Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **4** (2009), No. 2, pp. 119-188

GEOMETRIC ANALYSIS ON A STEP 2 GRUSIN OPERATOR

BY

CHIN-HUEI CHANG, DER-CHEN CHANG*, BERNARD GAVEAU,
PETER GREINER AND HSUAN-PEI LEE

Abstract

The Grusin operator $\Delta_G = \frac{1}{2}(\partial_x^2 + x^2\partial_y^2)$, $x, y \in \mathbb{R}$, is studied by Hamilton-Jacobi theory. In particular, we find all the geodesics of Δ_G of the induced nonholonomic geometry, construct a modified complex action f which allows us to obtain the heat kernel P_t of Δ_G. The small time asymptotics of P_t at all critical points of f are computed. Finally we discuss the connection between Δ_G and the subLaplacian of the 1-dimensional Heisenberg group.

1. Introduction

We are interested in the geometric and analytic properties of the step two Grusin operator

$$\Delta_G = \frac{1}{2}(X_1^2 + X_2^2),$$

where the vector fields X_1 and X_2 in $\mathbb{R}^2$ are given by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial y}, \quad (x, y) \in \mathbb{R}^2.$$

Received May 12, 2009.

Communicated by T. P. Liu.

AMS Subject Classification: Primary: 53C17; Secondary: 34K10, 35H20.

Key words and phrases: Grusin operator, geodesic, heat kernel, small time asymptotics.

*The second author is partially supported by a research grant from United State Army Research Office and a competitive research grant at Georgetown University.

Note that X_1 and X_2 are linearly independent everywhere except on the y-axis, where X_2 vanishes. Consequently, the operator Δ_G is elliptic except on the y-axis and the geometry of Δ_G is not Riemannian. On the other hand, $[X_1, X_2] = \frac{\partial}{\partial y}$, so Chow's theorem [6] holds, and any two points on the (x, y)-plane can be connected by a piecewise differentiable horizontal curve; a horizontal curve is a curve whose tangents are linear combinations of X_1 and X_2.

Our main tool is the Hamilton-Jacobi theory of bicharacteristics.(see [1], [2], [7]). The Hamiltonian function H associated with the symbol of $-\Delta_G$ is

$$H(x, y, \xi, \eta) = \frac{1}{2}\Big(\xi^2 + x^2\eta^2\Big)$$

where (x, y, ξ, η) are the coordinates of the cotangent bundle $T^*\mathbb{R}^2$. Note that by setting $\eta = 1$, H corresponds to the 1-dimensional harmonic oscillator in quantum mechanics. A geodesic is the projection of a bicharacteristic curve of H into the (x, y) plane. In order to obtain geodesics between two points (x_0, y_0) and (x, y) in $\mathbb{R}^2$, one solves the Hamiltonian system

$$\begin{aligned} \frac{dx}{ds} &= H_\xi = \xi, \qquad & \frac{d\xi}{ds} &= -H_x = -x\eta^2 \qquad \Big(\text{so } \frac{d^2x}{ds^2} = -x\eta^2\Big), \\ \frac{dy}{ds} &= H_\eta = x^2\eta, \qquad & \frac{d\eta}{ds} &= -H_\eta = 0, \end{aligned} \tag{2.4}$$

with boundary conditions given by

$$x(0) = x_0, \quad y(0) = y_0, \quad x(1) = x, \quad y(1) = y. \tag{2.5}$$

Since the operator Δ_G is translation invariant along the y-direction, one may assume that $y(0) = 0$, in other words, only the quantity $y - y_0$ matters. We shall mainly consider the case $y \geq 0$ which implies $\eta \geq 0$ by (2.4); the case $y \leq 0$ (so $\eta \leq 0$) is obatined by symmetry.

In solving (2.4) with $y \geq 0$ one finds that there are more than one bicharacteristic curve satisfying (2.5) and the projected curves in (x, y)-plane can be more than one, too. The length of a horizontal curve is defined in (4.26), (4.27). The length of the shortest geodesics is called the Carnot-Carathéodory distance. Here we consider all the geodesics between any two distinct points as Perry [11] showed that each geodesic contributed to the asymptotics of the resolvent in the Heisenberg group case.

From (2.4) one gets immediately that

$$x(s) = x_0 \cos(\eta s) + B \sin(\eta s), \tag{2.6}$$

with

$$B = \frac{x - x_0 \cos \eta}{\sin \eta}, \qquad \text{if } \eta \neq k\pi \tag{2.9}$$

and

$$y(s) = \eta \Big[\frac{1}{2}(x_0^2 + B^2)s + \frac{1}{4\eta}(x_0^2 - B^2)\sin(2\eta s) + \frac{x_0 B}{\eta}\sin^2(\eta s) \Big]. \tag{2.8}$$

Substituting (2.9) and $s = 1$ in (2.8) one has (cf. (2.10), (2.11), (2.12))

$$y = \frac{1}{4}[(x + x_0)^2 \frac{\eta + \sin \eta}{1 + \cos \eta} + (x - x_0)^2 \frac{\eta - \sin \eta}{1 - \cos \eta}] \tag{*}$$

Those η's satisfying (*) give rise to geodesics connecting $(x_0, 0)$ and (x, y). Clearly $\eta = k\pi$, $k = 1, 2, \ldots$ are not allowed in (*). However, if one puts $s = 1$ and $\eta = k\pi$ in (2.8), the geodesics with $\eta = k\pi$ can still be obtained (see 2.4.2), but the information is only partial. One has to study the behavior of the right hand side of (*) as a function of $\eta \in \mathbb{R}$ in order to have a systematic understanding of the geodesics. In so doing, one can count the number of geodesics, their multiplicity, and most important, one sees those geodesics which are not solutions of (*) are limits (in the sense of (2.23),(2.24)) of the geodesics given by solutions of (*). We call the geodesics coming from the solutions of (*) 'generic' and the rest 'exceptional'. As a result, one only needs to work with generic geodesics or the formula (*). A detailed description of all geodesics is contained in Theorem 2.6. One remarks that not all geodesics with $\eta = k\pi$, $k = 1, 2, \ldots$ are exceptional, see Section 2.6.1(II). Another interesting fact is that there are countably many geodesics connecting $(0, 0)$ and $(0, y)$ for any $y \neq 0$.

Imitating Riemannian case, one uses Hamilton-Jacobi theory to construct an action and then finds the heat kernel of Δ_G. As in our case, the geodesics connecting two distinct points may be multiple, so we uniformise with a parameter and the formal expression for the heat kernel is an integral over this parameter. It turns out that the action is the solution of the eiconal equation (cf.(3.24)). To get the heat kernel one has to find a proper contour to make the integral converge, and then one arrives at the modified complex action

$$f(\tau) = -i\tau y + \frac{\tau}{4}\left[(x+x_0)^2 \tanh\frac{\tau}{2} + (x-x_0)^2 \coth\frac{\tau}{2}\right], \ \tau \in \mathbb{C}. \qquad (3.28)$$

Note that f is holomorphic in $\mathbb{C} \setminus \{ik\pi \mid k = \pm 1, \pm 2, \ldots\}$.

The case $y \geq 0$ corresponds to $\mathrm{Im}\tau \geq 0$ and symmetrically $y \leq 0$ corresponds to $\mathrm{Im}\tau \leq 0$ and as before, we deal with $y \geq 0$ only.

Writing $\tau = u + iv$, we show that all the critical points of f in the upper half plane are exactly of the form iv with v satisfying (*). In fact, the equation $\frac{\partial \mathrm{Im} f}{\partial u}\big|_{u=0, v\geq 0} = 0$ is nothing but (*) with η replaced by v. One should mention here that η_j, $\eta_j \neq m\pi$ for some $m \in \mathbb{Z}$ and of multiplicity k, corresponds to a zero of order $k+1$ of $\mathrm{Im}f$ at $i\eta_j$. On the other hand, $\eta_j \equiv 0$ $\mathrm{mod}(\pi)$ and of multiplicity k, corresponds to a zero of order $k-1$ of $\mathrm{Im}f$ at $i\eta_j$ (here $k \geq 2$ always). Furthermore, we find a curve Γ (cf. Section 4.2) part of $\mathrm{Im}f = 0$, which possesses the property that $\mathrm{Re}f\big|_\Gamma > 0$ and $\mathrm{Re}f$ is strictly decreasing on $\Gamma^- = \Gamma \cap \{u + iv \mid u \leq 0\}$ and $\mathrm{Re}f$ is strictly increasing on $\Gamma^+ = \Gamma \cap \{u + iv \mid u \geq 0\}$. Together with the result $\mathrm{Re}f(0, \eta_j) = \frac{l_j^2}{2}$, where l_j is the length of the geodesic connecting $(x_0, 0)$ and (x, y) with $\eta = \eta_j$, (cf. Theorem 4.7), we show that the lengths of the geodesics are strictly increasing with respect to η_j.

With the help of the properties of the modified complex action f, one can easily verify the $P_t(x_0, x, y)$ of Section 3 is the heat kernel.

In Section 6 we compute the small time asymptotics for $P_t(x_0, x, y)$ at every critical point of f. In particular,

(I) for simple root η_j of (*), the expansion is of the form

$$(2\pi t)^{-\frac{3}{2}} e^{-\frac{f(i\eta_j)}{t}} \sum_{k=1}^{\infty} \alpha_k(\eta_j) t^{\frac{k}{2}};$$

(II) for $\eta_j \neq m\pi$ for some $m \in \mathbb{Z}$, η_j double roots of (*), one has

$$(2\pi t)^{-\frac{3}{2}} e^{-\frac{f(i\eta_j)}{t}} \sum_{k=1}^{\infty} \alpha_k(\eta_j) t^{\frac{k}{3}};$$

(III) for $\eta_j \equiv 0 \bmod(\pi)$, η_j is of multiplicity two, one has

$$(2\pi t)^{-\frac{3}{2}} e^{-\frac{jy\pi}{2t}} \sum_{k=0}^{\infty} \alpha_k(j) t^{k+\frac{1}{2}};$$

(IV) for $\eta_j \equiv 0 \bmod(\pi)$, η_j is of multiplicity three, one has

$$(2\pi t)^{-\frac{3}{2}} e^{-\frac{(j-1)y\pi}{2t}} \sum_{k=0}^{\infty} \alpha_k(j) t^{\frac{k}{2}+\frac{1}{4}}.$$

Thus small time asymptotics of P_t yield all geodesic lengths.

f is real analytic in x_0, x and y. It follows that the curve $\Gamma_{(x_0,x,y)}$ defined in Section 4 varies continously with respect to x_0, x and y for generic geodesics. As mentioned before, the exceptional geodesics are limits of generic ones, one has, for example, as x_0, $x \to 0$, the roots η_{2j-1}, η_{2j} of (*) with respect to $(x_0, 0)$ and (x, y) tend to $j\pi$, which corresponds to two distinct geodesics connecting $(0,0)$, $(0,y)$. We show that the sum of the first terms in the small time asymptotics with respect to $(x_0, 0)$, (x, y) at $i\eta_{2j-1}$ and at $i\eta_{2j}$ converges to the first term of the small time asymptotic with respect to $(0,0)$, $(0,y)$ at $i\eta_{2j-1} = i\eta_{2j} = ij\pi$. However, this is not true for higher order terms.

In the final section, we point out the relation between Grusin operator and the 1-dimensional Heisenberg group $\mathbf{H}_1$. We show how the fundamental solution of Δ_G can be obtained from the fundamental solution of $\Delta_{\mathbf{H}_1}$. We also show how to get $P_t(x_0, x, y)$ from the heat kernel of $\Delta_{\mathbf{H}_1}$. The relation between the geodesics of Δ_G and $\Delta_{\mathbf{H}_1}$ is described also. At the end, we write down the Wiener integral formula for $P_t(x_0, x, y)$ which shows the positivity of $P_t(x_0, x, y)$. Of course, this fact also follows from the positivity of the heat kernel of $\Delta_{\mathbf{H}_1}$ and the relation between it and $P_t(x_0, x, y)$ (cf. (7.2)).

On the other hand, the modified complex action for 1-dimensional Heisenberg group $\mathbf{H_1}$ is (cf. (7.1))

$$-i\eta y + \frac{\eta}{4}|x|^2 \coth\frac{\eta}{2}$$

where (x_1, x_2, y) are coordinates for $\mathbf{H_1}$ and $|x|^2 = x_1^2 + x_2^2$. And due to the group structure one can always take the origin as one of the end points.

Now in Grusin case with $x_0 = 0$, the modified complex action becomes

$$\begin{aligned}
& -i\tau y + \frac{\tau}{4}x^2(\coth\frac{\tau}{2} + \tanh\frac{\tau}{2}) \\
=& -i\tau y + \frac{\tau}{2}x^2 \coth\tau \\
=& -i\eta\frac{y}{2} + \frac{\eta}{4}x^2 \coth\frac{\eta}{2} \quad (\text{ by setting } \tau = \frac{\eta}{2}.)
\end{aligned}$$

Therefore the results in Sections 2 and 4 all apply to $\mathbf{H_1}$, except when $|x| = 0$, where there are uncountably many geodesics in $\mathbf{H_1}$ connecting $(0, 0, 0)$ and $(0, 0, y)$, $y \neq 0$; they are parametrized by S^1. By further study of the action, we shall give in a forthcoming paper an elementary proof of the positivity of the heat kernel for the Grusin operator and also for the heat kernel of $\Delta_{\mathbf{H_1}}$.

In conclusion, the modified complex action plays the central role in the construction of the heat kernel. We expect this will hold true for other subelliptic operators.

We thank Mr. You-Jen Chang for helping in preparing graphs of this paper.

2. SubRiemannian Geometry induced by the Grusin Operator

2.1. The definition of geodesics

In this section we study the geodesics determined by the Grusin operator in the (x, y)-plane. Recall

$$\Delta_G = \frac{1}{2}\Big(X_1^2 + X_2^2\Big) \tag{2.1}$$

where the vector fields X_1 and X_2 are given by

$$X_1 = \frac{\partial}{\partial x}, \qquad X_2 = x\frac{\partial}{\partial y}. \tag{2.2}$$

The Hamiltonian associated with the symbol of $-\Delta_G$ is

$$H(x, y, \xi, \eta) = \frac{1}{2}\Big(\xi^2 + x^2\eta^2\Big) \tag{2.3}$$

where we use (x, y, ξ, η) to denote the coordinate of the cotangent bundle $T^*\mathbb{R}^2$.

A bicharacteristic curve of H is the solution of the Hamiltonian system

$$\begin{aligned} \frac{dx}{ds} &= H_\xi = \xi, & \frac{d\xi}{ds} &= -H_x = -x\eta^2, \\ \frac{dy}{ds} &= H_\eta = x^2\eta, & \frac{d\eta}{ds} &= -H_\eta = 0. \end{aligned} \tag{2.4}$$

The geodesics are the projection to the base space $\mathbb{R}^2$ of the bicharacteristic curves of H in $T^*\mathbb{R}^2$. For our purpose, we first consider $s \in [0,1]$, and the initial conditions are given by

$$x(0) = x_0, \quad y(0) = y_0, \quad x(1) = x, \quad y(1) = y. \tag{2.5}$$

2.2. Reduction

Since $\frac{\partial}{\partial x}$ and $x\frac{\partial}{\partial y}$ are translation invariant in the y-direction, we will assume $y_0 = 0$ in the rest of this paper. Moreover, the system (2.4) is invariant under the transformation

$$(x, y, \xi, \eta) \mapsto (x, -y, \xi, -\eta),$$

thus it suffices to study $y \geq 0$ or $y \leq 0$ and we will assume $y \geq 0$ in the rest of our paper. It follows immediately from (2.4) that η is constant along any bicharacteristic curve of H. The choice $y \geq 0$ forces $\eta \geq 0$ by (2.4). First we give a rough idea about the solutions of the system (2.4) with initial conditions (2.5) for the following possible η's.

2.3. The case $\eta = 0$

When $\eta = 0$, the system (2.4) implies $\xi = \text{constant} = \xi(0) \doteq \xi_0$ and

$$x(s) = \xi_0 s + x_0, \qquad y(s) = y_0 = 0.$$

So the geodesic is a straight line segment on x-axis connecting $(x_0, 0)$ and $(x, 0)$. Conversely, if we assume $y = 0$, then by (2.4), we have

$$0 = y = y(1) = \eta \int_0^1 x^2(u)du.$$

This implies $\eta = 0$ or $x(s) \equiv 0$. But $x(s) \equiv 0$ implies $\xi = 0$ and $x_0 = x = 0$ by (2.4). This forces that the two end points coincide and η is arbitrary. In fact, $\{(0,0,0,\eta) | \, \eta \neq 0\}$ lies in the characteristic set of Δ_G. We exclude this trivial case. We conclude that

Lemma 2.1 *In* (2.4), *the constant η is zero if and only if the initial conditions are $(x_0, 0)$ and $(x, 0)$ with $x_0 \neq x$. The straight line connecting $(x_0, 0)$ and $(x, 0)$ is the only geodesics.*

2.4. The case $\eta > 0$

It follows from the proof of Lemma 2.1 that $y > 0$. In this case, we solve (2.4) to obtain

$$x(s) = x_0 \cos(\eta s) + B \sin(\eta s), \tag{2.6}$$

where

$$B = \frac{\xi_0}{\eta}, \qquad \xi_0 = \xi(0), \tag{2.7}$$

and

$$\begin{aligned} y(s) &= \eta \int_0^s x^2(u) du \\ &= \eta \Big[\frac{1}{2}(x_0^2 + B^2)s + \frac{1}{4\eta}(x_0^2 - B^2)\sin(2\eta s) + \frac{x_0 B}{\eta}\sin^2(\eta s) \Big]. \end{aligned} \tag{2.8}$$

Remark 2.2. Formula (2.6) implies if $\eta = k\pi$ for $k \in \mathbb{N}$, then $x = (-1)^k x_0$.

2.4.1. $\eta \neq k\pi, k \in \mathbb{N}$

In this case (2.5) and (2.6) yield

$$B = \frac{x - x_0 \cos\eta}{\sin\eta}. \tag{2.9}$$

Formula (2.8) then becomes

$$\begin{aligned} y =& y(1) \\ =& \frac{1}{2}\Big[(x^2 + x_0^2)\big(\frac{\eta - \sin\eta\, \cos\eta}{\sin^2\eta}\big) + 2xx_0\big(\frac{\sin\eta - \eta\cos\eta}{\sin^2\eta}\big) \Big] \end{aligned}$$

$$=\frac{1}{4}\Big[(x+x_0)^2\Big(\frac{(\eta+\sin\eta)(1-\cos\eta)}{\sin^2\eta}\Big)+(x-x_0)^2\Big(\frac{(\eta-\sin\eta)(1+\cos\eta)}{\sin^2\eta}\Big)\Big]$$

or more properly

$$y =\frac{1}{4}\Big[(x+x_0)^2\tilde{\mu}(\eta)+(x-x_0)^2\mu(\eta)\Big] \quad \text{where} \tag{2.10}$$

$$\tilde{\mu}(\eta) = \frac{\eta+\sin\eta}{1+\cos\eta} \qquad \text{and} \tag{2.11}$$

$$\mu(\eta) = \frac{\eta-\sin\eta}{1-\cos\eta}. \tag{2.12}$$

Those η's solving (2.10) give the geodesics between $(x_0, 0)$ and (x, y). Not all geodesics arise from the solutions of (2.10), for example, when $x = x_0 = 0$ and $y > 0$ we do not have any information from (2.10). The geodesics arise from the solutions of (2.10) will be called generic, and exceptional otherwise. Note that $\eta = k\pi$ can be a solution of (2.10), see 2.6.1 (II), (III). The properties of μ and $\tilde{\mu}$ will be investigated in 2.5.

2.4.2. $\boldsymbol{\eta = k\pi, k} \in \mathbb{N}$

By Remark 2.2, we must have $x = (-1)^k x_0$. Most geodesics in this case do not come from solving (2.10).

Setting $s = 1$ and $\eta = k\pi$ in (2.8) yields

$$y = \frac{k\pi}{2}(x_0^2 + B^2). \tag{2.13}$$

Solving for B, we get

$$B = \pm\Big(\frac{2y}{k\pi} - x_0^2\Big)^{\frac{1}{2}}. \tag{2.14}$$

(2.13) implies the following necessary condition for k

$$2y \geq k\pi x_0^2. \tag{2.15}$$

Thus the cases $x_0 \neq 0$ and $x_0 = 0$ are quite different. They will be treated separately in 2.6.2 and 2.6.3.

To find out all the geodesics, we need to solve (2.10), or to understand the properties of the functions μ and $\tilde{\mu}$.

2.5. Properties of μ, $\tilde{\mu}$ and the function F

For $a, b \in \mathbb{R}$, define

$$F(\eta) = a^2\tilde{\mu}(\eta) + b^2\mu(\eta). \tag{2.16}$$

We consider $\eta \geq 0$ only. The properties of μ and $\tilde{\mu}$ are summarized in the following lemma.

Lemma 2.3.

(a) *The functions μ and $\tilde{\mu}$ defined on $\eta \geq 0$ are positive functions vanishing at $\eta = 0$ only.*

(b) *The function μ (resp. $\tilde{\mu}$) has poles at $\eta = 2k\pi$ (resp. $\eta = (2k-1)\pi$), $k \in \mathbb{N}$.*

(c) *The function μ (resp. $\tilde{\mu}$) is strictly convex in each interval $(2(k-1)\pi, 2k\pi)$, (resp. $(0,\pi)$ and $((2k-1)\pi, (2k+1)\pi)$), $k \in \mathbb{N}$.*

(d) *On $(2k\pi, 2(k+1)\pi)$ (resp. $((2k-1)\pi, (2k+1)\pi)$, the function μ (resp. $\tilde{\mu}$) takes minimum at $\alpha'_k \in (2k\pi, (2k+1)\pi)$, (resp. $\alpha''_k \in ((2k-1)\pi, 2k\pi)$), $k \in \mathbb{N}$, where α'_k, (resp. α''_k) satisfies*

$$\tan\left(\frac{\alpha'_k}{2}\right) = \frac{\alpha'_k}{2}, \qquad \big(\text{resp.} \quad -\cot\left(\frac{\alpha''_k}{2}\right) = \frac{\alpha''_k}{2}\big), \quad k \in \mathbb{N}.$$

(e) *We have*

$$\mu(\alpha'_k) = \frac{\alpha'_k}{2}, \qquad \big(\textit{resp.} \quad \tilde{\mu}(\alpha''_k) = \frac{\alpha''_k}{2}\big), \quad k \in \mathbb{N},$$

hence

$$\begin{aligned} \cdots < \mu((2k-1)\pi) &= (k - \frac{1}{2})\pi < \mu(\alpha'_k) < \mu((2k+1)\pi) \\ &= (k + \frac{1}{2})\pi < \mu(\alpha'_{k+1}) < \cdots, \end{aligned}$$

(resp. $\cdots < \tilde{\mu}(2(k-1)\pi) = (k-1)\pi < \tilde{\mu}(\alpha''_k) < \tilde{\mu}(2k\pi) = k\pi < \tilde{\mu}(\alpha''_{k+1}) < \cdots$) for all $k \in \mathbb{N}$.

Proof. We just prove parts (c) and (e) for μ. The rest parts are straightforward.

Proof of (c). We have for $\eta \geq 0$, $\eta \neq k\pi$, $k \in \mathbb{N}$,

$$\mu''(z) = \frac{2\eta + \eta\cos\eta - 3\sin\eta}{(1-\cos\eta)^2}.$$

Let

$$\psi(\eta) \doteq 2\eta + \eta\cos\eta - 3\sin\eta = \eta(1+\cos\eta) + \eta - 3\sin\eta.$$

It suffices to prove $\psi(\eta) > 0$ for $0 < \eta < 3$. To do this, we differentiate $\psi(\eta)$ and get $\psi'(\eta) = 2 - \eta\sin\eta - 2\cos\eta$, $\psi''(\eta) = \sin\eta - \eta\cos\eta$, $\psi'''(\eta) = \eta\sin\eta$. Now $\psi'''(\eta)$ is strictly positive for $0 < \eta < 3$, this fact and $\psi''(0) = 0$ imply $\psi''(\eta) > 0$ for $0 < \eta < 3$. Again, $\psi'' > 0$ on $0 < \eta < 3$ and $\psi'(0) = 0$ imply that $\psi'(\eta) > 0$ for $0 < \eta < 3$. Repeat the same argument once more, we get $\mu''(\eta) > 0$ for $0 < \eta < 3$. The proof of the convexity of $\tilde{\mu}$ is even simpler. $\square$

Proof of (e). We concentrate on the function $\mu(\eta)$. The statement for $\tilde{\mu}$ can be proved similarly. From

$$\mu'(\eta) = \frac{2(1-\cos\eta) - \eta\sin\eta}{(1-\cos\eta)^2},$$

it follows that for $\eta \neq n\pi$, $n \in \mathbb{N}$, $\mu'(\eta) = 0 \Leftrightarrow \frac{\eta}{2} = \tan\frac{\eta}{2}$. When $\eta = \alpha_k'$, by repeatedly using $\tan\frac{\alpha_k'}{2} = \frac{\alpha_k'}{2}$, we have

$$\begin{aligned}
\mu(\alpha_k') &= \frac{\frac{\alpha_k'}{2} - \sin\frac{\alpha_k'}{2}\cos\frac{\alpha_k'}{2}}{\sin^2\frac{\alpha_k'}{2}} \\
&= \frac{1 - \cos^2\frac{\alpha_k'}{2}}{\sin\frac{\alpha_k'}{2}\cos\frac{\alpha_k'}{2}} \\
&= \frac{\sin^2\frac{\alpha_k'}{2}}{\sin\frac{\alpha_k'}{2}\cos\frac{\alpha_k'}{2}} \\
&= \tan\left(\frac{\alpha_k'}{2}\right) = \frac{\alpha_k'}{2}.
\end{aligned}$$ $\square$

The graphs of μ, $\tilde{\mu}$ and F are shown in Figures 2-1, 2-2 and 2-3.

Now we may prove the main theorem of this section.

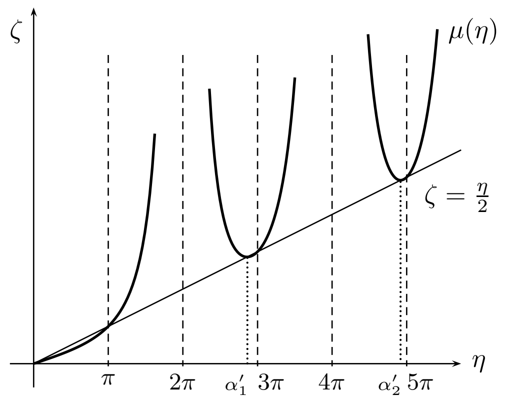

Figure 2-1

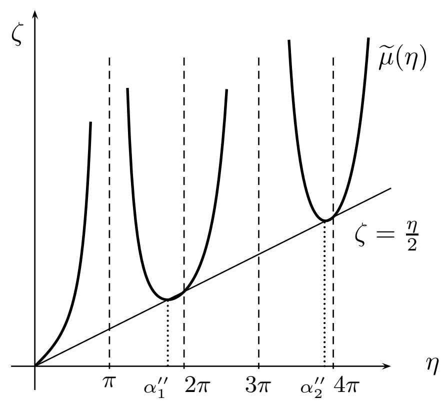

Figure 2-2

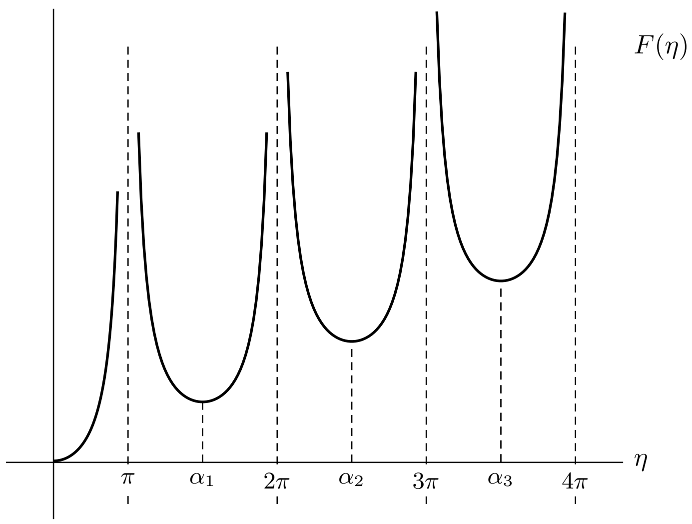

Figure 2-3

Theorem 2.4. *When $ab \neq 0$, the function $F(\eta) = a^2\tilde{\mu}(\eta) + b^2\mu(\eta)$ has the following properties:*

(a) *$F(\eta) \geq 0$ for $\eta \geq 0$ and $F(\eta) = 0$ if and only if $\eta = 0$;*

(b) *$F(\eta)$ has poles at $\eta = k\pi$ for $k \in \mathbb{N}$;*

(c) *$F(\eta)$ is strictly convex in each interval $((k-1)\pi, k\pi)$, $k \in \mathbb{N}$;*

(d) *in each interval $(k\pi, (k+1)\pi)$, $k \in \mathbb{N}$, $F(\eta)$ takes a unique minimum at α_k such that for $k \geq 1$, $\alpha_{2k-1} < \alpha_k''$, $\alpha_{2k} < \alpha_k'$ and*

$$F(\alpha_k) > \begin{cases} a^2\alpha_k' & \text{if } k \text{ is odd} \\ b^2\alpha_k'' & \text{if } k \text{ is even.} \end{cases}$$

So $F(\alpha_k)$ tends to infinity as $k \to \infty$;

(e) *for each $k \in \mathbb{N}$, $F(\alpha_k) < F(\alpha_{k+1})$.*

When $a = 0$, $b \neq 0$ (resp. $b = 0$, $a \neq 0$), F degenerates to $b^2\mu$ (resp. $a^2\tilde{\mu}$).

Proof. Properties (a)−(d) follows immediately from Lemma 2.3. Now we prove property (e). For a, $b \in \mathbb{R}$, $ab \neq 0$, recall

$$F(\eta) = a^2\tilde{\mu}(\eta) + b^2\mu(\eta) = a^2\frac{\eta + \sin\eta}{1 + \cos\eta} + b^2\frac{\eta - \sin\eta}{1 - \cos\eta} \geq 0.$$

It follows from Lemma 2.3 that $F(\eta) = 0$ at $\eta = 0$ only and is convex in $(k\pi, (k+1)\pi)$ for all $k = 0, 1, 2, \ldots$. Observe that

$$F(\eta) = \big(\eta g(\eta)\big)' = g(\eta) + \eta g'(\eta) \tag{2.17}$$

where

$$g(\eta) = \sin\eta\Big(\frac{a^2}{1 + \cos\eta} - \frac{b^2}{1 - \cos\eta}\Big).$$

Note that

$$g'(\eta) = \Big(\frac{a^2}{1 + \cos\eta} + \frac{b^2}{1 - \cos\eta}\Big) > 0, \qquad \forall\ \eta \in \mathbb{R} \setminus \{k\pi |\ k \in \mathbb{Z}\},$$

$g(\eta)$ is a periodic function with period 2π, and

$$g(\eta) = -g(2k\pi - \eta), \qquad \text{for } \ \eta \in (k\pi, (k+1)\pi). \tag{2.18}$$

The graph of $g(\eta)$ is shown in Figure 2-4.

Let $F(\eta)$ take the unique minima at α_k in the interval $(k\pi, (k+1)\pi)$ for $k = 1, 2, \ldots$. For $k \in \mathbb{N}$, $\eta \in (k\pi, (k+1)\pi)$, we have by (2.18)

$$g(\eta) = -g(2k\pi - \eta) \quad \text{and} \quad g'(\eta) = g'(2k\pi - \eta).$$

Put this and (2.18) into (2.17) to get

$$\begin{aligned} F(\eta) &= g(\eta) + \eta g'(\eta) = -g(2k\pi - \eta) + \eta g'(2k\pi - \eta) \\ &= g(2k\pi - \eta) + (2k\pi - \eta)g'(2k\pi - \eta) + 2\big[-g(2k\pi - \eta) \\ &\quad +(\eta - k\pi)g'(2k\pi - \eta)\big] \end{aligned}$$

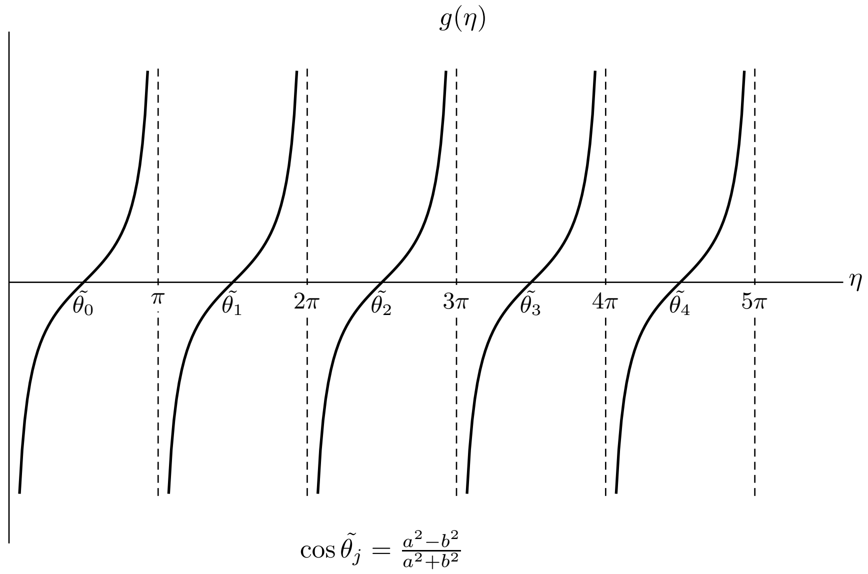

Figure 2-4

$$\begin{aligned} &= g(2k\pi-\eta)+(2k\pi-\eta)g'(2k\pi-\eta)+2\big[g(\eta)+(\eta-k\pi)g'(\eta)\big] \\ &= F(2k\pi-\eta)+2[g(\eta)+(\eta-k\pi)g'(\eta)]. \end{aligned} \tag{2.19}$$

When k is even, by the periodicity of g, the terms in the last bracket becomes

$$g(\eta)+(\eta-k\pi)g'(\eta)=g(\eta-k\pi)+(\eta-k\pi)g'(\eta-k\pi)=F(\eta-k\pi)>0$$

for $\eta\in(k\pi,(k+1)\pi)$. Therefore, for k even, $\eta\in(k\pi,(k+1)\pi)$ we have

$$F(\eta)=F(2k\pi-\eta)+2F(\eta-k\pi), \qquad \text{with} \quad 2k\pi-\eta\in((k-1)\pi,k\pi).$$

It follows that

$$F(\alpha_k)=F(2k\pi-\alpha_k)+2F(\alpha_k-k\pi)>F(2k\pi-\alpha_k)\geq F(\alpha_{k-1})$$

since $\alpha_k\in(k\pi,(k+1)\pi)$ and $F(\alpha_k-k\pi)>0$ for all $k\in\mathbb{N}$.

For odd integer k, the periodicity of g yields, for $\eta\in(k\pi,(k+1)\pi)$,

$$g(\eta)+(\eta-k\pi)g'(\eta)=g(\eta-(k-1)\pi)+(\eta-k\pi)g'(\eta-(k-1)\pi) \tag{2.20}$$

with $\eta - k\pi \in (0, \pi)$ and $\eta - (k-1)\pi \in (\pi, 2\pi)$. Direct computation shows that

$$g(\eta+\pi) + \eta g'(\eta+\pi) = a^2 \frac{\eta - \sin\eta}{1-\cos\eta} + b^2 \frac{\eta + \sin\eta}{1+\cos\eta} > 0$$

for all $\eta \in (0, \pi)$. Thus, by (2.19) we have $F(\eta) > F(2k\pi - \eta)$ for k odd and $\eta \in (k\pi, (k+1)\pi)$. Therefore

$$F(\alpha_k) > F(2k\pi - \alpha_k) \geq F(\alpha_{k-1})$$

since $2k\pi - \eta \in ((k-1)\pi, k\pi)$. This completes the proof of (e). □

2.6. The geodesics

2.6.1. The generic case

These are the geodesics connecting $(x_0, 0)$ and (x, y) which come from those η's solving (2.10):

$$y = \frac{1}{4}(x+x_0)^2 \tilde{\mu}(\eta) + \frac{1}{4}(x-x_0)^2 \mu(\eta) = F(\eta),$$

with $a = \frac{x+x_0}{2}$, $b = \frac{x-x_0}{2}$.

(I) $y > 0$, $x^2 \neq x_0^2$

By Remark 2.2, $\eta = k\pi$ can not be a solution here. Since $x^2 \neq x_0^2$, Theorem 2.4(e) implies that there exists $N \in \mathbb{N}$ such that $F(\alpha_{N-1}) \leq y < F(\alpha_N)$. When $F(\alpha_{N-1}) < y$ by Theorem 2.4 (a), (b), (c) there are $2N-1$ distinct solutions to (2.10). When $F(\alpha_{N-1}) = y$, α_{N-1} is counted as a solution of multiplicity two. In this sense, there are always $2N-1$ geodesics connecting $(x_0, 0)$ and (x, y).

(II) $y > 0$, $x = x_0 \neq 0$ (resp. $x = -x_0 \neq 0$)

Here the function $F(\eta)$ degenerates to $x_0^2 \tilde{\mu}(\eta)$ (resp. $x_0^2 \mu(\eta)$) due to $x = x_0$ (resp. $x = -x_0$).

We want to find those η satisfying $\frac{y}{x_0^2} = \tilde{\mu}(\eta)$ (resp. $\frac{y}{x_0^2} = \mu(\eta)$). By Lemma 2.3, there exists $N \in \mathbb{N}$ such that $\tilde{\mu}(\alpha''_{N-1}) \leq \frac{y}{x_0^2} < \tilde{\mu}(\alpha''_N)$ (resp. $\mu(\alpha'_{N-1}) \leq \frac{y}{x_0^2} < \mu(\alpha'_N)$). As in (I) we conclude that there are $2N-1$ geodesics connecting $(x_0, 0)$ and (x_0, y) (resp. $(x_0, 0)$ and $(-x_0, y)$). Note that (a) $\alpha''_N \in ((2N-1)\pi, 2N\pi)$ (resp. $\alpha'_N \in (2N\pi, (2N+1)\pi)$, (b) it is possible that $\eta = 2(N-1)\pi$ (resp. $\eta = (2N-1)\pi$). Clearly these geodesics are limits of geodesics in (I) when η is not an integral multiple of π. The fact that the geodesic corresponding to η, $\eta \equiv 0 \mod(\pi)$, is the limits of geodesics in (I), will be explained in Section 2.6.2.

(III) $y = 0$, $x \neq x_0$

This case lies between $y > 0$ (so $\eta > 0$), and $y < 0$ (so $\eta < 0$). Observe that obviously $\eta = 0$ is the only solution to (2.10) when $y = 0$. The explicit form of the geodesic is derived in 2.3. The point here is to show that its form is consistent with (2.6). For $x \neq x_0$ fixed and $y > 0$ small we have by (I) a solution $\eta \in (0, \pi)$ to (2.10). (2.9) can be written in the form

$$B\eta = \frac{\eta(x - x_0 \cos\eta)}{\sin\eta}.$$

Then put (2.6) as follows

$$x(s) = x_0 \cos(\eta s) + \frac{\eta(x - x_0\cos\eta)}{\sin\eta}\frac{\sin(\eta\, s)}{\eta s} \cdot s.$$

Let $y \to 0^+$ and so $\eta \to 0^+$. We have

$$x(s) = x_0 + (x - x_0)s,$$

which is exactly the form in Lemma 2.1.

2.6.2. The mild exceptional case $y > 0$, $x^2 = x_0^2 \neq 0$ with $\eta \equiv 0 \mod(\pi)$, η not a solution of (2.10)

These geodesics are called mild because only $\mu(\eta)$ or $\tilde{\mu}(\eta)$ disappears in $F(\eta)$ and there are only finitely many geodesics connecting $(x_0, 0)$ and (x, y), because of (2.15). We have briefly discussed how to obtain the geodesics with $\eta = k\pi$ in 2.4.2. However, that approach can not tell whether the geodesic is also a solution of (2.10) or not and thus the total number of geodesics is unclear. We are going to show that these geodesics are limits of generic ones. This will solve the above question. We only treat the case $x = x_0 \neq 0$. The case $x = -x_0 \neq 0$ can be done similarly.

Fix $y > 0$ and $x_0 \neq 0$. Observe that when $x = x_0$, for $\eta = k\pi$ to define a geodesic between $(x_0, 0)$ and (x_0, y), it is necessary that k is even by Remark 2.2. Also (2.15) must hold. We assume that $y > \frac{k\pi x_0^2}{2} = x_0^2\tilde{\mu}(k\pi)$ in the beginning. By Lemma 2.3, $\alpha''_{\frac{k}{2}} < k\pi$ and $\tilde{\mu}(\eta)$ is increasing for $\eta \in (\alpha''_{\frac{k}{2}}, (k+1)\pi)$. Given $\epsilon > 0$ such that $\frac{1}{2}(k\pi - \alpha''_{\frac{k}{2}}) > \epsilon > 0$, let $x \neq x_0$ and x close to x_0 to be determined later. Set $F(\eta) = \frac{1}{4}(x+x_0)^2\tilde{\mu}(\eta)+\frac{1}{4}(x-x_0)^2\mu(\eta)$ as before. By the definition of $\mu(\eta)$ and $\tilde{\mu}(\eta)$, there exists $\delta > 0$ such that

$$\begin{cases} \frac{1}{4}(x+x_0)^2\tilde{\mu}(\eta) & < \frac{k\pi x_0^2}{2} + \frac{1}{2}(y - \frac{k\pi x_0^2}{2}) = \frac{y}{2} + \frac{1}{4}k\pi x_0^2 \\ & \quad \text{if } |x - x_0| < \delta \text{ and } |\eta - k\pi| < 2\epsilon. \\ \frac{1}{4}(x-x_0)^2\mu(\eta) & < \frac{1}{2}(y - \frac{k\pi x_0^2}{2}) \\ & \quad \text{if } |x - x_0| < \delta \text{ and } \epsilon < |\eta - k\pi| < 2\epsilon. \end{cases}$$

It follows that if x is chosen such that $0 < |x - x_0| < \delta$, one has

$$\min_{\eta\in(k\pi+\epsilon,(k+1)\pi)} F(\eta) < y \quad \text{and} \quad \min_{\eta\in(k\pi-2\epsilon,k\pi-\epsilon)} F(\eta) < y.$$

The first inequality shows that $F(\alpha_k) < y$; whence by Theorem 2.4 there are solutions $\eta^{\pm}$ with $\eta^- < k\pi < \eta^+$ satisfying $y = F(\eta^{\pm})$. Furthermore, the above two inequalities imply that $|\eta^{\pm} - k\pi| < 2\epsilon$.

In conclusion we have proved

Lemma 2.5. *Given $y > 0$, $x_0 \neq 0$, k positive even integer so that $y > \frac{k\pi x_0^2}{2}$. For any $\epsilon > 0$, $\epsilon < \frac{1}{2}(k\pi - \alpha''_{\frac{k}{2}})$, there exists $\delta > 0$ such that for any x satisfying $0 < |x - x_0| < \delta$, there are two geodesics connecting $(x_0, 0)$ and (x, y) with $\eta = \eta^+$ or η^-, $\eta^- < k\pi < \eta^+$ and $|\eta^{\pm} - k\pi| < 2\epsilon$.*

Next, we rewrite (2.6) in the form

$$x(s) = A\sin(\eta s + \alpha), \tag{2.21}$$

where

$$A = \sqrt{x_0^2 + B^2} \quad \text{and} \quad \sin\alpha = \frac{x_0}{\sqrt{x_0^2 + B^2}}. \tag{2.22}$$

Now formula (2.8) becomes

$$y(s) = \eta A^2 \int_0^s \sin^2(\eta u + \alpha) du = \eta A^2 \Big[\frac{s}{2} - \frac{\sin 2(\eta s + \alpha)}{4\eta} + \frac{\sin(2\alpha)}{4\eta}\Big].$$

At $s = 1$, we have

$$\begin{aligned} y =& y(1) = \eta(x_0^2 + B^2)\Big[\frac{1}{2} - \frac{\sin 2(\eta + \alpha)}{4\eta} + \frac{\sin(2\alpha)}{4\eta}\Big] \\ =& \frac{\eta}{2}(x_0^2 + B^2)\Big[1 - \frac{\sin\eta\,\cos(2\alpha + \eta)}{\eta}\Big]. \end{aligned}$$

Solve for B and get

$$B = \pm\sqrt{\frac{2y}{\eta - \sin\eta\,\cos(\eta + 2\alpha)} - x_0^2}.$$

Note that B is a C^∞ function of η if $\frac{2y}{\eta - \sin\eta\,\cos(\eta+2\alpha)} - x_0^2 > 0$. The sign of B is determined by (2.7) or (2.9). Let x_0, y, $\eta = \eta^\pm$ be as in Lemma 2.5. As η approaches $k\pi$, we recover (2.14)

$$B = \pm\sqrt{\frac{2y}{k\pi} - x_0^2}.$$

Figures 2-5, 2-6 respectively illustrate geodesics for $x = x_0$ $\eta = 2\pi$ with $B > 0$ and with $B < 0$.

Thus when $2y > k\pi x_0^2$, as x tends to x_0, the two geodesics with $\eta = \eta^+$ or η^- tend to two distinct geodesics with $B > 0$ for one and with $B < 0$ for the other. When $2y = k\pi x_0^2$, we begin with $2\tilde{y} > k\pi x_0^2$ and $x = x_0$. As $\tilde{y}$ decreases to $\frac{k\pi x_0^2}{2}$, B tends to zero and the two geodesics merge to one. However, there is another geodesic from (I) merges together. So this geodesic has multiplicity three.

Let us look at (2.9) again. Instead of specifying that k is a positive even integer, we now use $2k\pi$. When $\eta^\pm \to 2k\pi^\pm$, $x - x_0 \to 0$ we have

$$\begin{aligned} \lim_{\substack{\eta^\pm \to 2k\pi^\pm \\ x - x_0 \to 0}} \frac{x - x_0}{\sin\eta^\pm} &= \lim_{\substack{\eta^\pm \to 2k\pi^\pm \\ x - x_0 \to 0}} \frac{x - x_0\cos\eta^\pm - x_0(1 - \cos\eta^\pm)}{\sin\eta^\pm} \\ &= \lim_{\substack{\eta^\pm \to 2k\pi^\pm \\ x - x_0 \to 0}} \frac{x - x_0\cos\eta^\pm}{\sin\eta^\pm} = B \end{aligned} \tag{2.23}$$

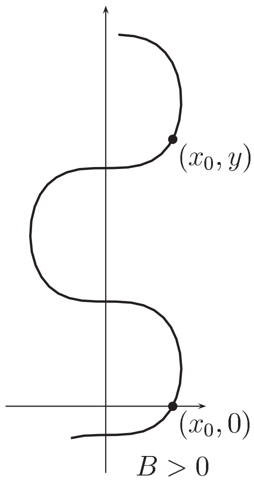

Figure 2-5

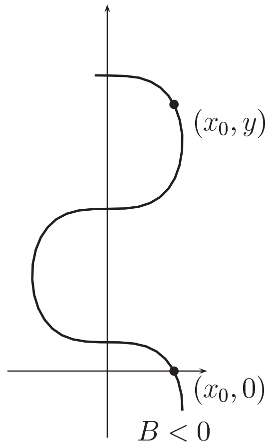

Figure 2-6

where

$$B > 0 \quad \text{if} \begin{cases} \eta^+ \to 2k\pi^+, & x > x_0, \text{ or} \\ \eta^- \to 2k\pi^-, & x < x_0, \end{cases}$$

and

$$B < 0 \quad \text{if} \begin{cases} \eta^+ \to 2k\pi^+, & x < x_0, \text{ or} \\ \eta^- \to 2k\pi^-, & x > x_0. \end{cases}$$

As for the case $x = -x_0$, we consider the limit $\eta^\pm \to (2k-1)\pi^\pm$, $x + x_0 \to 0$ and have

$$\begin{aligned} \lim_{\substack{\eta^\pm \to (2k-1)\pi \\ x+x_0 \to 0}} \frac{x+x_0}{\sin \eta^\pm} &= \lim_{\substack{\eta^\pm \to (2k-1)\pi^\pm \\ x+x_0 \to 0}} \frac{x - x_0 \cos \eta^\pm + x_0(1+\cos \eta^\pm)}{\sin \eta^\pm} \\ &= \lim_{\substack{\eta^\pm \to (2k-1)\pi^\pm \\ x+x_0 \to 0}} \frac{x - x_0 \cos \eta^\pm}{\sin \eta^\pm} = B \end{aligned} \tag{2.24}$$

where

$$B > 0 \quad \text{if} \begin{cases} \eta^+ \to (2k-1)\pi^+, & x + x_0 > 0, \text{ or} \\ \eta^- \to (2k-1)\pi^-, & x - x_0 < 0, \end{cases}$$

and

$$B<0 \quad \text{if} \quad \begin{cases} \eta^+ \to (2k-1)\pi^+, & x+x_0<0, \text{ or} \\ \eta^- \to (2k-1)\pi^-, & x+x_0>0. \end{cases}$$

Finally, we count the total number of geodesics connecting $(x_0, 0)$ and (x_0, y) (resp.$(-x_0, y)$). The number of mild exceptional geodesics here is $2(N-1)$ (resp. $2N$). In 2.6.1(II) we have $\alpha''_N \in ((2N-1)\pi, 2N\pi)$ (resp. $\alpha'_N \in (2N\pi, (2N+1)\pi)$). So there are $2N-1$ (resp. $2N-1$) generic geodesics coming from 2.6.1(II). Hence the total number of geodesics is $2N-1+2(N-1) = 4N-3 = 2(2N-1)-1$ (resp. $2N-1+2N = 4N-1 = 2(2N)-1$). This result agrees with 2.6.1(I).

Figures 2-7 and 2-8 illustrate the η's for both cases 2.6.1 (II) and 2.6.2. In Figure 2-7 η_1, η_2 and η_5 correspond to generic geodesics and η_3, η_4 correspond to mild exceptional geodesics. As y decreases to πx_0^2, one gets $\eta_3 = \eta_4 = \eta_5 = 2\pi$ which corresponds to a geodesic of multiplicity three.

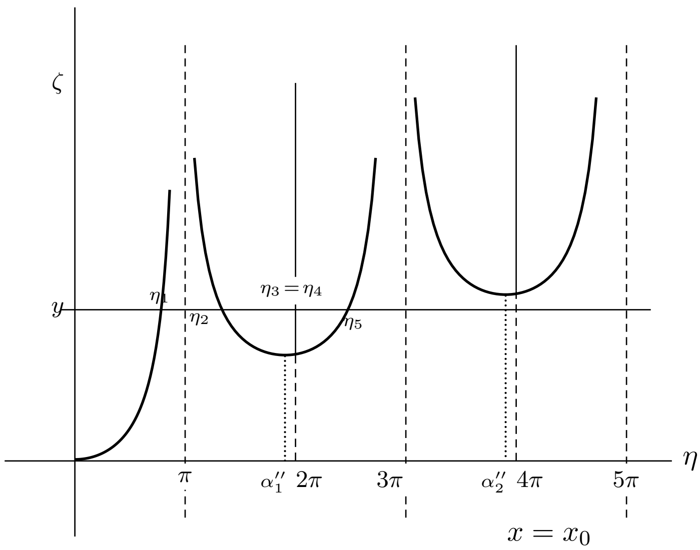

Figure 2-7

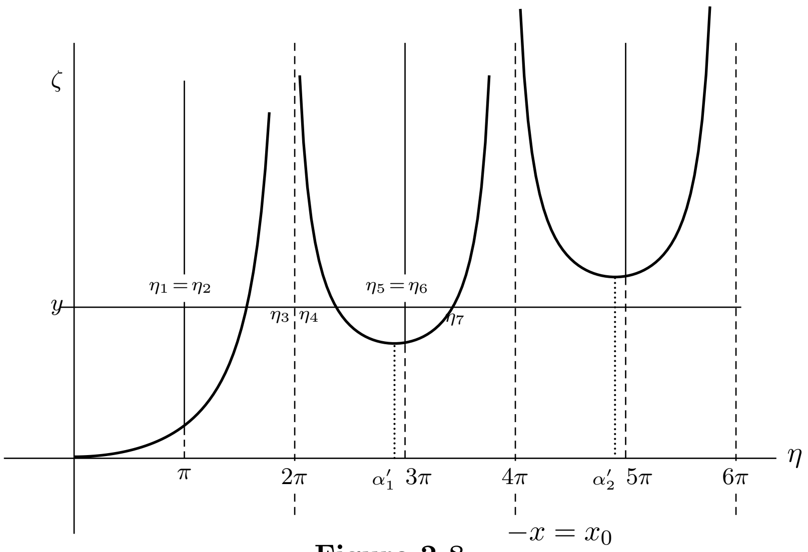

Figure 2-8

2.6.3. The exceptional case $y > 0$, $x = x_0 = 0$

Put $x_0 = 0$ in (2.22) (so $\alpha = 0$ automatically) and follow the same argument there. We get

$$B = \pm\sqrt{\frac{2y}{k\pi}},\ k = 1, 2, \cdots .$$

So there are infinitely many geodesics between $(0, 0)$ and $(0, y)$, $y > 0$, with $\eta_{2k-1} = \eta_{2k} = k\pi$, $k \in \mathbb{N}$. Explicitly,

$$\begin{cases} x(s) = \pm\sqrt{\frac{2y}{k\pi}} \sin(k\pi s) \\ y(s) = y\Big(s - \frac{\sin(2k\pi s)}{2k\pi}\Big), \end{cases} \quad k = 1, 2, 3, \ldots$$

So every $k\pi$ $k = 1, 2, \ldots$ corresponds to two distinct geodesics with the same length. Figure 2-9 illustrates the η's in this case.

2.6.4. The main Theorem on geodesics

In conclusion, we have the following theorem:

Theorem 2.6. *Given two points $(x_0, 0)$ and (x, y) in the plane with $y > 0$,*

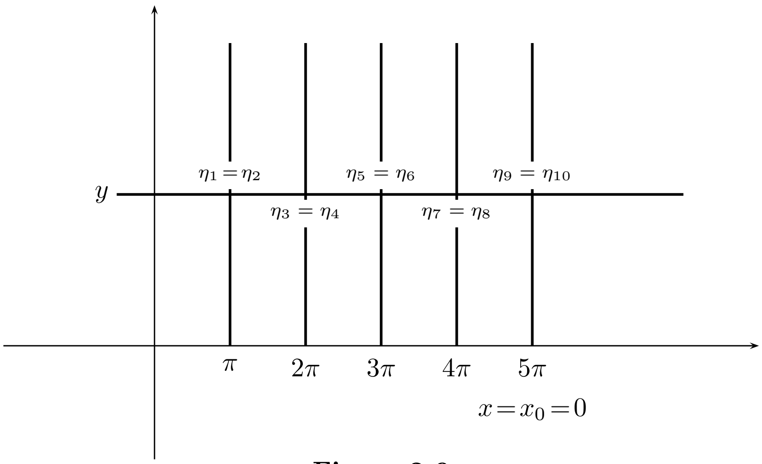

Figure 2-9

(1) *If $x \neq \pm x_0$ the equation (2.10) has finitely many solutions η_j, $j = 1, 2, \ldots, N$, N odd such that*

$$0 < \eta_1 < \pi < \eta_2 < \eta_3 < 2\pi < \cdots < \frac{N-1}{2}\pi < \eta_{N-1} \leq \eta_N < \frac{N+1}{2}\pi,$$

where $\eta_{N-1} = \eta_N$ occurs when $\eta_N = \alpha_{\frac{N-1}{2}}$.
The curves defined by

$$\mathcal{C}_j : \begin{cases} x(s) = x_0 \cos(\eta_j s) + \frac{x - x_0 \cos \eta_j}{\sin \eta_j} \sin(\eta_j s) \\ y(s) = \eta_j \int_0^s x^2(u) du \end{cases} \tag{2.25}$$

are all the geodesics connecting $(x_0, 0)$ and (x, y) in time $s \in [0, 1]$. Let ℓ_j denote the length of $\mathcal{C}_j$, one has $0 < \ell_1 < \ell_2 < \cdots < \ell_{N-1} \leq \ell_N$.

(2) *If $0 \neq x = x_0$, then there are finitely many geodesics $\mathcal{C}_j$, $j = 1, 2, \ldots, N$, N odd, connecting $(x_0, 0)$ and (x_0, y). The corresponding η_j's, $j = 1, 2, \ldots, N$ satisfy*

$$\begin{aligned} 0 <&\eta_1 < \pi < \eta_2 < \eta_3 = 2\pi = \eta_4 < \eta_5 < \cdots < \eta_{4k-2} < \eta_{4k-1} = 2k\pi = \eta_{4k} \\ <&\eta_{4k+1} < \cdots < \eta_{N-2} \leq \eta_{N-1} \leq \eta_N \leq \frac{N+1}{2}\pi, \end{aligned}$$

where $\eta_{N-2} < \frac{N-1}{2}\pi < \eta_{N-1} = \eta_N < \frac{N+1}{2}\pi$ occurs when $\eta_N = \alpha''_{\frac{N-1}{2}}$, $N \equiv 3 \bmod 4$; $\frac{N-3}{2}\pi < \eta_{N-3} < \eta_{N-2} = \eta_{N-1} = \eta_N < \frac{N+1}{2}\pi$ occurs when $\eta_N = \frac{N-1}{2}\pi$, $N \equiv 1 \bmod 4$ and $\eta_{N-3} < \eta_{N-2} = \eta_{N-1} < \eta_N < \frac{N+1}{2}\pi$

occurs when $\eta_{N-2} = \eta_{N-1} = \frac{N-1}{2}\pi$, $N \equiv 1 \bmod 4$.
When $\eta \neq 2k\pi$, *the corresponding geodesic is of the form* (2.25). *When* $y > k\pi x_0^2$ *we have* $\eta_{4k-1} = \eta_{4k} = 2k\pi$ *and the two geodesics take the form*

$$\mathcal{C}_{4k-1}, \mathcal{C}_{4k} : \begin{cases} x(s) = x_0 \cos(2k\pi s) \pm \frac{x_0}{\sqrt{2k\pi}} \sqrt{\frac{2y}{x_0^2} - 2k\pi} \sin(2k\pi s) \\ y(s) = 2k\pi \int_0^s x^2(u)du. \end{cases} \tag{2.26}$$

When $y = k\pi x_0^2$ *we have* $C_{N-2} = C_{N-1} = C_N$ *with* $N \equiv 1 \bmod 4$, *in other words* $k = \frac{N-1}{4}$, *and* $x(s) = x_0 \cos\left(\frac{N-1}{2}\pi s\right)$ *in* (2.26).
The length of the geodesic satisfies

$$\begin{aligned} 0 <& \ell_1 < \ell_2 < \ell_3 = \ell_4 < \cdots < \ell_{4k-2} < \ell_{4k-1} = \ell_{4k} \\ <& \ell_{4k+1} < \cdots < \ell_{N-2} \leq \ell_{N-1} \leq \ell_N. \end{aligned}$$

(3) *If* $0 \neq x = -x_0$, *then there are finitely many geodesics* $\mathcal{C}_j$, $j = 1, 2, \ldots, N$, N *odd, connecting* $(x_0, 0)$ *and* $(-x_0, y)$. *The corresponding* η_j*'s*, $j = 1, 2, \ldots, N$ *satisfy*

$$\begin{aligned} 0 <& \eta_1 = \pi = \eta_2 < \eta_3 < \eta_4 < \cdots < \eta_{4k-4} < \eta_{4k-3} = (2k-1)\pi = \eta_{4k-2} \\ <& \eta_{4k-1} < \cdots < \eta_{N-2} \leq \eta_{N-1} \leq \eta_N < \frac{N+1}{2}\pi, \end{aligned}$$

where $\eta_{N-2} < \eta_{N-1} = \eta_N$ *occurs when* $\eta_N = \alpha'_{\frac{N-1}{4}}$, $N \equiv 1 \bmod 4$ *and* $\eta_{N-2} = \eta_{N-1} = \eta_N$ *occurs when* $\eta_N = \frac{N-1}{2}\pi$, $N \equiv 3 \bmod 4$, *and* $\eta_{N-2} = \eta_{N_1} < \eta_N$ *occurs when* $\eta_{N-2} = \eta_{N_1} = \frac{N-1}{2}\pi$, $N \equiv 3 \bmod 4$.
When $\eta \neq (2k-1)\pi$, *the corresponding geodesic is of the form* (2.25). *When* $2y > (2k-1)\pi x_0^2$ *we have* $\eta_{4k-3} = \eta_{4k-2} = (2k-1)\pi$, *the two geodesics take the form*

$$\mathcal{C}_{4k-3}, \mathcal{C}_{4k-2} : \\ \begin{cases} x(s) = & x_0 \cos((2k-1)\pi s) \\ & \pm \frac{x_0}{\sqrt{(2k-1)\pi}} \sqrt{\frac{2y}{x_0^2} - (2k-1)\pi} \sin((2k-1)\pi s) \\ y(s) = & (2k-1)\pi \int_0^s x^2(u)du. \end{cases} \tag{2.27}$$

When $2y = (2k-1)\pi x_0^2$ *we have* $C_{N-2} = C_{N-1} = C_N$ *with* $x(s) = x_0 \cos\left(\frac{N-1}{2}\pi s\right)$ *in* (2.27), *i.e.* $k = \frac{N+1}{4}$.

The length ℓ_j satisfies

$$\begin{aligned}0 <&\ell_1 = \ell_2 < \ell_3 < \ell_4 < \cdots < \ell_{4k-4} < \ell_{4k-3} = \ell_{4k-2}\\ <&\ell_{4k-1} < \cdots < \ell_{N-2} \le \ell_{N-1} \le \ell_N.\end{aligned}$$

(4) *If $x = x_0 = 0$ $y > 0$, then there are infinitely many geodesics connecting $(0, 0)$ and $(0, y)$. The η_j's are of the form*

$$\begin{aligned}&\eta_1 = \pi = \eta_2 < \eta_3 = 2\pi = \eta_4 < \cdots < \eta_{4k-3} = (2k-1)\pi = \eta_{4k-2}\\ <&\eta_{4k-1} = 2k\pi = \eta_{4k} < \cdots .\end{aligned}$$

The geodesics take the form

$$\mathcal{C}_{2k-1}, \mathcal{C}_{2k} : \begin{cases} x(s) = \pm\sqrt{\frac{2y}{k\pi}}\sin(k\pi s)\\ y(s) = y\Big(s - \frac{\sin(2k\pi s)}{2k\pi}\Big)\end{cases} \tag{2.28}$$

where $\eta_{2k-1} = \eta_{2k} = k\pi$. The length of $\mathcal{C}_{2k-1}$ or $\mathcal{C}_{2k}$ is $\ell_{2k-1} = \ell_{2k} = k\pi y$, $k \in \mathbb{N}$.

(5) *If $y = 0$, $x_0 \neq x$ then there is a unique geodesic connecting $(x_0, 0)$ and $(x, 0)$ with*

$$\eta = 0, \qquad x(s) = x_0 + (x - x_0)s, \qquad y(s) \equiv 0.$$

All the geodesics in cases (2)–(5) are limits of the geodesics in case (1).

The statements concerning the length ℓ_j will follow from Corollary 4.6 and Theorem 4.7.

Remark 2.7. In the next section we will use geometric mechanics method to find the modified complex action which plays the key role in constructing the heat kernel of Δ_G. In that process we use only the generic geodesics which is adequate as justified by Theorem 2.6. The close relation between the modified complex action and the geometry discussed here will be seen more clearly in Sections 4-6. Only at that point one may say that the modified complex action indeed reflects the geometry of Δ_G.

3. The Modified Complex Action

3.1. Riemannian case

We recall the heat kernel for the Laplace-Beltrami operator,

$$\Delta = -\frac{1}{2}\sum_{j=1}^{n} X_j^* X_j = \frac{1}{2}\sum_{j=1}^{n} X_j^2 + \cdots .$$

Here $X_1, \ldots, X_n$ represent n linearly independent vector fields on an n-dimensional manifold $\mathcal{M}_n$. Assuming that $X = \{X_1, \ldots, X_n\}$ is an orthonormal set we obtain a volume element on $\mathcal{M}_n$ which yields X_j^*, the operator adjoint to X_j, $+\cdots$ stands for lower order terms. The heat kernel for Δ, at least locally, takes the form,

$$P_t(\mathbf{x}, \mathbf{x}_0) = \frac{1}{(2\pi t)^{\frac{n}{2}}} e^{-\frac{d(\mathbf{x},\mathbf{x}_0)^2}{2t}} \big(a_0 + a_1 t + a_2 t^2 + \cdots\big).$$

where $d(\mathbf{x}, \mathbf{x}_0)$ denotes for the Riemannian distance between $\mathbf{x}$ and $\mathbf{x}_0$ if the metric is induced by the orthonormal set $\{X_1, \ldots, X_n\}$. The a_j's are functions of $\mathbf{x}$ and $\mathbf{x}_0$. Note that

$$\frac{\partial}{\partial t}\Big(\frac{d(\mathbf{x}, \mathbf{x}_0)^2}{2t}\Big) + \frac{1}{2}\sum_{j=1}^{n}\Big(X_j \frac{d(\mathbf{x}, \mathbf{x}_0)^2}{2t}\Big)^2 = 0, \tag{3.1}$$

that is, $\dfrac{d(\mathbf{x}, \mathbf{x}_0)^2}{2t}$ is a solution of the Hamilton-Jacobi equation.

3.2. The classical action for Grusin operator

In the case of the Grusin operator

$$\Delta_G = \frac{1}{2}\Big(\frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}\Big)$$

we shall look for a heat kernel in the form

$$\frac{1}{t^\alpha} e^{-h}, \tag{3.2}$$

where, imitating (3.1), h is a solution of

$$\frac{\partial h}{\partial t} + \frac{1}{2}\Big(\frac{\partial h}{\partial x}\Big)^2 + \frac{1}{2}x^2\Big(\frac{\partial h}{\partial y}\Big)^2 = 0. \tag{3.3}$$

Thus we start with solving

$$\frac{\partial z}{\partial t} + H\Big(x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\Big) = 0 \tag{3.4}$$

for z, where

$$H(x, y, \xi, \eta) = \frac{1}{2}\xi^2 + \frac{1}{2}x^2\eta^2.$$

Normally one reduces this question to finding a solution of a system of ordinary differential equations as follows. Set

$$\tilde{H}(x, y, t, z, \xi, \eta, \gamma) = \gamma + H(x, y, \xi, \eta) = 0, \tag{3.5}$$

where $\xi = \frac{\partial z}{\partial x}$, $\eta = \frac{\partial z}{\partial y}$ and $\gamma = \frac{\partial z}{\partial t}$. We shall find the bicharacteristic curves which are solutions to the following system:

$$\begin{aligned}
\dot{x} =& \tilde{H}_\xi = \xi \qquad \text{where} \quad \dot{x} = \frac{dx}{ds}, \\
\dot{y} =& \tilde{H}_\eta = \eta x^2, \\
\dot{t} =& \tilde{H}_\gamma = 1, \\
\dot{\xi} =& -\tilde{H}_x = -\eta^2 x, \\
\dot{\eta} =& -\tilde{H}_y = 0, \\
\dot{\gamma} =& -\tilde{H}_t = 0, \\
\dot{z} =& \xi\tilde{H}_\xi + \eta\tilde{H}_\eta + \gamma\tilde{H}_\gamma.
\end{aligned} \tag{3.6}$$

With $0 \le s \le t$,

$$\begin{aligned}
\eta(s) =& \eta(0) = \eta = \text{constant}, \\
\gamma(s) =& \gamma = -H = \text{constant}, \\
t(s) =& s.
\end{aligned}$$

Here "constant" means "*constant along the bicharacteristic curve*". Observe that (3.6) is an extended system of (2.4). In particular,

$$\dot{z} = \xi \cdot \dot{x} + \eta \cdot \dot{y} - H. \tag{3.7}$$

From (3.6)

$$\ddot{x} = \dot{\xi} = -\eta^2 x, \qquad \text{or} \qquad \ddot{x} + \eta^2 x = 0,$$

so,

$$x(s) = x(0)\cos(\eta s) + \frac{\xi(0)}{\eta}\sin(\eta s), \qquad \text{if } \eta \neq 0, \tag{3.8}$$

(cf. (2.6), (2.7)). As for $y(s)$, one has

$$\begin{aligned}\dot{y}(s) &= \eta x^2(s)\\ &= \eta\Big[x^2(0)\Big(\frac{1}{2} + \frac{1}{2}\cos(2\eta s)\Big) + 2x(0)\frac{\xi(0)}{\eta}\sin(\eta s)\cos(\eta s)\\ &\quad + \frac{\xi^2(0)}{\eta^2}\Big(\frac{1}{2} - \frac{1}{2}\cos(2\eta s)\Big)\Big],\end{aligned}$$

hence,

$$\begin{aligned}y(s) - y(0) &= \eta\left[\frac{x^2(0)}{2}\Big(s + \frac{\sin(2\eta s)}{2\eta}\Big) + x(0)\frac{\xi(0)}{\eta^2}\sin^2(\eta s)\right.\\ &\quad \left.+\frac{1}{2}\frac{\xi^2(0)}{\eta^2}\Big(s + \frac{\sin(2\eta s)}{2\eta}\Big)\right]\\ &= \frac{\eta}{2}\Big(x^2(0) + \frac{\xi^2(0)}{\eta^2}\Big)s + \frac{1}{4}\Big(x^2(0) - \frac{\xi^2(0)}{\eta^2}\Big)\sin(2\eta s)\\ &\quad + \frac{x(0)}{2}\frac{\xi(0)}{\eta}\big(1 - \cos(2\eta s)\big).\end{aligned} \tag{3.9}$$

From (3.8), one has

$$\frac{\xi(0)}{\eta} = \frac{x(t) - x(0)\cos(\eta t)}{\sin(\eta t)}, \qquad \text{if } \sin(\eta t) \neq 0, \tag{3.10}$$

(cf. (2.9)). So we work with generic geodesics only, see Remark 2.7.

We introduce (3.10) into (3.9):

$$y(s) - y(0) = \frac{x^2(0)}{2}\Big(\eta s + \frac{1}{2}\sin(2\eta s)\Big)$$

$$+\frac{1}{2}\frac{x^2(t)-2x(t)x(0)\cos(\eta t)+x^2(0)\cos^2(\eta t)}{\sin^2(\eta t)}\Big(\eta s-\frac{1}{2}\sin(2\eta s)\Big)$$
$$+\frac{x(0)}{2}\frac{x(t)-x(0)\cos(\eta t)}{\sin(\eta t)}\Big(1-\cos(2\eta s)\Big).$$

Collecting terms containing $x^2(0)$, $x^2(t)$ and $x(0)x(t)$ respectively, we have

$$\begin{aligned} y(s)-y(0) \;&=\; \frac{x^2(0)}{4\sin^2(\eta t)}\Big\{2\eta s-\Big[\sin(2\eta t)-\sin(2\eta(t-s))\Big]\Big\} \\ &\quad+\frac{x^2(t)}{4\sin^2(\eta t)}(2\eta s-\sin(2\eta s)) \\ &\quad+\frac{x(0)x(t)}{2\sin^2(\eta t)}\left[\sin(2\eta s-\eta t)+\sin(\eta t)-2\eta s\cos(\eta t)\right]. \end{aligned}$$

From (3.7), one has

$$z(t)=z(0)+\int_0^t \dot{z}(s)\,ds=z(0)+S(t), \tag{3.11}$$

where $S(t)$ is the classical action. To simplify matters we set

$$x=x(t), \qquad y=y(t).$$

The classical action $S(t)$ is given by

$$\begin{aligned} S(t) \;&=\; \int_0^t \big(\xi\dot{x}+\eta\dot{y}-H\big)ds \\ &=\; \eta\big(y-y(0)\big)+\int_0^t \big(\xi^2(s)-H(s)\big)ds. \end{aligned} \tag{3.12}$$

Then (3.6) and (3.8) yield

$$\xi(s)=\dot{x}(s)=\xi(0)\cos(\eta s)-\eta x(0)\sin(\eta s).$$

H is constant along a bicharacteristic, so

$$H=H(0)=\frac{1}{2}\Big(\xi^2(0)+\eta^2x^2(0)\Big),$$

and

$$S(t) = \eta\big(y(t)-y(0)\big)+\int_0^t \left[\frac{\cos(2\eta s)}{2}\big(\xi^2(0)-\eta^2x^2(0)\big)-\eta x(0)\xi(0)\sin(2\eta s)\right] ds$$
$$= \eta\big(y-y(0)\big)+\frac{\sin(2\eta t)}{4\eta}\Big(\xi^2(0)-\eta^2x^2(0)\Big)+\eta x(0)\xi(0)\frac{\cos(2\eta t)-1}{2\eta}.$$

Replacing $\xi(0)$ by (3.10), one has

$$\begin{aligned}
&S(t)-\eta\big(y-y(0)\big)\\
&= \frac{\eta^2}{2}\left(\frac{x-x(0)\cos(\eta t)}{\sin(\eta t)}\right)^2\frac{\sin(2\eta t)}{2\eta}-\frac{1}{2}\eta^2x^2(0)\frac{\sin(2\eta t)}{2\eta}\\
&\quad +\eta^2x(0)\frac{x-x(0)\cos(\eta t)}{\sin(\eta t)}\frac{\cos(2\eta t)-1}{2\eta}\\
&= \frac{\eta}{4}\left[\Big(\frac{x-x(0)\cos(\eta t)}{\sin(\eta t)}\Big)^2\sin(2\eta t)-x^2(0)\sin(2\eta t)\right.\\
&\quad \left.-2x(0)\frac{x-x(0)\cos(\eta t)}{\sin(\eta t)}\big(1-\cos(2\eta t)\big)\right].
\end{aligned}$$

After simple computations, one has

$$S(t)=\eta\big(y-y(0)\big)-\frac{\eta}{4}\left(\big(x+x(0)\big)^2\tan\frac{\eta t}{2}-\big(x-x(0)\big)^2\cot\frac{\eta t}{2}\right).$$

3.3. The Hamilton-Jacobi equation

To find the solution of the Hamilton-Jacobi equation we still need $z(0)$, see (3.11). Instead, we shall substitute $S(t)$ of (3.12),

$$S=S(t,x,y,x_0,\eta),\qquad x_0=x(0),$$

into (3.4) and find the discrepancy. Recall

$$x(s)=x(s;x,y,x_0,\eta,t),\dots,\quad \text{etc.}$$

Then,

$$
\begin{aligned}
&\frac{\partial S}{\partial x}(t;x,y,x_0,\eta)\\
&=\int_0^t\Big[\frac{\partial\xi}{\partial x}\frac{dx}{ds}+\xi\frac{d}{ds}\frac{\partial x(s;x,y,x_0,\eta,t)}{\partial x}+\frac{\partial\eta}{\partial x}\frac{dy}{ds}+\eta\frac{d}{ds}\frac{\partial y(s;\cdots,t)}{\partial x}\\
&\quad-\frac{\partial H}{\partial\xi}\frac{\partial\xi}{\partial x}-\frac{\partial H}{\partial\eta}\frac{\partial\eta}{\partial x}-\frac{\partial H}{\partial x}\frac{\partial x(s;\cdots,t)}{\partial x}-\frac{\partial H}{\partial y}\frac{\partial y(s;\cdots,t)}{\partial x}\Big]ds\\
&=\int_0^t\Big(\xi\frac{d}{ds}\frac{\partial x(s;\cdots,t)}{\partial x}+\eta\frac{d}{ds}\frac{\partial y(s;\cdots,t)}{\partial x}+\dot{\xi}\frac{\partial x(s;\cdots,t)}{\partial x}+\dot{\eta}\frac{\partial y(s;\cdots,t)}{\partial x}\Big)ds\\
&=\int_0^t\frac{d}{ds}\left(\xi\frac{\partial x(s;\cdots,t)}{\partial x}+\eta\frac{\partial y(s;\cdots,t)}{\partial x}\right)ds\\
&=\xi(s)\frac{\partial x(s;\cdots,t)}{\partial x}\Big|_{s=0}^{s=t}+\eta(s)\frac{\partial y(s;\cdots,t)}{\partial x}\Big|_{s=0}^{s=t}.
\end{aligned}
$$

which yields

$$\frac{\partial S}{\partial x}(t;x,y,x_0,\eta)=\xi(t)-\eta(0)\frac{\partial y(0;x,y,x_0,\eta,t)}{\partial x}. \tag{3.13}$$

Similarly,

$$\frac{\partial S}{\partial y}(t;x,y,x_0,\eta)=\eta(t)-\eta(0)\frac{\partial y(0;x,y,x_0,\eta,t)}{\partial y}. \tag{3.14}$$

Moreover,

$$
\begin{aligned}
&\frac{\partial S}{\partial t}(t;\cdots)\\
&=\ \xi(t;\cdots)\dot{x}(t;\cdots)+\eta(t;\cdots)\dot{y}(t;\cdots)-H(t)\\
&\quad+\int_0^t\Big(\xi\frac{\partial}{\partial t}\frac{dx}{ds}+\eta\frac{\partial}{\partial t}\frac{dy}{ds}-\frac{\partial H}{\partial x}\frac{\partial x}{\partial t}-\frac{\partial H}{\partial y}\frac{\partial y}{\partial t}\Big)ds\\
&=\ \xi(t)\dot{x}(t)+\eta(t)\dot{y}(t)-H(t)+\int_0^t\Big(\xi\frac{d}{ds}\frac{\partial x}{\partial t}+\eta\frac{d}{ds}\frac{\partial y}{\partial t}+\dot{\xi}\frac{\partial x}{\partial t}+\dot{\eta}\frac{\partial x}{\partial t}\Big)ds \qquad (3.15)\\
&=\ \xi(t)\dot{x}(t)+\eta(t)\dot{y}(t)-H(t)+\int_0^t\frac{d}{ds}\Big(\xi\frac{\partial x}{\partial t}+\eta\frac{\partial y}{\partial t}\Big)ds\\
&=\ \xi(t)\dot{x}(t)+\eta(t)\dot{y}(t)-H(t)\\
&\quad+\xi(s)\frac{\partial x(s;\cdots)}{\partial t}\Big|_{s=0}^{s=t}+\eta(s)\frac{\partial y(s;\cdots)}{\partial t}\Big|_{s=0}^{s=t}.
\end{aligned}
$$

Now, $x_0 = x(0; \cdots, t)$ is fixed, and $x = x(t; \cdots, t)$ is also fixed, so

$$0 = \frac{d}{dt} x(t; \cdots, t) = \dot{x}(t) + \frac{\partial x}{\partial t}(s; \cdots, t)\Big|_{s=t},$$

and therefore,

$$\frac{\partial x}{\partial t}(s; \cdots, t)\Big|_{s=0}^{s=t} = -\dot{x}(t); \tag{3.16}$$

we note that $x(0) = x_0$ is fixed. On the other hand $y(0; \cdots)$ is not independent of t, so

$$\frac{\partial y}{\partial t}(s; \cdots, t)\Big|_{s=0}^{s=t} = -\dot{y}(t) - \frac{\partial y}{\partial t}(0; \cdots). \tag{3.17}$$

Consequently, one has

$$\frac{\partial S}{\partial t} = -H(t) - \eta(0)\frac{\partial y}{\partial t}(0; \cdots). \tag{3.18}$$

We set

$$h = \eta(0) y(0; \cdots) + S. \tag{3.19}$$

Then (3.13) and (3.14) yield

$$\frac{\partial h}{\partial x} = \xi(t), \qquad \frac{\partial h}{\partial y} = \eta(t), \tag{3.20}$$

and (3.18) gives us

$$0 = \frac{\partial h}{\partial t} + H(t) = \frac{\partial h}{\partial t} + H\big(x(t), y(t), \xi(t), \eta(t)\big).$$

In view of (3.20) we have found a solution of

$$\frac{\partial h}{\partial t} + H\Big(x(t), y(t), \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\Big) = \frac{\partial h}{\partial t} + H\Big(\nabla h\Big) = 0.$$

We note that

$$h = h(x, y, x_0, \eta(0), t).$$

Thus we have derived

Theorem 3.1. *When $\eta t \neq \pm k\pi$, $k = 1, 2, \ldots$,*

$$h = \eta y - \frac{1}{4}\eta\Big[(x + x_0)^2 \tan\frac{\eta t}{2} - (x - x_0)^2 \cot\frac{\eta t}{2}\Big] \tag{3.21}$$

is a solution of the Hamilton-Jacobi equation (3.3).

3.4. The eiconal equation and the modified complex action

We note that

$$h(x, y, x_0, \eta, t) = \frac{1}{t}h(x, y, x_0, \eta t, 1) \doteq \frac{1}{t}g(x, y, x_0, \eta t), \tag{3.22}$$

which we use to define g. According to (3.2) and Theorem 3.1, we look for a heat kernel in the form

$$\frac{1}{t^\alpha}e^{-h} = \frac{1}{t^\alpha}e^{-\frac{g}{t}}.$$

The heat kernel should not depend on η. So we use an age old technique to get rid of η by summing over it. Thus we shall look for a heat kernel in the following form:

$$P = \frac{1}{(2\pi t)^\alpha}\int_{\mathbb{R}} e^{-\frac{g(x,y,x_0,\lambda)}{t}}V(\lambda)d\lambda. \tag{3.23}$$

To simplify matters we summed over $\eta t = \lambda$; an extra t can always be absorbed in $t^{-\alpha}$, especially since we have not chosen α so far. $V(\lambda)$ is thrown in for good measure; we are free to do so and we shall need it.

Lemma 3.2. *$g(x, y, x_0, \lambda)$ is a solution of the eiconal equation*

$$\lambda\frac{\partial g}{\partial\lambda} + H\Big(x, y, \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\Big) = g. \tag{3.24}$$

Proof. One has

$$\frac{\partial h}{\partial t} = -\frac{1}{2}\left(\frac{\partial h}{\partial x}\right)^2 - \frac{1}{2}\,x^2\,\left(\frac{\partial h}{\partial y}\right)^2,$$

and (3.22) gives

$$\frac{\partial h}{\partial t} = -\frac{1}{t^2}g + \frac{1}{t}\eta\frac{\partial g}{\partial\lambda}, \qquad \lambda = \eta t.$$

The two right hand sides agree for all t, so we may as well set $t = 1$. Replacing η by λ we have derived the lemma. $\square$

It is time to fix the path of integration. Equations (3.21) and (3.22) yield

$$g(\lambda) = \lambda y - \frac{\lambda}{4}\left[(x+x_0)^2 \tan\frac{\lambda}{2} - (x-x_0)^2 \cot\frac{\lambda}{2}\right], \tag{3.25}$$

$\lambda \neq \pm k\pi$, $k = 1, 2, \ldots$. Still, this nonclassical action will give us all geodesics (see Remark 2.7).

To simplify the calculations we set

$$a^2 = (x+x_0)^2, \qquad b^2 = (x-x_0)^2.$$

Lemma 3.3. *Let* $\lambda = \theta\rho = \lambda_1 + i\lambda_2 \in \mathbb{C}$ *with* $\rho = |\lambda|$, $\lambda_1, \lambda_2 \in \mathbf{R}$. *Fix* θ. *Then*

$$\lim_{\rho\to\infty} Re\, g(\theta\rho) \;=\; \infty$$

for all (x, y) *off the canonical curve* $x^2 + x_0^2 = 0$ *if and only if* $\theta \in i\mathbb{R}$.

Proof. The content of the square bracket in (3.25) may be rewritten in the following form:

$$\begin{aligned}
\frac{a^2 \sin\frac{\lambda}{2}}{\cos\frac{\lambda}{2}} - \frac{b^2\cos\frac{\lambda}{2}}{\sin\frac{\lambda}{2}} &= \frac{a^2(1-\cos\lambda) - b^2(1+\cos\lambda)}{\sin\lambda}\\
&= \frac{a^2(1-\cos\lambda) - b^2(1+\cos\lambda)}{\sin\lambda\cdot\sin\bar\lambda}\sin\bar\lambda\\
&= \frac{a^2(1-\cos\lambda) - b^2(1+\cos\lambda)}{\frac{1}{2}\big(\cos(\lambda-\bar\lambda) - \cos(\lambda+\bar\lambda)\big)}\sin\bar\lambda\\
&= \frac{a^2(1-\cos\lambda) - b^2(1+\cos\lambda)}{\frac{1}{2}\big(\cosh(2\lambda_2) - \cos(2\lambda_1)\big)}\sin\bar\lambda\\
&= \frac{a^2(1-\cos\lambda) - b^2(1+\cos\lambda)}{\cosh^2\lambda_2 - \cos^2\lambda_1}\sin\bar\lambda.
\end{aligned}$$

Now a straightforward elementary calculation gives us

$$\begin{aligned} &\mathrm{Re}(g)(\lambda) \\ &= \lambda_1 y + \frac{1}{4}\left[a^2\frac{\lambda_2\sinh\lambda_2 - \lambda_1\sin\lambda_1}{\cosh\lambda_2 + \cos\lambda_1} + b^2\frac{\lambda_2\sinh\lambda_2 + \lambda_1\sin\lambda_1}{\cosh\lambda_2 - \cos\lambda_1}\right]. \end{aligned} \tag{3.26}$$

We write $\theta = \theta_1 + i\theta_2$, θ_1, $\theta_2 \in \mathbb{R}$.

(i) $\theta_1 = 0$, $\theta_2 = \pm 1$, so $\theta \in i\mathbb{R}$. When $\rho \approx \infty$,

$$\begin{aligned} \mathrm{Re}(g) &\approx \frac{1}{4}(a^2+b^2)\theta_2\rho\tanh(\theta_2\rho) \\ &\approx \frac{1}{2}(x^2+x_0^2)\rho \;\to\; \infty. \end{aligned}$$

(ii) $\theta_1 = \pm 1$, $\theta_2 = 0$ and $\theta \in \mathbb{R}$. Then

$$\mathrm{Re}(g) = \pm\rho y - \frac{\rho}{4}\Big(a^2\tan\frac{\rho}{2} - b^2\cot\frac{\rho}{2}\Big),$$

which is highly singular in $\rho \in \mathbb{R}$ when $x_0^2 + x^2 \neq 0$, otherwise

$$\mathrm{Re}(g) = \pm\rho y \;\to\; \pm(\mathrm{sgn})(y)\,\infty,$$

as $\rho \to \infty$.

(iii) $\theta_1 \neq 0$, $\theta_2 \neq 0$. In this case, for large ρ one has

$$\begin{aligned} \mathrm{Re}(g) &\approx \theta_1\rho y + \frac{1}{4}\Big(a^2\lambda_2\tanh\,\lambda_2 + b^2\lambda_2\coth\,\lambda_2\Big) \\ &\approx \theta_1\rho y + \frac{1}{4}(a^2+b^2)|\theta_2|\rho \\ &= \Big(\theta_1 y + \frac{1}{2}(x_0^2+x^2)|\theta_2|\Big)\rho. \end{aligned}$$

Choosing y so that

$$\big|\theta_1 y\big| \;>\; \frac{1}{2}(x_0^2+x^2)|\theta_2|,$$

we obtain

$$\mathrm{Re}(g) \;\to\; \big(\mathrm{sgn}(\theta_1 y)\big)\infty,$$

which can be $\pm\infty$ depending on the sign of y and θ_1. This completes the proof of the lemma. $\square$

So to have our integrand on its best behavior, our choice of integration path should end on the imaginary axis at $\pm\infty$. An obvious choice is the imaginary axis; this may also be forced on us, more or less, if the heat kernel is real. We make this choice and set

$$\lambda = -i\tau, \qquad \tau \in \mathbb{C}, \tag{3.27}$$

by convention, and

$$f(\tau) = g(-i\tau),$$

so

$$f(\tau) = -i\tau y + \frac{\tau}{4}\left[(x+x_0)^2 \tanh\frac{\tau}{2} + (x-x_0)^2 \coth\frac{\tau}{2}\right], \tag{3.28}$$

is the modified complex action.

We note that $f(\tau)$ is bounded near $\tau = 0$. Thus we look for a heat kernel in the form

$$P = \frac{1}{(2\pi t)^\alpha}\int_{\mathbb{R}} e^{-\frac{f(\tau)}{t}} V(\tau)d\tau, \tag{3.29}$$

where we used the shorthand

$$P = P_t = P_t(x_0, x, y).$$

$f(\tau)$ satisfies the eiconal equation (3.24):

$$\tau\frac{\partial f}{\partial \tau} + H\Big(x, y, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\Big) = f. \tag{3.30}$$

4. Properties of the Modified Complex Action

4.1. The critical points of the modified complex action f

Recall the modified complex action

$$f = -i\tau y + \frac{1}{4}\big(a^2\tau\tanh\frac{\tau}{2} + b^2\tau\coth\frac{\tau}{2}\big)$$

where $\tau = u + iv \in \mathbb{C}$, $a = x + x_0$ and $b = x - x_0$. Denote by f_1, f_2

respectively the real and imaginary part of f.

$$\begin{aligned} f_1(u,v) &= \mathrm{Re} f \\ &= vy+\frac{1}{4}\Big[a^2\frac{u\sinh u - v\sin v}{\cosh u+\cos v}+b^2\frac{u\sinh u+v\sin v}{\cosh u-\cos v}\Big]; \end{aligned} \tag{4.1}$$

$$\begin{aligned} f_2(u,v) &= \mathrm{Im} f \\ &= -uy+\frac{1}{4}\Big[a^2\frac{v\sinh u+u\sin v}{\cosh u+\cos v}+b^2\frac{v\sinh u-u\sin v}{\cosh u-\cos v}\Big]. \end{aligned} \tag{4.2}$$

In Section 3 we find the classical action S from the system (3.6) which contains (2.4) as a subsystem. Then we get from S successively the functions h, g and the modified complex action f. Hence it is not surprising that

$$\frac{\partial f}{\partial \tau}|_{u=0} = \frac{\partial f}{\partial u}|_{u=0} = \frac{\partial (if_2)}{\partial u}|_{u=0} = i[-y+\frac{1}{4}(a^2\tilde{\mu}(v)+b^2\mu(v))] \tag{4.3}$$

which is exactly i times (2.10). It follows from Theorem 2.6 that (4.3) vanishes at $v=\eta_j$, $j=1,\ldots,N$ for generic geodesics connecting $(x_0,0)$ and (x,y). So these geodesics correspond to the critical points of f. In fact, we will show that: For $v\geq 0$ (i.e. $y\geq 0$), all the critical points of f lie on the positive v-axis and are related to generic geodesics. Therefore the critical points of f is in one to one correspondence with the generic geodesics.

We first prove for the generic case, then give a brief discussion for the exceptional geodesics.

4.1.1. The generic case

(**A**) As f is holomorphic,

$$\frac{\partial f}{\partial \tau}=0 \quad \Leftrightarrow \quad \frac{\partial f_1}{\partial u}=\frac{\partial f_2}{\partial u}=0. \tag{4.4}$$

When $a=b=0$, $f(\tau)=-i\tau y$, so $\frac{\partial f}{\partial \tau}=-iy$ which vanishes if and only if $y=0$. In other words, this corresponds to the trivial initial conditions $x_0=x=y=0$. So we exclude this case and assume that $a^2+b^2>0$ and $y\geq 0$ (so $v\geq 0$ by (3.27)) in finding the set satisfying the right hand side of (4.4).

(**B**) We have

$$\begin{aligned}
4\frac{\partial f_1}{\partial u} \;=\; & \Big\{\frac{a^2}{(\cosh u+\cos v)^2} \\
& \times\Big[u+\sinh u\cosh u+u\cosh u\cos v+v\sinh u\sin v+\sinh u\cos v\Big]\Big\} \\
& +\Big\{\frac{b^2}{(\cosh u-\cos v)^2} \\
& \times\Big[u+\sinh u\cosh u-u\cosh u\cos v-v\sinh u\sin v-\sinh u\cos v\Big]\Big\}.
\end{aligned}$$

So $\frac{\partial f_1}{\partial u}=0$ implies that

$$\begin{aligned}
&\alpha^2\big(u+\sinh u\cosh u+u\cosh u\cos v+v\sinh u\sin v+\sinh u\cos v\big) \\
&+\beta^2\big(u+\sinh u\cosh u-u\cosh u\cos v-v\sinh u\sin v-\sinh u\cos v\big) \;=\; 0
\end{aligned}$$

where

$$\alpha^2=\frac{a^2}{(\cosh u+\cos v)^2},\qquad \beta^2=\frac{b^2}{(\cosh u-\cos v)^2}.$$

(**C**) When $v=0$, $\frac{\partial f_1}{\partial u}=0$ implies that

$$\begin{aligned}
&\alpha^2\big(u+\sinh u\cosh u+u\cosh u+\sinh u\big) \\
&+\beta^2\big(u+\sinh u\cosh u-u\cosh u-\sinh u\big)=0
\end{aligned}$$

and it is easy to see that $\frac{\partial f_2}{\partial u}\Big|_{v=0}=-y$.

As

$$\begin{aligned}
u+\sinh u\cosh u+u\cosh u+\sinh u&=(\sinh u+u)(\cosh u+1) \\
u+\sinh u\cosh u-u\cosh u-\sinh u&=(\sinh u-u)(\cosh u-1),
\end{aligned}$$

we see that they always take the same sign. Hence,

$$\frac{\partial f_1}{\partial u}=\frac{\partial f_2}{\partial u}=0 \quad \text{holds only when} \quad u=0 \text{ and } y=0.$$

(**D**) It remains to consider $v > 0$. First assume $u \geq 0$. Now $\frac{\partial f_1}{\partial u} = 0$ implies that

$$\begin{aligned} & v(\alpha^2-\beta^2)\sinh u \sin v \\ &\quad = -\alpha^2\big(u+\sinh u\cosh u + u\cosh u\cos v + \sinh u\cos v\big) \\ &\qquad -\beta^2\big(u+\sinh u\cosh u - u\cosh u\cos v - \sinh u\cos v\big). \end{aligned} \tag{4.5}$$

Since

$$\begin{aligned} & u+\sinh u\cosh u + u\cosh u\cos v + \sinh u\cos v \\ &\quad \geq u+\sinh u\cosh u - u\cosh u - \sinh u \\ &\quad = (\sinh u - u)(\cosh u - 1) \geq 0 \end{aligned}$$

and similarly

$$\begin{aligned} & u+\sinh u\cosh u - u\cosh u\cos v - \sinh u\cos v \\ &\quad \geq (\sinh u - u)(\cosh u - 1) \geq 0. \end{aligned}$$

As $v > 0$, (4.5) holds only if

$$(\alpha^2-\beta^2)\sin v \leq 0. \tag{4.6}$$

On the other hand,

$$\begin{aligned} \frac{\partial f_2}{\partial u} &= \frac{f_2}{u} \\ &+\frac{\alpha^2}{4}\Big[\Big(v\cosh u - \frac{v\sinh u}{u}\Big)(\cosh u+\cos v)-(u\sin v+v\sinh u)\sinh u\Big] \\ &+\frac{\beta^2}{4}\Big[\Big(v\cosh u - \frac{v\sinh u}{u}\Big)(\cosh u-\cos v)-(v\sinh u-u\sin v)\sinh u\Big]. \end{aligned} \tag{4.7}$$

Now consider the case $f_2 \leq 0$. Then $\frac{\partial f_2}{\partial u} = 0$ implies that

$$\begin{aligned} 0 \ \leq\ & \alpha^2\Big[v\Big(\cosh u - \frac{\sinh u}{u}\Big)(\cosh u+\cos v) - (v\sinh u + u\sin v)\sinh u\Big] \\ & +\beta^2\Big[v\Big(\cosh u - \frac{\sinh u}{u}\Big)(\cosh u-\cos v) - (v\sinh u - u\sin v)\sinh u\Big] \\ =\ & \alpha^2\Big[v\Big(\cosh^2 u - \frac{\sinh u\cosh u}{u} + \cosh u\cos v - \frac{\sinh u}{u}\cos v - \sinh^2 u\Big) \\ & -u\sin v\sinh u\Big] \end{aligned}$$

$$
\begin{aligned}
&+\beta^2\Big[v\Big(\cosh^2 u - \frac{\sinh u\cosh u}{u} - \cosh u\cos v + \frac{\sinh u}{u}\cos v - \sinh^2 u\Big) \\
&\quad +u\,\sin v\sinh u\Big].
\end{aligned}
$$

Therefore, after multiplying the whole formula by u we obtain

$$
\begin{aligned}
v\Big[&\alpha^2(u - \sinh u\cosh u + u\cosh u\cos v - \sinh u\cos v) \\
&\quad +\beta^2(u - \sinh u\cosh u - u\cosh u\cos v + \sinh u\cos v)\Big] \\
\geq\ &(\alpha^2-\beta^2)u^2\sin v\sinh u.
\end{aligned}
\tag{4.8}
$$

Multiply both sides of (4.8) by $(\alpha^2-\beta^2)\sinh u\sin v$, the inequality reverses because of (4.6). Then we substitute $v(\alpha^2-\beta^2)\sinh u\sin v$ by the right hand side of (4.5) to obtain

$$
\begin{aligned}
&(\alpha^2-\beta^2)^2u^2\sinh^2 u\sin^2 v \\
&\quad\geq -\Big\{\alpha^2\big[u(1+\cosh u\cos v)+\sinh u(\cosh u+\cos v)\big] \\
&\qquad +\beta^2\big[u(1-\cosh u\cos v)+\sinh u(\cosh u-\cos v)\big]\Big\} \\
&\qquad \times\Big\{\alpha^2\big[u(1+\cosh u\cos v)-\sinh u(\cosh u+\cos v)\big] \\
&\qquad +\beta^2\big[u(1-\cosh u\cos v)-\sinh u(\cosh u-\cos v)\big]\Big\} \\
&\quad = -\Big\{\alpha^4\big[u^2(1+\cosh u\cos v)^2-\sinh^2 u(\cosh u+\cos v)^2\big] \\
&\qquad +\beta^4\big[u^2(1-\cosh u\cos v)^2-\sinh^2 u(\cosh u-\cos v)^2\big] \\
&\qquad +2\alpha^2\beta^2\big[u^2(1-\cosh^2 u\cos^2 v)-\sinh^2 u(\cosh^2 u-\cos^2 v)\big]\Big\}.
\end{aligned}
$$

Collect terms containing u^2 on the right hand side then move it to the left hand side, we get

$$
\begin{aligned}
&u^2\big[\alpha^4(1+\cosh u\cos v)^2+\beta^4(1-\cosh u\cos v)^2 \\
&\quad +2\alpha^2\beta^2(1-\cosh^2 u\cos^2 v)+(\alpha^2-\beta^2)^2\sin^2 v\sinh^2 u\big] \\
&\geq \sinh^2 u\big[\alpha^4(\cosh u+\cos v)^2+\beta^4(\cosh u-\cos v)^2+2\alpha^2\beta^2(\cosh^2 u-\cos^2 v)\big].
\end{aligned}
$$

In other words,

$$
\begin{aligned}
&u^2\big[\alpha^4(\cosh u+\cos v)^2+\beta^4(\cosh u-\cos v)^2+2\alpha^2\beta^2(\sin^2 v-\sinh^2 u)\big] \\
&\geq \sinh^2 u \\
&\quad\times\big[\alpha^4(\cosh u+\cos v)^2+\beta^4(\cosh u-\cos v)^2+2\alpha^2\beta^2(\cosh^2 u-\cos^2 v)\big].
\end{aligned}
\tag{4.9}
$$

Since $(\cosh^2 u - \cos^2 v) - (\sin^2 v - \sinh^2 u) = 2\sinh^2 u \geq 0$, and $u^2 \leq \sinh^2 u$, the inequality (4.9) holds only when $u = 0$.

(**E**) The case $v > 0$, $u < 0$ and $f_2 \leq 0$ can be proved similarly. Actually, follow the same steps, we see (4.6) holds also. Then the inequality is reversed in (4.8). However, in this case $(\alpha^2 - \beta^2)\sinh u \sin v \geq 0$, so we get the same conclusion.

We have proved that when $v \geq 0$, $u \in \mathbb{R}$, all the critical points of f in $\{u + iv;\ f_2(u + iv) \leq 0\}$ lie on v-axis.

(**F**) Finally observe that f_2 is an odd function with respect to u. Therefore the critical points of f can only occur on the positive v-axis. In view of (4.3) we have proved the following theorem.

Theorem 4.1. *The critical points of the modified complex action f with $y \geq 0$ (and so $v \geq 0$) can occur only on the positive v-axis. In fact, they correspond exactly to the generic geodesics.*

4.1.2. Discussion on exceptional geodesics

In Section 2 we already know that the exceptional geodesics do not come from the solutions of (2.10) and the generic geodesics are dense. The function $F(\eta)$ is singular at $\eta = k\pi$, $k \in \mathbb{N}$ in general and is defined at these points only in the sense of (2.14), (2.23) and (2.24). The situation here is exactly the same. The modified complex action f is singular at $\tau = ik\pi$, $k \in \mathbb{N}$ in general. Suppose we take (2.14), (2.23) and (2.24) into account, f will be defined at these singular points. Also the critical points of f will include exceptional geodesics if we start from generic case and then taking limits in the sense of (2.14), (2.23) and (2.24). The details are left to the readers.

On the other hand, the volume form $V(\tau)$ constructed in Section 3 for the heat kernel of Δ_G is singular at $\tau = ik\pi$, $k \in \mathbb{N}$. This fact actually reflects the existence of the exceptional geodesics and will be seen clearer in Section 6.

4.2. The set $\{(u, v) : \mathrm{Im} f(u + iv) = f_2(u, v) = 0\}$ and the curve Γ

We follow the notations in 4.1. We need to consider the set $\{(u, v) : f_2(u, v) = 0,\ u \geq 0,\ v \geq 0\}$ only. The set $\{(u, v) :\ f_2(u, v) = 0,\ u \leq 0,\ v \geq$

$0\}$ can be obtained by symmetry. We then define the curve Γ which is crucial in proving the strict increase of the lengths of the geodesics claimed in Theorem 2.6 and the discussions in Sections 5 and 6.

We start with the following obvious fact

Lemma 4.2. *The function f_2 vanishes on v-axis except at $\tau = ik\pi$, $k \in \mathbb{N}$, which are simple poles of f.*

The next lemma is important.

Lemma 4.3. *Assume $y > 0$. Let $A = \{(u,v)|\ u \geq 0,\ v \geq 0,\ f_2(u,v) < 0\}$. Then $\frac{\partial f_2}{\partial u} < 0$ on $\bar{A}$ except those points $(0,\eta_j)$, $j = 1,\ldots,N$ with η_j's defined in Theorem* 2.6(1).

Proof. The lemma holds on positive u-axis since $\frac{\partial f_2}{\partial u}|_{v=0} = -y < 0$ by assumption. On $\bar{A} \cap \{u = 0\}$, we first assume that $x \neq \pm x_0$. By (4.3)

$$\frac{\partial f_2}{\partial u}|_{u=0} = -y + \frac{1}{4}F(v).$$

It follows from Theorem 2.4(c) that

$$\frac{\partial f_2}{\partial u}(0,v) < 0 \text{ if } 0 \leq v < \eta_1 \text{ or } \eta_{2j} < v < \eta_{2j+1},\ 1 \leq j \leq \frac{N-1}{2} \tag{4.10}$$

$$\frac{\partial f_2}{\partial u}(0,v) = 0 \text{ if } v = \eta_j,\ 1 \leq j \leq N \tag{4.11}$$

$$\frac{\partial f_2}{\partial u}(0,v) > 0 \text{ if } \eta_{2j-1} < v < \eta_{2j},\ 1 \leq j \leq \frac{N-1}{2} \text{ or } \eta_N < v. \tag{4.12}$$

Hence for positive u close to zero we have

$$f_2(u,v) < 0 \text{ if } v \text{ satisfies (4.10)} \tag{4.13}$$

$$f_2(u,v) > 0 \text{ if } v \text{ satisfies (4.12).} \tag{4.14}$$

Therefore $\bar{A} \cap \{u = 0\} = \{(0,v)|\ 0 \leq v \leq \eta_1,\ \eta_{2j} \leq v \leq \eta_{2j+1},\ 1 \leq j \leq \frac{N-1}{2}\}$ by (4.3) and by (4.10), (4.11) the lemma is true here. The assumption $x \neq \pm x_0$ can be dropped by the same argument in proving Theorem 2.6.

It remains to consider $u > 0$, $v > 0$. In view of (4.7) it suffices to prove

$$\sinh u\, \cosh u - u + (\sinh u - u\, \cosh u) \cos v + \frac{\sin v}{v} u^2 \sinh u\ >\ 0 \tag{4.15}$$

and

$$\sinh u\,\cosh u - u + (u\,\cosh u - \sinh u)\cos v - \frac{\sin v}{v}\,u^2\sinh u\ >\ 0. \quad (4.16)$$

Proof of (4.15). Note that

$$\sinh u < u\cosh u \qquad \text{for } u > 0. \quad (4.17)$$

(a) $\sin v \geq 0$. In this case

$$\begin{aligned}
&\sinh u\cosh u - u + (\sinh u - u\,\cosh u)\cos v + \frac{\sin v}{v}\,u^2\sinh u\\
\geq&\sinh u\cosh u - u + \sinh u - u\cosh u\\
=&(\sinh u - u)(1+\cosh u) > 0 \qquad \text{for } u > 0
\end{aligned}$$

(b) $\sin v < 0$. We first prove for $v < 2\pi$.

(b1) $\pi < v \leq \frac{3\pi}{2}$. Here $\cos v \leq 0$, so

$$\begin{aligned}
&\sinh u\cosh u - u + (\sinh u - u\,\cosh u)\cos v + \frac{\sin v}{v}\,u^2\sinh u\\
\geq&\sinh u\cosh u - u - \frac{u^2}{\pi}\sinh u = \sinh u\big(\cosh u - \frac{u^2}{\pi}\big) - u\\
\geq&\sinh u\big(1 + (\frac{1}{2} - \frac{1}{\pi})u^2\big) - u > 0 \quad \text{if } u > 0.
\end{aligned}$$

(b2) $\frac{3\pi}{2} < v < 2\pi$. Now $\cos v > 0$ and by (4.12)

$$\begin{aligned}
&\sinh u\cosh u - u + (\sinh u - u\,\cosh u)\cos v + \frac{\sin v}{v}\,u^2\sinh u\\
\geq&\sinh u\cosh u - u + \sinh u - u\cosh u - \frac{2}{3\pi}u^2\sinh u\\
=&\sinh u\cosh u + \sinh u - u(2\cosh^2\frac{u}{2}) - \frac{2}{3\pi}u^2\sinh u\\
=&2\cosh\frac{u}{2}\Big[\sinh\frac{u}{2}\big(1+\cosh u - \frac{2}{3\pi}u^2\big) - u\cosh\frac{u}{2}\Big]
\end{aligned} \quad (4.18)$$

which is 0 if $u = 0$.

On the other hand,

$$\begin{aligned}
&\frac{d}{du}\Big[\sinh\frac{u}{2}\big(1+\cosh u-\frac{2}{3\pi}u^2\big)-u\cosh\frac{u}{2}\Big]\\
=&\frac{1}{2}\cosh\frac{u}{2}\big(1+\cosh u-\frac{2}{3\pi}u^2\big)+\sinh\frac{u}{2}\big(\sinh u-\frac{4}{3\pi}u\big)-\cosh\frac{u}{2}-\frac{u}{2}\sinh\frac{u}{2}\\
=&\frac{1}{2}\cosh\frac{u}{2}\big(\cosh u-1-\frac{2}{3\pi}u^2\big)+\sinh\frac{u}{2}\Big(\sinh u+\big(\frac{4}{3\pi}+\frac{1}{2}\big)u\Big)>0,\ \text{if } u>0.
\end{aligned}$$

Therefore the last line of (4.18) is strictly positive if $u>0$. So (4.15) holds for $\frac{3\pi}{2}<v<2\pi$.

When $v>3\pi$ and $\sin v<0$, the proof is similar but easier. We omit it. This completes the proof of (4.15). □

Proof of (4.16).

(a) $\sin v\le 0$. In this case

$$\begin{aligned}
&\sinh u\cosh u-u+(u\cosh u-\sinh u)\cos v-\frac{\sin v}{v}u^2\sinh u\\
&\quad\ge\ \sinh u\cosh u-u-u\cosh u+\sinh u\\
&\quad=\ (1+\cosh u)(\sinh u-u)>0\quad\text{if }\ u>0.
\end{aligned}$$

(b) $\sin v>0$. We divide the proof into two cases.

(b1) $v>\pi$, $\sin v>0$. This implies that $v>2\pi$. Hence,

$$\begin{aligned}
&\sinh u\cosh u-u+(u\ \cosh u-\sinh u)\cos v-\frac{\sin v}{v}u^2\sinh u\\
&\quad\ge\ \sinh u\cosh u-u-u\cosh u+\sinh u-\frac{1}{2\pi}u^2\sinh u\\
&\quad\ge\ \sinh u-u+\cosh u\big(\underbrace{\sinh u-u-\frac{u^3}{2\pi}}_{>u^3(\frac{1}{3!}-\frac{1}{2\pi})>0}\big)>0,\quad\text{if } u>0.
\end{aligned}$$

In the last line we replace $u^2\sinh u$ by $u^3\cosh u$ which is larger by (4.17).

(b2) $0<v<\pi$. We have

$$\begin{aligned}
&\sinh u\cosh u-u+(u\ \cosh u-\sinh u)\cos v-\frac{\sin v}{v}u^2\sinh u\\
&\quad=\ \frac{\sinh(2u)}{2}-u+(u\cosh u-\sinh u)\cos v-\frac{\sin v}{v}u^2\sinh u
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{k=1}^{\infty}\frac{(2u)^{2k+1}}{(2k+1)!} + \Big(u\sum_{k=0}^{\infty}\frac{u^{2k}}{(2k)!} - \sum_{k=0}^{\infty}\frac{u^{2k+1}}{(2k+1)!}\Big)\cos v \\
&\quad -u^2\sum_{k=0}^{\infty}\frac{u^{2k+1}}{(2k+1)!}\frac{\sin v}{v}\Big) \\
&= \frac{1}{v}\sum_{k=1}^{\infty}\frac{u^{2k+1}}{(2k+1)!}\Big(2^{2k}v + 2kv\cos v - 2k(2k+1)\sin v\Big).
\end{aligned}
$$

It is easy to see that $2^{2k} - 2k - 2k(2k+1) > 0$ for $k \geq 3$. Furthermore,

$$
\begin{aligned}
&\frac{d}{dv}\Big(\underbrace{2^{2k}v + 2kv\cos v - 2k(2k+1)\sin v}_{\text{denote this function by } E(v)}\Big) \\
&= 2^{2k} + 2k\cos v - 2kv\sin v - 2k(2k+1)\cos v \\
&= 2^{2k} - (2k)^2\cos v - 2kv\sin v,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{d^2E(v)}{dv^2} &= (2k)^2\sin v - 2kv\cos v - 2k\sin v \\
&= 2k(2k-1)\sin v - 2kv\cos v \\
&\geq 2k(\sin v - v\cos v) > 0 \quad \text{for } 0 < v < \pi.
\end{aligned}
$$

Now $E(v) = 0$ if $v = 0$, and

$$
\frac{dE}{dv}\Big|_{v=0} = 2^{2k} - (2k)^2 = 0 \quad \text{for } k = 1, 2
$$

It follows that $E(v) > 0$ for $0 < v < \pi$, and the proof of (4.16) is therefore complete.

This completes the proof of Lemma 4.3. □

Assume that $y > 0$. For any $v_0 \geq 0$ fixed, the function $f_2(u, v_0)$ tends to $-\infty$ as u tends to ∞. Lemma 4.3 implies that there is a unique $u_0 \geq 0$ such that $f_2(u_0, v_0) = 0$, $f_2(u, v_0) < 0$ if $u > u_0$ and $f_2(u, v_0) \geq 0$ if $u \leq u_0$. Thus A is an unbounded simply connected set and $\partial A \setminus u$-axis is a curve $\{(u, v)\}$ so that u is a function of $v > 0$. We set

$$
\Gamma^{+} = \partial A \setminus \{(u, v) |\ v < \eta_1\} \qquad \text{if } y > 0.
$$

When $y = 0$,we set

$$\Gamma^+ = \{(u, 0)|\ u \geq 0\}.$$

Note that in this case $A = \emptyset$ and $\frac{\partial f_2}{\partial u}\Big|_{\Gamma^+} \equiv 0$.

The properties of Γ^+ are collected in the following proposition. We need consider $y > 0$ only. First assume that $x \neq x_0$.

Proposition 4.4.

(1) *Γ^+ is a C^∞ curve except at $(0, \eta_j)$, $j = 2, \ldots, N$.*

(2) *$\Gamma^+ = \bigcup_1^N \Gamma_j^+$ where*

$$\begin{aligned} \Gamma_{2j}^+ &= \{(0, v)|\ \eta_{2j} \leq v < \eta_{2j+1}\},\ 1 \leq j \leq \frac{N-1}{2} \\ \Gamma_{2j-1}^+ &= \{(u, v) \in \Gamma^+|\ \eta_{2j-1} \leq v < \eta_{2j}\},\ 1 \leq j \leq \frac{N-1}{2} \quad \textit{and} \\ \Gamma_N^+ &= \{(u, v) \in \Gamma^+|\ \eta_N \leq v\}. \end{aligned}$$

For u, v large Γ_N asymptotically tends to $uy = \frac{v}{2}(x_0^2 + x^2) = \frac{v}{4}(a^2 + b^2)$. Also $\{(u, v)|\ u \geq 0,\ v \geq 0\} \setminus A = \bigcup_{j=1}^{\frac{N+1}{2}} D_j$. Each D_j is simply connected. For $j = 1, \ldots, \frac{N-1}{2}$, D_j is bounded by Γ_{2j-1}^+ and v-axis, so is a bounded set. The boundary of $D_{\frac{N+1}{2}}$ is $\{(0, v)|\ v \geq \eta_N\} \cup \Gamma_N^+$ and so $D_{\frac{N+1}{2}}$ is unbounded.

(3) *At $i\eta_j$, $1 \leq j < N$ and $j = N$ if $\eta_{N-1} < \eta_N$,*

$$\frac{\partial f}{\partial \tau}(i\eta_j) = 0, \quad \textit{and} \quad \frac{\partial^2 f}{\partial \tau^2}(i\eta_j) \neq 0.$$

Γ^+ meets v-axis at $i\eta_j$, $j = 1, \ldots, N$ orthogonally.

(4) *If $\eta_{N-1} = \eta_N$, then $\eta_N = \alpha_{\frac{N-1}{2}}$ by Theorem 2.6(1). We have*

$$\frac{\partial f}{\partial \tau}(i\eta_N) = \frac{\partial^2 f}{\partial \tau^2}(i\eta_N) = 0, \quad \textit{and} \quad \frac{\partial^3 f}{\partial \tau^3}(i\eta_N) \neq 0.$$

In this case, Γ_{N-1}^+ degenerates to the point $(0, \eta_N)$ and Γ_{N-2}^+, Γ_N^+ and v-axis meet at $i\eta_N$ with three equal intersecting angles, i.e. $\frac{\pi}{3}$.

Proof. (1) follows from the implicit function theorem by Lemma 4.3.

(2) is clear from the discussions before the Proposition.

To prove (3), observe that η_j is not a local minimum of $\frac{\partial f_2}{\partial u}|_{u=0}$ for $1 \leq j < N$ or for $j = N$ if $\eta_{N-1} < \eta_N$. So, we have for these critical points,

$$\frac{\partial^2 f_2}{\partial u\,\partial v}(0, \eta_j) \neq 0,$$

or equivalently,

$$\frac{\partial^2 f}{\partial \tau^2}(i\eta_j) \neq 0.$$

The last statement of (3) follows from the fact that f_2 is harmonic there.

As for (4), if $\eta_{N-1} = \eta_N$, then η_N is a local minimum of $\frac{\partial f_2}{\partial u}(0, v)$ and $\eta_N = \alpha_{\frac{N-1}{2}}$ by Theorem 2.6(1). Hence

$$\frac{\partial^2 f_2}{\partial u\,\partial v}(0, \eta_N) = 0 \qquad \text{or equivalently} \qquad \frac{\partial^2 f}{\partial \tau^2}(i\eta_N) = 0.$$

By Theorem 2.4(c) F is strictly convex there, this and (4.3) imply the strong convexity of $\frac{\partial f_2}{\partial u}|_{u=0}$ near η_N, thus we have

$$\frac{\partial^3 f_2}{\partial u\,\partial v^2}(0, \eta_N) \neq 0 \qquad \text{or equivalently} \qquad \frac{\partial^3 f}{\partial \tau^3}(i\eta_N) \neq 0.$$

The last statement follows from the harmonicity of f_2. This proves the Proposition. $\square$

The orientation of Γ^+ starts from $i\eta_1$ to infinity. Let

$$\Gamma^- = \{\tau :\ \tau = u + iv \quad \text{such that} \quad -u + iv \in \Gamma^+\} \tag{4.19}$$

with orientation reversed. Then set

$$\Gamma = \Gamma^+ \cup \Gamma^- \tag{4.20}$$

The curve Γ is depicted in Figure 4-1 and 4-2.

Next we discuss Γ in limiting cases.

The case $y = 0$, $x_0 \neq x$ is obvious: it is just the u-axis. For the rest cases, the definition of Γ^+ still holds if we use the conventions for η_j in Theorem 2.6.

For example, the curve Γ in the case $y > 0$, $x_0 = x = 0$ consists of

$$\Gamma^+ = \{0^+ + iv : \ v \geq \pi\} \quad \text{and } \Gamma^- = \{0^- + iv : \ v \geq \pi\}$$

with usual orientation. In other words, we have $N = \infty$ and Γ^+_{2j-1} shrinks to the point $ij\pi$, $j = 1, 2, \ldots$.

The details of the other cases are omitted. See Figure 4-3—4-6 for the corresponding pictures of Γ.

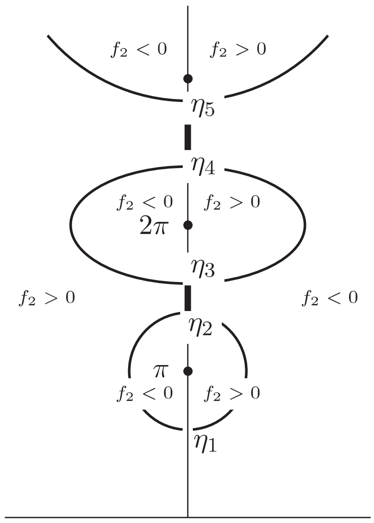

Figure 4-1

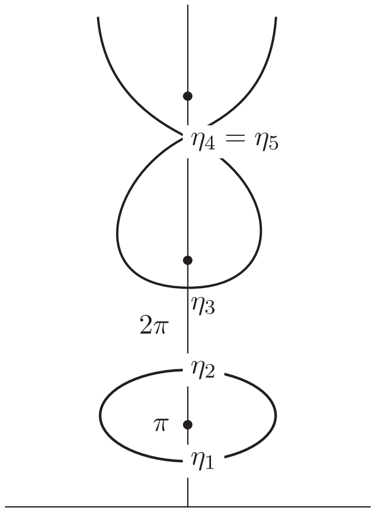

Figure 4-2

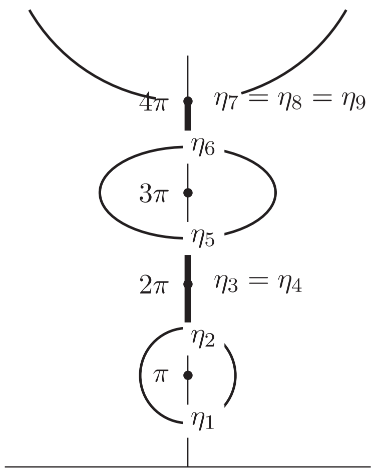

Figure 4-3

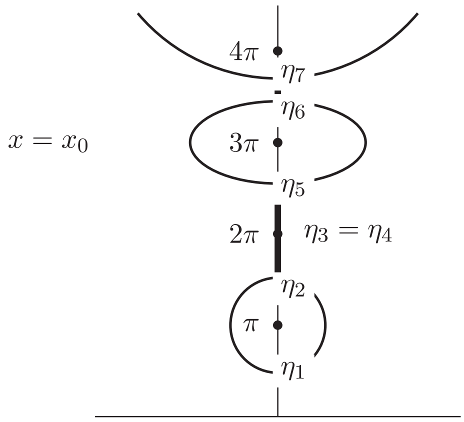

Figure 4-4

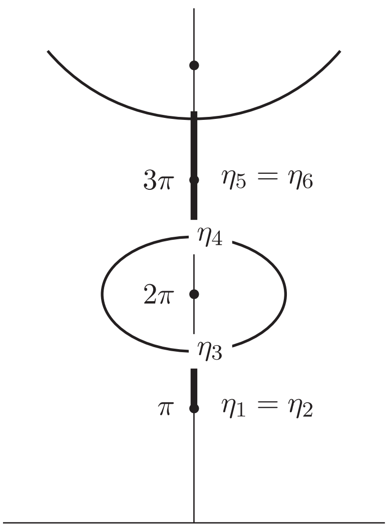

Figure 4-5

$x = -x_0$

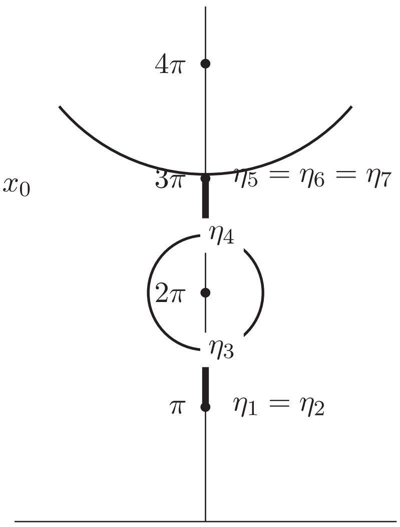

Figure 4-6

4.3. Monotonicity of Ref along the curve Γ^+

In this section we prove the following theorem.

Theorem 4.5. *The function $f_1 = Ref$ is positive and monotone increasing along Γ^+ which is defined in* 4.2.

Since $f_1(-u, v) = f_1(u, v)$, it follows that f_1 is monotone decreasing along Γ^-.

We shall give two proofs of this Theorem. When $y = 0$, Γ^+ is the positive u-axis and $\frac{\partial f_1}{\partial u}\big|_{\Gamma^+} > 0$ follows from 4.1.1(c). So we assume that $y > 0$ in the following proofs.

Proof. **Proof I:** First we deal with the case, $a^2 + b^2 > 0$.

(A) When $u = 0$, one has, by (4.10),

$$\frac{\partial f_1(0, v)}{\partial v} = -\frac{\partial f_2}{\partial u}\Big|_{u=0} > 0, \text{ if } 0 \le v < \eta_1, \text{ or } \eta_{2j} < v < \eta_{2j+1}, 1 \le j \le \frac{N-1}{2}.$$

Thus $f_1(0, v)$ is monotone increasing for v in the intervals $[0, \eta_1]$ and $[\eta_{2j}, \eta_{2j+1}]$,

$1 \le j \le \frac{N-1}{2}$, and so

$$f_1(0, \eta_{2j+1}) \ \ge\ f_1(0, \eta_{2j}) \quad \text{ if } \eta_{2j+1} > \eta_{2j}. \tag{4.21}$$

As $f_1(0,0) = \frac{b^2}{2} \ge 0$, we have

$$f_1(0, \eta_1) > 0 \qquad \text{since } \eta_1 > 0 \text{ if } y > 0. \tag{4.22}$$

(B) The case $u > 0$. On $\Gamma^+ \setminus \{u = 0\}$, u is a C^∞ function of v by 4.2. We differentiate $f_2(u(v), v) = 0$ with respect to v implicitly and obtain

$$u' = \frac{du}{dv} = -\frac{\frac{\partial f_2}{\partial v}}{\frac{\partial f_2}{\partial u}} \tag{4.23}$$

Note that by Lemma 4.3 $\frac{\partial f_2}{\partial u} < 0$ on $\Gamma^+ \setminus \{u = 0\}$.

Let now $r(v) = \mathrm{Re}(f)(u(v), v)$. Then by Cauchy-Riemann equations

$$\frac{dr}{dv} = u' \frac{\partial f_1}{\partial u} + \frac{\partial f_1}{\partial v} = \frac{(\frac{\partial f_2}{\partial u})^2 + (\frac{\partial f_2}{\partial v})^2}{-\frac{\partial f_2}{\partial u}}. \tag{4.24}$$

Therefore $\frac{dr}{dv} > 0$ on $\Gamma^+ \setminus \{u = 0\}$. As a consequence, f_1 is strictly increasing on $\Gamma^+ \setminus \{u = 0\}$. So the theorem holds for $a^2 + b^2 > 0$.

When $a^2 + b^2 = 0$, Γ^+ degenerates to $\{(0^+, v) | v \ge \pi\}$ and $f_1 = vy$. Obviously the theorem is also true. This completes our proof.

Proof II: By Lemma 4.3 and Theorem 4.1 we know that $(\frac{\partial f_2}{\partial u}, \frac{\partial f_2}{\partial v}) \ne 0$ on $\Gamma^+ \setminus \{(0, \eta_j) |\ j = 1, \ldots, N\}$. By Cauchy-Riemann equations $(\frac{\partial f_1}{\partial u}, \frac{\partial f_1}{\partial v}) = (\frac{\partial f_2}{\partial v}, -\frac{\partial f_2}{\partial u}) \ne 0$ on $\Gamma^+ \setminus \{(0, \eta_j) |\ j = 1, \ldots, N\}$ too. Since $f_2 < 0$ on the right hand part of Γ^+, $(\frac{\partial f_2}{\partial u}, \frac{\partial f_2}{\partial v})$ is perpendicular to Γ^+ and lies to the left of it. Thus $(\frac{\partial f_1}{\partial u}, \frac{\partial f_1}{\partial v})$ is tangent to Γ^+ and directs along the orientation of Γ^+. Therefore f_1 is strictly increasing on $\Gamma^+ \setminus \{(0, \eta_j) |\ j = 1, \ldots, N\}$ at the greatest rate. This completes the proof. □

As a consequence, we have for $1 \le j \le \frac{N-1}{2}$,

$$\mathrm{Re}(f)(0, \eta_{2j}) > \mathrm{Re}(f)(0, \eta_{2j-1}) \quad \text{ if } \eta_{2j} > \eta_{2j-1}. \tag{4.25}$$

Combining (4.21),(4.22) and (4.25), we have for $j = 1, 2, \ldots, N-1$

$$\mathrm{Re}(f)(0, \eta_{j+1}) \geq \mathrm{Re}(f)(0, \eta_j) > 0 \quad \text{if } \eta_{j+1} > \eta_j. \tag{4.26}$$

In summary, we have

Corollary 4.6. *Let η_j, $j = 1, \ldots, N$ be defined in Theorem* 2.6*. Then $0 \leq f_1(0, \eta_j) < f_1(0, \eta_{j+1})$, if $\eta_j < \eta_{j+1}$, $1 \leq j \leq N-1$.*

The following theorem gives the precise relation between $f_1(0, \eta_j)$ and the length ℓ_j of the geodesic C_j defined in Theorem 2.6.

Theorem 4.7. *We have $f_1(0, \eta_j) = \frac{\ell_j^2}{2}$ for all $1 \leq j \leq N$.*

So the statements concerning the ℓ_j's in Theorem 2.6 follows from Corollary 4.6 and Theorem 4.7. This finishes the proof of Theorem 2.6.

Proof. Let $\mathcal{C}_j$ be the geodesic defined in Theorem 2.6. We have

$$\dot{\mathcal{C}}_j(s) = (\dot{x}(s), \dot{y}(s)) = \dot{x}(s)\Big(\frac{\partial}{\partial x}\Big) + \frac{\dot{y}(s)}{x(s)}\Big(x\frac{\partial}{\partial y}\Big) = \dot{x}(s)X_1 + \frac{\dot{y}(s)}{x(s)}X_2.$$

Then ℓ_j is defined by the follwing

$$\begin{aligned} \ell_j &= \int_0^1 \Big[\dot{x}^2(s) + \Big(\frac{\dot{y}(s)}{x(s)}\Big)^2\Big]^{\frac{1}{2}} ds \\ &= \int_0^1 \Big[\xi^2(s) + x^2(s)\eta^2(s)\Big]^{\frac{1}{2}} ds \quad \text{(by (2.4))} \\ &= \int_0^1 (2H)^{\frac{1}{2}} ds = \sqrt{2H} \quad \text{(by (2.3))} \end{aligned} \tag{4.27}$$

since the energy H is constant along the bicharacteristic.

At $s = 0$, (2.6) and (2.7) yield

$$H = \frac{1}{2}(\xi^2 + x^2\eta^2) = \frac{\eta^2}{2}(x_0^2 + B^2) \tag{4.28}$$

Then by (2.9), (2.10), (2.11) and (2.12)

$$\begin{aligned}H =&\frac{\eta_j}{4}\Big[a^2(\tilde{\mu}(\eta_j) - \frac{\sin\eta_j}{1+\ \cos\eta_j}) + b^2(\mu(\eta_j) + \frac{\sin\eta_j}{1-\ \cos\eta_j})\Big]\\ =&\eta_j y + \frac{1}{4}\Big(\frac{-a^2\ \eta_j\ \sin\eta_j}{1+\cos\eta_j} + \frac{b^2\eta_j\ \sin\eta_j}{1-\cos\eta_j}\Big)\\ =&f_1(0,\eta_j) \qquad \text{(by (4.1))}\end{aligned} \tag{4.29}$$

if η_j solves (2.10). In other words, (4.29) holds for all η_j, $\eta_j \neq k\pi$. The case $\eta_j = k\pi$ can be checked in (4.27) directly with the help of (2.26), (2.27), (2.28). We omit the details. This proves the Theorem. $\square$

5. The Heat Kernel

We shall show that with an appropriate choice of $V(\tau)$, (3.29) does represent the heat kernel of Δ_G. By the definition of the heat kernel, one needs to show

$$\begin{cases}\Delta_G P_t - \frac{\partial}{\partial t}P_t = 0, & t>0,\\ \lim_{t\to 0} P_t(x,y,x_0) = \delta(x-x_0)\delta(y).\end{cases} \tag{5.1}$$

We start with the first assertion of (5.1)

$$\Big(\Delta_G - \frac{\partial}{\partial t}\Big)\frac{e^{-\frac{f}{t}}}{t^\alpha} = \frac{e^{-\frac{f}{t}}}{t^{\alpha+2}}\Big(H(\nabla f) - f\Big) - \frac{e^{-\frac{f}{t}}}{t^{\alpha+1}}\Big(\Delta_G f - \alpha\Big).$$

Then the eiconal equation (3.30) implies

$$\begin{aligned}\Big(\Delta_G - \frac{\partial}{\partial t}\Big)\frac{e^{-\frac{f}{t}}V(\tau)}{t^\alpha} &= \frac{e^{-\frac{f}{t}}}{t^{\alpha+1}}\tau\Big(-\frac{1}{t}\frac{\partial f}{\partial\tau}\Big)V - \frac{e^{-\frac{f}{t}}}{t^{\alpha+1}}\Big(\Delta_G f - \alpha\Big)V\\ &= -\frac{e^{-\frac{f}{t}}}{t^{\alpha+1}}\Big[\tau\frac{dV}{d\tau} + \Big(\Delta_G f - \alpha + 1\Big)V\Big] + \frac{\partial}{\partial\tau}\Big(\frac{\tau e^{-\frac{f}{t}}V(\tau)}{t^{\alpha+1}}\Big).\end{aligned}$$

Assuming

$$\lim_{\tau\to\pm\infty}\tau e^{-\frac{f}{t}}V(\tau) = 0, \tag{5.2}$$

one has

$$\begin{aligned}
&\Big(\Delta_G - \frac{\partial}{\partial t}\Big)\frac{1}{t^\alpha}\int_{\mathbb{R}} e^{-\frac{f}{t}}V(\tau)d\tau \\
=& -\frac{1}{t^{\alpha+1}}\int_{\mathbb{R}} e^{-\frac{f}{t}}\left[\tau\frac{dV}{d\tau}+\Big(\Delta_G f-\alpha+1\Big)V(\tau)\right]d\tau \\
=& \, 0
\end{aligned}$$

if $t > 0$ and

$$\tau\frac{dV}{d\tau}+\Big(\Delta_G f-\alpha+1\Big)V(\tau)=0. \tag{5.3}$$

From (3.28), one has

$$\Delta_G f \;=\; \frac{1}{2}\tau\coth\,\tau,$$

therefore (5.3) yields

$$\begin{aligned}
\tau\frac{dV}{d\tau}+\Big(\frac{1}{2}\tau\coth\,\tau-\alpha+1\Big)V(\tau)=0, \\
\frac{dV}{V} = \left(\frac{\alpha-1}{\tau}-\frac{1}{2}\coth\,\tau\right)d\tau,
\end{aligned}$$

so

$$\log\,V=(\alpha-1)\big(\log\,\tau+\log\,C\big)-\frac{1}{2}\log(\sinh\tau),$$

and we have derived

Lemma 5.1.

$$V(\tau)=\frac{(C\tau)^{\alpha-1}}{\sqrt{\sinh\tau}} \tag{5.4}$$

is the general solution of (5.3).

We need V holomorphic near $\tau = 0$ which forces us to choose $\alpha = n+\frac{1}{2}$, $n = 1, 2, \ldots$, and then

$$V(\tau)=C\tau^{n-1}\sqrt{\frac{\tau}{\sinh\tau}},$$

so we are looking at the following heat kernel

$$P_t = \frac{C}{(2\pi t)^{n+\frac{1}{2}}} \int_{\mathbb{R}} e^{-\frac{f(\tau)}{t}} \sqrt{\frac{\tau}{\sinh \tau}} \tau^{n-1} d\tau. \tag{5.5}$$

Here $(\frac{\tau}{\sinh \tau})^{\frac{1}{2}}$ is defined on $\mathbb{C} \backslash \bigcup_{k=0,1,2\cdots} [(2k+1)\pi i, 2(k+1)\pi i] \bigcup_{k=0,-1,-2,\ldots} [2(k-1)\pi i, (2k-1)\pi i]$. By writing $\tau = u + iv$, u, $v \in \mathbb{R}$

$$(\frac{\tau}{\sinh \tau})^{\frac{1}{2}} = (\frac{u^2+v^2}{\sinh^2 u \cos^2 v + \cosh^2 u \sin^2 v})^{\frac{1}{2}} e^{\frac{i}{2} \arg(\frac{\tau}{\sinh \tau})}, \tag{5.6}$$

where $\arg(\frac{\tau}{\sinh \tau}) = \arctan \left(\frac{v \sinh u \cos v - u \cosh u \sin v}{u \sinh u \cos v + v \cosh u \sin v} \right)$ for $u \in \mathbb{R}$, $|v| \leq \pi/2$ and this defines $(\frac{\tau}{\sinh \tau})^{\frac{1}{2}}$ on its domain.

By (4.1) and (5.6), one has (5.2) and so the first part of (5.1) holds. In fact, recall that in the definition of f we always have $vy \geq 0$; also $-\mathrm{Re}(f) \to -\infty$ as $|u| \to \infty$ for any v fixed, if $a^2 + b^2 = 2(x_0^2 + x^2) > 0$. So, in view of (5.6), (5.2) always holds as $|u| \to \infty$ for any v fixed. As a result, we may write (5.5) as

$$P_t = \frac{c}{(2\pi t)^{n+\frac{1}{2}}} \int_{-\infty+iv}^{\infty+iv} e^{-\frac{f(\tau)}{t}} \sqrt{\frac{\tau}{\sinh \tau}} \tau^{n-1} d\tau \tag{5.7}$$

for any $v \in (-\pi, \pi)$ fixed.

Furthermore, let B_+ be the subset of $\mathbb{C}$ between u-axis and the curve Γ defined by (4.20). We claim $\mathrm{Re}(f) > 0$ on $\bar{B}_+$ and $\mathrm{Re}(f) \to \infty$ on $\bar{B}_+$ as $|\tau| \to \infty$.

Proof. Recall that it is implicitly assumed that $(x_0, 0) \neq (x, y)$ and $y \geq 0$. Section 4.1 part (**C**) gives $\frac{\partial \mathrm{Re}(f)}{\partial u} \geq 0$ ($= 0$ only when $u = 0$ or $x = x_0 = 0$) on $u \geq 0$, $v = 0$. As $\mathrm{Re}(f)(0,0) = \frac{b^2}{2} = \frac{(x-x_0)^2}{2}$, one has $\mathrm{Re}(f) \geq 0$ and $\mathrm{Re}(f) \uparrow$ on positive u-axis. Next, Lemma 4.3 and Cauchy-Riemann equations give $\frac{\partial \mathrm{Re}(f)}{\partial v} > 0$ on $\bar{B_+} \cap \{(u,v) | \ u \geq 0\}$. Also, by Proposition 4.4(2) and the argument preceding (5.7) one concludes that $\mathrm{Re}(f) \to \infty$ on $\bar{B_+} \cap \{(u,v) | \ u \geq 0\}$ as $|\tau| \to \infty$. Since $\mathrm{Re}(f)(u,v) = \mathrm{Re}(f)(-u,v)$, the Claim is proved. $\square$

Therefore, when $y \geq 0$ one has

$$P_t = \frac{c}{(2\pi t)^{n+\frac{1}{2}}} \int_\gamma e^{-\frac{f(\tau)}{t}} \sqrt{\frac{\tau}{\sinh \tau}} \tau^{n-1} d\tau \tag{5.8}$$

where γ is any rectifiable curve homotopic to u-axis in $\bar{B}_+$. Note that $\mathrm{Re}(f)|_\gamma > 0$. In particular, when $\gamma = \Gamma$ one has $\mathrm{Im}(f) = 0$.

When $y \leq 0$, one deals with $v \leq 0$ and the corresponding result still holds. In conclusion, one has

lemma 5.2. *For each $(x, y) \in \mathbb{R}^2 \setminus \{(x_0, 0)\}$, there is a contour in (u, v) plane on which $Re(f) > 0$, $Re(f) \to \infty$ as $|\tau| \to \infty$ and (5.8) holds. In particular, there is a contour on which $Im(f) = 0$.*

Corollary 5.3. *One has*

$$\lim_{t \to 0^+} P_t(x_0, x, y) = 0 \tag{5.9}$$

uniformly for (x, y) in any compact subset of $\in \mathbb{R}^2 \setminus \{(x_0, 0)\}$.

Proof. Let W be a compact subset of $\mathbb{R}^2 \setminus \{(x_0, 0)\}$. Let $2\epsilon = dist((x_0, 0), W)$. Write $W = W_1 \cup W_2 \cup W_3$ where $W_1 = W \cap \{(x, y)|\ |y| \leq \epsilon\}$, $W_2 = W \cap \{(x, y)|\ y \geq \epsilon\}$, $W_3 = W \cap \{(x, y)|\ y \leq -\epsilon\}$. So W_1, W_2, W_3 are compact sets in $\mathbb{R}^2 \setminus \{(x_0, 0)\}$. For any $(x, y) \in W_1$, one has $|x - x_0| \geq \epsilon$; whence there exists a positive constant δ such that $\mathrm{Re}(f)|_{v=0} > \delta > 0$ for all $(x, y) \in W_1$. Therefore (5.9) holds for W_1.

The case W_3 is similar to W_2. So it suffices to prove (5.9) for W_2 and the corollary is proved. Let $(x, y) \in W_2$. Since $y \geq \epsilon$ it follows from our discussion in Sections 2 and 4 that the smallest critical point $\eta_1(x, y)$ of $f(x_0, x, y, \tau)$ is strictly positive. As W_2 is compact, one has $\eta_0 = \min_{(x,y) \in W_2} \eta_1(x, y) > 0$. Thus $\mathrm{Re}(f)|_{v=i\eta_0} > 0$ for all $(x, y) \in W_2$. Therefore (5.9) holds for W_2 too. □

We shall show that for all $t > 0$

$$\int_{\mathbb{R}^2} P_t(x_0, x, y) dx\ dy = \frac{V(0)c}{(2\pi t)^{n-1}}.$$

So by taking $c = 1$ and $n = 1$ the second part of (5.1) will follow in view of Corollary 5.3. By taking $v = 1$ or -1 in (5.7) we have,

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} dy \int_{-\infty+i\mathrm{sgn}(y)}^{+\infty+i\mathrm{sgn}(y)} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau \\
=&\int_{0}^{\infty} dy \int_{-\infty+i}^{+\infty+i} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau + \int_{-\infty}^{0} dy \int_{-\infty-i}^{+\infty-i} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau \\
=&\int_{0}^{\infty} dy \int_{-\infty}^{+\infty} e^{-\frac{f(u+i)}{t}} V(u+i) du + \int_{-\infty}^{0} dy \int_{-\infty}^{+\infty} e^{-\frac{f(u-i)}{t}} V(u-i) du \\
=&\int_{-\infty}^{\infty} V(u+i) du \int_{0}^{\infty} e^{-\frac{f(u+i)}{t}} dy + \int_{-\infty}^{\infty} V(u-i) du \int_{-\infty}^{0} e^{-\frac{f(u-i)}{t}} dy \\
=&\int_{-\infty}^{\infty} V(u+i) e^{-\frac{h(u+i)}{t}} \frac{t}{-iu} du + \int_{-\infty}^{\infty} V(u-i) e^{-\frac{h(u-i)}{t}} \frac{t}{+iu} du \\
=&i\int_{-\infty+i}^{\infty+i} V(z) e^{-\frac{h(z)}{t}} \frac{t}{z} dz - i\int_{-\infty-i}^{\infty-i} V(z) e^{-\frac{h(z)}{t}} \frac{t}{z} dz \\
=&2\pi t V(0) e^{-\frac{h(0)}{t}} \\
=&2\pi t V(0) e^{-\frac{(x-x_0)^2}{2t}}.
\end{aligned}
$$

where $f = -i\tau y + h$.

As

$$\int_{-\infty}^{\infty} e^{-\frac{(x-x_0)^2}{2t}} dx = \sqrt{2\pi t},$$

we obtain

$$\int_{\mathbb{R}^2} \int_{-\infty}^{\infty} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau = \big(2\pi t\big)^{\frac{3}{2}} V(0).$$

We have derived

Theorem 5.4. *The heat kernel for the subLaplacian of the Grusin operator is given by*

$$P_t(x_0, x, y) = \big(2\pi t\big)^{-\frac{3}{2}} \int_{\mathbb{R}} e^{-\frac{f(\tau)}{t}} \sqrt{\frac{\tau}{\sinh\tau}} d\tau. \tag{5.10}$$

Remark 5.5. Assuming V in (3.29) is analytic near 0 and $V(0) \neq 0$, set

$$V(\tau) = a_0 + a_1\tau + a_2\tau^2 + \cdots .$$

Substituting this into (5.3), one obtains

$$\begin{aligned}
&a_1\tau + \cdots + \Big(\frac{1}{2}\tau \coth \tau - \alpha + 1\Big)\big(a_0 + a_1\tau + a_2\tau^2 + \cdots\big) = 0,\\
&a_1\tau + \cdots + \Big(\frac{1}{2} + \cdots - \alpha + 1\Big)\big(a_0 + a_1\tau + a_2\tau^2 + \cdots\big) = 0,\\
&a_1\tau + \frac{3}{2}a_0 + \frac{3}{2}a_1\tau - \alpha a_0 - \alpha a_1\tau + \cdots = 0,
\end{aligned}$$

or,

$$\Big(\frac{3}{2} - \alpha\Big)a_0 + \mathcal{O}(\tau) = 0,$$

and $a_0 \neq 0$ implies

$$\alpha = \frac{3}{2}.$$

Remark 5.6. We note that

$$P_t(x_0, x, 0) = \frac{1}{(2\pi t)^{\frac{3}{2}}} \int_{-\infty}^{+\infty} e^{-\frac{h(\tau)}{t}} V(\tau)d\tau$$

is positive, hence $P_t(x_0, x, y)$ is also positive for small y, by continuity.

6. Small Time Asymptotics of the Heat Kernel

In this section we study the small time asymptotics of the heat kernel at every critical point of the complex action function f. For $y \geq 0$, we have

$$P_t(x_0, x, y) = (2\pi t)^{-\frac{3}{2}} \int_{\Gamma} e^{-\frac{f(\tau)}{t}} V(\tau)d\tau. \tag{6.1}$$

where Γ is defined by (4.20). As before, the critical points of f are denoted by η_j, $1 \leq j \leq N$. Let $\gamma_j = \Gamma_j^+ \cup \Gamma_j^-$, $1 \leq j \leq N$. When $\eta_j = \eta_{j+1}$ then γ_j degenerates to a point. Now, (6.1) can be written as

$$P_t(x_0, x, y) = (2\pi t)^{-\frac{3}{2}} \sum_{j=1}^{N} \int_{\gamma_j} e^{-\frac{f(\tau)}{t}} V(\tau)d\tau. \tag{6.2}$$

We want to find the small time asymptotics of the integrals

$$I_j' = \int_{\gamma_j} e^{-\frac{f(\tau)}{t}} V(\tau) d\tau \qquad j = 1, \ldots, N, \tag{6.3}$$

as $t \searrow 0$.

Recall that the function $f_1 = \mathrm{Re} f$ is strictly increasing on Γ^+. So in studying the small time asymptotics of I_j', we need only to consider the part of γ_j close to $i\eta_j$. Now, let $\varphi(\tau) \in C_0^\infty(\mathbb{C})$ satisfying

$$0 \le \varphi \le 1, \quad \varphi(\tau) = \begin{cases} 1 & |\tau| \le 1 \\ 0 & |\tau| \ge 2, \end{cases} \qquad \text{and } \varphi(\tau) = \varphi(e^{i\theta}\tau) \tag{6.4}$$

for all $\theta \in [0, 2\pi]$.

Set

$$\varphi_j(\tau) = \varphi\Big(\frac{\tau - i\eta_j}{\epsilon_j}\Big), \tag{6.5}$$

where $\epsilon_j > 0$ is to be specified case by case later on. The integral (6.3) is modified to

$$I_j = \int_{\gamma_j} e^{-\frac{f(\tau)}{t}} \varphi_j(\tau) V(\tau) d\tau \qquad j = 1, \ldots, N. \tag{6.6}$$

In the following lemma we collect the properties of $V(\tau)$ without proof.

Lemma 6.1. *For $u \in \mathbb{R}$, $v \ge 0$, $V(\tau)$ satisfies*

$$V(-u + iv) = \overline{V(u + iv)} \tag{6.7}$$

When η_j is not an integral multiple of π, $1 \le j \le N$, we write

$$V(0^+ + i\eta_j) = e^{i\beta_j} \Big(\frac{\eta_j}{|\sin \eta_j|}\Big)^{\frac{1}{2}}, \qquad 1 \le j \le N, \tag{6.8}$$

where

$$\begin{cases} \beta_j = 0 & \text{if } j \equiv 0 \text{ or } 1 \text{ mod}(8); \\ \beta_j = -\frac{\pi}{2} & \text{if } j \equiv 2 \text{ or } 3 \text{ mod}(8); \\ \beta_j = -\pi & \text{if } j \equiv 4 \text{ or } 5 \text{ mod}(8); \\ \beta_j = -\frac{3\pi}{2} & \text{if } j \equiv 6 \text{ or } 7 \text{ mod}(8). \end{cases}$$

Lemma 6.2. *For j odd, $1 \le j \le N$, and γ_j nondegenerate, let $\gamma_j^+ = \Gamma_j^+$. We write $\gamma_j^+ = \{u + i(v_j(u) + \eta_j)\}$ for γ_j^+ near $i\eta_j$. We have, when $i\eta_j$ is a double point of f,*

$$v_j'(0) = 0, \quad v_j''(0) = \frac{1}{3} \frac{\frac{\partial^3 f_1}{\partial u^2 \partial v}}{\frac{\partial^2 f_1}{\partial u^2}} \bigg|_{\tau = i\eta_j}, \quad v_j'''(0) = 0 \tag{6.9}$$

and when $i\eta_j$ is a triple point of f,

$$\begin{aligned} &v_j'(0) = 0, \\ &v_j''(0) = \frac{2}{9} \frac{\frac{\partial^4 f_1}{\partial u^2 \partial v^2}}{\frac{\partial^3 f_1}{\partial u^2 \partial v}} \bigg|_{\tau = i\eta_j}, \\ &v_j'''(0) = \sqrt{3} \Big(\frac{12 \frac{\partial^3 f_1}{\partial u^2 \partial v} \frac{\partial^5 f_1}{\partial u^2 \partial v^3} - 5 \big(\frac{\partial^4 f_1}{\partial u^2 \partial v^2}\big)^2}{135 \big(\frac{\partial^3 f_1}{\partial u^2 \partial v}\big)^2} \Big) \bigg|_{\tau = i\eta_j}. \end{aligned} \tag{6.10}$$

Lemma 6.2 follows from implicit differentiation.

6.1. γ_j nondegenerate and η_j is not an integral multiple of π

Here we have three cases.

6.1.1. j even

In (6.5) we choose ϵ_j such that $\eta_{j+1} \notin supp \varphi_j$. Now

$$\begin{aligned} I_j &= \int_{\eta_j}^{\eta_{j+1}} e^{\frac{-f_1(0,v)}{t}} \varphi_j(iv) V(0^+ + iv) i dv + \int_{\eta_{j+1}}^{\eta_j} e^{\frac{-f_1(0,v)}{t}} \varphi_j(iv) V(0^- + iv) i dv \\ &= 2\text{Re } e^{(\frac{\pi}{2} + \beta_j)i} \int_0^\infty e^{\frac{-f_1(0, v+\eta_j)}{t}} \varphi_j(i(v + \eta_j)) \Big(\frac{v + \eta_j}{|\sin(v + \eta_j)|} \Big)^{\frac{1}{2}} dv \end{aligned}$$

As $t \to 0$, we have

$$\begin{aligned} I_j \sim &- e^{-\frac{f_1(0,\eta_j)}{t}} \Big(\frac{\eta_j}{|\sin \eta_j|}\Big)^{\frac{1}{2}} \Big(\frac{2t}{a_2}\Big)^{\frac{1}{2}} \sin \beta_j \times \\ &\Big\{ \sqrt{\pi} + \frac{\sqrt{2}}{\sqrt{a_2}} \Big[\frac{1}{2}\Big(\frac{1}{\eta_j} - \cot \eta_j\Big) - \frac{a_3}{3a_2} \Big] t^{\frac{1}{2}} \\ &- \frac{\sqrt{\pi}}{a_2} \Big[\frac{a_3}{4a_2}\Big(\frac{1}{\eta_j} - \cot \eta_j\Big) + \frac{1}{8a_2}\Big(a_4 - \frac{5}{3}\frac{a_3^2}{a_2}\Big) \\ &- \frac{1}{8}\Big(2 + 3\cot^2 \eta_j - \frac{1}{\eta_j^2} - 2\frac{\cot \eta_j}{\eta_j}\Big) \Big] t + \cdots \Big\}, \end{aligned} \tag{6.11}$$

where $a_2 = \frac{\partial^2 f_1}{\partial v^2}\big|_{i\eta_j}$, $a_3 = \frac{\partial^3 f_1}{\partial v^3}\big|_{i\eta_j}$, $a_4 = \frac{\partial^4 f_1}{\partial v^4}\big|_{i\eta_j}$. Note that $a_2 > 0$ by the fact that f_1 is strictly increasing on Γ^+ and $i\eta_j$ is a double point of f.

Remark 6.3. By Lemma 6.1, $I_j \equiv 0$ if $j \equiv 0$ or $4 \bmod(8)$.

6.1.2. j odd, $j \le N$ and $\eta_{N-1} < \eta_N$ if $j = N$

So η_j is a double point of f. Here we choose ϵ_j in (6.5) such that $v_j(u)$ in Lemma 6.2 is a C^∞ function of $u \in [0, \epsilon_j]$. Now

$$I_j = 2\mathrm{Re} \int_{\gamma_j^+} e^{-\frac{f_1(u,v_j(u))}{t}} \varphi_j(\tau) V(\tau) d\tau$$

And as $t \to 0$, we have

$$\begin{aligned} I_j \sim\ & e^{-\frac{f_1(0,\eta_j)}{t}} \Big(\frac{\eta_j}{|\sin \eta_j|}\Big)^{\frac{1}{2}} \Big(\frac{2t}{b_2}\Big)^{\frac{1}{2}} \\ &\times \mathrm{Re}\Big(e^{i\beta_j} \Big\{ \sqrt{\pi} - i\frac{\sqrt{2}}{\sqrt{b_2}} \Big[\frac{1}{2}\Big(\frac{1}{\eta_j} - \cot \eta_j\Big) - \frac{b_3}{3b_2} \Big] t^{\frac{1}{2}} \\ &+ \frac{\sqrt{\pi}}{b_2} \Big[\frac{b_3}{4b_2}\Big(\frac{1}{\eta_j} - \cot \eta_j\Big) - \frac{1}{8b_2}\Big(b_4 + \frac{5}{3}\frac{b_3^2}{b_2}\Big) \\ &- \frac{1}{8}\Big(2 + 3\cot^2 \eta_j - \frac{1}{\eta_j^2} - 2\frac{\cot \eta_j}{\eta_j}\Big) \Big] t + \cdots \Big\} \Big), \end{aligned} \tag{6.12}$$

where $b_2 = \frac{\partial^2 f_1}{\partial u^2}\big|_{i\eta_j}$, $b_3 = \frac{\partial^3 f_1}{\partial u^2 \partial v}\big|_{i\eta_j}$, $b_4 = \frac{\partial^4 f_1}{\partial u^4}\big|_{i\eta_j}$. By the same reason as in 6.1.1, b_2 is a strictly positive number.

6.1.3. j odd, $j = N$ and $\eta_N = \eta_{N-1} = \alpha_{\frac{N-1}{2}} > \eta_{N-2}$

Here ϵ_j is chosen as in 6.1.2. Note that η_N is a triple point of f. So we need (6.10) in calculating the small time asymptotics of I_N. Similar to 6.1.2

$$I_N = 2\mathrm{Re}\int_{\gamma_N^+} e^{-\frac{f_1(\tau)}{t}}\varphi_N(\tau)V(\tau)d\tau$$

As $t\to 0$,

$$\begin{aligned} I_N \sim\ & \frac{2}{3}t^{\frac{1}{3}}\Big(\frac{9\sqrt{3}}{4b_3}\Big)^{\frac{1}{3}}e^{-\frac{f_1(0,\eta_N)}{t}}\Big(\frac{\eta_N}{|\sin\eta_N|}\Big)^{\frac{1}{2}}\mathrm{Re}\Big(e^{i\beta_N}\\ &\times\Big\{\Gamma(\frac{1}{3})+\Gamma(\frac{2}{3})\Big(\frac{9\sqrt{3}}{4b_3}\Big)^{\frac{1}{3}}\Big[-\frac{i}{2}\Big(\frac{1}{\eta_N}-\cot\eta_N\Big)+\frac{b_4}{b_3}\Big(-\frac{2}{9}i+\frac{1}{3\sqrt{3}}\Big)\Big]t^{\frac{1}{3}}\\ &-\Big(\frac{9\sqrt{3}}{4b_3}\Big)^{\frac{2}{3}}\Big[\Big(\frac{b_4}{b_3}\Big)^2\Big(\frac{i}{6\sqrt{3}}+\frac{1}{144}\Big)+\frac{b_4}{b_3}\Big(\frac{1}{\eta_N}-\cot\eta_N\Big)\Big(\frac{1}{6}+\frac{i}{4\sqrt{3}}\Big)\\ &+\frac{1}{8}\Big(2+3\cot^2\eta_N-\frac{1}{\eta_N^2}-2\frac{\cot\eta_N}{\eta_N}\Big)-\frac{b_5}{b_3}\Big(\frac{2i}{15\sqrt{3}}-\frac{1}{20}\Big)\Big]t^{\frac{2}{3}}\\ &+\cdots\Big\}\Big), \end{aligned}\tag{6.13}$$

where $b_3=\frac{\partial^3 f_1}{\partial u^2\partial v}\big|_{i\eta_N}$, $b_4=\frac{\partial^4 f_1}{\partial u^4}\big|_{i\eta_N}$, $b_5=\frac{\partial^5 f_1}{\partial u^4\partial v}\big|_{i\eta_N}$. Note that $b_3>0$.

Remark 6.4. Observe that in 6.1.1 and 6.1.2 all the powers in t are multiples of $\frac{1}{2}$ which is due to the fact that $i\eta_j$ is a double point of f. For the similar reason, the powers of t in 6.1.3 are multiples of $\frac{1}{3}$.

6.2. $\eta_j\equiv 0 \bmod(\pi)$, $j<N$

The cases include $a=b=0$ and $a=0$ or $b=0$ with $j<N$. For j odd, γ_j degenerates to the point $\frac{j+1}{2}\pi$; hence we only consider j even with $\eta_j=\frac{j}{2}\pi$. Observe that f is regular at $i\eta_j$, while V is singular at $i\eta_j$.

6.2.1. $y>0$, $a=b=0$

We choose ϵ_j such that $supp\varphi_j\subset\{\tau\mid|\tau-i\eta_j|<\frac{1}{2}\}$. Now

$$\begin{aligned} I_j &= \int_{\gamma_j} e^{-\frac{vy}{t}}\varphi_j(\tau)V(\tau)d\tau\\ &= 2e^{-\frac{j\pi y}{2t}}\mathrm{Re}\Big\{e^{(\frac{\pi}{2}+\beta_j)i}\int_0^\pi e^{-\frac{vy}{t}}\varphi_j\Big(i\big(v-\frac{j\pi}{2}\big)\Big)\Big(\frac{\frac{j\pi}{2}+v}{|\sin\ v|}\Big)^{\frac{1}{2}}dv\Big\} \end{aligned}$$

Remark 6.5.

(i) For the same reason as Remark 6.3 we have $I_j \equiv 0$ if $j \equiv 0$ or $4 \bmod(8)$.

(ii) The integrand in I_j is integrable.

As $t \to 0$

$$\begin{aligned} I_j \ \sim \ & -\big(\frac{2j\pi t}{y}\big)^{\frac{1}{2}} e^{-\frac{j\pi y}{2t}} \sin\ \beta_j \big\{\sqrt{\pi} + \frac{t}{2j\sqrt{\pi} y} \\ & + \frac{\sqrt{\pi}}{16y^2}\big(1 - \frac{6}{j^2\pi^2}\big)t^2 + \cdots\big\}. \end{aligned} \tag{6.14}$$

6.2.2. $y > 0,\ x = x_0 \neq 0,\ j < N\ (b = 0)$

In this case, $I_j \equiv 0$ by Remark 6.3 or Remark 6.5(i).

6.2.3. $y > 0,\ x = -x_0 \neq 0,\ j < N\ (a = 0)$

Here φ_j is chosen as in 6.2.1 and $j \equiv 2$ or $6 \bmod(8)$, or $\eta_j \equiv \pi \mod(2\pi)$. As $t \to 0$

$$\begin{aligned} I_j \ \sim \ & -\big(\frac{j\pi t}{c}\big)^{\frac{1}{2}} e^{-\frac{j\pi y}{2t}} \sin\ \beta_j \big\{2\sqrt{\pi} + \frac{1}{c}\big(\frac{2}{j\sqrt{\pi}} + \frac{3\sqrt{\pi}x_0^2}{c}\big)t \\ & + \frac{\sqrt{\pi}}{2c^2}\big[1 - \frac{6}{j^2\pi^2} + \frac{30x_0^2}{c}\big(\frac{1}{j\pi} + \frac{j\pi}{12}\big) + \frac{105x_0^4}{2c^2}\big]t^2 + \cdots\big\}. \end{aligned} \tag{6.15}$$

where $c = 2y - \frac{j}{2}\pi x_0^2 = 2\frac{\partial f_1}{\partial v}\big|_{i\eta_j} > 0$ since $j < N$. Note that $\frac{\partial^2 f_1}{\partial v^2}\big|_{i\eta_j} = -x_0^2$, $\frac{\partial^3 f_1}{\partial v^3}\big|_{i\eta_j} = -\frac{j}{4}\pi x_0^2$.

Remark 6.6. In (6.14) and (6.15), the term $t^{\frac{1}{2}}$ in front of the braces comes from V. The gap of the powers in t inside the braces is one which is due to the fact that f is regular at $i\eta_j$.

6.3. $j = N,\ \eta_N = \eta_{N-1} = \eta_{N-2} = \frac{N-1}{2}\pi > \eta_{N-3},\ a = 0$ or $b = 0$

This case is a mixture of 6.1 and 6.2. Here η_N is a double point of f and V is singular at η_N too. ϵ_j is chosen as in 6.1.2. We have

$$I_N = 2\mathrm{Re} \int_{\gamma_N^+} e^{-\frac{f_1(\tau)}{t}} \varphi_N(\tau) V(\tau) d\tau.$$

As $t \to 0$,

$$\begin{aligned} I_N \sim\ & 4\Big(\frac{(N-1)\pi}{b_2}\Big)^{\frac{1}{2}} e^{-\frac{(N-1)\pi y}{2t}} t^{\frac{1}{4}} \mathrm{Re}\ e^{-i\frac{N-2}{4}\pi} \\ & \times\Big\{1 + i\frac{\sqrt{\pi}}{2\sqrt{b_2}}\Big(\frac{-1}{(N-1)\pi} + \frac{b_3}{4b_2}\Big)t^{\frac{1}{2}} + \frac{2}{b_2}\Big[-\frac{b_4}{6b_2} - \frac{1}{12} \\ & + \frac{5b_3}{12(N-1)\pi b_2} - \frac{25}{96}\frac{b_3^2}{b_2^2} + \frac{1}{2(N-1)^2\pi^2}\Big]t + \cdots\Big\}, \end{aligned} \tag{6.16}$$

where b_k are defined in 6.1.2.

Remark 6.7. In (6.16), the power $\frac{1}{4}$ in front of the braces is due to the singularity of V and the fact that $i\eta_N$ is a double point of f. The gap $\frac{1}{2}$ of the powers of t in the braces of (6.16) is due to the fact that $i\eta_N$ is a double of f.

6.4. Some observations

We want to find out the possible relations among the different types of small time asymptotics discussed in the previous sections. The cases 6.1.3 and 6.3 are of distinct types. We can only look at the rest cases.

In the following, we assume, for simplicity, that y is fixed and let a or b tends to zero.

(A) In 6.2, by Remark 6.5(i) we see that when $j \equiv 0$ or 4 mod(8), both 6.2.1 and 6.2.2 are identically zero. So in this case 6.2.1 is the limit of 6.2.2 as $x_0 \to 0$. As for 6.2.3, if we let $x_0 \to 0$ in (6.15), it is easy to see that the limit is (6.14). Therefore, when $j \equiv 2$ or 6 mod(8), 6.2.1 is the limiting case of 6.2.3 as $x_0 \to 0$.

We then seek the relations between 6.1 and 6.2.

(B) Suppose η_{2j-1}, η_{2j} are not intgral multiples of π and $\eta_{2j-1} \nearrow j\pi$ (resp. $\eta_{2j} \searrow j\pi$) as $a^2 \searrow 0$ when j is odd or $b^2 \searrow 0$ when j is even. Necessarily, η_{2j-1} (resp. η_{2j}) is a double point of f. Direct computation shows that

$$\frac{\partial^k f}{\partial \tau^k}\Big|_{i\eta_{2j-1}} = O\big(|\eta_{2j-1} - j\pi|^{-k+1}\big) \qquad \text{as } a^2 \searrow 0,\ k = 1, 2, \ldots. \tag{6.17}$$

$$\Big(\text{resp. } \frac{\partial^k f}{\partial \tau^k}\Big|_{i\eta_{2j}} = O\big(|\eta_{2j} - j\pi|^{-k+1}\big) \qquad \text{as } b^2 \searrow 0,\ k = 1, 2, \ldots\Big) \tag{6.18}$$

We write down a few derivatives of f:

$$\begin{aligned}\frac{\partial^2 f}{\partial \tau^2}\Big|_{i\eta_j} &= \frac{1}{4}\big(a^2\sec^2\frac{v_j}{2}+b^2\csc^2\frac{v_j}{2}\big)\\ &\quad +\frac{v_j}{8}\big(a^2\tan\frac{v_j}{2}\sec^2\frac{v_j}{2}-b^2\cot\frac{v_j}{2}\csc^2\frac{v_j}{2}\big).\end{aligned} \tag{6.19}$$

$$\begin{aligned}\frac{\partial^3 f}{\partial \tau^3}\Big|_{i\eta_j} &= \frac{3i}{8}\big(-a^2\tan\frac{v_j}{2}\sec^2\frac{v_j}{2}+b^2\cot\frac{v_j}{2}\csc^2\frac{v_j}{2}\big)\\ &\quad -\frac{iv_j}{8}\Big[a^2\big(\frac{1}{2}\sec^4\frac{v_j}{2}+\tan^2\frac{v_j}{2}\sec^2\frac{v_j}{2}\big)\\ &\quad +b^2\big(\frac{1}{2}\csc^4\frac{v_j}{2}+\cot^2\frac{v_j}{2}\csc^2\frac{v_j}{2}\big)\Big].\end{aligned} \tag{6.20}$$

$$\begin{aligned}\frac{\partial^4 f}{\partial \tau^4}\Big|_{i\eta_j} &= -\frac{1}{2}\Big[a^2\big(\frac{1}{2}\sec^4\frac{v_j}{2}+\tan^2\frac{v_j}{2}\sec^2\frac{v_j}{2}\big)\\ &\quad +b^2\big(\frac{1}{2}\csc^4\frac{v_j}{2}+\cot^2\frac{v_j}{2}\csc^2\frac{v_j}{2}\big)\Big]\\ &\quad +\frac{v_j}{8}\Big[-a^2\big(2\tan\frac{v_j}{2}\sec^4\frac{v_j}{2}+\tan^3\frac{v_j}{2}\sec^2\frac{v_j}{2}\big)\\ &\quad +b^2\big(2\cot\frac{v_j}{2}\csc^4\frac{v_j}{2}+\cot^3\frac{v_j}{2}\csc^2\frac{v_j}{2}\big)\Big].\end{aligned} \tag{6.21}$$

From (6.19)– (6.21), for example, the coefficient of $t^{\frac{1}{2}}$ in the braces of (6.11) (resp. (6.12)) is not null, it blows up at the order

$$|\eta_{2j}-j\pi|^{-\frac{1}{2}} \qquad (\text{resp. } |\eta_{2j-1}-j\pi|^{-\frac{1}{2}}) \tag{6.22}$$

as $\eta_{2j}\searrow j\pi$ (resp. $\eta_{2j-1}\nearrow j\pi$). As a result, we can not find a connection between (6.11), (6.12) and the small time asymptotics in 6.2 for these terms.

(C) However, let us look at the first term in (6.11) and (6.12).

For j even, we have as $j\searrow\frac{j\pi}{2}$,

$$\lim_{\eta_j\searrow\frac{j\pi}{2}}|a_2\sin\eta_j| = \eta_j B_j^2, \qquad \text{where } B_j^2=\frac{4y}{j\pi}-x_0^2>0, \tag{6.23}$$

by (6.19) and (2.14).

For j odd, we have as $j\nearrow\frac{j+1}{2}\pi$,

$$\lim_{\eta_j\nearrow\frac{j+1}{2}\pi}|b_2\sin\eta_j| = \eta_j B_j^2, \tag{6.24}$$

Thus, in view of (6.23), the first of (6.11) becomes, as $\eta_{2j}\searrow j\pi$, (or, $b\to 0$

if j is even and $a \to 0$ if j is odd),

$$-e^{-\frac{j\pi y}{t}}\Big(\frac{2t}{\frac{2y}{j\pi}-x_0^2}\Big)^{\frac{1}{2}}(\sin\beta_{2j})\sqrt{\pi}. \tag{6.25}$$

Similarly, as $\eta_{2j-1} \nearrow j\pi$, the first of (6.12) becomes,

$$e^{-\frac{j\pi y}{t}}\Big(\frac{2t}{\frac{2y}{j\pi}-x_0^2}\Big)^{\frac{1}{2}}\sqrt{\pi}\mathrm{Re}\ e^{i\beta_{2j-1}}. \tag{6.26}$$

By Lemma 6.1, $\beta_{2j}+\frac{\pi}{2}=\beta_{2j-1}$, or we get (6.25)= (6.26). It follows that the sum of (6.25) and (6.26) is

$$-2e^{-\frac{j\pi y}{t}}\Big(\frac{2t}{\frac{2y}{j\pi}-x_0^2}\Big)^{\frac{1}{2}}(\sin\beta_{2j})\sqrt{\pi}. \tag{6.27}$$

Observe that

(i) (6.27) equals the first term of I_{2j} in (6.15) when $a=0$.

(ii) (6.27) equals 0 when $2j \equiv 0$ or 4 mod(8). So it is the same as 6.2.2.

(iii) When $a=b=0$, or $x_0=0$ in (6.27), the first term of I_{2j} in (6.14) is exactly (6.27).

6.5. Conclusions

We have computed the first three terms of the small time asymptotics of the heat kernel of Grusin operator at every critical point of the complex action f. We also find out some relations between these coefficients. We may say that

(i) all the geodesics determine the geometry;

(ii) all the informations of the geodesics are contained in the complex function f;

(iii) one can then use f to construct the fundmental solution and the heat kernel of the Grusin operator;

(iv) a thorough understanding of the action f enables one to compute the small time asymptotics of the heat kernel at every critical point of f;

(v) the information contained in f is revealed in these asymptotic expansions.

7. Connection with the Heisenberg sub-Laplacian and Brownian Motion

As we have seen in Section 1, the Grusin operator is

$$\Delta_G = \frac{1}{2}(X_1^2 + X_2^2) = \frac{1}{2}\Big[\big(\frac{\partial}{\partial x}\big)^2 + \big(x\frac{\partial}{\partial y}\big)^2\Big].$$

We can compare it to the operator

$$\mathcal{L} = \frac{1}{2}\big(\tilde{X}_1^2 + \tilde{X}_2^2\big) = \frac{1}{2}\Big[\big(\frac{\partial}{\partial x_1} + \frac{x_2}{2}\frac{\partial}{\partial t}\big)^2 + \big(\frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial t}\big)^2\Big]$$

on the Heisenberg group $\mathbf{H}_1$. It is easy to see that the operator $\mathcal{L}$ is left-invariant translation under the group law on $\mathbf{H}_1$:

$$(x_1, x_2, t) \circ (y_1, y_2, s) = \big(x_1 + y_1, x_2 + y_2, t + s + \frac{1}{2}(x_2 y_1 - x_1 y_2)\big).$$

The fundamental solution for $\mathcal{L}$ with singularity at the origin is

$$K(x_1, x_2, t; 0, 0, 0) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{\operatorname{csch}(\tau)d\tau}{(x_1^2 + x_2^2)\coth(\tau) - i4t}.$$

Set

$$x_1 = x, \quad x_2 = z, \quad t = \frac{xz}{2} - y$$

Then the operator $\mathcal{L}$ transforms to

$$\Delta_{H_1} = \frac{1}{2}\Big[\big(\frac{\partial}{\partial x}\big)^2 + \big(x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\big)^2\Big]$$

Δ_{H_1} is translation invariant in y and z. Hence it suffices to have the singularity at $(x_0, 0, 0)$.

$$(x_0, 0, 0)^{-1} \cdot (x, z, \frac{xz}{2} - y) = \big(x - x_0, z, \frac{(x_0 + x)z}{2} - y\big).$$

Therefore,

$$\begin{aligned}
&\tilde{K}(x, z, y; x_0, 0, 0) = K(x_1, x_2, t; x_0, 0, 0) \\
&= \int_{\mathbb{R}} \frac{\pi^{-2}\operatorname{csch}(\eta)}{[(x - x_0)^2 + z^2]\coth(\eta) - i[2(x_0 + x)z - 4y]} d\eta.
\end{aligned}$$

Note that the fundamental solution for Δ_G may be obtained from the fundamental solution for Δ_{H_1}, using the Hadamard method of descent, by integrating the $\tilde{K}(x,z,y;x_0,0,0)$ for Δ_{H_1} with respect to z. Recall

$$\int_{\mathbb{R}} \frac{d\lambda}{a\lambda^2+b\lambda+c} = \frac{2\pi \mathrm{sgn}(a)}{\sqrt{4ac-b^2}}, \quad \lambda \in \mathbb{R}$$

if $a \neq 0$ and $a\lambda^2+b\lambda+c \neq 0$ $\forall \lambda \in \mathbb{R}$. Hence

$$\begin{aligned} &K_G(x,x_0,y) \\ =&\frac{1}{\pi}\int_{\mathbb{R}} \frac{\big[\sinh(\eta)\cosh(\eta)\big]^{-1/2} d\eta}{\sqrt{(x-x_0)^2\coth(\eta)+4iy+(x+x_0)^2\tanh(\eta)}}. \end{aligned}$$

We may also look at this connection by the other method. Let $\mathbf{H}_1$ be the Heisenberg group whose Lie algebra $\mathbf{h}$ has a basis $\{\tilde{X}_1, \tilde{X}_2, T\}$ with the bracket relation $[\tilde{X}_1, \tilde{X}_2] = -T$. Then

$$-\frac{1}{2}\big(\tilde{X}_1^2+\tilde{X}_2^2\big)$$

is the sub-Laplacian on $\mathbf{H}_1$. Let $\mathbf{N}_{\tilde{X}_2} = \langle \tilde{X}_2 \rangle = \{a\tilde{X}_2\}_{a\in\mathbb{R}}$ be a subgroup generated by the element $\tilde{X}_2$. The map $\rho : \mathbf{H}_1 \to \mathbb{R}^2$ defined by

$$\begin{aligned} \rho : \mathbf{H}_1 \to \mathbb{R}^2 \cong \mathbf{h} \ni g =& x_1\tilde{X}_1 + x_2\tilde{X}_2 + zT \\ =& (x_1,x_2,z) \mapsto (u,v) \in \mathbb{R}^2 \end{aligned}$$

where

$$u = x_1, \qquad v = z - \frac{1}{2}x_1x_2$$

realizes the projection map

$$\mathbf{H}_1 \cong \mathbb{R}^3 \to \mathbf{N}_{\tilde{X}_2} \setminus \mathbf{H}_1 \cong \mathbb{R}^2.$$

In fact, this is a principal bundle and the trivialization is given by the map

$$\mathbf{N}_{\tilde{X}_2} \times (\mathbf{N}_{\tilde{X}_2} \setminus \mathbf{H}_1) \cong \mathbb{R}\times\mathbb{R}^2 \ni (a;u,v) \mapsto (x_1,x_2,z) \in \mathbb{R}^3 \cong \mathbf{H}_1$$

where

$$(a;u,v) \mapsto \big(u, a, v + \frac{1}{2}au\big).$$

So the operator $\Delta_{H_1} = \frac{1}{2}\big(\tilde{X}_1^2+\tilde{X}_2^2\big)$ on $\mathbf{H}_1$ and Grusin operator Δ_G commute each other through the map ρ:

$$\Delta_{H_1} \circ \rho^* = \rho^* \circ \Delta_G.$$

The heat kernel $P_{H_1}(t,g) \in C^\infty(\mathbb{R}_+ \times \mathbf{H}_1)$ is given by

$$P_{H_1}(t,g) = P_{H_1}(t,x_1,x_2,z) = \frac{1}{(2\pi t)^2} \int e^{\frac{4i\eta z - \eta \coth \frac{\eta}{2}(x_1^2+x_2^2)}{4t}} \frac{\eta}{2\sinh\frac{\eta}{2}} d\eta \tag{7.1}$$

Hence

$$\begin{aligned} &\int_{-\infty}^{+\infty} P_{H_1}(t,(x_1,x_2,z),(u,a,v+\frac{1}{2}ua))\ da \\ &= \ P_G(t,(x_1,z-\frac{1}{2}x_1x_2),(u,v)) \end{aligned} \tag{7.2}$$

that is, the fiber integration of the function $P_{H_1}(t,g,h)$ along the fiber of the map ρ gives the heat kernel of the Grusin operator.

7.1. About the bicharacteristics and geodesics

Consider now the bicharacteristics of Δ_G and of Δ_{H_1}. We have the following table.

1. Thus a bicharacteristic of $\mathbf{H}_1$ such that $\xi_2 \equiv 0$ projects by

$$(x_1(s),x_2(s),y(s),\xi_1(s),\xi_2(s),\eta) \ \rightarrow \ (x(s)=x_1(s),y(s),\xi(s)=\xi_1(s),\eta)$$

on a bicharacteristic of Grusin operator.

Table 1. Comparison of the Hamitonian systems between Δ_G and Δ_{H_1}

$(x_1,x_2,y) \in \mathbf{H}_1,\ \Delta_{H_1}$	$(x=x_1,y) \in \mathbb{R}^2,\ \Delta_G$
$H = \frac{1}{2}(\xi_1^2 + (\xi_2+x_1\eta)^2)$	$H = \frac{1}{2}(\xi^2 + x^2\eta^2)$
$\frac{dx_1}{ds} = \xi_1,\ \ \frac{d\xi_1}{ds} = -(\xi_2+x_1\eta)\eta$	$\frac{dx}{ds} = \xi,\ \ \frac{d\xi}{ds} = -x\eta^2$
$\frac{dx_2}{ds} = \xi_2 + x_1\eta,\ \ \frac{d\xi_2}{ds} = 0$	
$\frac{dy}{ds} = (\xi_2+x_1\eta)x_1,\ \ \frac{d\eta}{ds} = 0$	$\frac{dy}{ds} = x^2\eta,\ \ \frac{d\eta}{ds} = 0$

2. Conversely, a bicharacteristic of Grusin operator is the projection of a bicharacteristic of $\mathbf{H}_1$ with $\xi_2 = 0$. And these are the only possibilities.

7.2. About the Brownian motion for Grusin operator

The stochastic process for the Grusin operator is

$$x_\omega(t) = x_0 + b_\omega(t), \qquad y_\omega(t) = \int_0^t x_\omega(u) d\beta_\omega(u)$$

Here b_ω and β_ω are independent Brownian motions starting from 0 so that $x_\omega(0) = x_0$ and $y_\omega(0) = 0$ (which we can always assume).

The heat kernel for Grusin operator is

$$\begin{aligned} P_t(x, y \big| x_0, 0) =& E\Big\{\delta(x - x_\omega(t))\delta(y - y_\omega(t))\Big\} \\ =& E\Big\{\delta(x - x_0 - b_\omega(t))\delta(y - \int_0^t x_\omega(u) d\beta_\omega(u))\Big\}. \end{aligned}$$

Write

$$\delta(y) = \int e^{iy\eta} \frac{d\eta}{2\pi}$$

as the inverse Fourier transform of the function 1. Thus

$$\begin{aligned} & P_t(x, y \big| x_0, 0) \\ =& \int \frac{d\eta}{2\pi} e^{iy\eta} E\Big(\delta(x - x_0 - b_\omega(t)) \exp(-i\eta \int_0^t x_\omega(u) d\beta_\omega(u))\Big). \end{aligned} \tag{7.3}$$

The expectation E is the Wiener integral over $b_\omega(t)$ and $\beta_\omega(t)$. In the formula (7.3), one can first integrate over the Brownian motion β, due to the fact that $x_\omega(u)$ is independent of $\beta_\omega(u)$, so that

$$E_\beta\Big(\exp\big(-i\eta \int_0^t x_\omega(u) d\beta_\omega(u)\big)\Big) = \exp\Big(-\frac{1}{2}\eta^2 \int_0^t (x_\omega(u))^2 du\Big)$$

and consequently we have reduced the heat kernel to

$$P_t(x, y \big| x_0, 0) = \int \frac{d\eta}{2\pi} e^{iy\eta} E\Big\{\exp\big(-\frac{1}{2}\eta^2 \int_0^t (x_\omega(u))^2 du\big) \delta(x - x_0 - b_\omega(t))\Big\}$$

with $x_\omega(u) = x_0 + b_\omega(u)$. Then the Wiener integral becomes a path integral which is quadratic in the exponential and can always explicitly calculated, one way or another.

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Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan, ROC.

E-mail: pino@math.sinica.edu.tw

Department of Mathematics, Georgetown University, Washington DC 20057-0001, USA.
E-mail: chang@math.georgetown.edu

Laboratoire Analyse et Physique Mathématique, 14 avenue Félix Faure, 75105 Paris, France.
E-mail: bernard.gaveau@upmc.fr

Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan, ROC.
E-mail: greiner@math.sinica.edu.tw, greiner@math.toronto.edu

Institute of Mathematics, Academia Sinica, Taipei 11529, Taiwan, ROC.
E-mail: hplee@math.sinica.edu.tw