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EXP-FUNCTION METHOD FOR SOLVING HIGHER-ORDER BOUNDARY VALUE PROBLEMS

BY

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Abstract

In this paper, we apply a relatively new technique which is called the exp-function method for solving higher order boundary value problems which arise in various physical phenomena of applied and engineering sciences. The proposed method proves to be very accurate and efficient for solving such problems.

1. Introduction

This paper is devoted to the study of higher-order boundary value problem which are known to arise in the study of astrophysics, hydrodynamic and hydro magnetic stability, fluid dynamics, astronomy, beam and long wave theory , engineering and applied physics, see [2]-[6], [14]-[19], [22], [24]-[26]. If a uniform magnetic field is applied across the fluid in the same direction as that of gravity, then the instability may sets in as over stability which can be modeled by a twelfth or eighth-order boundary value problem; whereas the instability which occur as ordinary convection can be modeled by a tenth-order boundary value problem. We would like to point out that the eighth-order boundary value problems arise in the torsinal vibration of uniform beam, see [2]-[6], [14]-[19], [22], [24]-[26] and the references therein. The boundary value problems of higher order have been investigated due

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to their mathematical importance and the potential for applications in diversified applied sciences. Several techniques including the finite-difference, polynomial, non-polynomial spline and decomposition have been developed for solving such type of problems, see [1]-[29]. Most of these techniques have their inbuilt deficiencies, like divergence of the results at the points adjacent to the boundary and calculation of the so-called Adomian's polynomials. To rectify these difficulties, He and Wu [9] developed another method, which is called the exp-function method, to find solitary, periodic and compacton like solutions of nonlinear differential equations, see [1], [7]-[10], [12], [13], [20], [21], [23], [27]-[29]. The exp-function method has been applied for solving KdV, high-dimensional nonlinear evolution equation, Burgers equations, combined KdV, mKdV and various other physical problems, see [1], [7]-[10], [12], [13], [20], [21], [23], [27]-[29]. The basic motivation of this paper is to extend the applications of this powerful technique for solving higher-order nonlinear boundary value problems. Several examples are given to verify the efficiency and accuracy of the proposed algorithm. It is worth mentioning that we obtained exact solutions for all the higher-order nonlinear boundary value problems. We have also obtained the soliton solutions for a reaction diffusion problem. The numerical results are very encouraging. We have demonstrated that the exp-function method can be viewed as an alternative method to variational iteration, homotopy perturbation and decomposition methods for the implementation and efficiency.

2. Exp-function Method

To convey an idea of the exp-function method, we consider the general nth-order boundary value problem of the type

$$y^{(n)}(x) = f(x, y), \tag{1}$$

with boundary conditions

$$y^{(j-2)}(a_i) = A_i, \quad i = 1, 2, \ldots, \frac{n}{2}, \qquad y^{(j-2)}(b_i) = B_i, \quad j = 2, 4, \ldots, n$$

The exp-function method, developed by He and Wu [9], is based on the assumption that solution of nonlinear ordinary differential equations an be expressed in the following form

$$y(x)=\frac{\sum_{n=-c}^{d}a_n\exp[nx]}{\sum_{m=-p}^{q}b_m\exp[mx]}, \tag{2}$$

where p, q, c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. Equation (2) can be re-written as the following alternate and useful form

$$y(x)=\frac{a_c\exp[cx]+\cdots+a_{-d}\exp[-dx]}{b_p\exp[px]+\cdots+b_{-q}\exp[-qx]}. \tag{3}$$

To determine the values of c and p, we balance the linear terms of higher order of the equation with the highest order nonlinear terms.

3. Numerical Applications

In this section, we apply the exp-function method, as developed in Section 3, for solving the higher-order boundary value problems. Numerical results are very encouraging showing the complete reliability and efficiency of the proposed method.

Example 3.1.([16, 17, 26]) Consider the nonlinear boundary value problem of eighth-order

$$y^{(viii)}(x)=e^{-x}y^2(x),\quad 0<x<1, \tag{4}$$

with boundary conditions

$$y(0)=y''(0)=y^{(iv)}(0)=y^{(vi)}(0)=1,\quad y(1)=y''(1)=y^{(iv)}(1)=y^{(vi)}(1)=e.$$

The exact solution for this problem is

$$y(x)=e^x. \tag{5}$$

We suppose that the solution of the boundary value problem can be expressed in the following form

$$y(x)=\frac{a_c\exp[cx]+\cdots+a_{-d}\exp[-dx]}{b_p\exp[px]+\cdots+b_{-q}\exp[-qx]}.$$

The appropriate simplification would yield

$$y^{(viii)} = \frac{c_1 \exp[(255p + c)x] + \cdots}{c_2 \exp[256px] + \cdots}, \tag{6}$$

and

$$y^2 = \frac{c_3 \exp[2cx] + \cdots}{c_4 \exp[2px] + \cdots} = \frac{c_3 \exp[(254p + 2c)x] + \cdots}{c_4 \exp[256px] + \cdots}, \tag{7}$$

where c_i are determined coefficients only for simplicity. Balancing the highest order of exp-function in (6) and (7), we have

$$255p + c = 254p + 2c, \tag{8}$$

which in turn gives

$$p = c. \tag{9}$$

The values of d and q can also be determined by balancing the linear term of the lowest order

$$y^{(viii)} = \frac{\cdots + d_1 \exp[(-255q - d)x]}{\cdots + d_2 \exp[-256qx]}, \tag{10}$$

and

$$y^2 = \frac{\cdots + d_3 \exp[-2dx]}{\cdots + d_x \exp[-2qx]} = \frac{\cdots + d_3 \exp[(-254q - 2d)x]}{\cdots + d_4 \exp[-256qx]}, \tag{11}$$

where d_i are determined coefficients only for simplicity. Now, balancing the lowest order of exp-function in (10) and (11), we have

$$-255q - d = -254q - 2d, \tag{12}$$

which in turn gives

$$q = d. \tag{13}$$

Case 3.1.1. $p = c = 1$, and $q = d = 1$. Equation (3) reduces to

$$y(x) = \frac{a_1 \exp[x] + a_0 + a_{-1} \exp[-x]}{b_1 \exp[x] + b_0 + b_{-1} \exp[-x]}. \tag{14}$$

Substituting (14) into (4), we have

$$\frac{1}{A}\Big[c_8\exp(8x)+c_7\exp(7x)+c_6\exp(6x)+c_5\exp(5x)+c_4\exp(4x)$$
$$+c_3\exp(3x)+c_2\exp(2x)+c_1\exp(x)+c_0+c_{-1}\exp(-x)$$
$$+c_{-2}\exp(-2x)+c_{-3}\exp(-3x)+c_{-4}\exp(-4x)+c_{-5}\exp(-5x)$$
$$+c_{-6}\exp(-6x)+c_{-7}\exp(-7x)+c_{-8}\exp(-8x)+c_{-9}\exp(-9x)$$
$$+c_{-10}\exp(-10x)\Big]=0, \tag{15}$$

where $A=(b_1\exp(x)+b_0+b_{-1}\exp(-x))^9$, C_i $(i=-10,-9,-8,\dots,6,7,8)$ are constants obtained by Maple 7.

Equating the coefficients of $\exp(nx)$ to be zero, we obtain

$$\Big\{c_{-10}=0,\ c_{-9}=0,\ c_{-8}=0,\ c_{-7}=0,\ c_{-6}=0,\ c_{-5}=0,\ c_{-4}=0,$$
$$c_{-3}=0,\ c_{-2}=0,\ c_{-1}=0,\ c_0=0,\ c_1=0,\ c_2=0,\ c_3=0,$$
$$c_4=0,\ c_5=0,\ c_6=0,\ c_7=0,\ c_8=0.\Big\} \tag{16}$$

Solution of (15) will yield

$$\Big\{a_{-1}=0,\quad b_1=0,\quad a_0=0,\quad b_0=a_1,\quad b_{-1}=0,\quad a_1=a_1.\Big\} \tag{17}$$

Consequently, the exact solution is obtained $y(x)=e^x$.

Case 3.1.2. $p=c=2$, and $q=d=1$. Equation (3) reduces to

$$y(x)=\frac{a_2\exp[2x]+a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_2\exp[2x]+b_1\exp[x]+b_0+b_{-1}\exp[-x]}. \tag{18}$$

Proceeding as before, we obtain

$$\Big\{a_{-1}=0,\quad a_0=0,\quad a_1=b_0,\quad a_2=a_2,$$
$$b_{-1}=0,\quad b_0=b_0,\quad b_1=a_2,\quad b_2=0.\Big\} \tag{19}$$

$$y(x)=\frac{a_2e^{(2x)}+b_0e^{(x)}}{a_2e^{(x)}+b_0}=\frac{e^{(x)}(a_2e^{(x)}+b_0)}{(a_2e^{(x)}+b_0)},\ \text{where } (a_2e^{(x)}+b_0)\neq 0.$$

Consequently, the exact solution is obtained as $y(x)=e^x$.

Example 3.2.([16, 17, 25]) Consider the following nonlinear boundary value problem of tenth-order

$$y^{(x)}(x) = e^{-x}y^2(x), \quad 0 < x < 1 \tag{20}$$

with boundary conditions

$$\begin{aligned} y(0) &= 1, \qquad y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = 1, \\ y(1) &= e, \qquad y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = e. \end{aligned}$$

The exact solution for this problem is

$$y(x) = e^x. \tag{21}$$

We suppose that the solution of the boundary value problem can be expressed in the following form

$$y(x) = \frac{a_c \exp[cx] + \cdots + a_{-d} \exp[-dx]}{b_p \exp[px] + \cdots + b_{-q} \exp[-qx]}.$$

Proceeding as before, we obtain $p = c$, $q = d$.

Case 3.2.1. $p = c = 1$, and $q = d = 1$. Equation (3) reduces to

$$y(x) = \frac{a_1 \exp[x] + a_0 + a_{-1} \exp[-x]}{b_1 \exp[x] + b_0 + b_{-1} \exp[-x]}. \tag{22}$$

Substituting (22) into (20), we have

$$\begin{aligned} \frac{1}{A}\Big[&c_{10}\exp(10x)+c_9\exp(9x)+c_8\exp(8x)+c_7\exp(7x)+c_6\exp(6x)+c_5\exp(5x) \\ &+c_4\exp(4x)+c_3\exp(3x)+c_2\exp(2x)+c_1\exp(x)+c_0+c_{-1}\exp(-x) \\ &+c_{-2}\exp(-2x)+c_{-3}\exp(-3x)+c_{-4}\exp(-4x)+c_{-5}\exp(-5x) \\ &+c_{-6}\exp(-6x)+c_{-7}\exp(-7x)+c_{-8}\exp(-8x)+c_{-9}\exp(-9x) \\ &+c_{-10}\exp(-10x)+c_{-11}\exp(-11x)+c_{-12}\exp(-12x)\Big] = 0, \end{aligned} \tag{23}$$

where $A = (b_1 \exp(x) + b_0 + b_{-1} \exp(-x))^{11}$, C_i $(i = -12, -11, -10, \ldots, 8, 9, 10)$ are constants obtained by Maple 7. Equating the coefficients of $\exp(nx)$

to be zero, we obtain

$$\Big\{c_{-12}=0,\ c_{-11}=0,\ c_{-10}=0,\ c_{-9}=0,\ c_{-8}=0,\ c_{-7}=0,\ c_{-6}=0,\\ c_{-5}=0,\ c_{-4}=0, c_{-3}=0,\ c_{-2}=0,\ c_{-1}=0,\ c_{0}=0,\ c_{1}=0,\ c_{2}=0,\\ c_{3}=0,\ c_{4}=0,\ c_{5}=0,\ c_{6}=0,\ c_{7}=0,\ c_{8}=0,\ c_{9}=0,\ c_{10}=0.\Big\} \quad (24)$$

Solution of (23) will yield

$$\{a_{-1}=0, \quad b_1=0, \quad a_0=0, \quad b_0=b_0, \quad b_{-1}=0, \quad a_1=b_0.\} \quad (25)$$

Consequently, the exact solution is obtained as $y(x)=e^x$.

Case 3.2.2. For $p=c=2$, and $q=d=1$, equation (3) reduces to

$$y(x)=\frac{a_2\exp[2x]+a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_2\exp[2x]+b_1\exp[x]+b_0+b_{-1}\exp[-x]}. \quad (26)$$

Proceeding as before, we obtain

$$\Big\{a_{-2}=0, \quad a_{-1}=a_{-1}, \quad a_0=0, \quad a_1=0, \quad a_2=0,\\ b_{-2}=0, \quad b_0=0, \quad b_0=a_{-1}, \quad b_2=0.\Big\}$$

Consequently, the exact solution is obtained as $y(x)=e^x$.

Example 3.3.([16, 17, 18, 26]) Consider the following nonlinear boundary value problem of twelfth-order

$$y^{(xii)}=2e^x y^2(x)+y'''(x), \quad 0<x<1 \quad (26)$$

with boundary conditions

$$y(0)=y''(0)=y^{(iv)}(0)=y^{(vi)}(0)=y^{(viii)}(0)=y^{(x)}(0)=1,\\ y(1)=y''(1)=y^{(iv)}(1)=y^{(vi)}(1)=y^{(viii)}(1)=y^{(x)}(1)=e^{-1}.$$

The exact solution for this problem is

$$y(x)=e^{-x}. \quad (27)$$

We assume that the solution of the above boundary value problem can be expressed in the form

$$y(x)=\frac{a_c\exp[cx]+\cdots+a_{-d}\exp[-dx]}{b_p\exp[px]+\cdots+b_{-q}\exp[-qx]}. \tag{28}$$

Proceeding as before, we obtain $p=c$, $q=d$.

Case 3.3.1. For $p=c=1$, and $q=d=1$, equation (3) reduces to

$$y(x)=\frac{a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_1\exp[x]+b_0+b_{-1}\exp[-x]}. \tag{29}$$

Substituting (29) into (26), we have

$$\begin{aligned}
&\frac{1}{A}\Big[c_{14}\exp(14x)+c_{13}\exp(13x)+c_{12}\exp(12x)+c_{11}\exp(11x)+c_{10}\exp(10x)\\
&\quad+c_9\exp(9x)+c_8\exp(8x)+c_7\exp(7x)+c_6\exp(6x)+c_5\exp(5x)+c_4\exp(4x)\\
&\quad+c_3\exp(3x)+c_2\exp(2x)+c_1\exp(x)+c_0+c_{-1}\exp(-x)+c_{-2}\exp(-2x)\\
&\quad+c_{-3}\exp(-3x)+c_{-4}\exp(-4x)+c_{-5}\exp(-5x)+c_{-6}\exp(-6x)\\
&\quad+c_{-7}\exp(-7x)+c_{-8}\exp(-8x)+c_{-9}\exp(-9x)+c_{-10}\exp(-10x)\\
&\quad+c_{-11}\exp(-11x)+c_{-12}\exp(-12x)\Big]=0,
\end{aligned} \tag{30}$$

where $A=(b_1\exp(x)+b_0+b_{-1}\exp(-x))^{13}$, C_i $(i=-12,-11,-10,\ldots,12,$ $13,\ 14)$ are constants obtained by Maple 7. Equating the coefficients of $\exp(nx)$ to be zero, we obtain

$$\begin{aligned}
&\Big\{c_{-12}=0,\ c_{-11}=0,\ c_{-10}=0,\ c_{-9}=0,\ c_{-8}=0,\ c_{-7}=0,\ c_{-6}=0,\\
&\quad c_{-5}=0,\ c_{-4}=0, c_{-3}=0,\ c_{-2}=0,\ c_{-1}=0,\ c_0=0,\ c_1=0,\ c_2=0,\\
&\quad c_3=0,\ c_4=0,\ c_5=0,\ c_6=0,\ c_7=0,\ c_8=0,\ c_9=0,\ c_{10}=0,\ c_{11}=0,\\
&\quad c_{12}=0,\ c_{13}=0,\ c_{14}=0.\Big\}
\end{aligned}$$

Solution of (30) will yield

$$\{a_{-1}=0,\quad b_1=a_0,\quad a_0=a_0,\quad b_0=0,\quad b_{-1}=0,\quad a_1=0.\}$$

Consequently, the exact solution is obtained as $y(x)=e^{-x}$.

Case 3.3.2. For $p = c = 2$, and $q = d = 1$, equation (3) reduces to

$$y(x) = \frac{a_2 \exp[2x] + a_1 \exp[x] + a_0 + a_{-1} \exp[-x]}{b_2 \exp[2x] + b_1 \exp[x] + b_0 + b_{-1} \exp[-x]}. \tag{31}$$

Proceeding as before, we obtain

$$\Big\{a_{-1} = 0, \quad a_0 = a_0, \quad a_1 = 0, \quad a_2 = 0,$$
$$b_{-1} = 0, \quad b_0 = 0, \quad b_1 = a_0, \quad b_2 = 0.\Big\}$$

Consequently, the exact solution is obtained as $y(x) = e^{-x}$.

Example 3.4.([16, 17, 18, 26]) Consider the nonlinear boundary value problem of twelfth-order

$$y^{(xii)} = \frac{1}{2} e^{-x} y^2(x), \quad 0 < x < 1, \tag{32}$$

with boundary conditions

$$y(0) = y''(0) = y^{(iv)}(0) = y^{(vi)}(0) = y^{(viii)}(0) = y^{(x)}(0) = 2,$$
$$y(1) = y''(1) = y^{(iv)}(1) = y^{(vi)}(1) = y^{(viii)}(1) = y^{(x)}(1) = 2e.$$

The exact solution for this problem is

$$y(x) = 2e^x. \tag{33}$$

We assume that the solution of the above boundary value problem can be expressed in the form

$$y(x) = \frac{a_c \exp[cx] + \cdots + a_{-d} \exp[-dx]}{b_p \exp[px] + \cdots + b_{-q} \exp[-qx]}. \tag{34}$$

Proceeding as before, we obtain $p = c$, $q = d$.

Case 3.4.1. For $p = c = 1$, and $q = d = 1$ equation (3) reduces to

$$y(x) = \frac{a_1 \exp[x] + a_0 + a_{-1} \exp[-x]}{b_1 \exp[x] + b_0 + b_{-1} \exp[-x]}. \tag{35}$$

Substituting (35) into (32), we have

$$\begin{aligned}
&\frac{1}{A}\Big[c_{12}\exp(12x)+c_{11}\exp(11x)+c_{10}\exp(10x)+c_9\exp(9x)+c_8\exp(8x)\\
&\quad+c_7\exp(7x)+c_6\exp(6x)+c_5\exp(5x)+c_4\exp(4x)+c_3\exp(3x)+c_2\exp(2x)\\
&\quad+c_1\exp(x)+c_0+c_{-1}\exp(-x)+c_{-2}\exp(-2x)+c_{-3}\exp(-3x)\\
&\quad+c_{-4}\exp(-4x)+c_{-5}\exp(-5x)+c_{-6}\exp(-6x)+c_{-7}\exp(-7x)\\
&\quad+c_{-8}\exp(-8x)+c_{-9}\exp(-9x)+c_{-10}\exp(-10x)+c_{-11}\exp(-11x)\\
&\quad+c_{-12}\exp(-12x)+c_{-13}\exp(-13x)+c_{-14}\exp(-14x)\Big]=0, \qquad (36)
\end{aligned}$$

where $A=(b_1\exp(x)+b_0+b_{-1}\exp(-x))^{13}$, C_i $(i=-14,-13,-12,\dots,10,$ 11, 12) are constants obtained by Maple 7.

Equating the coefficients of $\exp(nx)$ to be zero, we obtain

$$\begin{aligned}
&\Big\{c_{-14}=0,\ c_{-13}=0,\ c_{-12}=0,\ c_{-11}=0,\ c_{-10}=0,\ c_{-9}=0,\ c_{-8}=0,\\
&\quad c_{-7}=0,\ c_{-6}=0,\ c_{-5}=0,\ c_{-4}=0, c_{-3}=0,\ c_{-2}=0,\ c_{-1}=0,\ c_0=0,\\
&\quad c_1=0,\ c_2=0,\ c_3=0, c_4=0,\ c_5=0,\ c_6=0,\ c_7=0,\ c_8=0,\ c_9=0,\\
&\quad c_{10}=0,\ c_{11}=0,\ c_{12}=0.\Big\}
\end{aligned}$$

Solution of (36) will yield

$$\Big\{a_{-1}=0,\quad b_1=0,\quad a_0=0,\quad b_0=b_0,\quad b_{-1}=0,\quad a_1=2b_0.\Big\}$$

Consequently, the exact solution is obtained as $y(x)=2e^x$.

Case 3.4.2. For $p=c=2$, and $q=d=1$, equation (3) reduces to

$$y(x)=\frac{a_2\exp[2x]+a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_2\exp[2x]+b_1\exp[x]+b_0+b_{-1}\exp[-x]}.$$

Proceeding as before, we obtain

$$\begin{aligned}
&\Big\{a_{-1}=0,\quad a_0=2b_{-1},\quad a_1=2b_0,\quad a_2=2b_1,\\
&\quad b_{-1}=b_{-1},\quad b_0=b_0,\quad b_1=b_1,\quad b_2=0.\Big\} \qquad (37)
\end{aligned}$$

Equation (37) leads to the following solution

$$y(x)=\frac{2b_1e^{(2x)}+2b_0e^x+2b_{-1}}{b_1e^x+b_0+b_{-1}e^{-x}},\quad \text{where}\quad b_1e^x+b_0+b_{-1}e^{-x}\neq 0.$$

Consequently, the exact solution is obtained as $y(x)=2e^x$.

Example 3.5.([11]) Consider the following reaction diffusion equation [11]

$$y''(x)+y^n(x)=0,\quad 0<x<L. \tag{38}$$

with boundary conditions $y(0)=y(L)=0$.

Equation (38) can be re-written in the following equivalent form:

$$y'''(x)+ny^{n-1}(x)y'(x)=0, \tag{39}$$

with boundary conditions $y(0)=0$, $y'(0)=\alpha$, $y(L)=0$, where is arbitrary constant.

Case 3.1. For $p=c=1$, and $q=d=1$, equation (39) reduces to

$$y(x)=\frac{a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_1\exp[x]+b_0+b_{-1}\exp[-x]}. \tag{40}$$

Substituting (40) into (39) with , we have

$$\frac{1}{A}\Big[C_3\exp(3x)+C_2\exp(2x)+C_1\exp(x)+C_0+C_{-1}\exp(-x)$$
$$+C_{-2}\exp(-2x)+C_{-3}\exp(-3x)\Big]=0, \tag{41}$$

where $A=(b_1\exp(x)+b_0+b_{-1}\exp(-x))^4$, C_i $(i=-3,-2,\dots,2,3)$ are constants obtained by Maple 7.

Equating the coefficients of $\exp(nx)$ to zero, we obtain

$$\Big\{C_{-3}=0,\quad C_{-2}=0,\quad C_{-1}=0,\quad C_0=0,\quad C_1=0,\quad C_2=0,\quad C_3=0.\Big\}$$

Solving the system, we obtain

$$\left\{a_{-1}=-\frac{1}{2}b_{-1},\ b_1=\frac{1}{4}\frac{b_0^2}{b_{-1}},\ a_0=\frac{5}{2}b_0,\ b_0=b_0,\ b_{-1}=b_{-1},\ a_1=-\frac{1}{8}\frac{b_0^2}{b_{-1}}.\right\}$$

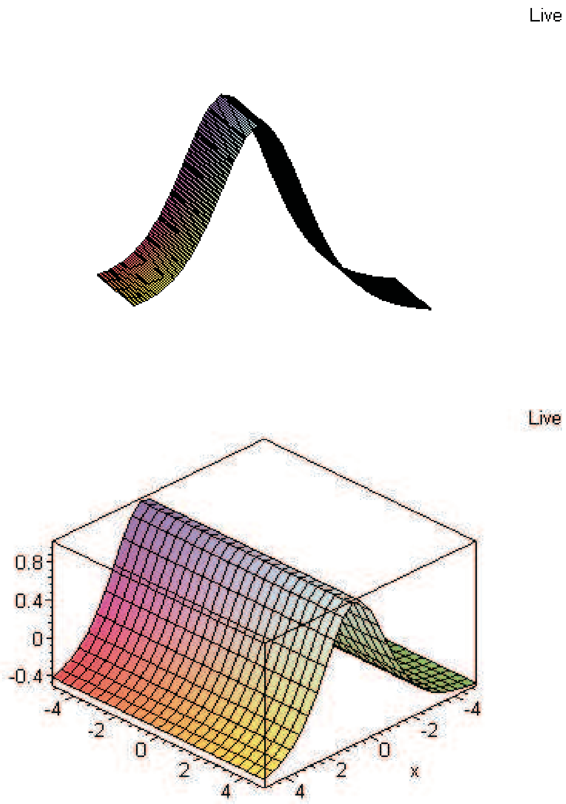

Figure 3.5.1.

The soliton solutions of equation (39) are given as:

$$y(x)=\frac{-\frac{1}{8}\frac{b_0^2}{b_{-1}}e^x+\frac{5}{2}b_0-\frac{1}{2}b_{-1}e^{-x}}{\frac{1}{4}\frac{b_0^2}{b_{-1}}e^x+b_0+b_{-1}e^{-x}}. \tag{42}$$

Figure 3.5.1 depicts the soliton solutions of equation (39).

Case 3.2. For $p=c=2$, and $q=d=2$, the trial function of equation (39) becomes

$$y(x)=\frac{a_2\exp[2x]+a_1\exp[x]+a_0+a_{-1}\exp[-x]}{b_2\exp[2x]+b_1\exp[x]+b_0+b_{-1}\exp[-x]}. \tag{43}$$

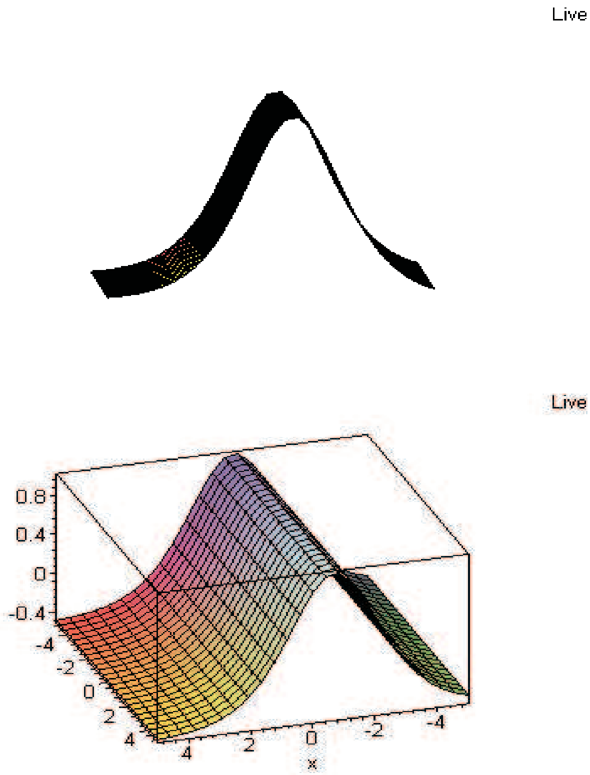

Figure 3.5.2.

Proceeding as before, we obtain

$$\left\{a_{-1}=0,\quad b_1=b_1,\quad a_1=\frac{5}{2}b_1,\quad b_0=\frac{1}{4}\frac{b_1^2}{b_2},\right.$$
$$\left.b_{-1}=0,\quad a_0=-\frac{1}{8}\frac{b_1^2}{b_2},\quad b_2=b_2,\quad a_2=-\frac{1}{2}b_2.\right\} \tag{44}$$

Equation (44) gives the following solution

$$y(x)=\frac{-\frac{1}{2}b_2e^{2x}+\frac{5}{2}b_1e^{x}-\frac{1}{8}\frac{b_1^2}{b_2}e^{x}}{b_2e^{2x}+b_1e^{x}+\frac{1}{4}\frac{b_1^2}{b_2}}.$$

Figure 3.5.2 depicts the soliton solutions of equation (39).

Remark 3.1. It is worth mentioning that exact solutions are obtained for all the higher-order non-linear boundary value problems, where as soliton solutions are calculated in case of reaction-diffusion equation.

4. Conclusion

In this paper, we applied the exp-function method for solving the higher-order nonlinear boundary value problems. The proposed method was applied for finding the exact solution of the higher-order boundary value problems. We have also considered an example for the reaction diffusion equation and obtained soliton solutions. The results are very promising.

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