{-1}e^{-O(1)t} + e^{-O(1)\epsilon^2 t}\Big]e^{-|\lambda_2^- x|} & \text{for } x \le 1, \\ O(1)\epsilon^2\rho^3(t) + \epsilon^{-1}e^{-O(1)\epsilon t} & \text{for } x \ge 1. \end{cases} \end{aligned} \tag{6.17}$$

From (6.15), (6.16), and (6.17) we have that

$$\boldsymbol{j}_1 \le O(1)\frac{\delta^3}{\epsilon^3}\Big[\theta^1(x,t;\lambda_2^-,E) + \psi^{3/2}(x,t;\lambda_2^-)\Big]. \tag{6.18}$$

By integration by parts,

$$\begin{aligned} \boldsymbol{j}_2 &= O(1)\int_0^t\int_R \|\Theta_{1y}(y,\sigma)\|\Big|\partial_y\Big(\theta^1(x-y,t-\sigma;\lambda_2^-,4)\Theta_1^1(y,\sigma)^2\Big)\Big| dy d\sigma \\ &= O(1)\int_0^t\int_R \theta^2(x-y,t-\sigma;\lambda_2^-,E)\|\Theta_{1y}(y,\sigma)\|\Theta_1^1(y,\sigma)^2 dy d\sigma \\ &\quad + O(1)\int_0^t\int_R \theta^1(x-y,t-\sigma;\lambda_2^-,E)\|\Theta_{1y}(y,\sigma)\|\Theta_{1y}^1(y,\sigma)\Theta_1^1(y,\sigma) dy d\sigma. \end{aligned} \tag{6.19}$$

Similar to (6.18), we have

$$\boldsymbol{j}_2 \le O(1)\frac{\delta^3}{\epsilon^3}\Big[\theta^1(x,t;\lambda_2^-,E) + \psi^{3/2}(x,t;\lambda_2^-)\Big]. \tag{6.20}$$

By the equation for $\Theta_1^1(x,t)$ in (3.2c), we have

$$\begin{aligned}\boldsymbol{j}_3 &= O(1)\int_0^t\int_R \theta^1(x-y,t-\sigma;\lambda_2^-,4)\Big(\Theta_1^1(y,\sigma)\mathscr{E}_1^-(y,\sigma)+\Theta_{1y}^1(y,\sigma)^2\Big)dyd\sigma\\ &\equiv \boldsymbol{j}_3^1+\boldsymbol{j}_3^2. \end{aligned}\tag{6.21}$$

Since

$$\Theta_1^1(x,t)\cdot|\mathscr{E}_1^-(x,t)|\le \delta^2\epsilon^2\rho^2(t)\varrho(x),$$

by Lemma 5.12 with $\lambda=\lambda_2^-$, $D=O(1)$, and $(\alpha,\beta)=(2,1)$, we have

$$\chi_-\cdot\boldsymbol{j}_3^1\le O(1)\frac{\delta^2}{\epsilon}\chi_-\cdot\psi^2(x,t;\lambda_2^-).\tag{6.22}$$

In the evaluation of $\boldsymbol{j}_3^2$, we need to use (6.7); and break the function $\eta(y,\sigma)^2$ $(1+|y|)^{-1}$ for $y\le 0$ as follows

$$\begin{aligned}&\eta(y,\sigma)^2(1+|y|)^{-1}\\ &\le O(1)\begin{cases}0 & \text{for } y\le\lambda_1^-\sigma+\sqrt{\sigma},\\ \Big[\sqrt{|y-\lambda_1^-\sigma|}|y-\lambda_1^-\sigma|\Big]^{-2}\sigma^{-1} & \text{for } y\in[\lambda_1^-\sigma+\sqrt{\sigma},\lambda_1^-\sigma/2],\\ \rho^2(\sigma)(|y|+1)^{-1} & \text{for } y\in[\lambda_1^-\sigma/2,0].\end{cases}\end{aligned}$$

Then, by Lemma 5.8 with $(\alpha_1,\alpha_2,\beta)=(2,2,1)$, we have that for $x\le 1$

$$\begin{aligned}&\int_0^t\int_{\lambda_1^-\sigma/2}^0\theta^1(x-y,t-\sigma;\lambda_2^-,D)(|y|+1)^{-1}\eta^2(y,\sigma)dyd\sigma\\ &\le O(1)\begin{cases}\frac{1}{\epsilon}\Big[\theta^1(x,t;\lambda_2^-,D)+\rho^1(|x-\lambda_2^-|)\Big] & \text{for } (\frac{\lambda_1^-}{2}-\lambda_2^-)t\le x\le 0,\\ e^{-\frac{(x-\lambda_2^-t)^2}{D't}}\Big(\frac{1}{\epsilon\sqrt{t}}+\frac{1}{(1+\epsilon^2t)^{-2}}\Big) & \text{for } x\le(\frac{\lambda_1^-}{2}-\lambda_2^-)t.\end{cases}\end{aligned}\tag{6.23}$$

For $y\in[\lambda_1^-\sigma,\lambda_1^-\sigma/2]$,

$$(|y|+1)^{-1}\eta^2(y,\sigma)\le O(1)\sigma^{-1}\psi^3(y,\sigma;\lambda_1^-)\epsilon^{-4}\le O(1)\psi^4(y,\sigma;\lambda_1^-).$$

Then, by Lemma 5.4 with $(\alpha,\beta)=(4,1)$ will have that

$$\int_0^t\int_{\lambda_1^-\sigma}^{\lambda_1^-\sigma/2}\theta^1(x-y,t-\sigma;\lambda_2^-,4)\frac{\eta^2(y,\sigma)}{|y|}dyd\sigma$$

$$\leq\ O(1)\frac{1}{\epsilon^4}\Big[\theta^1(x,t;\lambda_2^-,D)+\psi^{3/2}(x,t;\lambda_2^-)\Big].$$

So, we have that

$$\chi_-\boldsymbol{j}_3^2\leq O(1)\frac{\delta^2}{\epsilon^4}\Big[\theta^1(x,t;\lambda_2^-,D')+\psi^{3/2}(x,t;\lambda_2^-)\Big].$$

For the term $\boldsymbol{i}_{22}^2(x,t)$, by Lemma 5.1 with $(\alpha,\beta)=(2,2)$ and Lemma 5.3 with $(\alpha,\beta)=(3,2)$ it follows

$$\boldsymbol{i}_{22}^2(x,t)\leq O(1)\delta^2\Big[\theta^1(x,t;\lambda_2^-,E)+\psi^{3/2}(x,t;\lambda_2^-)\Big].$$

As a consequence of the estimates of $\boldsymbol{i}_{ij}^2$, there exist c_0 and c_1 which are independent of ϵ and $\bar{\epsilon}$ such that

$$\mathbb{D}_1^2(x,t)\equiv\chi_-\bar{\mathbb{D}}_1^2(x,t)\leq c_0\bar{\epsilon}\begin{cases}\delta\Big[\theta^1(x,t;\lambda_2^-,E)+\psi^{3/2}(x,t;\lambda_2^-)\Big] & \text{for } x\leq 1,\\ 0 & \text{for } x\geq 1,\end{cases}\tag{6.24}$$

where $\bar{\epsilon}=\delta/\epsilon^4$.

This also yields that

$$\begin{cases}\mathscr{E}_2(x,t)\leq c_1\bar{\epsilon}\delta\rho^1(t)\varrho(x),\\ \mathscr{E}_2^-(x,t)\leq c_1\bar{\epsilon}\delta\epsilon\rho^1(t)\varrho(x).\end{cases}$$

By similar procedures, we will have that

$$\begin{aligned}&|\partial_x\mathbb{D}_1^2(x,t)|\\ &\quad\leq\ O(1)\bar{\epsilon}\begin{cases}\delta\Big[\frac{1}{\sqrt{t}}+\varrho(x)\Big]\Big(\theta^1(x,t;\lambda_2^-,E)+\psi^{3/2}(x,t;\lambda_2^-)\Big) & \text{for } x\leq 1,\\ 0 & \text{for } x\geq 1.\end{cases}\end{aligned}$$

For the term $\mathbb{D}_1^1(x,t)$, we have the representation

$$\mathbb{D}_1^1(x,t)=\sum_{1\leq i,j\leq 2}\boldsymbol{i}_{ij}^1(x,t)+\int_0^t\int_R\theta^1(x-y,t-\sigma;\lambda_1^-,4)\mathscr{E}_2^-(y,\sigma)dyd\sigma.$$

The ways of estimating $\boldsymbol{i}_{11}^1$ is similar to that in (6.13), so we omit the calculation, here. The method to estimate $\boldsymbol{i}_{12}^1$ is similar to that for obtaining $\boldsymbol{i}_{21}^2$

in (6.12). We omit it, too. We have that

$$|\boldsymbol{i}_{11}^1|, |\boldsymbol{i}_{12}^1|, |\boldsymbol{i}_{21}^1| \le O(1)\delta\left(\frac{\delta}{\epsilon}\psi^{3/2}(x,t;\lambda_1^-) + \delta\theta^1(x,t;\lambda_1^-)\right).$$

The way of estimating $\boldsymbol{i}_{22}^1(x,t)$ is rather simpler compared to the estimate of $\boldsymbol{i}_{11}^2(x,t)$ in the following sense:

$$\begin{aligned}\boldsymbol{i}_{22}^1(x,t) \ &\le\ O(1)\delta^2 \int_0^t \int_{y<1} |\partial_y \theta^1(x-y,t-\sigma;\lambda_1^-,4)| \\ &\qquad \times\Big[\theta^2(y,\sigma;\lambda_2^-,4) + \psi^3(y,\sigma;\lambda_2^-)\Big] dy d\sigma \\ &\le\ O(1)\delta^2 \begin{cases} \theta^{3/2}(x,t;\lambda_1^-,E) + \psi^{3/2}(x,t;\lambda_1^-) & \text{for } x \le \lambda_1^- t + \sqrt{t}, \\ \theta^1(x,t;\lambda_1^-,E) + \rho^1(x-\lambda_2^- t) & \text{for } x \in [\lambda_1^- t, 0], \\ e^{-\frac{|\lambda_1^- - \lambda_2^-|\, x}{2}}\rho^1(t) & \text{for } x \ge 0. \end{cases}\end{aligned}$$

By Lemma 5.13 with $\nu = 1$ and $(\alpha,\beta) = (1,1)$, it follows

$$\begin{aligned}&\int_0^t \int_R \theta^1(x-y,t-\sigma;\lambda_1^-,4)\mathscr{E}_2^-(y,\sigma) dy d\sigma \\ &\quad \le\ O(1)\delta\bar{\epsilon} \begin{cases} \theta^1(x,t;\lambda_1^-,E) & \text{for } x \le \lambda_1^- t - \sqrt{t}, \\ \theta^1(x,t;\lambda_1^-,E) + \epsilon\rho^1(x-\lambda_1^- t) & \text{for } \lambda_1^- t + \sqrt{t} \le x \le 1, \\ \rho^1(t) e^{-|(\lambda_1^- - \lambda_2^-)x|/2} & \text{for } x \ge 1. \end{cases}\end{aligned}$$

So, we conclude that

$$\mathbb{D}_1^1(x,t) \le O(1)\delta\bar{\epsilon} \begin{cases} \theta^1(x,t;\lambda_1^-,E) + \psi^{3/2}(x,t;\lambda_1^-) & \text{for } x \le \lambda_1^- t + \sqrt{t}, \\ \theta^1(x,t;\lambda_1^-,E) + \epsilon\eta(x,t) & \text{for } \lambda_1^- t + \sqrt{t} \le x \le 0, \\ \epsilon\rho^1(t) e^{-\frac{|\lambda_1^- - \lambda_2^-|x}{2}} & \text{for } x \ge 0, \end{cases} \tag{6.25}$$

and similarly,

$$\begin{aligned}&|\partial_x \mathbb{D}_1^1(x,t)| \\ &\quad \le\ O(1)\delta\bar{\epsilon} \begin{cases} \frac{1}{\sqrt{t}}\Big[\theta^1(x,t;\lambda_1^-,E) + \psi^{3/2}(x,t;\lambda_1^-)\Big] & \text{for } x \le \lambda_1^- t + \sqrt{t}, \\ \frac{1}{\sqrt{|x|+1}}\Big[\theta^1(x,t;\lambda_1^-,E) + \epsilon\eta(x,t)\Big] & \text{for } \le \lambda_1^- t + \sqrt{t} \le x \le 0, \\ \epsilon\rho^1(t) e^{-\frac{|\lambda_1^- - \lambda_2^-|x}{2}} & \text{for } x \ge 0, \end{cases}\end{aligned}$$

So, the theorem is true for $\mathbb{D}_i$ when $i=1$. Through the above procedure for obtaining the estimates for $\|\mathbb{D}_1(x,t)\|$, one can show the theorem is true for all $i\geq 1$. $\square$

Proof of Theorem A.

Proof. If we can find a constant $\mathscr{C}$ to satisfy (4.29), then this theorem is proved. So, we may assume the existence of such a constant $\mathscr{C}$ which satisfies

$$\mathscr{C}\leq|\log\epsilon|. \tag{6.26}$$

Combining (4.29) and (4.28), we obtain a global ansatz for the behavior of $\mathbb{V}$ in space and time. Then, through the Green functions $\theta^1(x-y,t-\sigma;\lambda_i^-,4)$ one has the following integral representation of $\mathbb{V}^i$ for $i=1,2$:

$$\mathbb{V}^i(x,t)=\int_R\theta^1(x-y;t;\lambda_i^-,4)\mathbb{V}^i(y,0)dy+\Big[\mathbb{I}_1^i[\Theta]+\mathbb{I}_2^i[\bar\gamma]+\mathbb{I}_3^i[\bar\gamma]+\sum_{j=4}^6\mathbb{I}_j^i[\mathbb{V}]\Big](x,t), \tag{6.27}$$

We introduce a partition of unit matrix to localize the structure of shock:

$$\mathbf{P}_1(x)=\chi_+(x-J_0)r_2(u_-)\cdot l_2(u_-),\quad \mathbf{P}_2=1-\mathbf{P}_1.$$

We will regroup the representation for $\mathbb{V}^1(x,t)$ as that in (4.18)

$$\begin{aligned}
\mathbb{V}^1(x,t) &= \mathbb{V}^1_\pi(x,t)+\mathbb{V}^1_{\pi^\perp}(x,t),\\
\mathbb{V}^1_\pi(x,t) &\equiv \Big(\mathbb{I}_3^1[\bar\gamma]+\mathbb{I}_5^1[\mathbb{V}_\pi]+\bar{\mathbb{I}}_6^1[\mathbb{V}_\pi]\Big)(x,t),\\
\mathbb{V}^1_{\pi^\perp}(x,t) &\equiv \int_R\theta^1(x-y,t;\lambda_1^-,4)\mathbb{V}^1(y,0)dy\\
&\quad+\Big[\mathbb{I}_1^1[\Theta]+\mathbb{I}_2^1[\bar\gamma]+\mathbb{I}_4^1[\boldsymbol{P}_1\mathbb{V}]\Big](x,t)\\
&\quad+\Big[\mathbb{I}_4^1[\boldsymbol{P}_2\mathbb{V}]+\Big(\mathbb{I}_5^1[\mathbb{V}]-\mathbb{I}_5^1[\mathbb{V}_\pi]\Big)+\Big(\mathbb{I}_6^1[\mathbb{V}]-\bar{\mathbb{I}}_6^1[\mathbb{V}_\pi]\Big)\Big](x,t).
\end{aligned}\tag{6.28}$$

By Lemma P with $\epsilon_0=\delta$ we have such a $\mathscr{C}_0=O(1)$ to satisfy (4.29b).

In the representation of $\mathbb{V}^1_{\pi^\perp}$ in (6.28), the functional $\mathbb{I}_1^1[\Theta]+\mathbb{I}_2^1[\bar\gamma]+\mathbb{I}_4^1[\boldsymbol{P}_1\mathbb{V}]$ is independent of the constant $\mathscr{C}$. This functional will give the dominant ansatz of $\mathbb{V}^1_{\pi^\perp}$.

We proceed to establish the estimates for $\mathbb{V}_{\pi^\perp}$.

By Theorem DW and (4.3),

$$\|\boldsymbol{I}_1[\Theta](x,t)\| \le O(1)\delta\epsilon\rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))}.$$

By Lemma 5.13 with $(\alpha,\beta)=(1,2)$, $\nu=\lambda_2^-$, and $\lambda=\lambda_1^-$, it follows that

$$|\mathbb{I}_1^1[\Theta](x,t)| \le O(1)\delta \begin{cases} \min\left(\frac{1}{\sqrt{t}},\sqrt{\epsilon}\right)\bar{\Gamma}^1(t)\theta^1(x,t;\lambda_1^-,D) \\ \quad +\min\left(\frac{1}{\sqrt{|x|+1}},\sqrt{\epsilon}\right)\eta^1(x,t) & \text{for } x\le 0, \\ \sqrt{\epsilon}\rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))} & \text{for } x\ge 0, \end{cases} \tag{6.29}$$

Apply Lemma 5.13 with $(\alpha,\beta)=(1,2)$ and $\nu=\lambda_2^-$ and $\nu=1$ to the estimates in (4.7) for $\boldsymbol{I}_2[\bar{\gamma}]$ to yield

$$|\mathbb{I}_2^1[\bar{\gamma}](x,t)| \le O(1)\delta \begin{cases} \epsilon\bar{\Gamma}^2(t)\theta^2(x,t;\lambda_1^-,D)+\frac{\epsilon}{\sqrt{|x|+1}}\eta^1(x,t) & \text{for } x\le 0, \\ \epsilon^{3/2}\rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))} & \text{for } x\ge 0. \end{cases} \tag{6.30}$$

By (4.28a) for $\mathbb{V}^2$,

$$|l_1(u_-)\cdot\boldsymbol{I}_4[\boldsymbol{P}_1\mathbb{V}](x,t)| \le O(1)|\log\epsilon|\epsilon\delta\rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))}.$$

Then, by Lemma 5.13 with $(\alpha,\beta)=(1,2)$ and $\lambda=\lambda_1^-$ it yields

$$\begin{aligned} &|\mathbb{I}_4^1[\boldsymbol{P}_1\mathbb{V}](x,t)| \\ &\le O(1)|\log\epsilon|\delta \begin{cases} \min\left(\sqrt{\epsilon},\frac{1}{\sqrt{|x|+1}}\right)\eta^1(x,t) \\ \quad +\min\left(\sqrt{\epsilon},\frac{1}{\sqrt{t}}\right)\bar{\Gamma}^1(t)\theta^1(x,t;\lambda_1^-,D) & \text{for } x\le 0, \\ \sqrt{\epsilon}\rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))} & \text{for } x\ge 0. \end{cases} \end{aligned} \tag{6.31}$$

The estimates (6.29), (6.30), and (6.31) yield that

$$\Big(|\mathbb{I}_1^1[\Theta]|+|\mathbb{I}_2^1[\bar{\gamma}]|+|\mathbb{I}_4^1[\boldsymbol{P}_1\mathbb{V}]\Big)(x,t)$$

$$\le O(1)\begin{cases} \delta\sqrt{\epsilon}|\log\epsilon|\rho^1(t)e^{-\frac{\lambda_2^-|x|(1+O(\epsilon))}{2}} & \text{for } x\ge 0,\\ \delta\bar{\Gamma}^1(t)|\log\epsilon|\min\Big(\sqrt{\epsilon},\frac{1}{\sqrt{t}}\Big)\theta^1(x,t;\lambda_1^-,D) & \\ \quad+\delta|\log\epsilon|\min\Big(\sqrt{\epsilon},\frac{1}{\sqrt{|x|+1}}\Big)\eta(x,t) & \text{for } \lambda_1^-t+\sqrt{t}\le x\le 0,\\ \delta\bar{\Gamma}^1(t)|\log\epsilon|\min\Big(\sqrt{\epsilon},\frac{1}{\sqrt{t}}\Big)\theta^1(x,t;\lambda_1^-,D) & \text{for } x\le\lambda_1^-t+\sqrt{t}, \end{cases} \tag{6.32}$$

For the other functionals, we substitute (4.29a) and (4.29b) with $\mathscr{C}\le|\log\epsilon|$ into $\mathbb{I}_4^1[\boldsymbol{P}_2\mathbb{V}]$, $\mathbb{I}_5^1[\mathbb{V}]-\mathbb{I}_5^1[\mathbb{V}_\pi]$, and $\mathbb{I}_6^1[\mathbb{V}]-\mathbb{I}_6^1[\mathbb{V}_\pi]$. Due to the choice $J_0=\epsilon^{-1}|\log\epsilon|J$ with $J\gg 1$, we have that

$$|e^{-\lambda_2^-|x|(1+O(\epsilon))}\chi_-(x-J_0)\mathbb{V}^2(x,t)|\ll\delta\epsilon t^{-3/2}e^{-3\lambda_2^-|x|(1+O(\epsilon))/4}.$$

On the other hand,

$$\begin{aligned} e^{-\lambda_2^-|x|(1+O(\epsilon))}\|\mathbb{V}^1_{\pi^\perp}(x,t)\| &\le O(1)\mathscr{C}\sqrt{\epsilon}|\log\epsilon|\delta\rho^1(t)e^{-3\lambda_2^-|x|(1+O(\epsilon))/4}\\ &\le O(1)\sqrt{\epsilon}|\log\epsilon|^2\delta\rho^1(t)e^{-3\lambda_2^-|x|(1+O(\epsilon))/4}.\\ e^{-\lambda_2^-|x|(1+O(\epsilon))}\|\mathbb{V}^1_{\pi}(x,t)\| &\le O(1)\mathscr{C}\frac{\delta^2|\log\epsilon|^3}{\epsilon^3}\rho^1(t)e^{-3\lambda_2^-|x|(1+O(\epsilon))/4}\\ &\le O(1)\frac{\delta^2|\log\epsilon|^4}{\epsilon^3}\rho^1(t)e^{-3\lambda_2^-|x|(1+O(\epsilon))/4}. \end{aligned}$$

By combining the above three estimates, we have

$$\begin{aligned} \|\boldsymbol{P}_2\mathbb{V}(x,t)\| &\le e^{-\lambda_2^-|x|(1+O(\epsilon))}\Big(|\mathbb{V}^1(x,t)|+|\chi_-(x-J_0)||\mathbb{V}^2(x,t)|\Big)\\ &\le O(1)\delta|\log\epsilon|^2\sqrt{\epsilon}\rho^1(t)e^{-3\lambda_2^-|x|(1+O(\epsilon))/4}. \end{aligned}$$

Then, by Lemma 5.13 with $(\alpha,\beta)=(1,2)$ and $\lambda=\lambda_1^-$, $\nu=\frac{3\lambda_2^-}{4}$, we have

$$\begin{aligned} &|\mathbb{I}_4^1[\boldsymbol{P}_2\mathbb{V}](x,t)|\\ &\le O(1)\delta|\log\epsilon|^2\sqrt{\epsilon}\begin{cases} \min\Big(\sqrt{\epsilon},\frac{1}{\sqrt{|x|+1}}\Big)\eta^1(x,t) & \\ \quad+\min\Big(\sqrt{\epsilon},\frac{1}{\sqrt{t}}\Big)\bar{\Gamma}^1(t)\theta^1(x,t;\lambda_1^-,D) & \text{for } x\le 0,\\ \sqrt{\epsilon}\rho^1(t)e^{-3\lambda_2^-|x|(1+O(\epsilon))/4} & \text{for } x\ge 0. \end{cases} \end{aligned} \tag{6.33}$$

For the term $\mathbb{I}_5^1[\mathbb{V}] - \mathbb{I}_5^1[\mathbb{V}_\pi]$, it follows

$$\begin{aligned}
&\Big| l_1(u_-)\cdot(\boldsymbol{I}_5[\mathbb{V}] - \boldsymbol{I}_5[\mathbb{V}_\pi])(x,t)\Big| \\
&= \Big| l_1(u_-)\cdot \boldsymbol{I}_5[\mathbb{V}_{\pi^\perp}](x,t)\Big| \\
&\le O(1)\mathscr{C}|\log\epsilon|\delta^2\Big[\frac{\theta^3(x,t;\lambda_1^-,D)}{\epsilon^2} + \frac{\theta^{5/2}(x,t;\lambda_1^-,D)}{\epsilon^3} + \eta^1(x,t)^2\Big] \\
&\le O(1)|\log\epsilon|^2\delta^2\Big[\frac{\theta^3(x,t;\lambda_1^-,D)}{\epsilon^2} + \frac{\theta^{5/2}(x,t;\lambda_1^-,D)}{\epsilon^3} + \eta^1(x,t)^2\Big]. \qquad (6.34)
\end{aligned}$$

This yields that

$$\Big|(\mathbb{I}_5^1[\mathbb{V}]-\mathbb{I}_5^1[\mathbb{V}_\pi])(x,t)\Big| \le O(1)\begin{cases} \frac{|\log\epsilon|^2\delta^2\theta^{3/2}(x,t;\lambda_1^-,D)}{\epsilon^3} \text{ for } x\le\lambda_1^- t+\sqrt{t}, \\ \frac{\delta^2|\log\epsilon|^2}{\epsilon^3|x-\lambda_1^-t|(1+\epsilon|x-\lambda_1^-t|)} \text{ for } \lambda_1^-t+\sqrt{t}\le x\le 0, \\ \frac{|\log\epsilon|^2\delta^2}{\epsilon^2}\rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))} \text{ for } x\ge 0. \end{cases} \qquad (6.35)$$

For the term $\mathbb{I}_6^1[\mathbb{V}] - \bar{\mathbb{I}}_6^1[\mathbb{V}_\pi]$,

$$\begin{aligned}
\Big|\boldsymbol{N}[\mathbb{V}] - \bar{\boldsymbol{N}}[\mathbb{V}_\pi]\Big| &\le \Big|\boldsymbol{N}[\mathbb{V}] - \bar{\boldsymbol{N}}[\mathbb{V}]\Big| + \Big|\bar{\boldsymbol{N}}[\mathbb{V}] - \bar{\boldsymbol{N}}[\mathbb{V}_\pi]\Big| \\
&\le O(1)\Big(|\mathbb{V}|^3 + |\mathbb{V}_{\pi^\perp}|\cdot|\mathbb{V}|\Big). \qquad (6.36)
\end{aligned}$$

By assuming that $\delta \ll \epsilon^6$, we have that

$$|\mathbb{V}|^3(x,t) \le O(1)\delta^2\epsilon\begin{cases} \theta^3(x,t;\lambda_1^-,D) \text{ for } x\le\lambda_1^-t+\sqrt{t}, \\ \theta^3(x,t;\lambda_1^-,D) + \psi^{\frac{9}{2}}(x,t;\lambda_1^-) \\ \quad + \psi^{\frac{9}{3}}(x,t;\lambda_1^-) \ \text{ for } \lambda_1^-t+\sqrt{t}\le x\le 0, \\ t^{-9/2}\ e^{-\lambda_2^-|x|(1+O(\epsilon))} \text{ for } x\ge 0, \end{cases} \qquad (6.37)$$

$$|\mathbb{V}|\cdot|\mathbb{V}_{\pi^\perp}| \le O(1)\begin{cases} \delta^2\epsilon^2\theta^{\frac{5}{2}}(x,t;\lambda_1^1,D) + \delta\epsilon\theta^3(x,t;\lambda_1^-,D) \\ \text{for } x\le\lambda_1^-t+\sqrt{t}, \\ \delta^2\epsilon^2\theta^{\frac{5}{2}}(x,t;\lambda_1^-,D) + \delta\epsilon\psi^3(x,t;\lambda_1^-) \\ \text{for } x\in[\lambda_1^-t+\sqrt{t},-J_0], \\ \delta\epsilon t^{-3} \text{ for } x\in[-J_0,0], \\ \delta\epsilon t^{-3}e^{-\lambda_2^-|x|(1+O(\epsilon))} \text{ for } x\ge 0. \end{cases} \qquad (6.38)$$

The above two estimates yield that

$$\Big|(\mathbb{I}_6^1[\mathbb{V}]-\bar{\mathbb{I}}_6^1[\mathbb{V}_\pi]\Big|(x,t)\ll\Big|(\mathbb{I}_5^1[\mathbb{V}]-\mathbb{I}_5^1[\mathbb{V}_\pi]\Big|(x,t). \tag{6.39}$$

So, (6.32), (6.33), (6.35), (6.39) conclude that

$$\begin{aligned}&|\mathbb{V}^1_{\pi^\perp}(x,t)|\\ &\le\ O(1)\begin{cases}\delta\sqrt{\epsilon}|\log\epsilon|\rho^1(t)e^{-\frac{\lambda_2^-|x|(1+O(\epsilon))}{2}}\ \text{for}\ x\ge 0,\\ \delta\bar{\Gamma}^1(t)|\log\epsilon|\min\Big(\sqrt{\epsilon},\frac{1}{\sqrt{t}}\Big)\theta^1(x,t;\lambda_1^-,D)\\ \quad+\delta|\log\epsilon|\min\Big(\sqrt{\epsilon},\frac{1}{\sqrt{|x|+1}}\Big)\eta(x,t)\\ \quad+\frac{\delta^2|\log\epsilon|^2}{\epsilon^3}\Big\{|x-\lambda_1^-t|\Big(1+\epsilon|x-\lambda_1^-t|^{\frac12}\Big)\Big\}^{-1}\\ \quad\ \text{for}\ \lambda_1^-t+\sqrt{t}\le x\le 0,\\ \delta\bar{\Gamma}^1(t)|\log\epsilon|\min\Big(\sqrt{\epsilon},\frac{1}{\sqrt{t}}\Big)\theta^1(x,t;\lambda_1^-,D)\\ \quad+\frac{\delta^2|\log\epsilon|^3}{\epsilon^3}\theta^{3/2}(x,t;\lambda_1^-,D')\\ \quad\ \text{for}\ x\le\lambda_1^-t+\sqrt{t},\end{cases}\end{aligned} \tag{6.40}$$

For estimating $\mathbb{V}^2$, we need to use the estimate (6.40) and (4.29b) with $\mathscr{C}_0=O(1)$; and assume $\mathscr{C}\le|\log\epsilon|$ for (4.29c). Then, we decompose $\mathbb{I}_1^2[\Theta]$, $\mathbb{I}_5^2[\mathbb{V}_\pi]$, and $\bar{\mathbb{I}}_6^2[\mathbb{V}]-\bar{\mathbb{I}}_6^2[\mathbb{V}_\pi]$:

$$\begin{aligned}\mathbb{I}^2_{1;ij}[\Theta]&\equiv\int_0^t\int_R\theta^1(x-y,t-\sigma;\lambda_2^-,4)\\ &\quad\times\Big(\Theta^i\Theta^j l_2(u_-)f''(\mathbb{A})(r_i(u_-),r_j(u_-))\Big)_y(y,\sigma)dyd\sigma,\\ \mathbb{I}^2_{5;i}[\mathbb{V}_\pi]&\equiv\int_0^t\int_R\theta^1(x-y,t-\sigma;\lambda_2^-,4)\\ &\quad\times\Big(\Theta^i\mathbb{V}^1_\pi l_2(u_-)f''(\mathbb{A})(r_i(u_-),r_1(u_-))\Big)_y(y,\sigma)dyd\sigma,\\ \bar{\mathbb{I}}^2_{6;i}[\mathbb{V},\mathbb{V}_\pi]&\equiv\int_0^t\int_R\theta^1(x-y,t-\sigma;\lambda_2^-,4)\\ &\quad\times\Big(\mathbb{V}^i_{\pi^\perp}\mathbb{V}^1_\pi l_2(u_-)f''(\mathbb{A})(r_i(u_-),r_1(u_-))\Big)_y(y,\sigma)dyd\sigma,\end{aligned}$$

Then, regroup (6.27) as follows

$$\begin{aligned}\mathbb{V}^2(x,t) &= \int_R \theta^1(x-y,t;\lambda_2^-,4)\mathbb{V}^2(y,0)dy \\ &\quad +\Big(\mathbb{I}^2_{1;11}[\Theta]+\mathbb{I}^2_{5,1}[\mathbb{V}_\pi]+\bar{\mathbb{I}}^2_6[\mathbb{V}_\pi]+2\bar{\mathbb{I}}^2_{6;1}[\mathbb{V},\mathbb{V}_\pi]\Big)(x,t) \\ &\quad +\sum_{\substack{1\le i,j\le 2\\(i,j)\neq(1,1)}} \mathbb{I}^2_{1;ij}[\Theta](x,t)+\Big(\mathbb{I}^2_2[\bar\gamma]+\mathbb{I}^2_3[\bar\gamma]+\mathbb{I}_4[\mathbb{V}]\Big)(x,t) \\ &\quad +\Big(\mathbb{I}^2_5[\mathbb{V}_{\pi^\perp}]+\mathbb{I}^2_{5,2}[\mathbb{V}_\pi]+2\bar{\mathbb{I}}^2_{6;2}[\mathbb{V},\mathbb{V}_\pi]+\bar{\mathbb{I}}^2_6[\mathbb{V}_{\pi^\perp}] \\ &\quad +(\mathbb{I}^2_6[\mathbb{V}]-\bar{\mathbb{I}}^2_6[\mathbb{V}])\Big)(x,t). \qquad (6.41)\end{aligned}$$

The procedure for estimating $\sum_{\substack{1\le i,j\le 2\\(i,j)\neq(1,1)}} \mathbb{I}^2_{1;ij}[\Theta]+(\mathbb{I}^2_2[\bar\gamma]+\mathbb{I}^2_3[\bar\gamma]+\mathbb{I}_4[\mathbb{V}])+(\mathbb{I}^2_5[\mathbb{V}_{\pi^\perp}]$ $+\mathbb{I}^2_{5,2}[\mathbb{V}_\pi]+2\bar{\mathbb{I}}^2_{6;2}[\mathbb{V},\mathbb{V}_\pi]+\bar{\mathbb{I}}^2_6[\mathbb{V}_{\pi^\perp}]+(\mathbb{I}^2_6[\mathbb{V}]-\bar{\mathbb{I}}^2_6[\mathbb{V}]))$ essentially is identical to the procedure for obtaining (4.29a). We omit the details and just state the result:

$$\begin{aligned}&\Big|\mathbb{I}^2_2[\bar\gamma]+\mathbb{I}^2_3[\bar\gamma]+\mathbb{I}_4[\mathbb{V}]\Big|(x,t) \\ &\le O(1)\begin{cases}\delta\epsilon^{J/2}\rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))/2} \text{ for } \lambda_1^- t\le x\le -J_0,\\ \delta\epsilon^{J/2}\rho^1(t)e^{-\lambda_2^-|x|(1+O(\epsilon))/2} \text{ for } x\le\lambda_1^- t,\end{cases} \qquad (6.42)\end{aligned}$$

$$\begin{aligned}&\Big|\sum_{\substack{1\le i,j\le 2\\(i,j)\neq(1,1)}} \mathbb{I}^2_{1;ij}[\Theta]+\mathbb{I}^2_5[\mathbb{V}_{\pi^\perp}]+\mathbb{I}^2_{5,2}[\mathbb{V}_\pi]+2\bar{\mathbb{I}}^2_{6;2}[\mathbb{V},\mathbb{V}_\pi] \\ &\qquad +\bar{\mathbb{I}}^2_6[\mathbb{V}_{\pi^\perp}]+(\mathbb{I}^2_6[\mathbb{V}]-\bar{\mathbb{I}}^2_6[\mathbb{V}])\Big|(x,t) \\ &\qquad\le O(1)\begin{cases}\frac{\delta^2|\log\epsilon|^2}{\epsilon^4}\psi^{3/2}(x,t;\lambda_2^-) \text{ for } \lambda_1^- t\le x\le -J_0,\\ \frac{\delta^2|\log\epsilon|^2}{\epsilon^4}\theta^3(x,t;\lambda_1^-,D) \text{ for } x\le\lambda_1^- t.\end{cases} \qquad (6.43)\end{aligned}$$

We need to apply the procedure for obtaining $\boldsymbol{i}^2_{11}$ in (6.14) to $\mathbb{I}^2_{1;11}[\Theta]+\mathbb{I}^2_{5,1}[\mathbb{V}_\pi]+\bar{\mathbb{I}}^2_6[\mathbb{V}_\pi]+2\bar{\mathbb{I}}^2_{6;1}[\mathbb{V},\mathbb{V}_\pi]$ in order to obtain a sufficient decaying rate of $\mathbb{V}^2(-J_0,t)$. However, we need to modify the procedure in order to obtain the optimal rate for $\bar{\mathbb{I}}^2_{6;1}[\mathbb{V},\mathbb{V}_\pi]$. So, we need to rewrite $\partial_y l_2(u_-)f''(\mathbb{A})(r_1(u_-),$

$r_1(u_-))\mathbb{V}^1_\pi\mathbb{V}^1_{\pi^\perp}$ as follows

$$\begin{aligned}
&\partial_y\big[l_2(u_-)f''(\mathbb{A})(r_1(u_-),r_1(u_-))\mathbb{V}^1_\pi\mathbb{V}^1_{\pi^\perp}\big]\\
&\quad= \frac{\partial_\sigma+\lambda_1^-\partial_y-\partial_y^2}{\lambda_1^- - \lambda_2^-}\big[l_2(u_-)f''(\mathbb{A})(r_1(u_-),r_1(u_-))\mathbb{V}^1_\pi\mathbb{V}^1_{\pi^\perp}\big]\\
&\qquad-\frac{\partial_\sigma+\lambda_2^-\partial_y-\partial_y^2}{\lambda_1^- - \lambda_2^-}\big[l_2(u_-)f''(\mathbb{A})(r_1(u_-),r_1(u_-))\mathbb{V}^1_\pi\mathbb{V}^1_{\pi^\perp}\big]. \qquad (6.44)
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial_\sigma+\lambda_1^-\partial_y-\partial_y^2}{\lambda_1^- - \lambda_2^-}\big[l_2(u_-)f''(\mathbb{A})(r_1(u_-),r_1(u_-))\mathbb{V}^1_\pi\mathbb{V}^1_{\pi^\perp}\big] \qquad (6.45)\\
&= O(1)\epsilon^2 e^{-\lambda_2^-|y|(1+O(\epsilon))}\mathbb{V}^1_\pi\mathbb{V}^1_{\pi^\perp}+O(1)l_1(u_-)\left(\boldsymbol{I}_3[\bar\gamma]+(\boldsymbol{I}_0[\mathbb{V}_\pi]+\bar{\boldsymbol{N}}[\mathbb{V}_\pi])_y\right)\mathbb{V}^1_{\pi^\perp}\\
&\quad+\partial_y\Big[O(1)\ l_1(u_-)\left(\boldsymbol{I}_0[\mathbb{V}_{\pi^\perp}]+\boldsymbol{I}_1[\Theta]+\boldsymbol{I}_2[\bar\gamma]+\boldsymbol{I}_4[\mathbb{V}]+\boldsymbol{N}[\mathbb{V}]-\bar{\boldsymbol{N}}[\mathbb{V}_\pi]\right)\mathbb{V}^1_\pi\\
&\quad+O(1)\mathbb{V}^1_{\pi^\perp}\partial_y\mathbb{V}^1_\pi\Big]+O(1)l_1(u_-)(\boldsymbol{I}_0[\mathbb{V}_{\pi^\perp}]+\boldsymbol{I}_1[\Theta]+\boldsymbol{I}_2[\bar\gamma]+\boldsymbol{I}_4[\mathbb{V}]+\boldsymbol{N}[\mathbb{V}]\\
&\quad-\bar{\boldsymbol{N}}[\mathbb{V}_\pi])\partial_y\mathbb{V}^1_\pi+O(1)\epsilon^2e^{-\lambda_2^-|y|(1+O(\epsilon))}l_1(u_-)(\boldsymbol{I}_0[\mathbb{V}_{\pi^\perp}]+\boldsymbol{I}_1[\Theta]+\boldsymbol{I}_2[\bar\gamma]\\
&\quad+\boldsymbol{I}_4[\mathbb{V}]+\boldsymbol{N}[\mathbb{V}]-\bar{\boldsymbol{N}}[\mathbb{V}_\pi])\mathbb{V}^1_\pi+2\big[l_2(u_-)f''(\mathbb{A})(r_1(u_-),r_1(u_-))\mathbb{V}^1_{\pi^\perp}\partial_y^2\mathbb{V}^1_\pi\big].
\end{aligned}$$

By (6.44) and (6.45),

$$\begin{aligned}
&\bar{\mathbb{I}}^2_{6;1}[\mathbb{V},\mathbb{V}_\pi]\\
&\quad= O(1)\int_0^t\int_R\theta^1(x-y,t-\sigma;\lambda_2^-,4)\Big[\epsilon^2e^{-\lambda_2^-|y|(1+O(\epsilon))}\mathbb{V}^1_\pi\mathbb{V}^1_{\pi^\perp}\\
&\qquad+\mathbb{V}^1_{\pi^\perp}\partial_y\mathbb{V}^1_\pi+l_1(u_-)\cdot\Big\{(\boldsymbol{I}_3[\bar\gamma]+(\boldsymbol{I}_0[\mathbb{V}_\pi]+\bar{\boldsymbol{N}}[\mathbb{V}_\pi])_y)\ \mathbb{V}^1_{\pi^\perp}\\
&\qquad+\left(\boldsymbol{I}_0[\mathbb{V}_{\pi^\perp}]+\boldsymbol{I}_1[\Theta]+\boldsymbol{I}_2[\bar\gamma]+\boldsymbol{I}_4[\mathbb{V}]+\boldsymbol{N}[\mathbb{V}]-\bar{\boldsymbol{N}}[\mathbb{V}_\pi]\right)\partial_y\mathbb{V}^1_\pi\\
&\qquad+\epsilon^2\ e^{-\lambda_2^-|y|(1+O(\epsilon))}\ l_1(u_-)(\boldsymbol{I}_0[\mathbb{V}_{\pi^\perp}]+\boldsymbol{I}_1[\Theta]+\boldsymbol{I}_2[\bar\gamma]+\boldsymbol{I}_4[\mathbb{V}]\\
&\qquad+\boldsymbol{N}[\mathbb{V}]-\bar{\boldsymbol{N}}[\mathbb{V}_\pi])\mathbb{V}^1_\pi+f''(\mathbb{A})(r_1(u_-),r_1(u_-))\mathbb{V}^1_{\pi^\perp}\partial_y^2\mathbb{V}^1_\pi\Big\}\Big](y,\sigma)\ dyd\sigma\\
&\qquad+O(1)\int_0^t\int_R\partial_y\theta^1(x-y,t-\sigma;\lambda_2^-,4)l_1(u_-)\Big[\boldsymbol{I}_0[\mathbb{V}_{\pi^\perp}]+\boldsymbol{I}_1[\Theta]\\
&\qquad+\boldsymbol{I}_2[\bar\gamma]+\boldsymbol{I}_4[\mathbb{V}]+\boldsymbol{N}[\mathbb{V}]-\bar{\boldsymbol{N}}[\mathbb{V}_\pi]\Big]\mathbb{V}^1_\pi(y,\sigma)\ dy\ d\sigma. \qquad (6.46)
\end{aligned}$$

Finally, by the technical lemmas in Section 5 we can have that

$$\bar{\mathbb{I}}^2_{6;1}[\mathbb{V},\mathbb{V}_\pi](x,t)\ \ll\ \delta\ \epsilon^{J/2}\ \rho^1(t)\ e^{-\lambda_2^-|x|(1+O(\epsilon))/2}$$

$$+O(1)\begin{cases}\frac{\delta^2|\log\epsilon|^2}{\epsilon^4}\psi^{3/2}(x,t;\lambda_2^-) & \text{for } x\in[\lambda_1^- t,-J_0],\\ \theta^3(x,t;\lambda_1^-,D) & \text{for } x\le\lambda_1^- t.\end{cases} \tag{6.47}$$

The rest of the proof for (4.29c) is similar to the proof for (4.29a). We omit it; and conclude there exists $\mathscr{C}=O(1)$ to satisfy (4.29a).
The estimate (4.29d) is a consequence of (4.28) and (4.29) in the region $x\ge -J_0$. □

Proof of Theorem B.

Proof. In this theorem, we need to show that there exists a constant $C=O(1)$ to satisfy (4.28a) and (4.28b).

By substituting (4.29a), (4.29b), and (4.29c) into (4.10). Then, it results in the existence of $C_1=O(1)$ to satisfy (4.28b):

$$|\bar{\gamma}'(t)|\le C_1\left(\frac{\delta^2|\log\epsilon|^2}{\epsilon^5}\,t^{-1}\left[1+\epsilon\sqrt{t}\right]^{-1}+\delta\,\epsilon^{\frac{J}{2}}\rho^1(t)\right), \tag{6.48}$$

Now, we need to consider the variable $\mathbb{W}(x,t)$. The relationship between $\mathbb{V}(x,t)$ and $\mathbb{W}(x,t)$ is due to (4.11):

$$\mathbb{W}_x(x,t)\equiv\mathbb{V}(x,t).$$

This yields that

$$\begin{aligned}\mathbb{W}_x^i(x,t)&\equiv\mathbb{V}^i(x,t)+O(1)\,\epsilon^2|\mathbb{W}|(x,t)\,e^{-\lambda_2|x|(1+O(1)\epsilon)},\\ \mathbb{W}_x^i(-J_0,t)&\equiv\mathbb{V}^i(-J_0,t)+O(1)\,\epsilon^{2+J}|\mathbb{W}|(-J_0,t).\end{aligned} \tag{6.49}$$

Since we want to have a sharper estimate for $\mathbb{V}^2(x,t)$ only, by the relationship in (4.12) the structure of $\mathbb{V}^2$ is dominated by $\mathbb{W}_x^2$. We are interested in the variables $\mathbb{W}_x^2(x,t)$ and $\mathbb{W}^2(x,t)$ only.

From (4.15), we have the data for $\mathbb{W}^2(-J_0,t)$:

$$\mathbb{W}^2(-J_0,t)=O(1)\,\epsilon^{J+1}\,|\mathbb{W}|(-J_0,t). \tag{6.50}$$

From (4.14), we have the equation for $\mathbb{W}^2(x,t)$:

$$\partial_t\mathbb{W}^2+\lambda_2(\mathbb{A})\partial_x\mathbb{W}^2-\partial_x^2\mathbb{W}^2=\mathscr{S}_1(x,t)+\mathscr{S}_2(x,t), \tag{6.51}$$

where the function $\mathscr{S}_1(x,t)$ and $\mathscr{S}_2(x,t)$ are

$$\mathscr{S}_1(x,t) \equiv l_2(u_-)\cdot \boldsymbol{I}_2[\bar\gamma]. \tag{6.52}$$

$$\begin{aligned}|\mathscr{S}_2(x,t)| \;\le\; & O(1)\,\big[\epsilon^2\,|\mathbb{W}|(x,t)\,(|H'(t)|+|\bar\gamma'(t)|)+\epsilon^2\,|\mathbb{V}|(x,t)\\ &+\epsilon^3\,|\mathbb{W}|(x,t)\big]\,e^{-\lambda_2^-|x|(1+O(\epsilon))}\big[|(\boldsymbol{N}+\boldsymbol{I}_0)[\mathbb{V}]|+|\boldsymbol{I}_1[\Theta]|\big](x,t)\\ &+|\bar\gamma'(t)\,l_2(u_-)\cdot(u_+-\phi(x-H(t)-\bar\gamma(t)))|\end{aligned} \tag{6.53}$$

Under the condition (4.28a) with $C\le|\log\epsilon|$ and the conditions (4.29b) and (4.29d), for $x\ge -J_0$ the functions $\mathscr{S}_1(x,t)$ and $\mathscr{S}_2(x,t)$ satisfy

$$\begin{aligned}&|\mathscr{S}_2(x,t)|\\ &\le O(1)\,\big[\delta^2\,|\log\epsilon|\rho^2(t)+\delta\epsilon^2|\log\epsilon|^2\rho^1(t)+\delta^2\,|\log\epsilon|^4\,\rho^2(t)\big]\,e^{-\lambda_2^-|x|(1+O(\epsilon))}\\ &\quad+O(1)\cdot\begin{cases}\dfrac{\delta^4|\log\epsilon|^8}{\epsilon^{10}}t^{-2}(1+\epsilon\sqrt{t})^{-2}+\delta^2\,|\log\epsilon|^2\,\rho^2(t)+\delta^2\rho^2(t)\\ +\dfrac{\delta^2|\log\epsilon|^2}{\epsilon^4}t^{-1}(1+\epsilon\sqrt{t})^{-1}\ \text{for } x\in[-J_0,0],\\ \Big[\dfrac{\delta^4|\log\epsilon|^8}{\epsilon^{10}}t^{-2}(1+\epsilon\sqrt{t})^{-2}+\delta^2\,|\log\epsilon|^2\,\rho^2(t)+\delta^2\rho^2(t)\\ +\dfrac{\delta^2|\log\epsilon|^2}{\epsilon^4}t^{-1}(1+\epsilon\sqrt{t})^{-1}\Big]\,e^{-\lambda_2^-|x|(1+O(\epsilon))}\ \text{for } x\ge 0.\end{cases}\end{aligned} \tag{6.54}$$

We consider that

$$\int_0^t\int_{-J_0}^\infty G_-(x,t;y,\sigma)\big[\partial_\sigma\mathbb{W}^2+\lambda_2(\mathbb{A})\partial_y\mathbb{W}^2-\partial_y^2\mathbb{W}^2-\mathscr{S}_1-\mathscr{S}_2\big](y,\sigma)\,dyd\sigma=0.$$

Here, $G_-(x,t;y,\sigma)$ is the green function of (LB) given in (5.34). Then, through integration by parts and (5.35):

$$\begin{aligned}&\mathbb{W}^2(x,t)-\int_{-J_0}^\infty G_-(x,t;y,\sigma)\mathbb{W}^2(y,0)dy+\int_0^t\int_{-J_0}^\infty T(x,t;y,\sigma)\mathbb{W}^2(y,\sigma)dyd\sigma\\ &\quad+\int_0^t\big[G_-(x,t;-J_0,\sigma)\,\mathbb{W}^2_y(-J_0,\sigma)-\partial_yG_-(x,t;-J_0,\sigma)\,\mathbb{W}^2(-J_0,\sigma)\big]\,d\sigma\\ &\quad-\int_0^t\int_{-J_0}^\infty G_-(x,t;y,\sigma)\,(\mathscr{S}_1+\mathscr{S}_2)(y,\sigma)dyd\sigma=0,\end{aligned} \tag{6.55}$$

where $T(x,t;y,\sigma)$ is the truncation error of $G_-(x,t;y,\sigma)$ defined in (5.35).

By (5.36), we have that

$$\begin{aligned}
&\int_0^t \int_{-J_0}^{\infty} T_x(x,t;y,\sigma)\mathbb{W}^2(y,\sigma)\, dyd\sigma \\
&= \int_0^t \int_{-J_0}^{\infty} O(1)\epsilon^3\, e^{-\lambda_2^-|y|(1+O(\epsilon))}\partial_x G_-(x,t;y,\sigma)\, \mathbb{W}^2(y,\sigma)\, dyd\sigma \\
&\quad + \int_0^t \int_0^{\infty} O(1)\epsilon^2\, e^{-\lambda_2^-|y|(1+O(\epsilon))}\partial_y G_-(x,t;y,\sigma)\, \mathbb{W}_y^2(y,\sigma)\, dyd\sigma \\
&\quad + \int_0^t \int_{-J_0}^{0} O(1)\epsilon^2\partial_y G_-(x,t;y,\sigma)\, \mathbb{W}_y^2(y,\sigma)\, dyd\sigma. \qquad (6.56)
\end{aligned}$$

In the RHS of (6.55), the variable $\mathbb{W}^2(y,\sigma)$ satisfies (4.29d); the variables $\mathbb{W}^2(-J_0,t)$ and $\mathbb{W}_y^2(-J_0,t)$ satisfy (6.50) and (6.49) with the function $|\mathbb{W}|(-J_0,t)$ satisfying (4.29d). So, combined with (6.56) we have that

$$|\mathbb{W}_x^2|(x,t) \le \int_{-J_0}^{\infty} |\partial_x G_-(x,t;y,\sigma)\, \mathbb{W}^2(y,0)|\, dy + O(1)\big[\mathbb{T}+\mathbb{B}+\mathbb{S}_1+\mathbb{S}_2\big](x,t), \qquad (6.57)$$

where

$$\begin{aligned}
\mathbb{T}(x,t) \;&\equiv\; \int_0^t \int_{-J_0}^{\infty} \epsilon^2|\log\epsilon||\partial_x G_-(x,t;y,\sigma)|\delta\rho^1(\sigma)e^{-\lambda_2^-|y|(1+O(\epsilon))}\, dyd\sigma \\
&\quad + \int_0^t \int_{-J_0}^{\infty} \epsilon^2\partial_y G_-(x,t;y,\sigma)|\log\epsilon|\delta\rho^1(\sigma)\, e^{-\lambda_2^-|y|(1+O(\epsilon))/2}dyd\sigma, \\
\mathbb{B}(x,t) \;&\equiv\; \int_0^t |\partial_x G_-(x,t;-J_0,\sigma)|\Bigg[\frac{\delta^2|\log\epsilon|^2}{\epsilon^4}\big[\sigma\,(1+\epsilon\sqrt{\sigma})\big]^{-1} \\
&\quad +\epsilon^{J+1}|\log\epsilon|\delta\rho^1(\sigma)\Bigg]\, d\sigma \\
&\quad + \int_0^t |\partial_x\partial_y G_-(x,t;-J_0,\sigma)\, \epsilon^J|\log\epsilon|\delta\rho^1(\sigma)\, d\sigma, \qquad (6.58) \\
\mathbb{S}_1(x,t) \;&\equiv\; \int_0^t \int_{-J_0}^{\infty} |\partial_x G_-(x,t;y,\sigma)\, \mathscr{S}_1(y,\sigma)|\, dyd\sigma, \\
\mathbb{S}_2(x,t) \;&\equiv\; \int_0^t \int_{-J_0}^{\infty} |\partial_x G_-(x,t;y,\sigma)\, \mathscr{S}_2(y,\sigma)|\, dyd\sigma.
\end{aligned}$$

By Lemma 5.17,

$$|\mathbb{T}(x,t)| \le O(1)\, \epsilon|\log\epsilon|\, \delta e^{-\lambda_2^-|x|/2}\, \rho^1(t).$$

By (5.38)

$$\int_{-J_0}^{\infty} |\partial_x G_-(x,t;y,0)|\mathbb{W}^2(y,0)dy \le \delta\rho^1(t)\ e^{-\lambda_2^-|x|/2}.$$

By Lemma 5.16,

$$\mathbb{B}(x,t) \le O(1)\cdot$$
$$\begin{cases} \left(\frac{\delta^2\ |\log\epsilon|^4}{\epsilon^4} + \delta\ \epsilon^{J-2}\ |\log\epsilon|^3\right)\ \left[(t+1)(1+\epsilon\sqrt{t+1})\right]^{-1} \text{ for } x\in[-J_0+1,0] \\ e^{-\lambda_2^-|x|(1+O(\epsilon))}\left(\frac{\delta^2|\log\epsilon|^4}{\epsilon^4}+\delta\ \epsilon^{J-2}|\log\epsilon|^3\right)\left[(t+1)(1+\epsilon\sqrt{t+1})\right]^{-1} \text{ for } x\ge 0. \end{cases}$$

By Lemma 5.17,

$$\mathbb{S}_1(x,t) \le K_0\ \delta\ \rho^1(t)\ e^{-\frac{\lambda_2^-|x|}{2}}.$$

By Lemma 5.17 and Lemma 5.15, we have

$$\begin{aligned} |\mathbb{S}_2(x,t)| \ &\le\ O(1)\ \epsilon|\log\epsilon|\delta\rho^1(t)\ e^{-\lambda_2|x|/2} \\ &+O(1)\ \begin{cases} 0 & \text{for } x\ge 0, \\ \dfrac{\delta^2|\log\epsilon|^3}{\epsilon^5(t+1)(1+\epsilon\sqrt{t+1})} & \text{for } x\in[-J_0,0]. \end{cases} \end{aligned}$$

The above five estimates yields estimates for $\mathbb{W}_x^2$. By the relation (4.12) with $\|\mathbb{W}(x,t)\|$ from (4.29d) and with $\|\mathbb{V}(x,t)\|$ from (4.29) and (4.28), we conclude the estimate of $\mathbb{V}^2(x,t)$ for Theorem B. The estimates $\mathbb{W}^2(x,t)$ is a by-product. We omit it. □

References

1. J. Goodman, Nonlinear asymptotic stability of viscous shock profiles for conservation laws, *Arch. Rat. Mech. Anal.*, **95** (1986), 325-344.

2. J. Goodman and Z. Xin, Viscous limits for piecewise smooth solutions to systems of conservation laws, *Arch. Rational Mech. Anal.*, **121** (1992), 235-265.

3. D. Hoff and T.-P. Liu, The inviscid limit for the Navier-Stokes equations of compressible isentropic flow with shock data, *Indiana Univ. Math. J.*, **38** (1989), 861-915.

4. S, Kawashima and A. Matsumura, Asymptotic stability of travelling wave solutions of systems of one-dimensional gas motion, *Comm. Math. Phys.*, **101** (1985), 97-127.

5. G. Kreiss and H.-O. Kreiss, Stability of systems of viscous conservation laws, *Commun. Pure and Appl. Math.* vol. 1998.

6. T.-P. Liu, Nonlinear stability of shock waves for viscous conservation laws, *Amer. Math. Soc. Memoir*, No 328. Amer. Math. Soc.: Providence 1985.

7. T.-P. Liu, Pointwise convergence to shock waves for viscous conservation laws, *Comm. Pure and Appl. Math.* vol. L. No. 11, (1997), 1113-1182.

8. T.-P. Liu and Z. Xin, Pointwise decay to contact discontinuities for systems of viscous conservation laws, *Asian J. Math.*, **1**(1997), No 1, 34-84.

9. T.-P. Liu and S.-H. Yu, Propagation of a Stationary Shock Layer in the Presence of a Boundary, *Arch. Rat. Mech. Anal.*, **139** (1997), 57-82.

10. T.-P. Liu and S.-H. Yu, Continuum Shock Profiles for Discrete Conservation Laws, I. Construction, *Commun. Pure and Appl. Math.*, **52**(1999), No. 1, 85-127.

11. T.-P. Liu and S.-H. Yu, Continuum shock profiles for discrete conservation laws, II: stability, *Commun. Pure and Appl. Math.*, **52**(1999), No. 9, 1047-1073.

12. T.-P. Liu and S.-H. Yu, Nonlinear Stability of Weak Detonation Waves for a Combustion Model, *Comm. Math. Phys.*, **204**(1999), No. 3, 551-586.

13. T.-P. Liu and S.-H. Yu, Riemann problem for viscous conservation laws, Under preparation.

14. T.-P. Liu and D. Y. Zeng, Time-asymptotic behavior of wave propagation around a viscous shock profile, *Comm. Math. Phys.*, **290**(2009), 23-82.

15. T.-P. Liu and K. Zumbrun, Nonlinear Stability of an undercompressive shock of the complex Burgers' equation, *Commun. Math. Phys.*, **86** (1993), No. 1, 163-163.

16. T.-P. Liu and K. Zumbrun, On nonlinear stability of general undercompressive viscous shock waves, *Comm. Math. Phys.*, **174** (1995), No. 2, 319-345.

17. S. Nishibata and S.-H. Yu, The asymptotic behavior of the hyperbolic conservation laws with relaxation on the quarter plane, *SIAM J. Appl. Math.* (1997).

18. A. Szepessy and Z. Xin, *Nonlinear stability of viscous shock waves, Arch. Rat. Mech. Anal.*, **122** (1993), 53-103.

19. G. B. Whitham, *The Linear and Nonlinear Waves*, Wiley, New York, 1974.

20. S.-H. Yu, Zero dissipation limit to solutions with shocks for systems of hyperbolic conservation laws, *Arch. Rat. Mech. Anal.*, **146** (1999), 275-370.

Institute of Mathematics, Academia Sinica, Taipei and Department of Mathematics, Stanford University.
E-mail: liu@math.stanford.edu

Department of Mathematics, National University of Singapore.
E-mail: matysh@nus.edu.sg