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ON THE LIMIT SETS OF SPHERICAL CR MANIFOLDS

BY

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Abstract

In this paper we shall study the limit sets of groups acting on the boundary of the visibility manifolds. As an application, we study the developing maps of compact spherical CR manifolds.

Introduction

Let $\mathbb{H}_{\mathbb{C}}^{n+1}$ be the simply connected complex complete hyperbolic space of (complex) dimension $n+1$ endowed with the group of isometries $\mathrm{Iso}(\mathbb{H}_{\mathbb{C}}^{n+1}) = \mathrm{PU}(n+1,1) \rtimes \langle\tau\rangle$ where τ is an anti-holomorphic involution. Then $\mathbb{H}_{\mathbb{C}}^{n+1}$ has a compactification whose boundary is the $(2n+1)$-dimensional sphere S^{2n+1} on which the Lie group $\mathrm{Iso}(\mathbb{H}_{\mathbb{C}}^{n+1})$ extends to an analytic action. It is known that the group $\mathrm{PU}(n+1,1)$ acts as CR transformations on S^{2n+1}.

Given a subgroup G of $\mathrm{PU}(n+1,1)$, we have a limit set $L(G)$ defined by Chen-Greenberg [2], (compare [5] more generally). On the other hand, when a discontinuous group Γ acts on S^{2n+1}, there is a limit set $\Lambda(\Gamma)$ for which Γ acts properly discontinuously on the domain $S^{2n+1} - \Lambda(\Gamma)$. (See [14].) In Section 3, we prove the following. Compare Theorem 3.1 for the visibility manifolds more generally. (See also [11].)

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Theorem A. *Let* Γ *be a discrete subgroup of* $\mathrm{PU}(n+1,1)$. *Then*

$$L(\Gamma)=\Lambda(\Gamma).$$

In particular, if the domain of discontinuity $\Omega = S^{2n+1} - L(\Gamma) \neq \emptyset$, *then* Γ *acts properly discontinuously on* Ω.

Note that the complex involution τ of $\mathrm{Iso}(\mathbb{H}_{\mathbb{C}}^{n+1})$ is not a CR diffeomorphism. However, the above equality still holds for discrete subgroups of $\mathrm{Iso}(\mathbb{H}_{\mathbb{C}}^{n+1})$.

In Section 4, we determine homogeneous spherical CR space forms related to the homogeneous Sasakian space forms (cf. [1]). Let G be the group of *pseudo-Hermitian transformations* of a simply connected spherical CR manifold X of dimension $2n+1$ (cf. [12]). When G acts transitively on X, we shall determine the limit set $L(G)$ in S^{2n+1}, see Proposition 4.1.

In Section 5, we apply Theorem A to show the several properties of limit sets of holonomy groups and developing maps of spherical CR manifolds. In Subsection 5.1, we obtain

Theorem B. *Let* M *be a* $(2n+1)$*-dimensional compact spherical* CR *manifold. If the developing image misses a point of* S^{2n+1}, *then* $\mathrm{dev} : \tilde{M} \to S^{2n+1}$ *is a covering map onto its image.*

It is known that the link of an isolated singular point admits a canonical CR-structure (cf. [18], [6]). In particular, the 3-dimensional Brieskorn manifold $M(p,q,r) = S^5 \cap \{(z_1,z_2,z_3) \in \mathbb{C}^3 \mid z_1^p + z_2^q + z_3^r = 0\}$ admits a CR-structure. Moreover, there is an S^1-invariant CR-structure (cf. [13]). In Subsection 5.2, we study spherical CR-structures on $M(p,q,r)$.

Theorem C. *Let* $M(p,q,r)$ *be the* 3*-dimensional Brieskorn manifold. Put* $\kappa = p^{-1}+q^{-1}+r^{-1}-1$. *Then* $M(p,q,r)$ *admits a spherical homogeneous* CR*-structure such that the holonomy group* Γ *is discrete with* $L(\Gamma)=\emptyset$, $\{\infty\}$ *or* $\mathcal{S}^1$ *according as* $\kappa > 0$, $\kappa = 0$, *or* $\kappa < 0$ *respectively. In the case that* $\kappa < 0$, Γ *can be chosen to be indiscrete. Here* $\mathcal{S}^1$ *is a smooth circle in* S^3. *Moreover, the developing map is a covering onto* $S^3 - L(\Gamma)$.

See Theorem 5.2 for more details. In the last section, as an application, we shall study the limit set of the boundary of a pseudo-Riemannian manifold of negative curvature.

1. Preliminaries

1.1. Limit set of geometric model

Let X be a locally compact Hausdorff space on which a discontinuous group Γ acts topologically. We shall define the limit sets for Γ action on X successively. The notion of limit sets here is due to Kulkarni (cf. [14]). We consider the following sets.

$$\Lambda_0 = \text{the closure of the set } \{x \in X|\, \Gamma_x \text{ is an infinite subgroup}\}.$$

$$\Lambda_1 = \text{the closure of the set of cluster points of } \{\gamma y \mid \gamma \in \Gamma\} \quad (\forall\, y \in X - \Lambda_0).$$

$$\Lambda_2 = \text{the closure of the set of cluster points of } \{\gamma K \mid \gamma \in \Gamma\} \quad (\forall \text{ compact subset } K \subset X - \{\Lambda_0 \cup \Lambda_1\}).$$

Definition 1.1. The set $\Lambda = \Lambda(\Gamma) = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2$ is said to be the limit set of Γ. The set $\Omega(\Gamma) = X - \Lambda(\Gamma)$ is called the domain of the discontinuity for Γ.

It is the fundamental result [14] that

Proposition 1.1. If $\Omega \neq \emptyset$, then Γ acts properly discontinuously on Ω. In particular Γ is discrete.

1.2. Visibility manifolds

A Hadamard manifold Y is a complete, simply connected Riemannian manifold of dimension $n \geq 2$ having sectional curvature $k \leq 0$. Denote by d the distance function on Y. Given two geodesics $\alpha, \beta\colon (-\infty, \infty) \to Y, \alpha$ and β are asymptotic if there exists a number L such that $d(\alpha(t), \beta(t)) \leq L (t \geq 0)$. Denote by $\alpha(\infty)$ an asymptote class (equivalence class of asymptotic relation) of α. Note there is another asymptote class $\alpha(-\infty)$. An asymptotic class of geodesics of Y is said to be a point at infinity. Let ∂Y be the set of all points at infinity. The union $\bar{Y} = Y \cup \partial Y$ together with the cone topology is homeomorphic to the closed ball. (Compare [5].) Let $\mathrm{Iso}(Y)$ be the group of isometries of Y. Setting $h(\alpha(\infty)) = (h\alpha)(\infty)$ for each element $h \in \mathrm{Iso}(Y)$, h extends to a homeomorphism of $\bar{Y}$ onto itself.

Definition 1.2. If a Hadamard manifold Y satisfies that *for any points $x \neq y$ in ∂Y there exists at least one geodesic joining x and y*, then Y is said to be a *visibility manifold*.

For brevity, any Riemannian manifold (also orbifold) whose universal cover is isometric to a visibility manifold is also called a visibility manifold. We recall the several results from [5].

Proposition 1.2. Let Y be a visibility manifold. If a non-elliptic isometry g fixes distinct points x and y of ∂Y, then it translates a geodesic joining x to y.

Definition 1.3. Let Γ be a subgroup of $\mathrm{Iso}(Y)$. Points x, y in ∂Y, not necessarily distinct, are said to be dual relative to Γ provided that given any neighbourhood U, V of x and y respectively in $\bar{Y}$ there exists $g \in \Gamma$ such that $g(\bar{Y} - U) \subseteq V$ and hence $g^{-1}(\bar{Y} - V) \subseteq U$.

Proposition 1.3. *If x, y are dual points in ∂Y, then there exists a sequence $\{g_n\} \leq \Gamma$ such that $g_n^{-1}(p) \to x$ and $g_n(p) \to y$ as $n \to \infty$ for any point $p \in Y$.*

Proposition 1.4. *Let Y be a visibility manifold. For $x, y \in \partial Y$, let $\{g_n\} \leq \Gamma$ be a sequence such that for $p \in Y$ we have $g_n(p) \to y$ and $g_n^{-1}(p) \to x$ as $n \to \infty$. Then x and y are dual. Moreover if U and V are neighbourhoods in $\bar{Y}$ of x and y respectively, then for n sufficiently large $g_n(\bar{Y} - U) \subseteq V$ and $g_n^{-1}(\bar{Y} - V) \subseteq U$.*

2. Limit Sets of Visibility Manifold

Let Y be a Hadamard manifold for which ∂Y is called the boundary sphere of Y. When Γ is a discontinuous group of $\mathrm{Iso}(Y)$, we can discuss the limit set $\Lambda = \Lambda(\Gamma)$ and a properly discontinuous action of Γ on $\partial Y - \Lambda$. We introduce another limit set for a subgroup (not necessarily discontinuous) of isometries of Y acting topologically on ∂Y.

Definition 2.1. Let G be a subgroup of $\mathrm{Iso}(Y)$. The limit set $L(G)$ is defined to be the set of cluster points of the orbits $G \cdot p$ in ∂Y $(p \in Y)$;

$$L(G) = \overline{G \cdot p} \cap \partial Y$$

We must show that the above definition does not depend on the choice of p. Let $p, q \in Y$. Then there is a geodesic connecting p to q which we denote by $[p, q]$ in Y. Similarly if there is a geodesic between $x, y \in \partial Y$ with $x \neq y$, we write also $[x, y]$ in $\bar{Y}$.

Lemma 2.1. *Given any two points p, q in Y,*

$$\overline{G \cdot p} \cap \partial Y = \overline{G \cdot q} \cap \partial Y,$$

$L(G)$ is well defined.

Proof. For a sequence $\{g_i\}_{i \in \mathbb{N}}$ of G, suppose $\lim\limits_{i \to \infty} g_i p = x \in \partial Y$ and $\lim\limits_{i \to \infty} g_i q = y \in \partial Y$. Let d be the distance function on Y. Note that d is invariant under $\mathrm{Iso}(Y)$. Suppose $x \neq y$. As $[p, q]$ is a geodesic segment in Y, it follows

$$\infty > \mathrm{length}[p, q] = d(p, q) = d(g_i p, g_i q) = \mathrm{length}[g_i p, g_i q] \to \infty.$$

This contradiction shows $x = y$ or $L(G)$ is independent of the choice of points in Y. $\square$

Lemma 2.2.(Minimality [2]) *Let Λ be any G-invariant closed subset in ∂Y. If Λ contains more than one point, $L(G) \subset \Lambda$.*

Proof. Choose distinct points $\{x, y\}$ from Λ. Let $z = \lim\limits_{i \to \infty} g_i p \in L(G)$ for $p \in Y$. As Λ is G-invariant, it is sufficient to show that $\lim\limits_{i \to \infty} g_i x = z$ or $\lim\limits_{i \to \infty} g_i y = z$. If α is a geodesic between x and y, then $g_i \alpha$ is a geodesic between $g_i x$ and $g_i y$. Choose a point q in $\mathrm{Int}\alpha$. As $g_i q \in g_i \alpha$ $(\forall\, i)$, the sequence $g_i q$ converges to a point on the closure of the geodesic between $\lim\limits_{i \to \infty} g_i x$ and $\lim\limits_{i \to \infty} g_i y$ (possibly $\lim\limits_{i \to \infty} g_i x = \lim\limits_{i \to \infty} g_i y$). On the other hand, the point $w = \lim\limits_{i \to \infty} g_i q$ belongs to ∂Y and thus it is either one of the endpoints

$\{\lim_{i\to\infty} g_i x, \lim_{i\to\infty} g_i y\}$, say $w = \lim_{i\to\infty} g_i x \in \Lambda$. Hence by Lemma 2.1, $w = \lim_{i\to\infty} g_i p = z$ □

We have the following properties of the limit sets.

Proposition 2.1.

(1) *If G' is a subgroup of finite index, then $L(G') = L(G)$.*

(2) $L(\bar{G}) = L(G)$.

(3) *If a normal subgroup H of G has no common fixed point and satisfies that $L(H) \neq \emptyset$, then $L(H) = L(G)$.*

The proofs in [2] work similarly.

Proposition 2.2. *Let N be a normal subgroup of G. Suppose that one of the following conditions is satisfied:*

(i) $L(N) = \emptyset$.

(ii) *N is abelian and the elements of G do not have the common fixed point in ∂Y.*

Then N leaves each point in $L(G)$ fixed.

Proof. (i) If $L(N) = \emptyset$ then by Property (2), $L(\bar{N}) = \emptyset$. It follows that $\bar{N}$ is compact which has a fixed point in Y. Let X be a subset of Y fixed by every element of N. Since N is normal in G, X is invariant under G. Now let $x \in L(G)$ and $p \in X$. There is a sequence $\{g_k\} \leq G$ such that $\lim_{k\to\infty} g_k(p) = x$. Since $g_k(p) \in X$ is fixed by N, the same is true for $x = \lim_{k\to\infty} g_k(p)$.

(ii) Suppose $L(N) \neq \emptyset$. By the condition (ii), Property (3) implies that $L(N) = L(G)$. Let $x \in L(G)$ and $p \in Y$. Then there is a sequence $\{n_k\} \leq N$ such that $\lim_{k\to\infty} n_k(p) = x$. Each $n \in N$ satisfies that $n(q) = \lim_{k\to\infty} nn_k(p) = \lim_{k\to\infty} n_k(n(p)) = q$. □

3. Limit Sets

Suppose that Y is a visibility manifold (cf. Definition 1.2). Put $X = \partial Y$. We prove the following. (Compare [11].)

Theorem 3.1. *Let Γ be a discrete subgroup of the isometry group* $\mathrm{Iso}(Y)$. *Then*

$$L(\Gamma) = \Lambda(\Gamma).$$

In particular, if the domain of discontinuity $\Omega = X - L(\Gamma) \neq \emptyset$, then Γ acts properly discontinuously on Ω.

Proof of Theorem 3.1. First we prove some general facts.

Lemma 3.1.

(1) $\Lambda_0 \subset L(\Gamma)$.

(2) $\Lambda_1 \subset L(\Gamma)$.

Proof. Let $a \in \Lambda_0$. The point a is fixed by some element $\gamma \in \Gamma$. Define $b = \lim_{n\to\infty} \gamma^n p \in X$ which is a fixed point of γ $(p \in Y)$. By Proposition 1.2, there exists a geodesic α joining a and b which is translated by γ. Then either $\alpha(\infty) = \lim_{n\to\infty} \gamma^n \alpha(0)$ or $\alpha(-\infty) = \lim_{n\to\infty} \gamma^{-n}\alpha(0)$ which is equal to a, thus $a \in L(\Gamma)$ by definition of $L(\Gamma)$.

Let $a_1 \in \Lambda_1$. Then $a_1 = \lim_{i\to\infty} g_i z$ $(z \in X - \Lambda_0)$. Consider the dual points (cf. Proposition 1.4); $x = \lim_{i\to\infty} g_i p$, $x' = \lim_{i\to\infty} g_i^{-1} p$ $(p \in Y)$. If $z = x'$, then $z \in L(\Gamma)$ which is Γ-invariant, closed so $a_1 = \lim_{i\to\infty} g_i z \in \overline{L(\Gamma)} = L(\Gamma)$. If $z \neq x'$, then choose a neighborhood $V \subset X$ of x' such that $z \notin V$. For any neighborhood of x in X, the dual points satisfy that

$$g_i(X - V) \subset U \quad (i > N \ \text{ for sufficiently large } N).$$

As $z \in X - V$, this implies that $\lim_{i\to\infty} g_i z = x$. Since $a_1 = \lim_{i\to\infty} g_i z$ as above, it follows that $a_1 = x \in L(\Gamma)$. In each case, we obtain that $\Lambda_1 \subset L(\Gamma)$. $\square$

Lemma 3.2. *Suppose that $\Lambda_2 \ni a_2$. By definition, there exist a sequence $\{g_i\}$ of Γ and some compact set $K \subset X - \{\Lambda_0 \cup \Lambda_1\}$ such that every neighborhood W of a_2 meets $g_i K$ for $i > N$ for sufficiently large N. If*

$$K \cap L(\Gamma) = \emptyset, \tag{3.1}$$

then $a_2 \in L(\Gamma)$.

Proof. Let $b = \lim_{i\to\infty} g_i p$, $b' = \lim_{i\to\infty} g_i^{-1} p$ be the dual points. If $a_2 \neq b$, then choose a neighborhood $U \ni b$ and a neighborhood $W \ni a_2$ such that

$$U \cap W = \emptyset. \tag{3.2}$$

On the other hand, we can find a neighborhood $V \ni b'$ such that

$$V \cap K = \emptyset. \tag{3.3}$$

For this, if every neighborhood of b' has nontrivial intersection with K, then $b' \in \bar{K} = K$, so that $L(\Gamma) \cap K \neq \emptyset$. This contradicts our hypothesis (3.1). By duality, we have $g_i(X - V) \subset U$, $i > N$ for sufficiently large N. Therefore, as $K \subset X - V$ by (3.3), $g_i K \subset U$. From (3.2), $g_i K \cap W = \emptyset$. This contradicts the condition of the lemma. Hence $a_2 = b$ as above so that $a_2 \in L(\Gamma)$. $\square$

The proof of Theorem 3.1 splits into two cases that Λ_0 consists of one point or consists of more than one point.

Case I. Suppose that $\Lambda_0 = \{a\}$.

Subcase 1. Λ_1 consists of a single point. If so, Λ_1 is fixed by Γ, so it belongs to Λ_0, i.e. $\Lambda_1 = \{a\}$.

We show that $L(\Gamma) = \{a\}$. If $x \in L(\Gamma)$ is any point, then there exists a sequence $\{\gamma_i\}$ of Γ such that $\lim_{i\to\infty} \gamma_i p = x$ $(p \in Y)$. Take the dual point $x' = \lim_{i\to\infty} \gamma_i^{-1} p$. If $x' \neq a$, there exist a neighborhood V of x' and a neighborhood W of a in X such that $V \cap W = \emptyset$. For any neighborhood U of x, it follows that for a sufficiently large N

$$\gamma_i(X - V) \subset U \ (\forall\ i > N). \tag{3.4}$$

As $a \in X - V$, this implies that $\lim_{i\to\infty} \gamma_i a = x$. But a is fixed by any element of Γ by the hypothesis of **Case I**. It follows that $a = x$. If $x' = a$, then choose $z \in X - \Lambda_0$ where $\Lambda_0 = \{a\}$. For a neighborhood V of x' such that $z \in X - V$ and any neighborhood U of x, the equation (3.4) holds, i.e. $\lim_{i\to\infty} \gamma_i z = x$. On the other hand, $\lim_{i\to\infty} \gamma_i z \in \Lambda_1$ by definition. As $\Lambda_1 = \{a\}$, it follows that $x = a$. Hence $L(\Gamma) = \{a\}$.

Moreover, in this case, since $L(\Gamma) = \{a\}$, for any compact set $K \subset X - \{\Lambda_0 \cup \Lambda_1\}$ where $\Lambda_0 \cup \Lambda_1 = \{a\}$, it follows that $K \cap L(\Gamma) = \emptyset$ which satisfies the condition (3.1) of Lemma 3.2. Thus $\Lambda_2 \subset L(\Gamma)$. As $\Lambda_1 \subset \Lambda_2 \neq \emptyset$ when we take a point as K. So $\Lambda_2 = L(\Gamma)$.

Subcase 2. Λ_1 contains more than one point. Then by Minimality, $L(\Gamma) \subset \Lambda_1$. By (2) of Lemma 3.1, we have $L(\Gamma) = \Lambda_1$. As $L(\Gamma) \subset \Lambda_0 \cup \Lambda_1$, $K \subset X - \{\Lambda_0 \cup \Lambda_1\}$ satisfies (3.1), Lemma 3.2 implies that $\Lambda_2 \subset L(\Gamma)$. As $\Lambda_1 \subset \Lambda_2$, $L(\Gamma) = \Lambda_2$.

Under the assumption $\Lambda_0 = \{a\}$, we conclude that

$$\begin{cases} \text{Subcase 1} \;\; \Lambda_0 = \Lambda_1 = \Lambda_2 = L(\Gamma) = \{a\}. \\ \text{Subcase 2} \;\; \Lambda_0 = \{a\} \subset \Lambda_1 = \Lambda_2 = L(\Gamma). \end{cases} \tag{3.5}$$

In particular, $L(\Gamma) = \Lambda$.

Case II. Suppose that Λ_0 contains more than one point. Then by Minimality $L(\Gamma) \subset \Lambda_0$ so that $L(\Gamma) = \Lambda_0$ by (1) of Lemma 3.1. Note that Λ_1, Λ_2 contain more than one point respectively. For this, if Λ_1 is a single point $\{a_1\}$ which is fixed by Γ, then the above argument of **Subcase 1** can be applied to show that $L(\Gamma) = \{a_1\}$, being contradiction. In particular, Λ_1 contains more than one point so that $\Lambda_1 = L(\Gamma)$. As $\Lambda_1 \subset \Lambda_2$, Λ_2 also contains more than one point and so $L(\Gamma) \subset \Lambda_2$. Since $K \subset X - \{\Lambda_0 \cup \Lambda_1\} = X - L(\Gamma)$, it follows from Lemma 3.2 that $\Lambda_2 \subset L(\Gamma)$, i.e. $\Lambda_2 = L(\Gamma)$. Hence in this case,

$$\Lambda_0 = \Lambda_1 = \Lambda_2 = L(\Gamma).$$

This proves Theorem 3.1. $\square$

Proof of Theorem A in Introduction. When we take $Y = \mathbb{H}_{\mathbb{C}}^{n+1}$ and $X = S^{2n+1}$, the result follows from Theorem 3.1. $\square$

4. Homogeneous Space Forms

4.1. Homogeneous *CR* space forms after Burns-Shnider

Suppose that M is a strictly pseudoconvex CR manifold of dimension $2n+1$. Let $\mathrm{Aut}_{CR}(M)$ be the group of all CR-automorphism of M onto itself (cf. [3]). Burns and Shnider [1] have classified (simply connected) spherical CR manifolds M of dimension $2n+1$ under the condition that $\mathrm{Aut}_{CR}(M)$ acts transitively.

Since $\mathrm{Aut}_{CR}(M)$ acts transitively on M, it is easily checked that the developing map dev : $\tilde{M} \to S^{2n+1}$ is a covering map onto its image $\Omega \subset S^{2n+1}$. Here $\tilde{M}$ is the universal covering space of M. By the result of Burns and Shnider, Ω is determined as follows:

Theorem 4.1.

(a) S^{2n+1}, $S^{2n+1} - S^{2k-1}$ $(k = 1, \ldots, n)$.

(b) $S^{2n+1} - S^n$.

(c) $\mathcal{N}$, $\mathcal{N} - \mathcal{N}_k$ $(k = 1, \ldots, n)$.

Note that $\mathcal{N} = S^{2n+1} - \{\infty\}$ where ∞ is the *point at infinity* of S^{2n+1}. We shall explain $\mathcal{N}_k$ in the next subsection where $\mathcal{N} - \mathcal{N}_k = S^{2n+1} - S^{2(n-k+1)}$ $(k = 1, \ldots, n)$.

Recall that the group of CR-transformations $\mathrm{Aut}_{CR}(S^{2n+1}) = \mathrm{PU}(n+1, 1)$ for the spherical CR manifold S^{2n+1}.

4.2. Homogeneous Sasakian space forms

We shall explain the above theorem in connection with Sasakian structure. Let η be a contact form representing a strictly pseudoconvex CR structure $(\mathrm{Null}\,\eta, J)$ on M. Then (η, J) is a pseudo-Hermitian structure. Denote by

$$\mathrm{Psh}(M, \eta) = \{f \in \mathrm{Diff}(M) \mid f_* \circ J = J \circ f_*, f^*\eta = \eta\}$$

the group of pseudo-Hermitian transformations of M. $\mathrm{Psh}(M, \eta)$ is a subgroup of $\mathrm{Aut}_{CR}(M)$. For the contact form η, the Reeb field ξ is a vector field satisfying $\eta(\xi) = 1, d\eta(\xi, X) = 0$ $(\forall\ X \in TM)$. In terms of CR-structure, the relation between Sasakian manifolds and pseudo-Hermitian manifolds is mentioned as follows (cf. [12]):

Definition 4.1. A Sasakian manifold is referred to a pseudo-Hermitian manifold (η, J) whose Reeb field ξ generates a one-parameter subgroup of $\mathrm{Psh}(M, \eta)$.

In other words, the Sasakian metric is the Riemannian metric $g_\eta = \eta \cdot \eta + d\eta(J\ ,)$ where ξ is Killing.

Case (a) Since S^{2k-1} is viewed as the boundary of k-dimensional complex hyperbolic space $\mathbb{H}_{\mathbb{C}}^k$, the sphere complement $S^{2n+1} - S^{2k-1}$ is the homogeneous Riemannian manifold:

$$\mathrm{P}(\mathrm{U}(k,1) \times \mathrm{U}(n-k+1))/\mathrm{U}(k) \times \mathrm{U}(n-k).$$

There is a Riemannian submersion:

$$S^1 \to S^{2n+1} - S^{2k-1} \to \mathbb{H}_{\mathbb{C}}^k \times \mathbb{CP}^{n-k} \tag{4.1}$$

where $\mathbb{H}_{\mathbb{C}}^k = \mathrm{PU}(k,1)/\mathrm{U}(k)$ and the $(n-k)$-dimensional complex projective space $\mathbb{CP}^{n-k} = \mathrm{PU}(n-k+1)/\mathrm{U}(n-k)$. The above principal bundle is the Sasakian fibration (i.e. a standard pseudo-Hermitian manifold).

In particular, when $k = n$, $S^{2n+1} - S^{2n-1}$ is identified with the quadric

$$V_{-1}^{2n+1} = \{(z_1, \ldots, z_{n+1}) \in \mathbb{C}^{n+1} \mid |z_1|^2 + \cdots + |z_n|^2 - |z_{n+1}|^2 = -1\}.$$

The $(2n+1)$-dimensional manifold V_{-1}^{2n+1} is not simply connected because it is a principal S^1-bundle over $\mathbb{H}_{\mathbb{C}}^n$. Denote by $\tilde{V}_{-1}^{2n+1}$ the simply connected manifold. Noting that $\mathrm{P}(\mathrm{U}(n,1) \times \mathrm{U}(1)) = \mathrm{U}(n,1)$, there is the corresponding lift $\mathrm{U}(n,1)^\sim$ of $\mathrm{U}(n,1)$ to $\tilde{V}_{-1}^{2n+1}$ acting transitively on $\tilde{V}_{-1}^{2n+1}$.

As a consequence, for the group $\mathsf{G}_{k,1} = \mathrm{P}(\mathrm{U}(k,1) \times \mathrm{U}(n-k+1))$, we obtain that

$$L(\mathsf{G}_{k,1}) = S^{2k-1} \ (k = 1, \ldots, n). \tag{4.2}$$

Case (b) Let $\mathrm{PO}(k+1,1)$ be the isometries of the real $(k+1)$-dimensional hyperbolic space $\mathbb{H}_{\mathbb{R}}^{k+1}$. $\mathrm{PO}(k+1,1)$ is naturally embedded into $\mathrm{PU}(k+$

$1,1)$. As $(\mathrm{PO}(k+1,1), S^k)$ is the subgeometry of $(\mathrm{PU}(n+1,1), S^{2n+1})$ $(k=0,\ldots,n)$, we obtain that

$$\begin{aligned} \mathrm{Aut}_{CR}(S^{2n+1}-S^k) &= \mathrm{P}(\mathrm{O}(k+1,1)\times \mathrm{U}(n-k)). \\ L(\mathrm{PO}(k+1,1)) &= S^k \ (k=0,\ldots,n). \end{aligned} \tag{4.3}$$

Since S^n is the boundary of the real hyperbolic space $\mathbb{H}^{n+1}_{\mathbb{R}}$, the unit sphere bundle $T_1\mathbb{H}^{n+1}_{\mathbb{R}}$ is CR-equivalent to the complement $S^{2n+1}-S^n$ on which the group of CR-transformations is $\mathrm{P}(\mathrm{O}(n+1,1)\cdot S^1) = \mathrm{PO}(n+1,1)$. $\mathrm{PO}(n+1,1)$ acts transitively on $S^{2n+1}-S^n$ such that there is the fibration:

$$\begin{array}{ccccc} \frac{\mathrm{O}(n+1)}{\mathrm{O}(n)} & \longrightarrow & \frac{\mathrm{PO}(n+1,1)}{\mathrm{O}(n)} & \longrightarrow & \frac{\mathrm{PO}(n+1,1)}{\mathrm{O}(n+1)} \\ || & & || & & || \\ S^n & \longrightarrow & S^{2n+1}-S^n & \longrightarrow & \mathbb{H}^{n+1}_{\mathbb{R}}. \end{array} \tag{4.4}$$

For $k\neq n$, $S^{2n+1}-S^k$ does not have a transitive group in view of (4.3).

Remark 4.1. From (4.4), when $n=1$, $T_1\mathbb{H}^2_{\mathbb{R}} = S^3-S^1$ is the unit circle bundle: $S^1 \to T_1\mathbb{H}^2_{\mathbb{R}} \to \mathbb{H}^2_{\mathbb{R}}$ which is Sasakian. The unit circle bundle $T_1S^n = \mathrm{O}(n+1)/\mathrm{O}(n-1)$ is also Sasakian as well as $T_1\mathbb{H}^2_{\mathbb{R}}$:

$$\mathrm{SO}(2) \to \mathrm{O}(n+1)/\mathrm{O}(n-1) \to \mathrm{O}(n+1)/(\mathrm{SO}(2)\times \mathrm{O}(n-1)).$$

However, there is no (regular) Sasakian structure on $T_1\mathbb{R}^n = \mathbb{R}^n\times S^{n-1}$.

(c) The group $\mathcal{N} = \mathcal{R}\times\mathbb{C}^n$ (with the usual group law) denotes the Heisenberg nilpotent Lie group and take $\mathcal{N}_k = \mathcal{R}\times(\mathbb{R}\times\mathbb{C}^{n-k})$ which is the typical $2(n-k+1)$-dimensional nilpotent Lie subgroup of $\mathcal{N}$ $(k=1,\ldots,n)$. (For $k=1$, $\mathcal{N}-\mathcal{N}_1$ consists of two components.) Recall that $G = \mathcal{N}\rtimes(\mathrm{U}(n)\times\mathbb{R}^+)$ is the subgroup of $\mathrm{PU}(n+1,1) = \mathrm{Aut}_{CR}(S^{2n+1})$ whose elements stabilizes the *point at infinity* ∞ of S^{2n+1}. Let

$$G_k = (\mathcal{R}\times(\mathbb{C}^{k-1}\times\mathbb{R}\times\mathbb{C}^{n-k}))\rtimes(\mathrm{U}(k-1)\times\{1\}\times\mathrm{U}(n-k)\times\mathbb{R}^+)$$

be the subgroup of G. Then they have shown that the complement $\mathcal{N}-\mathcal{N}_k$ (if $k=1$, one of the components) admits a simply transitive subgroup $(\mathcal{R}\times(\mathbb{C}^{k-1}\times\mathbb{R}\times\mathbb{C}^{n-k}))\rtimes\mathbb{R}^+$.
If ω_∞ is the usual contact form of $\mathcal{N}$, then it is noted that $\mathrm{Psh}(\mathcal{N},\omega_\infty) =$

$\mathcal{N} \rtimes \mathrm{U}(n)$ and the multiplicative group $\mathbb{R}^+$ of G acts as homothetic transformations of ω_∞. In fact, if $f = \lambda \in \mathbb{R}^+$, then $f^*\omega_\infty = \lambda^2 \cdot \omega_\infty$ on $\mathcal{N}$. This holds similarly for $\mathcal{N} - \mathcal{N}_k$ such that $\mathrm{Aut}_{CR}(\mathcal{N} - \mathcal{N}_k) = G_k$, while it follows

$$\mathrm{Psh}(\mathcal{N} - \mathcal{N}_k) = \big(\mathcal{R} \times (\mathbb{C}^{k-1} \times \mathbb{R} \times \mathbb{C}^{n-k})\big) \rtimes \big(\mathrm{U}(k-1) \times \{1\} \times \mathrm{U}(n-k)\big),$$

so that $\mathrm{Psh}(\mathcal{N} - \mathcal{N}_k) = \mathrm{Psh}(\mathcal{N} - \mathcal{N}_k, \omega_\infty)$ is not transitive on $\mathcal{N} - \mathcal{N}_k$.
In this case,

$$L(\mathcal{N}_k) = \{\infty\} \ (k = 1, \dots, n), \ \ L(\mathcal{N}) = \{\infty\}. \tag{4.5}$$

We have the following result concerning homogeneous Sasakian space forms (homogeneous standard pseudo-Hermitian space forms).

Proposition 4.1.

(i) *For* **(a), (b)***, choosing the canonical form ω on Ω,* $\mathrm{Psh}(\Omega, \omega) = \mathrm{P}(\mathrm{U}(k,1) \times \mathrm{U}(n-k+1))$*, or* $\mathrm{PO}(n+1,1)$ *which is the transitive group of pseudo-Hermitian transformations of Ω respectively. In each case, there is the principal S^1-bundle of Sasakian space form:*

$$\begin{aligned} &S^1 \to S^{2n+1} - S^{2k-1} \to \mathbb{H}^k_{\mathbb{C}} \times \mathbb{CP}^{n-k} \ (k = 1, \dots, n), \\ &S^1 \to S^3 - S^1 \to \mathbb{H}^2_{\mathbb{R}}. \end{aligned} \tag{4.6}$$

(ii) *For* **(c)***, there is no transitive group of pseudo-Hermitian transformations on $\mathcal{N} - \mathcal{N}_k$ except for $\mathcal{N}$ where* $\mathrm{Psh}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(n)$*. There is the principal bundle of Sasakian space form:*

$$\mathcal{R} \to \mathcal{N} \to \mathbb{C}^n. \tag{4.7}$$

This settles a classification of all simply connected spherical homogeneous Sasakian space forms;

Corollary 4.1. *Let (Ω, ω) be a spherical pseudo-Hermitian manifold with transitive group of pseudo-Hermitian transformations. If $\tilde{\Omega}$ is the universal covering of Ω with its lift $\tilde{\omega}$, then $\tilde{\Omega}$ is either one of the following manifolds equipped with the canonical form indicated in* **(a), (b), (c)***:*

(a′) S^{2n+1}, $S^{2n+1} - S^{2k-1}$ $(k = 1, \dots, n-1)$, $\tilde{V}^{2n+1}_{-1}$.

(b$'$) $S^{2n+1}-S^n$.

(c$'$) $\mathcal{N}$.

As a consequence, if (M,η) is a simply connected homogeneous spherical CR manifold, then the developing map induces a CR diffeomorphism $\mathrm{dev}: M \to \tilde{\Omega}$. Defining $\tilde{\omega} = \mathrm{dev}^{-1*}\eta$, this implies that $(M, \mathrm{Psh}(M,\eta))$ is $(\tilde{\Omega}, \mathrm{Psh}(\tilde{\Omega},\tilde{\omega}))$ where $\mathrm{Psh}(\tilde{\Omega},\tilde{\omega})$ maps onto $\mathrm{Psh}(\Omega,\omega)$ which is a transitive subgroup of $\mathrm{Aut}_{CR}(S^{2n+1})$.

Remark 4.2. For $n=1$, note that both V^3_{-1} and $T_1\mathbb{H}^2_{\mathbb{R}}$ $(= S^3 - S^1)$ are diffeomorphic but different pseudo-Hermitian (or CR-) structures because they have different transitive groups $\mathrm{U}(1,1)$ and $\mathrm{PO}(2,1)^0$.

5. Application to Spherical CR Manifolds

5.1. Developing maps

In this subsection, we shall prove Theorem B in Introduction. In Subsection 4.1, we know that a closed subgroup of $\mathrm{Aut}_{CR}(S^{2n+1}-S^{2k-1}) = \mathrm{P}(\mathrm{U}(k,1)\times \mathrm{U}(n-k+1))$ acts properly on $S^{2n+1}-S^{2k-1}$ because it admits a homogeneous Riemannian metric (cf. **Case (a)**). On the other hand, there is no homogeneous Riemannian metric on $S^{2n+1}-S^k$ except for $k=n$. For $k\neq n$, $\mathrm{Aut}_{CR}(S^{2n+1}-S^k) = \mathrm{P}(\mathrm{O}(k+1,1)\times \mathrm{U}(n-k))$ is not transitive on $S^{2n+1}-S^k$ from (4.3). However we prove the following.

Lemma 5.1. *Any closed subgroup of* $\mathrm{P}(\mathrm{O}(k+1,1)\times \mathrm{U}(n-k))$ *acts properly on* $S^{2n+1}-S^k$ $(k=0,\ldots,n)$.

Proof. If $\mathrm{Sim}(\mathcal{N}) = \mathcal{N} \rtimes (\mathrm{U}(n)\times \mathbb{R}^+)$ is the Heisenberg similarity group of $\mathrm{PU}(n+1,1)$, then $\mathbb{R}^k \rtimes \mathbb{R}^+$ is the maximal noncompact solvable subgroup of $\mathrm{PO}(k+1,1)\cap \mathrm{Sim}(\mathcal{N}) = \mathrm{Sim}(\mathbb{R}^k)$. Here we can identify:

$$S^k = \mathbb{R}^k \cup \{\infty\} \subset S^{k+1} = (\mathcal{R} \rtimes \mathbb{R}^k)\cup\{\infty\} \subset S^{2n+1} = \mathcal{N}\cup\{\infty\}, etc.$$

Then it follows

$$S^{2n+1}-S^k = \mathcal{N}\cup\{\infty\} - \mathbb{R}^k\cup\{\infty\} = \mathcal{N}-\mathbb{R}^k = \mathcal{R}\times\mathbb{C}^n - 0\times\mathbb{R}^k \quad (5.1)$$

in which $\mathbb{R}^k \rtimes \mathbb{R}^+$ acts as

$$t \cdot \lambda(s, z) = (\lambda^2 s, \lambda z + t) \tag{5.2}$$

where $\lambda \in \mathbb{R}^+, t \in \mathbb{R}^k, s \in \mathcal{R}, z \in \mathbb{C}^n$.

We prove that the group $\mathbb{R}^k \rtimes \mathbb{R}^+$ acts properly on $S^{2n+1} - S^k = \mathcal{R} \times \mathbb{C}^n - 0 \times \mathbb{R}^k$. Given a sequence $\{g_i = t_i \cdot \lambda_i\} \leq \mathbb{R}^k \rtimes \mathbb{R}^+$ where $t_i = (t_i^1, \ldots, t_i^k, 0, \ldots, 0) \in \mathbb{R}^k$, $\lambda_i \in \mathbb{R}^+$ and that

$$(s_i, z_i) = (s_i, (z_i^1, \ldots, z_i^n)) \to (s, z) = (s, (z^1, \ldots, z^n)) \in \mathcal{R} \times \mathbb{C}^n - 0 \times \mathbb{R}^k$$

$(i \to \infty)$, suppose that

$$g_i(s_i, z_i) \to (u, w) = (u, w^1, \ldots, w^n) \in \mathcal{R} \times \mathbb{C}^n - 0 \times \mathbb{R}^k.$$

As

$$g_i(s_i, z_i) = (\lambda_i^2 s_i, (\lambda_i z_i^1 + t_i^1, \ldots, \lambda_i z_i^k + t_i^k, \lambda_i z_i^{k+1}, \ldots, \lambda_i z_i^n),$$

it follows that

(1) $\lambda_i^2 s_i \to u$.
(a) $(\lambda_i z_i^1 + t_i^1, \ldots, \lambda_i z_i^k + t_i^k) \to (w^1, \ldots, w^k)$.
(2) $(\lambda_i z_i^{k+1}, \ldots, \lambda_i z_i^n) \to (w^{k+1}, \ldots, w^n)$.

Suppose that $s \neq 0$. As $s_i \to s$, we may assume that $s_i^{-1} \to s^{-1}$ for sufficiently large i. By (1), it follows that $\lambda_i^2 \to us^{-1}$. Thus $\{\lambda_i\}$ converges to some $\lambda \in \mathbb{R}^+$. As $(\lambda_i z_i^1 + t_i^1, \ldots, \lambda_i z_i^k + t_i^k) \to (w^1, \ldots, w^k)$ by (2), $t_i = (t_i^1, \ldots, t_i^k) \to (w^1 - \lambda z^1, \ldots, w^k - \lambda z^k)$, so $\{t_i\}$ converges to an element in $\mathbb{R}^k$. Hence $\{g_i = t_i \cdot \lambda_i\}$ converges in $\mathbb{R}^k \rtimes \mathbb{R}^+$.

Suppose that $s = 0$ in $(s, z) = (s, (z^1, \ldots, z^n)) \in \mathcal{R} \times \mathbb{C}^n - 0 \times \mathbb{R}^k$. If some z^ℓ $(\ell = k+1, \ldots, n)$ is not zero, then we can assume $(z_i^\ell)^{-1} \to (z^\ell)^{-1}$. As $\lambda_i z_i^\ell \to w^\ell$ by (3), it follows again that $\lambda_i \to w^\ell (z^\ell)^{-1}$, i.e. $\{\lambda_i\}$ converges. By the same argument, $\{t_i\}$ and hence $\{g_i\}$ converges.

Suppose that all $z^\ell = 0$ $(\ell = k+1, \ldots, n)$, i.e.

$$z = (z^1, \ldots, z^k, 0 \ldots, 0) = (x^1 + \mathbf{i}y^1, \ldots, x^k + \mathbf{i}y^k, 0, \ldots, 0).$$

Since $(s, z) = (0, (z^1, \ldots, z^k, 0 \ldots, 0)) \in \mathcal{R} \times \mathbb{C}^n - 0 \times \mathbb{R}^k$, there is some $z^m = x^m + \mathbf{i}y^m$ with $y^m \neq 0$. We can assume $(y_i^m)^{-1} \to (y^m)^{-1}$ for sufficiently large i. As $\lambda_i z_i^m + t_i^m = (\lambda_i x_i^m + t_i^m) + \mathbf{i}\lambda_i y_i^m \to w^m = a^m + \mathbf{i}b^m$, it follows that $\lambda_i y_i^m \to b^m$. Then $\lambda_i \to b^m(y^m)^{-1}$ so $\{\lambda_i\}$ converges.

In each case, the sequence $\{g_i\}$ converges in $\mathbb{R}^k \rtimes \mathbb{R}^+$. Hence it acts properly on $S^{2n+1} - S^k = \mathcal{R} \times \mathbb{C}^n - 0 \times \mathbb{R}^k$.

Let $\mathrm{P}(\mathrm{O}(k+1,1)\times \mathrm{U}(n-k)) = (\mathbb{R}^k \rtimes \mathbb{R}^+)\cdot K$ be the decomposition where K be the maximal compact Lie subgroup. Given a sequence $\{g_i\} \leq \mathrm{P}(\mathrm{O}(k+1,1) \times \mathrm{U}(n-k))$ and $p_i \to p \in S^{2n+1} - S^k$, suppose $g_i p_i \to q \in S^{2n+1} - S^k$ $(i\to\infty)$. Let $g_i = h_i \cdot k_i$ from the decomposition. As $k_i \in K$, we may assume $k_i \to k$ for some $k \in K$. Then $k_i p_i \to kp$ and $g_i p_i = h_i \cdot (k_i p_i) \to q$ as above. If we note that $h_i \in \mathbb{R}^k \rtimes \mathbb{R}^+$, $\{h_i\}$ converges to some $h \in \mathbb{R}^k \rtimes \mathbb{R}^+$. It follows that $g_i \to h \cdot k$ and hence $\mathrm{P}(\mathrm{O}(k+1,1) \times \mathrm{U}(n-k))$ acts properly on $S^{2n+1} - S^k$. So does any closed subgroup of $\mathrm{P}(\mathrm{O}(k+1,1) \times \mathrm{U}(n-k))$. $\square$

We characterize the developing map. (See [11].)

Theorem 5.1. *Let M be a $(2n+1)$-dimensional compact spherical CR manifold. If the developing image misses a point from S^{2n+1}, then* $\mathrm{dev} : \tilde{M} \to S^{2n+1}$ *is a covering map onto its image.*

Proof. Let $\mathrm{dev}\colon \tilde{M} \to S^{2n+1}$ be a developing map and $\rho\colon \pi_1(M) \to \mathrm{PU}(n+1,1)$ a holonomy map such that $\mathrm{dev}(\gamma x) = \rho(\gamma)\,\mathrm{dev}(x)$ $(\gamma \in \pi_1(M))$. Put $\Lambda = S^{2n+1} - \mathrm{dev}(\tilde{M})$ which is not empty by our hypothesis. Let $\Gamma = \rho(\pi_1(M)) \leq \mathrm{PU}(n+1,1)$ be the holonomy group.

Step 1. If Λ consists of just one point, say $\infty \in S^{2n+1}$, then Γ stabilizes $\{\infty\}$. Recall that the Heisenberg similarity group $\mathrm{Sim}(\mathcal{N}) = \mathcal{N} \rtimes (\mathrm{U}(n)\times\mathbb{R}^+)$ is the full subgroup of $\mathrm{PU}(n+1,1)$ which stabilizes $\{\infty\}$. It follows that $\Gamma \leq \mathrm{Sim}(\mathcal{N})$. As $\mathcal{N} = S^{2n+1} - \{\infty\}$, we have the developing pair:

$$(\rho, \mathrm{dev})\colon (\pi_1(M), \tilde{M}) \to (\mathrm{Sim}(\mathcal{N}), \mathcal{N}).$$

There exists a canonical affine connection on $\tilde{M}$ induced from that of $\mathcal{N}$ by the map dev. If $\tilde{M}$ is geodesically complete, then $\mathrm{dev} : \tilde{M} \to \mathcal{N}$ is an affine diffeomorphism. Then M is diffeomorphic to the orbit space $\mathcal{N}/\Gamma$ where Γ is a discrete subgroup of $\mathrm{Sim}(\mathcal{N})$. It is easy to see that $\Gamma \leq \mathrm{E}(\mathcal{N}) = \mathcal{N} \rtimes \mathrm{U}(n)$ is a discrete uniform subgroup and so $\mathcal{N}/\Gamma$ is an infranilmanifold.

On the other hand, if $\tilde{M}$ is not complete, it follows from the proof of Fried [8] (see also [11]) that there exists a Γ- invariant closed subset J in $\mathcal{N}$, as

$J \neq \emptyset$ we may assume $0 \in J$ up to conjugate. For $\gamma \in \Gamma$, let $x = \lim_{i\to\infty} \gamma^i 0 \in J$. Since Γ fixes $\{\infty\}$, the points $\{x, \infty\}$ are dual points by γ. Suppose J has a point y different from x. Choose neighbourhoods $x \in U$, $\infty \in V$ in S^{2n+1} such that $y \notin U$. By proposition 1.4, $\gamma^i(S^{2n+1} - U) \subset V$ $(i\to\infty)$. It follows that $\lim_{i\to\infty} \gamma^i y = \infty$. Since J is a Γ- invariant closed subset and $y \in J$, $\infty = \lim_{i\to\infty} \gamma^i y \in J$. As $J \subset \mathcal{N}$, this is a contradiction. Hence, J consists of a single point (i.e. $x = 0$). Then Γ leaves $\{0, \infty\}$ so that $\Gamma \leq \mathrm{U}(n) \times \mathbb{R}^+$, and $\mathrm{dev}\colon \tilde{M} \to \mathcal{N} - \{0\} = S^{2n+1} - \{0, \infty\} \approx S^{2n} \times \mathbb{R}^+$ is a diffeomorphism. It follows that a finite cover of M is diffeomorphic to a Hopf manifold $S^{2n} \times S^1$.

Step 2. Suppose that Λ contains more than one point. By Minimality, note that $L(\Gamma) \subset \Lambda$. Put $\pi = \pi_1(M)$.

Step 2-(i). If Γ is discrete, it follows from Theorem A that Γ acts properly discontinuously on $S^{2n+1} - L(\Gamma)$. The developing pair reduces to the following:

$$(\rho, \mathrm{dev})\colon (\pi, \tilde{M}) \to (\Gamma, S^{2n+1} - L(\Gamma)).$$

As $S^{2n+1} - L(\Gamma)$ admits a Γ-invariant Riemannian metric by properness of Γ, $\tilde{M}$ admits a π-invariant Riemannian metric such that dev is a local isometry. Since the quotient $M = \tilde{M}/\pi$ is compact, $\mathrm{dev}\colon \tilde{M} \to S^{2n+1} - L(\Gamma)$ is a covering map.

Step 2-(ii). Suppose Γ is not discrete in $\mathrm{PU}(n+1, 1)$. Let $H = \bar{\Gamma}^0$ which is the identity component of the closure of Γ in $\mathrm{PU}(n+1, 1)$.

Case(1). H is compact. Then H fixes a totally geodesic subspace $\mathbb{H}^k_{\mathbb{C}}$ $(0 \leq k \leq n)$ up to conjugate in $\mathbb{H}^{n+1}_{\mathbb{C}}$. If $\mathbb{H}^0 = \{0\}$, then H fixes the unique point 0 in $\mathbb{H}^{n+1}_{\mathbb{C}}$. As Γ normalizes H, Γ fixes $\{0\}$ also. Hence Γ belongs to the stabilizer $\mathrm{Iso}(\mathbb{H}^{n+1}_{\mathbb{C}})_0 = \mathrm{U}(n+1)$ at 0. As $\mathrm{U}(n+1)$ is maximal compact in $\mathrm{PU}(n+1, 1)$, this implies that $\mathrm{dev}\colon \tilde{M} \to S^{2n+1}$ is a covering. M is diffeomorphic to the spherical space form S^{2n+1}/Γ, where Γ is a finite subgroup of $\mathrm{U}(n+1)$. For $k \neq 0$, as H is compact $L(H) = \partial\mathbb{H}^k_{\mathbb{C}} = S^{2k-1}$. As the closure $\bar{\Gamma}$ normalizes H, (3) of proposition 2.1 implies that $L(\bar{\Gamma}) = S^{2k-1}$. In this case, the developing pair reduces to the following:

$$(\rho, \mathrm{dev})\colon (\pi_1(M), \tilde{M}) \to (\Gamma, S^{2n+1} - S^{2k-1}).$$

In this case it follows from (a) of Subsection 4.1 that the subgroup Aut_{CR} $(S^{2n+1} - S^{2k-1})$ of $\mathrm{PU}(n+1,1)$ is $\mathrm{P}(\mathrm{U}(k,1) \times \mathrm{U}(n-k+1))$. As stated earlier, $S^{2n+1} - S^{2k-1}$ is the homogeneous Riemannian manifold such that $\Gamma \leq \mathrm{P}(\mathrm{U}(k,1) \times \mathrm{U}(n-k+1))$. Hence, $\mathrm{dev}\colon \tilde{M} \to S^{2n+1} - S^{2k-1}$ is a covering map. Moreover, $S^{2n+1} - S^{2k-1}$ is simply connected whenever $k \neq n$. dev becomes a diffeomorphism so that Γ would be discrete in $\mathrm{P}(\mathrm{U}(k,1) \times \mathrm{U}(n-k+1)) \leq \mathrm{PU}(n+1,1)$. This contradicts the hypothesis of **Step 2-(ii)**. As a consequence, we have

$$(\rho, \mathrm{dev})\colon (\pi, \tilde{M}) \to (\Gamma, S^{2n+1} - S^{2n-1}) \tag{5.3}$$

is a covering. (Compare Proposition 5.1.)

Case(2). H is noncompact. If H contains a connected normal solvable subgroup, then H contains a nontrivial abelian subgroup $\mathcal{A}$. By (ii) of Proposition 2.2, either H has the common fixed point, say $\{\infty\}$ in S^{2n+1} or $\mathcal{A}$ fixes $L(H)$ pointwisely. The former case shows $H \leq \mathrm{Sim}(\mathcal{N})$. For the latter case, if $L(H)$ consists of a single point, say $\{\infty\}$, then H fixes $\{\infty\}$ so $H \leq \mathrm{Sim}(\mathcal{N})$. If $L(H)$ contains more than one point, say $\{0, \infty\}$ at least, then $\mathcal{A}$ fixes $\{0, \infty\}$ pointwisely as above which follows that $\mathcal{A} \leq \mathrm{U}(n) \times \mathbb{R}^+$. This implies $\mathcal{A}$ fixes $\{0, \infty\}$ exactly. Hence $L(H) = \{0, \infty\}$ by (3) of Proposition 2.1. Therefore, $H \leq \mathrm{U}(n) \times \mathbb{R}^+$. This reduces to **Step 1**, however Γ would be discrete by the classification of **Step 1**. As a consequence, either this case does not occur or H must be semisimple. By the classification of connected semisimple groups of $\mathrm{PU}(n+1,1)$ (cf. [2]), it follows that $H = \mathrm{P}(\mathrm{U}(k,1) \times \mathrm{U}(n-k+1))$ or $\mathrm{P}(\mathrm{O}(k+1,1) \times \mathrm{U}(n-k))$.
In each case, $L(H) = S^{2k-1}$ $(k = 1, \ldots, n)$ or S^k $(k = 0, \ldots, n)$. In particular note that $L(\bar{\Gamma}) = S^{2k-1}$ or S^k respectively. By Lemma 5.1 and (a) of Subsection 4.1, there is a $\bar{\Gamma}$-invariant Riemannian metric on $S^{2n+1} - L(\bar{\Gamma})$. As $L(\bar{\Gamma}) \subset \Lambda$, we have :

$$(\rho, \mathrm{dev})\colon (\pi, \tilde{M}) \to (\Gamma, S^{2n+1} - L(\bar{\Gamma})).$$

If $L(\bar{\Gamma}) = S^{2k-1}$ and $k \neq n$, then $\mathrm{dev}\colon \tilde{M} \to S^{2n+1} - S^{2k-1}$ is diffeomorphic. In particular, Γ would be discrete so it does not occur. If $L(\bar{\Gamma}) = S^k$ and $n \neq 1$ or $n = 1,\ k = 0$, $\mathrm{dev}\colon \tilde{M} \to S^{2n+1} - S^k$ is diffeomorphic, again this case does not occur. Therefore we arrive at the following conclusions:

(i) $(\rho, \mathrm{dev})\colon (\pi, \tilde{M}) \to (\Gamma, S^{2n+1} - S^{2n-1})$ is an equivariant covering map such that $\bar{\Gamma} = \mathrm{U}(n,1)$. Here $S^{2n+1} - S^{2n-1} = V_{-1}^{2n+1}$.

(ii) $(\rho, \mathrm{dev})\colon (\pi, \tilde{M}) \to (\Gamma, S^{3} - S^{1})$ is an equivariant covering map such that $\bar{\Gamma} = \mathrm{PO}(2,1)$. Here $S^3 - S^1 = T_1\mathbb{H}^2_{\mathbb{R}}$.

We now prove that cases **(i)** and **(ii)** above do not occur. Let

$$1 \to \mathcal{Z}\mathrm{U}(n,1) \to \mathrm{U}(n,1) \to \mathrm{PU}(n,1) \to 1$$

be the extended exact sequence where $\mathcal{Z}\mathrm{U}(n,1)$ is the center S^1. This lifts to the central extension :

$$1 \to \mathbb{R} \to \mathrm{U}(n,1)^{\sim} \xrightarrow{q} \mathrm{PU}(n,1) \to 1. \tag{5.4}$$

As $\mathrm{dev}\colon \tilde{M} \to V_{-1}^{2n+1}$ is the covering, this lifts to a diffeomorphism :

$$(\tilde{\rho}, \tilde{\mathrm{dev}})\colon (\pi_1(M), \tilde{M}) \to (\tilde{\Gamma}, \tilde{V}_{-1}^{2n+1})$$

where $\tilde{\rho}(\pi) = \tilde{\Gamma} = \tilde{\mathrm{dev}} \cdot \pi \cdot \tilde{\mathrm{dev}}^{-1} \leq \mathrm{U}(n,1)^{\sim}$ because $(\mathrm{U}(n,1), V_{-1}^{2n+1})$ lifts to $(\mathrm{U}(n,1)^{\sim}, \tilde{V}_{-1}^{2n+1})$. There is the commutative diagram:

$$\begin{array}{ccccccc} & & \tilde{\Gamma} & \leq & \mathrm{U}(n+1)^{\sim} & & \\ & {}^{\tilde{\rho}}\nearrow & \downarrow\tilde{p} & & \tilde{p}\downarrow & \searrow^{q} & \\ \pi & \xrightarrow{\rho} & \Gamma & \leq & \mathrm{U}(n+1) & \xrightarrow{p} & \mathrm{PU}(n,1) \end{array} \tag{5.5}$$

Since $\tilde{\mathrm{dev}}$ is a diffeomorphism, $\tilde{\Gamma}$ is a discrete subgroup. It follows from the exact sequence (5.4) that $\overline{q(\tilde{\Gamma})}^0$ is solvable (cf. [17]). On the other hand, $p(\bar{\Gamma}) \leq \overline{q(\tilde{\Gamma})}$ by the commutativity of (5.5). As $\bar{\Gamma} = \mathrm{U}(n,1)$ in this case from (i), $\overline{q(\tilde{\Gamma})}^0 = \mathrm{PU}(n,1)$ which is a contradiction.

For (ii), there is an equivariant diffeomorphism:

$$(\tilde{\rho}, \tilde{\mathrm{dev}})\colon (\pi_1(M), \tilde{M}) \to (\tilde{\Gamma}, \widetilde{S^3 - S^1}).$$

Similarly, $\tilde{\rho}(\pi) = \tilde{\Gamma} = \tilde{\mathrm{dev}} \cdot \pi \cdot \tilde{\mathrm{dev}}^{-1} \leq \widetilde{\mathrm{PO}}(2,1)$ is discrete. As $\tilde{p}(\tilde{\Gamma}) = \Gamma$, the closure $\bar{\Gamma}^0$ is solvable again but this is impossible because $\bar{\Gamma} = \mathrm{PO}(2,1)$. As a consequence, **Case (2)** does not occur. $\square$

For the rest of this subsection, we show **Case (1)** that H is compact can occur.

Proposition 5.1. *There exists a developing pair*

$$(\rho, \mathrm{dev})\colon (\pi, \tilde{M}) \to (\Gamma, S^{2n+1} - S^{2n-1})$$

such that $\bar{\Gamma}^0 = S^1$ *and* $L(\Gamma) = S^{2n-1}$. *Here* S^1 *is the center* $\mathcal{Z}\mathrm{U}(n,1)$.

Proof. Consider the commutative diagram:

$$\begin{array}{ccccc}
\mathbb{Z} & = & \mathbb{Z} & & \\
\downarrow & & \downarrow & & \\
\mathbb{R} & \longrightarrow & \mathrm{U}(n,1)^{\sim} & \xrightarrow{q} & \mathrm{PU}(n,1) \\
\downarrow & & \downarrow{\scriptstyle \tilde{p}} & & || \\
S^1 & \longrightarrow & \mathrm{U}(n,1) & \xrightarrow{p} & \mathrm{PU}(n,1)
\end{array} \tag{5.6}$$

Let $\Gamma \leq \mathrm{U}(n,1)$ be a discrete cocompact subgroup such that $p(\Gamma)$ is also discrete cocompact. (For example, choose the fundamental group of a compact complex hyperbolic manifold $\mathbb{H}_{\mathbb{C}}^n/Q$. By the exact sequence $1 \to \mathbb{Z}_{n+1} \to \mathrm{SU}(n,1) \xrightarrow{p} \mathrm{PU}(n,1) \to 1$, then put $\Gamma = p^{-1}(Q)$.)
Put $\tilde{\Gamma} = \tilde{p}^{-1}(\Gamma)$ in the vertical sequence above. Then there is a central group extension:

$$1 \to \mathbb{Z} \to \tilde{\Gamma} \xrightarrow{q} Q \to 1. \tag{5.7}$$

Here we put $Q = q(\tilde{\Gamma}) = p(\Gamma)$ which is discrete in $\mathrm{PU}(n,1)$. This group extension gives a cocycle $[f] \in H^2(Q, \mathbb{Z})$. Let $[a \cdot f] \in H^2(Q, \mathbb{R})$ for some irrational number a. Taking the set $a\mathbb{Z} \times Q$ with group law:

$$(a \cdot m, \alpha)(a \cdot n, \beta) = (a(m + n + f(\alpha, \beta)), \alpha\beta),$$

this gives a group $\tilde{\Gamma}_a$ for which there is a central extension:

$$1 \to a\mathbb{Z} \to \tilde{\Gamma}_a \xrightarrow{q} Q \to 1. \tag{5.8}$$

As is noted that $(a \cdot m, \alpha) = (a \cdot m, 1) \cdot (0, \alpha) \in \mathbb{R} \cdot \tilde{\Gamma} \leq \mathrm{U}(n,1)^{\sim}$, $\tilde{\Gamma}_a$ is a subgroup of $\mathrm{U}(n,1)^{\sim}$. Here we chose a normalized 2-cocycle f, i.e. $f(1, \alpha) = f(\alpha, 1) = 0$. Note that $\tilde{\Gamma}_a$ is discrete because both $a\mathbb{Z}$ and Q are discrete in $\mathrm{U}(n,1)^{\sim}$ and $\mathrm{PU}(n,1)$ respectively. We know that $\mathrm{U}(n,1)^{\sim}$ acts properly on $\widetilde{S^{2n+1}-S^{2n-1}}$ as well as $\mathrm{U}(n,1)$ does on $V_{-1}^{2n+1}=S^{2n+1}-S^{2n-1}$. Hence $\tilde{\Gamma}_a$ acts properly discontinuously on $\widetilde{S^{2n+1}-S^{2n-1}}$.

Put $M=\widetilde{S^{2n+1}-S^{2n-1}}/\tilde{\Gamma}_a$. Then M is a spherical CR manifold whose developing map is the projection:

$$\tilde{p}:\widetilde{S^{2n+1}-S^{2n-1}}\to S^{2n+1}-S^{2n-1}$$

and the holonomy group is $\tilde{p}(\tilde{\Gamma}_a)\leq \mathrm{U}(n,1)$. As $\tilde{p}(a\mathbb{Z})\leq S^1$ for the irrational number a, it follows that $\overline{\tilde{p}(a\mathbb{Z})}=S^1$. By (5.6), there is the exact sequence: $1\to\tilde{p}(a\mathbb{Z})\to\tilde{p}(\tilde{\Gamma}_a)\to q(\tilde{\Gamma}_a)=Q\to 1$ such that Q is discrete as before. This shows $\overline{\tilde{p}(\tilde{\Gamma}_a)}^{0}=\overline{\tilde{p}(a\mathbb{Z})}=S^1$ and $\overline{\tilde{p}(\tilde{\Gamma}_a)}$ is cocompact in $\mathrm{U}(n,1)$. In particular, we have $L(\tilde{p}(\tilde{\Gamma}_a))=S^{2n-1}$. □

Remark 5.1. This gives an indiscrete holonomy group on the contrary to the result of [7] for $n=1$.

5.2. Spherical CR-structures on $M(p,q,r)$

It is known that the link of an isolated singular point admits CR-structure [18], [4]. In this connection, CR homogeneous space forms are discussed in [6]. See also [1] for the homogeneous spherical CR space forms. In this subsection, we give spherical homogeneous CR-structures on the Brieskorn manifold $M(p,q,r)$ whose holonomy group is not necessarily discrete. Theorem C of Introduction is stated more precisely:

Theorem 5.2. *Let $M(p,q,r)$ be the 3-dimensional Brieskorn manifold. Put $\kappa=p^{-1}+q^{-1}+r^{-1}-1$. Then $M(p,q,r)$ admits a spherical CR structure whose holonomy group Γ satisfies that $L(\Gamma)=\emptyset$, $\{\infty\}$, or $\mathcal{S}^1$ according as $\kappa>0$, $\kappa=0$, or $\kappa<0$ respectively. Here $\mathcal{S}^1$ is a geometric circle in S^3. Moreover, the developing map* dev *is a covering of the universal cover of $M(p,q,r)$ onto $S^3-L(\Gamma)$.*

(1) *The above spherical CR-structure on $M(p,q,r)$ is homogeneous so that* dev *is a diffeomorphism for $\kappa>0$, $\kappa=0$ and an infinite cyclic covering map for $\kappa<0$.*

(2) *For $\kappa<0$, there is a homogeneous spherical CR-structure whose holonomy group is indiscrete.*

Proof. Recall that Milnor [15] has proved that there exists a 3-dimensional simply connected Lie group G such that $M(p,q,r)$ is diffeomorphic to $\Pi\backslash G$ where Π is a discrete cocompact subgroup of G acting from the left. According to the sign $\kappa = p^{-1}+q^{-1}+r^{-1}-1$, G is as follows:

(i) $\kappa > 0$. $G = \mathrm{SU}(2)$ and Π is a finite subgroup.
(ii) $\kappa = 0$. $G = \mathcal{N}$, the Heisenberg Lie group and Π is a cocompact subgroup.
(iii) $\kappa < 0$. $G = \widetilde{\mathrm{SL}(2,\mathbb{R})}$, the universal covering of $\mathrm{PSL}(2,\mathbb{R})$ and Π is a cocompact subgroup.

It suffices to construct a developing pair:

$$(\rho, \mathrm{dev}) : (\Pi, G) \longrightarrow (\mathrm{PU}(2,1), S^3)$$

Case (i) $G = \mathrm{SU}(2)$ is identified with S^3 by the orbit map $\mathrm{dev}(g) = gx$ where x is the point $(1,0,0,0) \in S^3$. Obviously, dev is equivariant with respect to the inclusion $\mathrm{SU}(2) \to \mathrm{PU}(2,1)$ so that $\Pi\backslash G = \Pi\backslash S^3$. In this case, dev is diffeomorphism and $L(\Pi) = \emptyset$.

Case (ii) As we have already seen that $S^3 - \{\infty\} = \mathcal{N}$ is the Heisenberg Lie group, it follows that $\Pi\backslash G = \Pi\backslash\mathcal{N}$. In this case, dev is diffeomorphism and $L(\Pi) = \{\infty\}$.

For **Case (i), (ii)**, $\Pi\backslash G$ is a homogeneous spherical CR space form.

Case (iii) We first give two homogeneous spherical CR-structures on $\Pi\backslash G$. Recall that $\mathrm{PSL}(2,\mathbb{R})$ is isomorphic to $\mathrm{PO}(2,1)^0$ or to $\mathrm{PU}(1,1)$ as Lie groups respectively. Suppose $\varphi\colon \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PO}(2,1)^0$ is an isomorphism. Let $\tilde\varphi\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \widetilde{\mathrm{PO}(2,1)}^0$ be the isomorphism of the universal covering groups. As $\widetilde{\mathrm{PO}(2,1)}^0$ acts transitively on $\widetilde{S^3 - S^1} = \widetilde{T_1\mathbb{H}^2_{\mathbb{R}}}$, choosing $\tilde x \in \widetilde{S^3 - S^1}$, we have a diffeomorphism:

$$\widetilde{\mathrm{dev}}\colon G = \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \widetilde{S^3 - S^1} \tag{5.9}$$

such that $\widetilde{\mathrm{dev}}(g) = \tilde\varphi(g)\cdot\tilde x$. Since $\widetilde{\mathrm{dev}}$ is equivariant with respect to $\tilde\varphi$, it follows that $\Pi\backslash G \cong \tilde\varphi(\Pi)\backslash\widetilde{S^3 - S^1}$ which admits a developing map as the projection :

$$(\rho, \mathrm{dev})\colon (\Pi, G) \to (\mathrm{PO}(2,1), T_1\mathbb{H}^2_{\mathbb{R}})$$

such that the limit set $L(\rho(\Pi)) = S^1 = \partial\mathbb{H}^2_{\mathbb{R}}$.

Suppose that $\psi\colon \mathrm{PSL}(2,\mathbb{R}) \cong \mathrm{PU}(1,1)$ is another isomorphism. As there is a finite covering: $\mathbb{Z}_2 \to \mathrm{SU}(1,1) \to \mathrm{PU}(1,1)$, we have an isomorphism $\tilde{\psi}\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \widetilde{\mathrm{SU}(1,1)}$ of the universal covering groups. As $\mathrm{PU}(1,1)$ acts transitively on $V^3_{-1} = S^3 - S^1$, $\widetilde{\mathrm{SU}(1,1)}$ acts simply transitively on $\widetilde{S^3 - S^1}$. There is a diffeomorphism ($\tilde{x} \in \widetilde{S^3 - S^1}$) :

$$\widetilde{\mathrm{dev}}\colon \widetilde{\mathrm{SL}(2,\mathbb{R})} \longrightarrow \widetilde{S^3 - S^1}$$

defined by $\mathrm{dev}(g) = \tilde{\psi}(g)\cdot\tilde{x}$ which is equivariant with respect to $\tilde{\psi}$. Therefore,

$$\Pi\backslash G \cong \tilde{\psi}(\Pi)\backslash\widetilde{S^3 - S^1}.$$

Hence $\Pi\backslash G$ admits a developing pair

$$(\rho, \mathrm{dev}) : (\Pi, G) \longrightarrow (\mathrm{PU}(1,1), S^3 - S^1)$$

as before. In particular, $L(\rho(\Pi)) = S^1 = \partial\mathbb{H}^1_{\mathbb{C}}$. In each case, the limit circle is diffeomorphic to a geometric circle $\mathcal{S}^1$.

Next we give a homogeneous spherical CR-structure on $\Pi\backslash G$ but the holonomy group is indiscrete. Recall that there is the central group extension:

$$1 \to \mathbb{Z} \to G = \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathrm{PSL}(2,\mathbb{R}) \overset{\psi}{\cong} \mathrm{PU}(1,1) \to 1. \tag{5.10}$$

Let $\widetilde{\mathrm{SL}(2,\mathbb{R})} = \tilde{K}\cdot AN$ be the Iwasawa decomposition where we put

$$\tilde{K} = \{e^r \mid r \in \mathsf{R}\}$$

which is the lift of $K = \mathrm{SO}(2) \leq \mathrm{PSL}(2,\mathbb{R})$ to $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. (Here R stands for $\mathbb{R}$ while $\tilde{K}$ is viewed as a multiplicative group.) Let the infinite group Z act on $\mathsf{R} \times \widetilde{\mathrm{SL}(2,\mathbb{R})}$ as

$$(n, (r, g)) = (n + r, e^{-n}\cdot g).$$

Since $\mathbb{Z}$ of (5.10) is a center of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$, Z is a central subgroup of $\mathsf{R} \times$

$\widetilde{\mathrm{SL}(2,\mathbb{R})}$. We form the Lie group G with central subgroup R;

$$\mathsf{G} = \mathsf{R} \underset{\mathsf{Z}}{\times} \widetilde{\mathrm{SL}(2,\mathbb{R})}.$$

Let $\mathrm{diag}\mathsf{R} = \{(r, e^{-r}) \in \mathsf{R} \times \tilde{K}\}$ which is invariant under Z by the above action. Then G has the compact subgroup $\mathsf{K} = \mathsf{Z}\backslash\mathrm{diag}\mathsf{R}$ isomorphic to S^1. The natural inclusion $\iota\colon G \to \mathsf{G}$ defined by $\iota(g) = [(0,g)]$ induces an equivariant diffeomorphism $\bar{\iota}\colon G \to \mathsf{G}/\mathsf{K}$ such that

$$\bar{\iota}\colon \Pi\backslash G \cong \iota(\Pi)\backslash\mathsf{G}/\mathsf{K}.$$

Put $\tilde{\Gamma} = \iota(\Pi)$. Let $\tau : \mathsf{G} \to \mathsf{G}$ be the isomorphism defined by

$$\tau([r,x]) = [a \cdot r, x].$$

Using the same notations of the proof of Proposition 5.1 for $n = 1$, it is easily checked that τ maps $\tilde{\Gamma}$ isomorphically onto $\tilde{\Gamma}_a$ (cf. (5.7), (5.8)). The equivariant map τ induces a diffeomorphism:

$$\Pi\backslash G \xrightarrow{\bar{\iota}} \tilde{\Gamma}\backslash\mathsf{G}/\mathsf{K} \xrightarrow{\bar{\tau}} \tilde{\Gamma}_a\backslash\mathsf{G}/\tau(\mathsf{K}).$$

On the other hand, as G is identified with $\mathrm{U}(1,1)^{\sim}$ which acts transitively on $\widetilde{S^3 - S^1}$ so that

$$\mathsf{G}/\tau(\mathsf{K}) = \mathrm{U}(1,1)^{\sim}/\tau(\mathsf{K}) = \widetilde{S^3 - S^1}.$$

Passing to $(\tilde{\Gamma}_a, \mathsf{G})$ and projecting down to $S^3 - S^1$ by $\tilde{p}$, we get a developing pair

$$(\rho, \mathrm{dev})\colon (\Pi, G) \to (\mathrm{PU}(1,1), S^3 - S^1)$$

such that $\rho(\Pi) = \tilde{p}(\tilde{\Gamma}_a)$ is indiscrete in $\mathrm{PU}(1,1)$ with $L(\rho(\Pi)) = S^1$. $\square$

Remark 5.2. When our spherical CR structure on $M(p,q,r)$ is homogeneous, by Corollary 4.1 or Theorem 5.1, the developing pair is unique. This occurs for $\kappa \geq 0$. When $\kappa < 0$, a finite cover $L(p,q,r)$ of $M(p,q,r)$ is a nontrivial circle bundle over an oriented closed surface. Take a compact real hyperbolic surface $\mathbb{H}^2_{\mathbb{R}}/\Gamma_1$ and a compact complex hyperbolic line $\mathbb{H}^1_{\mathbb{C}}/\Gamma_2$. Sewing along the common geodesic circle $S^1 = \mathbb{H}^1_{\mathbb{R}}/\mathbb{Z}$ gives a closed surface Σ. It is shown in [10] that there is a faithful representation of $\pi_1(\Sigma) = \Gamma$

into $\mathrm{PU}(2,1)$ whose limit set $L(\Gamma)$ is a topological (non-rectifiable) circle. Moreover $S^3 - L(\Gamma)/\Gamma$ is a *nontrivial* circle bundle over Σ. It might be possible that some finite cover $L(p,q,r)$ is diffeomorphic to $S^3 - L(\Gamma)/\Gamma$.

Problem 5.1. *Does $M(p,q,r)$ admit a spherical CR-structure whose limit set of the holonomy group is a topological circle ?*

In other words, let $F \to L(p,q,r) \to M(p,q,r)$ be a covering for some finite group F. Does F preserve the spherical CR-structure of $L(p,q,r)$?

See the deformation of complex hyperbolic discrete subgroups and spherical CR manifolds for [19], [9].

6. Limit Set of Real Pseudo-Hyperbolic Manifolds

As an application, we shall define the limit set on the boundary of the pseudo-Riemannian manifold. Put

$$V_-^{n,2} = \{x \in \mathbb{R}^{n+2} \mid \mathcal{B}(x,x) = x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - x_{n+2}^2 < 0\}.$$

If $P_{\mathbb{R}} : \mathbb{R}^{n+2} - \{0\} \to \mathbb{RP}^{n+1}$ is the canonical projection, then the real pseudo-hyperbolic space $\mathbb{H}_{\mathbb{R}}^{n,1}$ is defined to be $P_{\mathbb{R}}(V_-^{n,2})$. For this, the $n+1$-dimensional quadrics

$$V_{-1}^{n,2} = \{x \in \mathbb{R}^{n+2} \mid x_1^2 + \cdots + x_n^2 - x_{n+1}^2 - x_{n+2}^2 = -1\}$$

with Lorentz metric g is the complete pseudo-Riemannian manifold of signature $(n,1)$ and of constant curvature -1. Since $P_{\mathbb{R}}(V_-^{n,2}) = P_{\mathbb{R}}(V_{-1}^{n,2})$ and $P_{\mathbb{R}} : V_{-1}^{n,2} \to \mathbb{H}_{\mathbb{R}}^{n,1}$ is a two-fold covering, $\mathbb{H}_{\mathbb{R}}^{n,1}$ is a complete pseudo-Riemannian manifold of signature $(n,1)$ and of curvature -1. The action $\mathrm{O}(n,2)$ on $V_-^{n,2}$ induces an action on $\mathbb{H}_{\mathbb{R}}^{n,1}$. The kernel of this action is the center $\mathbb{Z}/2 = \{\pm 1\}$ whose quotient is called the real pseudo-hyperbolic group $\mathrm{PO}(n,2)$. We recall the projective compactification of $\mathbb{H}_{\mathbb{R}}^{n,1}$ by taking the closure $\overline{\mathbb{H}_{\mathbb{R}}^{n,1}}$ in $\mathbb{RP}^{n+1}$. Consider the commutative diagram:

$$\begin{array}{ccccc}
\mathbb{R}^* & \longrightarrow & (\mathrm{GL}(n+2,\mathbb{R}), \mathbb{R}^{n+2} - \{0\}) & \xrightarrow{P} & (\mathrm{PGL}(n+2,\mathbb{R}), \mathbb{RP}^{n+1}) \\
\| & & \bigcup & & \bigcup \\
\mathbb{R}^* & \longrightarrow & (\mathrm{O}(n,2), V_-^{n,2} \cup V_0^{n,1}) & \xrightarrow{P} & (\mathrm{PO}(n,2), \mathbb{H}_{\mathbb{R}}^{n,1} \cup S^{n-1,1})
\end{array}$$

It follows that

$$\overline{\mathbb{H}_{\mathbb{R}}^{n,1}} = \mathbb{H}_{\mathbb{R}}^{n,1} \cup S^{n-1,1}.$$

From this viewpoint, it is easy to check that the pseudo-hyperbolic action of $\mathrm{PO}(n,2)$ on $\mathbb{H}_{\mathbb{R}}^{n,1}$ extends to *conformal action* of $S^{n-1,1}$. We use the complex coordinate so that

$$\mathbb{H}_{\mathbb{R}}^{n,1} = \begin{cases} \{[z_1,\dots,z_m,w] \in \mathbb{C}^{m+1} \mid |z_1|^2+\cdots+|z_m|^2-|w|^2=-1\} \\ \qquad (n=2m). \\ \{[z_1,\dots,z_m,x,w] \in \mathbb{C}^m \times \mathbb{R} \times \mathbb{C} \mid \\ |z_1|^2+\cdots+|z_m|^2+x^2-|w|^2=-1\} \\ \qquad (n=2m+1). \end{cases} \tag{6.1}$$

Embed S^1 into the subgroup $T^{m+1} \leq \mathrm{O}(n,2)$ as follows.

$$S^1 = \begin{cases} (e^{\mathbf{i}\theta},\dots,e^{\mathbf{i}\theta};e^{\mathbf{i}\theta}) & n=2m, \\ (e^{\mathbf{i}\theta},\dots,e^{\mathbf{i}\theta},1;e^{\mathbf{i}\theta}) & n=2m+1. \end{cases}$$

Then S^1 acts properly on $\overline{\mathbb{H}_{\mathbb{R}}^{n,1}}$ such that

$$t([z_1,\dots,z_m,(x),w]) = [e^{\mathbf{i}\theta}z_1,\dots,e^{\mathbf{i}\theta}z_m,(x),e^{\mathbf{i}\theta}w] \quad (t \in S^1).$$

Since $w \neq 0$, it acts freely so that there is the principal bundle:

$$S^1 \to \overline{\mathbb{H}_{\mathbb{R}}^{n,1}} \xrightarrow{P} \bar{W} \tag{6.2}$$

where we put $W = \mathbb{H}_{\mathbb{R}}^{n,1}/S^1$ and $\partial W = S^{n-1,1}/S^1$. Put

$$p = (z_1,\dots,z_m,(x),w)$$

where (x) means either empty or x depending on whether $n=2m$ or $n=2m+1$ and $|z_1|^2+\cdots+|z_m|^2+(x^2)-|w|^2=-1$. Since S^1 acts freely on $V_{-1}^{n,2}$ as above, S^1 induces the vector field V on $V_{-1}^{n,2}$ such that

$$V_p = (\mathbf{i}z_1,\dots,\mathbf{i}z_m,(0),\mathbf{i}w).$$

It follows that $\mathcal{B}(V_p,V_p) = |z_1|^2+\cdots+|z_m|^2-|w|^2 = -1-(x^2)$. Let ξ be the vector field on $\mathbb{H}_{\mathbb{R}}^{n,1}$ with $P_*(V)=\xi$. Therefore, $g(\xi,\xi)=\mathcal{B}(V_p,V_p) < -1$. Hence P induces a Riemannian metric $\hat{g}$ on W such that $P_* : TS^{1\perp} \to TW$

is an isometry at each point of $\mathbb{H}_{\mathbb{R}}^{n,1}$. In particular, $P : (\mathbb{H}_{\mathbb{R}}^{n,1}, g) \longrightarrow (W, \hat{g})$ is a pseudo-Riemannian submersion. Choose an orthonormal frame $\{e_i\}$ on $\mathbb{H}_{\mathbb{R}}^{n,1}$ with respect to g. It follows from the O'Neill's formula (cf. [16]) that

$$4k(P_* e_i, P_* e_j) = K(e_i, e_j) + \frac{3}{4} g([e_i, e_j]^F, [e_i, e_j]^F)$$

where $[e_i, e_j]^F$ is the summand of $[e_i, e_j]$ to the fiber TS^1. In particular, $g([e_i, e_j]^F, [e_i, e_j]^F) < 0$ so that $4k(P_* e_i, P_* e_j) \leq -1$ where $K(e_i, e_j) = -1$. Hence $(W, \hat{g})$ is a complete simply connected Riemannian manifold of strictly negative curvature.

Let $\hat{G} \leq \mathrm{Iso}(W)$ be a subgroup. Note that W is a visibility manifold. So the limit set of $\hat{G}$ is defined to $L(\hat{G}) = \overline{\hat{G} \cdot p} \cap \partial W$ for some $p \in W$. It follows as before.

Proposition 6.1. *Let $\hat{\Lambda}$ be a $\hat{G}$-invariant closed subset of ∂W. If $\hat{\Lambda}$ contains more than one point, then $L(\hat{G}) \subset \hat{\Lambda}$.*

Definition 6.1. Let S^1 be the subgroup of $\mathrm{O}(n, 2)$ as above. Put the centralizer

$$\mathcal{Z}(S^1) = \{g \in \mathrm{O}(n, 2) \mid gt = tg \ (\forall\, t \in S^1)\}.$$

Put $\hat{\mathcal{Z}}(S^1) = \mathcal{Z}(S^1)/S^1$ which acts on W.

It is easy to see the following.

Proposition 6.2. *$P : (\mathcal{Z}(S^1), \mathbb{H}_{\mathbb{R}}^{n,1}) \to (\hat{\mathcal{Z}}(S^1), W)$ is equivariant. In particular, the group $\hat{\mathcal{Z}}(S^1)$ acts as isometries of $(W, \hat{g})$.*

Here comes the definition of limit set for $\mathbb{H}_{\mathbb{R}}^{n,1}$.

Definition 6.2. Let G be a subgroup of $\mathcal{Z}(S^1)$ and $p \in \mathbb{H}_{\mathbb{R}}^{n,1}$. The limit set of G is defined by the intersection $\mathcal{L}(G) = \overline{G \cdot S^1(p)} \cap S^{n-1,1}$.

We must prove the following.

Lemma 6.1. *The limit set $\mathcal{L}(G)$ does not depend on the choice of points in $\mathbb{H}_{\mathbb{R}}^{n,1}$.*

Proof. Let $p, q \in \mathbb{H}_{\mathbb{R}}^{n,1}$ so $P(p), P(q) \in W$. Given a sequence $\{g_i\} \leq \mathcal{Z}(S^1)$, suppose that $\lim g_i p = x, \lim g_i q = y \in \partial W$. It suffices to prove that

$\lim g_i t\cdot p = \lim g_i q$ for some $t\in S^1$. Since it follows $P(g_ip) = \hat{g}_iP(p)$ as above, there is a unique geodesic $[\hat{g}_iP(p), \hat{g}_iP(q)]$ in W. On the other hand, the length of $[\hat{g}_iP(p), \hat{g}_iP(q)]$ is equal to that of $[P(p), P(q)]$, so we see that $\lim \hat{g}_iP(p) = \lim \hat{g}_iP(q) \in \partial W$ whenever they approach the boundary of W. As $P(x) = P(\lim g_ip) = P(\lim g_iq) = P(y)$, $y = t\cdot x$ for some $t\in S^1$. Hence $\lim g_it\cdot p = \lim g_iq$. $\square$

Proposition 6.3.(Minimality) *Let Λ be a G-invariant closed subset in $S^{n-1,1}$. Suppose Λ is invariant under the above S^1-action. If the quotient Λ/S^1 contains more than one point, then $\mathcal{L}(G)\subset\Lambda$.*

Proof. Choose two points x, y from Λ such that they are distinct in Λ/S^1. As $P(x), P(y)$ are also distinct points in ∂W, there is a unique geodesic $[P(x), P(y)]\subset\bar{W}$. Choose a point $P(w)\in[P(x), P(y)]$ for some $p\in\mathbb{H}^{n,1}_{\mathbb{R}}$. For any infinite number of elements $\{g_i\}\leq G$, suppose that $\lim g_iw \in \mathcal{L}(G)\subset S^{n-1,1}$. As $\hat{g}_iP(w)\in[\hat{g}_iP(x), \hat{g}_iP(y)]$, it follows that $\lim\hat{g}_iP(w) = \lim\hat{g}_iP(x)$ or $\lim\hat{g}_iP(w) = \lim\hat{g}_iP(y)$ because the geodesics $[\hat{g}_iP(x), \hat{g}_iP(y)]$ converges to either point of ∂W or a geodesic $[\lim\hat{g}_iP(x), \lim\hat{g}_iP(y)]$. In each case, since $\lim\hat{g}_iP(x), \lim\hat{g}_iP(y)\in P(\Lambda)$ by Proposition 6.1. It implies that $P(\lim g_iw) = \lim\hat{g}_iP(w)\in P(\Lambda)$ so $\lim g_iw\in S^1\cdot\Lambda = \Lambda$. Hence $\mathcal{L}(G)\subset\Lambda$. $\square$

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