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SUBCLASSES OF MEROMORPHICALLY P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS ASSOCIATED WITH A LINEAR OPERATOR

BY

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Abstract

The aim of this paper is to investigate various inclusion relations and some other properties of a certain subclasses $\Lambda_{p,k}(\alpha, A, B, n)$ and $\Lambda^*_{p,k}(\alpha, A, B, n)$ of meromorphically p-valent analytic funtions with negative coefficients which are defined here by means of a linear operator. The familier concept of neighborhoods of analytic functions is also extended and applied to the functions considered here. We also derive many interesting results for the modified Hadamard products of functions belonging to the class $\Lambda^*_{p,k}(\alpha, A, B, n)$.

1. Introduction

Let Σ_p denote the class of functions $f(z)$ in the form :

$$f(z) = z^{-p} + \sum_{k=0}^{\infty} a_k z^k \qquad (p \in N = \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic and p-valent in the punctured unit disk $U^* = \{z : 0 < |z| < 1\} = U\backslash\{0\}$. If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is

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subordinate to $g(z)$, written symbolically as follows :

$$f \prec g \qquad or \qquad f(z) \prec g(z) \qquad (z \in U),$$

if there exists a Schwarz function $\omega(z)$ in U such that

$$f(z) = g(\omega(z)) \qquad (z \in U).$$

Let Λ_p denote the subclass of $\sum_p$ cosisting of functions of the form :

$$f(z) = z^{-p} - \sum_{k=0}^{\infty} |a_k|\, z^k \qquad (p \in N = \{1, 2, 3, \ldots\}). \tag{1.2}$$

For function $f(z)$ in the class Λ_p, we define a linear operator D^n by :

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= z^{-p} - \sum_{k=0}^{\infty}(p+k+1)\,|a_k|\, z^k = \frac{(z^{p+1} f(z))'}{z^p}, \end{aligned}$$

and (in general)

$$\begin{aligned} D^n f(z) &= D(D^{n-1} f(z)) = z^{-p} - \sum_{k=0}^{\infty}(p+k+1)^n\,|a_k|\, z^k, \\ (f(z) &\in \Lambda_p, n \in N_0 = N \cup \{0\}). \end{aligned} \tag{1.3}$$

Then it is easily verified that [4],

$$z(D^n f(z))' = D^{n+1} f(z) - (p+1) D^n f(z) \quad (f(z) \in \Lambda_p, n \in N_0, p \in N). \tag{1.4}$$

The linear operator D^n was considered by Uralegaddi and Somanatha [14] when $p = 1$, Aouf and Hossan [4] present several results involving the operator D^n for $p \in N$.

Making use of the operator D^n, we say that $f(z) \in \Lambda_p$ is in the class $\Lambda_{p,k}(\alpha, A, B, n)$ if it satisfies the following subordnation condition :

$$\frac{(D^{n+1} f(z))'}{(D^n f(z))'} \prec \frac{1 + [B - (A-B)(p-\alpha)]z}{1 + Bz} \qquad (z \in U) \tag{1.5}$$

$$(0 \le \alpha < p, -1 \le A < B \le 1, 0 < B \le 1, p \in N, n \in N_0),$$

or, equivalenty, by using (1.4), if it satisfies the following subordination condition :

$$1+\frac{z(D^nf(z))''}{(D^nf(z))'} \prec -\frac{p+[pB+(A-B)(p-\alpha)]z}{1+Bz} \qquad (z\in U), \tag{1.6}$$

or, equivalenty, if the following inequality holds true :

$$\left|\frac{1+\frac{z(D^nf(z))''}{(D^nf(z))'}+p}{B(1+\frac{z(D^nf(z))''}{(D^nf(z))'})+[pB+(A-B)(p-\alpha)]}\right|<1 \quad (z\in U). \tag{1.7}$$

Furthermore, we say that a function $f(z)\in\Lambda_{p,k}(\alpha,A,B,n)$ is in the analogous class $\Lambda^*_{p,k}(\alpha,A,B,n)$ if $f(z)$ is of the following form :

$$f(z)=z^{-p}-\sum_{k=p}^{\infty}|a_k|\,z^k \qquad (p\in N). \tag{1.8}$$

We note that :

(i) $\Lambda^*_{p,k}(\alpha,\beta A,\beta B,0)=\Lambda^*_{p,k}(\alpha,\beta,A,B)$ $(0\le\alpha<p,\ 0<\beta\le 1,\ -1\le A<B\le 1\,,0<B\le 1)$ (Aouf and Shammaky [5]);

(ii) $\Lambda^*_{1,k}(\alpha,\beta A,\beta B,0)=\sum_k(\alpha,\beta,A,B)(0\le\alpha<1,0<\beta\le 1,-1\le A<B\le 1,0<B\le 1)$(Srivastava et al. [12]);

(iii) $\Lambda^*_{1,k}(\alpha,-1,1,0)=\Lambda^*_k(\alpha)$,where $\Lambda^*_k(\alpha)$ is the class of meromorphically convex functions of order α with negative coefficients, which was studied by Uralegaddi and Ganigi [13].

Also we observe that :

(i) $\Lambda^*_{p,k}(\alpha,-\beta,\beta,n)=\Lambda^*_{p,k}(\alpha,\beta,n)$

$$=\left\{f(z)\in\Lambda_p \text{ and } \left|\frac{1+\frac{z(D^nf(z))''}{(D^nf(z))'}+p}{1+\frac{z(D^nf(z))''}{(D^nf(z))'}-p+2\alpha}\right|<\beta \right.$$
$$\left.(z\in U,0\le\alpha<p,\ 0<\beta\le 1,p\in N,n\in N_0)\right\}; \tag{1.9}$$

(ii) $\Lambda^*_{p,k}(\alpha,-\beta,\beta,0)=\Lambda^*_{p,k}(\alpha,\beta)$ $(0\le\alpha<p,0<\beta\le 1,p\in N)$, is the class of meromorphic p-valent convex functions of order α $(0\le\alpha<p)$ and type β $(0<\beta\le 1)$ with negative coefficients.

In this paper we investigate the various important properties and characteristics of the classes $\Lambda_{p,k}(\alpha, A, B, n)$ and $\Lambda^*_{p,k}(\alpha, A, B, n)$. Following the recent investigations by Altintas et al. [3, p.1668] , we extend the concept of neighborhoods of analytic functions, which was considered earlier by (for example) Goodman [6] and Ruscheweyh [10] to meromorphically functions, belonging to the classes $\Lambda_{p,k}(\alpha, A, B, n)$ and $\Lambda^*_{p,k}(\alpha, A, B, n)$. We also derive many interesting results for the modified Hadamard products of functions belonging to the class $\Lambda^*_{p,k}(\alpha, A, B, n)$.

2. Inclusion Properties of the Class $\Lambda_{p,k}(\alpha, A, B, n)$

We begin by recalling the following lemma.

Lemma 1. (Jack [7]). *Let the (nonconstant) function $w(z)$ be analytic in U with $w(0) = 0$. If $|w(z)|$ is attains its maximum value on the circle $|z| = r < 1$ at a point $z_0 \in U$, then*

$$z_0 w'(z_0) = \gamma w(z_0),$$

where γ is a real number and $\gamma \geq 1$.

Theorem 1. *If*

$$1 \geq \frac{(B-A)(p-\alpha)}{(1-B)}$$

$(0 \leq \alpha < p, -1 \leq A < B < 1, 0 < B < 1, p \in N, n \in N_0)$.

Then $\Lambda_{p,k}(\alpha, A, B, n+1) \subset \Lambda_{p,k}(\alpha, A, B, n)$.

Proof. Let $f(z) \in \Lambda_{p,k}(\alpha, A, B, n+1)$ and suppose that

$$1 + \frac{z(D^n f(z))''}{(D^n f(z))'} = -\frac{p + [pB + (A-B)(p-\alpha)]w(z)}{1 + Bw(z)}, \tag{2.1}$$

where the function $w(z)$ is either analytic or meromorphic in U, with $w(0) = 0$. Then from (1.5), we have

$$\frac{(D^{n+1} f(z))'}{(D^n f(z))'} = \frac{1 + [B - (A-B)(p-\alpha)]w(z)}{1 + Bw(z)}. \tag{2.2}$$

By differrentiating both side of (2.2) with respect to z logarithmically and using (2.1) again, we have

$$\begin{aligned} 1+\frac{z(D^{n+1}f(z))''}{(D^{n+1}f(z))'} &= -\frac{p+[pB+(A-B)(p-\alpha)]w(z)}{1+Bw(z)} \\ &\quad -\frac{(A-B)(p-\alpha)zw'(z)}{(1+Bw(z))(1+[B-(A-B)(p-\alpha)]w(z))}. \end{aligned} \tag{2.3}$$

If we suppose now that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=1 \qquad (z_0\in U), \tag{2.4}$$

and applying Lemma 1, we find that

$$z_0w'(z_0)=\gamma w(z_0) \ \ (\gamma\geq 1). \tag{2.5}$$

Writing $w(z_0)=e^{i\theta}(0\leq\theta\leq 2\pi)$ and setting $z=z_0$ in (2.3), we find after computations that

$$\begin{aligned} &\left|\frac{1+\frac{z(D^{n+1}f(z))''}{(D^{n+1}f(z))'}+p}{B(1+\frac{z(D^{n+1}f(z))''}{(D^{n+1}f(z))'})+[pB+(A-B)(p-\alpha)]}\right|^2_{z=z_0}-1 \\ &= \left|\frac{(1+\gamma)+[B-(A-B)(p-\alpha)]e^{i\theta}}{1+[(1-\gamma)B-(A-B)(p-\alpha)]e^{i\theta}}\right|^2-1=\frac{\Omega(\theta)}{|1+[(1-\gamma)B-(A-B)(p-\alpha)]e^{i\theta}|^2}, \end{aligned}$$

where

$$\Omega(\theta)=\gamma^2(1-B^2)+2\gamma[1+B^2-B(A-B)(p-\alpha)]+2\gamma\cos\theta[2B-(A-B)(p-\alpha)] \tag{2.6}$$

$(0\leq\alpha<p,-1\leq A<B<1,0<B<1,0\leq\theta\leq 2\pi,\gamma\geq 1,p\in N)$.

Then by hypothesis, we have

$$\Omega(0) = \gamma^2(1-B^2)+2\gamma(1+B)[(1+B)-(A-B)(p-\alpha)]\geq 0,$$

and

$$\Omega(\pi) = \gamma^2(1-B^2)+2\gamma(1-B)[(1-B)+(A-B)(p-\alpha)]\geq 0,$$

which, together, show that

$$\Omega(\theta)\geq 0 \qquad (0\leq\theta\leq 2\pi) \tag{2.7}$$

which obviously contradicts our hypothesis that $f(z) \in \Lambda_{p,k}(\alpha, A, B, n+1)$. Thus we must have $|w(z)| < 1$ $(z \in U)$ and from (2.1), we conclude that $f(z) \in \Lambda_{p,k}(\alpha, A, B, n)$, which evidently complete the proof of Theorem 1.

Theorem 2. *Let $f(z) \in \Lambda_{p,k}(\alpha, A, B, n)$, then the function $F_\lambda(z)$ given by*

$$F_\lambda(z) = \frac{\lambda}{z^{\lambda+p}} \int_0^z t^{\lambda+p-1} f(t) dt \tag{2.8}$$

$(0 \le \alpha < p,\, p \in N,\, \mathrm{Re}(\lambda) \ge \frac{(B-A)(p-\alpha)}{1-B},\, -1 \le A < B < 1, 0 < B < 1)$, *also belonges to the class $\Lambda_{p,k}(\alpha, A, B, n)$.*

Proof. It is easily seen from (1.3) and (2.8) that

$$z(D^n F_\lambda(z))' = \lambda(D^n f(z)) - (\lambda + p)(D^n F_\lambda(z)). \tag{2.9}$$

Suppose that $f(z) \in \Lambda_{p,k}(\alpha, A, B, n)$ and put

$$1 + \frac{z(D^n F_\lambda(z))''}{(D^n F_\lambda(z))'} + p = \frac{(B-A)(p-\alpha)w(z)}{1+Bw(z)}, \tag{2.10}$$

where $w(z)$ is either analytic or meromorphic in U, with $w(0) = 0$.

Then by using (2.9) and (2.10), we find after some computations that

$$\begin{aligned} 1 + \frac{z(D^n f(z))''}{(D^n f(z))'} + p = {} & \frac{(B-A)(p-\alpha)w(z)}{1+Bw(z)} \\ & - \frac{(A-B)(p-\alpha)zw'(z)}{(\lambda + [\lambda B - (A-B)(p-\alpha)]w(z))(1+Bw(z))}. \end{aligned} \tag{2.11}$$

Now we follow the lines of proof of Theorem 1 and assume that (2.4) and (2.5) holds true. Thus by writing $w(z_0)$ in (2.11), we obtain,

$$\begin{aligned} & \left| \frac{1 + \frac{z(D^n f(z))''}{(D^n f(z))'} + p}{B(1 + \frac{z(D^n f(z))''}{(D^n f(z))'} + p) + [pB + (A-B)(p-\alpha)]} \right|^2_{z=z_0} - 1 \\ & = \frac{\Omega(\theta)}{|\lambda + [B(\lambda - \gamma) - (A-B)(p-\alpha)e^{i\theta}]|^2}, \end{aligned} \tag{2.12}$$

where

$$\begin{aligned}\Omega(\theta) &= \gamma^2(1-B^2)+2\gamma[(1+B^2)\mathrm{Re}(\lambda)-B(A-B)(p-\alpha) \\ &\quad +2\gamma\cos\theta[2B\mathrm{Re}(\lambda)-(A-B)(p-\alpha)]\end{aligned} \tag{2.13}$$

$(-1 \leq A < B < 1,\ 0 < B < 1,\ \gamma \geq 1,\ 0 \leq \theta < \pi)$.

Then, by hypothesis, we have

$$\Omega(0) = \gamma^2(1-B^2)+2\gamma(1+B)[(1+B)\mathrm{Re}(\lambda)-(A-B)(p-\alpha)] \geq 0,$$

and

$$\Omega(\pi) = \gamma^2(1-B^2)+2\gamma(1-B)[(1-B)\mathrm{Re}(\lambda)+(A-B)(p-\alpha)] \geq 0,$$

which, together, show that

$$\Omega(\theta) \geq 0 \qquad (0 \leq \theta \leq 2\pi). \tag{2.14}$$

In view of $(2.14), (2.12)$ would obviously contradict our hypothesis that $f(z) \in \Lambda_{p,k}(\alpha, A, B, n)$. Hence, we must have

$$|w(z)| < 1 \quad (z \in U),$$

and we conclude from (2.10) that $F_\lambda(z) \in \Lambda_{p,k}(\alpha, A, B, n)$, where $F_\lambda(z)$ given by (2.8). This compelete the proof of Theorem 2. □

Theorem 3. *The following equivalence holds true for the class* $\Lambda_{p,k}(\alpha, A, B, n)$,

$$f(z) \in \Lambda_{p,k}(\alpha, A, B, n) \Leftrightarrow F_1(z) \in \Lambda_{p,k}(\alpha, A, B, n+1), \tag{2.15}$$

where $F_1(z)$ *is given by* (2.8) *with* $\lambda = 1$.

Proof. In view of (2.9) and (1.4), we have

$$\begin{aligned}D^n f(z) &= z(D^n F_1(z))' + (1+p)D^n F_1(z) \\ &= (D^{n+1}F_1(z)),\end{aligned}$$

which yields (2.15). □

3. Basic Properties of the Class $\Lambda^*_{p,k}(\alpha, A, B, n)$

Throughout this section we assume that,

$$A + B \geq 0 \quad (-1 \leq A < B \leq 1, 0 < B \leq 1).$$

We first determine a necessary and sufficient condition for a fuction $f(z) \in \Lambda_p$ of the form (1.8) to be in the class $\Lambda^*_{p,k}(\alpha, A, B, n)$.

Theorem 4. *Let the function $f(z)$ be given by* (1.8). *Then $f(z) \in \Lambda^*_{p,k}(\alpha, A, B, n)$ if and only if*

$$\sum_{k=p}^{\infty}(k+p+1)^n(\frac{k}{p})G(\alpha, A, B, p, k)\,|a_k| \leq H(\alpha, A, B, p), \tag{3.1}$$

where

$$G(\alpha, A, B, p, k) = [k(1+B) + p(1+A) + (B-A)\alpha]\,, \tag{3.2}$$

and

$$H(\alpha, A, B, p) = (B-A)(p-\alpha). \tag{3.3}$$

Proof. Suppose that the function $f(z)$ given by (1.8) is in the class $\Lambda^*_{p,k}(\alpha, A, B, n)$.

Then from (1.8) and (1.7), we have

$$\begin{aligned}
&\left|\frac{z(D^n f(z))'' + (1+p)(D^n f(z))'}{B(1 + z(D^n f(z))'') + [pB + (A-B)(p-\alpha)](D^n f(z))'}\right| \\
&= \left|\frac{-\sum\limits_{k=p}^{\infty} k(k+p)(k+p+1)^n\,|a_k|\,z^{n+k}}{(B-A)(p-\alpha)p - \sum\limits_{k=p}^{\infty} k(k+p+1)^n[kB + (B-A)\alpha + Ap]\,|a_k|\,z^{k+p}}\right| \\
&< 1 \quad (z \in U).
\end{aligned} \tag{3.4}$$

Since $|\mathrm{Re}\{z\}| \leq |z|$ for any z, choosing z to be real and letting $z \to 1^-$

through real value, (3.4) yields

$$\sum_{k=p}^{\infty} k(k+p)(k+p+1)^n |a_k| \leq (B-A)(p-\alpha)p - \sum_{k=p}^{\infty} k(k+p+1)^n [Bk+(B-A)\alpha+Ap] |a_k|, \quad (3.5)$$

which leads us immdiatly to the coefficient inequality (3.1).

Next in order to prove the converse we assume that the inequality (3.1) holds true.Then we observe that

$$\left|\frac{z(D^n f(z))'' + (1+p)(D^n f(z))'}{B(1+z(D^n f(z))'') + [pB+(A-B)(p-\alpha)](D^n f(z))'}\right| \leq \frac{\sum_{k=p}^{\infty} k(k+p)(k+p+1)^n |a_k|}{(B-A)(p-\alpha)p - \sum_{k=p}^{\infty} k(k+p+1)^n [Bk+(B-A)\alpha+Ap] |a_k|} < 1 \qquad (z \in U). \quad (3.6)$$

Hence by maximum modulus theorem, we have $f(z) \in \Lambda_{p,k}(\alpha, A, B, n)$. This completes the proof of Theorem 4. □

Corollary 1. *Let the function $f(z)$ given* (1.8). *If $f(z) \in \Lambda_{p,k}(\alpha, A, B, n)$, then*

$$|a_k| \leq \frac{pH(\alpha, A, B, p)}{k(k+p+1)^n G(\alpha, A, B, p, k)} \quad (k \geq p, \quad p \in N). \quad (3.7)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = z^{-p} - \frac{pH(\alpha, A, B, p)}{k(k+p+1)^n G(\alpha, A, B, p, k)} z^k \qquad (k \geq p, \quad p \in N). \quad (3.8)$$

Remark 1.

(i) Putting $n = 0, p = 1$, and replacing A by βA and B by βB $(0 < \beta \leq 1)$ in Theorem 4, we obtain the result obtained by Srivastava et al. ([12, Theorem 1]);

(ii) Putting $n = 0, p = 1$, $A = 1$ and $B = -1$ in Theorem 4, we obtain the result obtained by Uralegaddi and Ganigi ([13, Theorem 1]).

Next we prove the following growth and distortion properties for the class $\Lambda^*_{p,k}(\alpha, A, B, n)$.

Theorem 5. *If a function $f(z)$ defined by* (1.8) *is in the class $\Lambda^*_{p,k}(\alpha, A, B, n)$, then*

$$\left[\frac{(p+m-1)!}{(p-1)!} - \frac{p!H(\alpha, A, B, p)}{(p-m)!(2p+1)^n G(\alpha, A, B, p)} r^{2p}\right] r^{-p-m}$$
$$\leq \left|f^{(m)}(z)\right| \leq \left[\frac{(p+m-1)!}{(p-1)!} + \frac{p!H(\alpha, A, B, p)}{(p-m)!(2p+1)^n G(\alpha, A, B, p)} r^{2p}\right] r^{-p-m}$$
$$(0 < |z| = r < 1, 0 \leq m < p). \tag{3.9}$$

The result is sharp for function $f(z)$ given by

$$f(z) = z^{-p} - \frac{H(\alpha, A, B, p)}{(2p+1)^n G(\alpha, A, B, p)} z^p \quad (p \in N). \tag{3.10}$$

Proof. For $f(z) \in \Lambda^*_{p,k}(\alpha, A, B, n)$, we find from Theorem 4, that

$$\begin{aligned}\frac{(2p+1)^n G(\alpha, A, B, p)}{p!} \sum_{k=p}^{\infty} k!\,|a_k| &\leq \sum_{k=p}^{\infty} (\frac{k}{p})(k+p+1)^n G(\alpha, A, B, p, k)\,|a_n| \\ &\leq H(\alpha, A, B, p),\end{aligned}$$

which yields,

$$\sum_{k=p}^{\infty} k!\,|a_k| \leq \frac{p!H(\alpha, A, B, p)}{(2p+1)^n G(\alpha, A, B, p)}. \tag{3.11}$$

Now by differentiating $f(z)$ in (1.8) m times, we have

$$f^{(m)}(z) = (-1)^m \frac{(p+m-1)!}{(p-1)!} z^{-p-m} - \sum_{k=p}^{\infty} \frac{k!}{(k-m)!} |a_k|\, z^{k-m} \tag{3.12}$$
$$(m \in N_0\ ,\ p \in N, \quad m < p).$$

Theorem 5 follows easily from (3.11) and (3.12) and the sharpness of each inequality in (3.9) satisfies by the function $f(z)$ given by (3.10). □

Remark 2.

(i) Putting $n = 0, p = 1$, and replacing A by βA and B by βB $(0 < \beta \leq 1)$ in Theorem 5 (with $m = 0$ and $m = 1$), we obtain the results obtained by Srivastava et al. ([12, Theorem 3]);

(ii) Putting $n = 0$, $p = 1$, $A = 1$ and $B = -1$ in Theorem 5 (with $m = 0$ and $m = 1$), we obtain the results obtained by Uralegaddi and Ganigi ([13, Theorem 3]).

In the end of this section we determine the radii of meromorphically p-valent starlikeness of order $\gamma\,(0 \leq \gamma < p)$ and meromorphically p-valent convexity of order $\gamma\,(0 \leq \gamma < p)$ for functions in the class $\Lambda^*_{p,k}(\alpha, A, B, n)$.

Theorem 6. *Let the function $f(z)$ defined by* (1.8) *be in the class $\Lambda^*_{p,k}(\alpha, A,\ B, n)$. Then*

(i) *$f(z)$ is meromorphically p-valent starlike of order γ $(0 \leq \gamma < p)$ in $|z| < r_1$, where*

$$r_1 = \inf_{k \geq p} \left\{ (k+p+1)^n \frac{k(p-\gamma)G(\alpha, A, B, p, k)}{p(k+\gamma)H(\alpha, A, B, p)} \right\}^{\frac{1}{k+p}}; \tag{3.13}$$

(ii) *$f(z)$ is meromorphically p-valent convex of order γ $(0 \leq \gamma < p)$ in $|z| < r_2$, where*

$$r_2 = \inf_{k \geq p} \left\{ (k+p+1)^n \frac{(p-\gamma)G(\alpha, A, B, p, k)}{(k+\gamma)H(\alpha, A, B, p)} \right\}^{\frac{1}{k+p}}. \tag{3.14}$$

Each of these results is sharp for the function $f(z)$ given by (3.8).

Proof. (i) From the definition (1.8), we easily get

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} - p + 2\gamma} \right| \leq \frac{\sum\limits_{k=p}^{\infty} (k+p)\,|a_k|\,|z|^{k+p}}{2(p-\gamma) - \sum (k-p+2\gamma)\,|a_k|\,|z|^{k+p}}.$$

Thus, we have the desired inequlity :

$$\left| \frac{\frac{zf'(z)}{f(z)} + p}{\frac{zf'(z)}{f(z)} - p + 2\gamma} \right| \leq 1 \qquad (0 \leq \gamma < p; p \in N), \tag{3.15}$$

if

$$\sum_{k=p}^{\infty}(\frac{k+\gamma}{p-\gamma})\,|a_k|\,|z|^{k+p} \leq 1. \tag{3.16}$$

Hence, by Theorem 4, (3.16) will be true if

$$(\frac{k+\gamma}{p-\gamma})\,|z|^{k+p} \leq (k+p+1)^n(\frac{k}{p})\frac{G(\alpha,A,B,p,k)}{H(\alpha,A,B,p)} \quad (k \geq p, p \in N). \tag{3.17}$$

This last inequality (3.17) leads us immediatly to the disc $|z| < r_1$, where r_1 is given by (3.13).

(ii) In order to prove the second assertion of Throrem 6 we find from the definition (1.8) that

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\gamma}\right| \leq \frac{\sum_{k=p}^{\infty} k(k+p)\,|a_k|\,|z|^{k+p}}{2p(p-\gamma)-\sum_{k=p}^{\infty}(k-p+2\gamma)\,|a_k|\,|z|^{k+p}}.$$

Thus, we have the desired inequlity :

$$\left|\frac{1+\frac{zf''(z)}{f'(z)}+p}{1+\frac{zf''(z)}{f'(z)}-p+2\gamma}\right| \leq 1 \qquad (0 \leq \gamma < p; p \in N), \tag{3.18}$$

if

$$\sum_{k=p}^{\infty}\frac{k(k+\gamma)}{p(p-\gamma)}\,|a_k|\,|z|^{k+p} \leq 1. \tag{3.19}$$

Hence, by Theorem 4, (3.19) will be true if

$$\frac{k(k+\gamma)}{p(p-\gamma)})\,|z|^{k+p} \leq \frac{k}{p}(k+p+1)^n\frac{G(\alpha,A,B,p,k)}{H(\alpha,A,B,p)} \quad (k \geq p,\, p \in N). \tag{3.20}$$

This last inequality (3.20) leads us immediatly to the disc $|z| < r_2$, where r_2 is given by (3.14), each assertion is sharp for the function $f(z)$, given by (3.8). This completes the proof of Theorem 6.

4. Neighborhoods and Partial Sums

Following the earlier work (based upon the familiar concept of neighbourhoods of analytic functions) by Goodman [6] and Rusheweyh [10] and (more recently) by Altinatas et al. ([1] , [2] and [3]) and Liu and Srivastava ([8] and [9]), we begin by introducing here the δ-neighborhood of a function $f(z) \in \Lambda_p$ of the form (1.2) by means of the definition given below :

$$\begin{aligned} N_\delta(f) \; = \; & \Big\{ g \in \Lambda_p : g(z) = z^{-p} - \sum_{k=0}^{\infty} |b_k| \, z^{\kappa} \text{ and} \\ & \sum_{k=0}^{\infty} \frac{k(k+p+1)^n G(\alpha, |A|, B, p, k)}{p \;\; H(\alpha, A, B, p)} \, ||a_k| - |b_k|| \leq \delta \Big\} \qquad (4.1) \\ & (0 \leq \alpha < p, p \in N, \; -1 \leq A < B \leq 1 \; , \; 0 < B \leq 1 \, , \; \delta > 0). \end{aligned}$$

Theorem 7. *Let $\delta > 0$. If $f(z) \in \Lambda_p$ given by* (1.2), *satisfies the following condition*

$$\frac{f(z) + \varepsilon z^{-p}}{1 + \varepsilon} \in \Lambda_{p,k}(\alpha, A, B, n), \tag{4.2}$$

for any complex numder ε such that $|\varepsilon| < \delta$, then $N_\delta(f) \subset \Lambda_{p,k}(\alpha, A, B, n)$.

Proof. It is easily seen from (1.7) that $g(z) \in \Lambda_{p,k}(\alpha, A, B, n)$ if and only if, for any complex number σ, $|\sigma| = 1$, we have

$$\frac{1 + \frac{z(D^n g(z))'{}'}{(D^n g(z))'} + p}{B(1 + \frac{z(D^n g(z))'{}'}{(D^n g(z))'} + p) + (A - B)(p - \alpha)} \neq \sigma \;\; (z \in U),$$

which is equivalent to

$$\frac{g(z) * h(z)}{z^{-p}} \neq 0 \qquad\qquad (z \in U), \tag{4.3}$$

where, for covenience,

$$h(z) = z^{-p} - \sum_{k=0}^{\infty} c_k z^k,$$

$$c_k = k(k+p+1)^n \frac{\sigma(B-A)\alpha - k(1 - \sigma B) - p(1 - \sigma A)}{\sigma p(B - A)(p - \alpha)}. \tag{4.4}$$

it follows from (4.4), that

$$\begin{aligned}|c_k| &\leq k(k+p+1)^n \frac{(B-A)\alpha + k(1+B) + p(1+|A|)}{p(B-A)(p-\alpha)}. \\ &= k(k+p+1)^n \frac{G(\alpha, |A|, B, p, k)}{pH(\alpha, A, B, p)}.\end{aligned}$$

If $f(z) \in \Lambda_p$ given by (1.2) satisfies (4.2), then (4.3) yields

$$\left|\frac{f(z) * h(z)}{z^{-p}}\right| \geq \delta \qquad (z \in U). \tag{4.5}$$

Now, if we suppose that,

$$\varphi(z) = z^{-p} - \sum_{k=0}^{\infty} |d_k| z^k \in N_\delta(f), \tag{4.6}$$

we easily seen that

$$\begin{aligned}\left|\frac{[\varphi(z) - f(z)] * h(z)}{z^{-p}}\right| &= \left|\sum_{k=0}^{\infty}(|d_k| - |a_k|)c_k z^{k+p}\right| \\ &\leq |z| \sum_{k=0}^{\infty} k(k+p+1)^n \frac{G(\alpha, |A|, B, p, k)}{pH(\alpha, A, B, p)} ||d_k| - |a_k|| < \delta.\end{aligned}$$

Thus for any number σ such that $|\sigma| = 1$, we have

$$\frac{\varphi(z) * h(z)}{z^{-p}} \neq 0 \qquad (z \in U),$$

which implies that $\varphi(z) \in \Lambda_{p,k}(\alpha, A, B, n)$.This complete the proof of Theorem 7. □

We now define the $\delta-$neighbourhood of a function $f(z) \in \Lambda_p$ of the form (1.8) as follows :

$$\begin{aligned}N_\delta^*(f) = \Big\{ g(z) \in \Lambda_p :\ & g(z) = z^{-p} - \sum_{k=p}^{\infty} |b_k| z^k \quad \text{and} \\ & \sum_{k=p}^{\infty} \frac{k(k+p+1)^n G(\alpha, |A|, B, p, k)}{pH(\alpha, A, B, p)} ||a_k| - |b_k| \leq \delta| \Big\}.\end{aligned} \tag{4.7}$$

Theorem 8. *Let $0 \leq A < 1$. If the function $f(z)$, given by (1.8), is in the class $\Lambda^*_{p,k}(\alpha, A, B, n+1)$, then*

$$N^*_\delta(f) \subset \Lambda^*_{p,k}(\alpha, A, B, n) \quad (\delta = \frac{2p}{2p+1}).$$

The result is sharp in the sense that δ can not be increased.

Proof. Using the same method as in the proof of Theorem 7, we can show that

$$\begin{aligned} h(z) &= z^{-p} - \sum_{k=p}^{\infty} c_k z^k \\ &= z^{-p} - \sum_{k=p}^{\infty} k(k+p+1)^n \frac{\sigma(B-A)\alpha - k(1-\sigma B) - p(1-\sigma A)}{p\sigma(B-A)(p-\alpha)} z^k. \end{aligned} \tag{4.8}$$

If $f(z) \in \Lambda^*_{p,k}(\alpha, A, B, n+1)$ is given by (1.8), we obtain

$$\begin{aligned} \left|\frac{f(z) * h(z)}{z^{-p}}\right| &= \left|1 - \sum_{k=p}^{\infty} c_k \,|a_k|\, z^{k+p}\right| \\ &\geq 1 - \frac{1}{2p+1} \sum_{k=p}^{\infty} \frac{k(k+p+1)^{n+1} G(\alpha, A, B, p, k)}{pH(\alpha, A, B, p)} |a_k| \\ &\geq \frac{2p}{2p+1} = \delta \end{aligned}$$

by appealing to the assertion (3.1) of Theorem 4. The remaining part of the proof of Theorem 8 is similar to that of Theorem 7, and we skip the details involved.

To show the sharpness of the assertion of Theorem 8, we consider the functions $f(z)$ and $g(z)$ given by

$$f(z) = z^{-p} - \frac{H(\alpha, A, B, p)}{(2p+1)^{n+1} G(\alpha, A, B, p)} z^p \in \Lambda^*_{p,k}(\alpha, A, B, n+1) \tag{4.9}$$

and

$$\begin{aligned} g(z) &= z^{-p} - \Big[\frac{H(\alpha, A, B, p)}{(2p+1)^{n+1} G(\alpha, A, B, p)} \\ &\qquad + \frac{H(\alpha, A, B, p)\delta'}{(2p+1)^n G(\alpha, A, B, p)}\Big] z^p \quad (\delta' > \frac{2p}{2p+1}). \end{aligned} \tag{4.10}$$

Clearly, the function $g(z)$ belongs to $N_\delta^*(f)$. On the other hand, we find from Theorem 4 that $g(z)$ is not in the class $\Lambda_{p,k}^*(\alpha, A, B, n)$. This complete the proof of Theorem 8. □

Next, we prove the following result.

Theorem 9. *Let $f(z) \in \Lambda_p$ be given by* (1.2) *and define the partial sums $s_1(z)$ and $s_m(z)$ as follows:*

$$s_1(z) = z^{-p}$$

and

$$s_m(z) = z^{-p} - \sum_{k=0}^{m-2} |a_k|\, z^k \quad (m \in N \backslash \{1\}). \tag{4.11}$$

Suppose also that

$$\sum_{k=0}^{\infty} d_k\, |a_k| \leq 1 \quad (d_k = \frac{k(k+p+1)^n G(\alpha, |A|, B, p, k)}{pH(\alpha, A, B, p)}), \tag{4.12}$$

then we have

(i) $f(z) \in \Lambda_{p,k}(\alpha, A, B, n)$,

(ii) $$\operatorname{Re}\left\{\frac{f(z)}{s_m(z)}\right\} > 1 - \frac{1}{d_{m-1}} \quad (z \in U,\ m \in N), \tag{4.13}$$

and

(iii) $$\operatorname{Re}\left\{\frac{s_m(z)}{f(z)}\right\} > \frac{d_{m-1}}{1 + d_{m-1}} \quad (z \in U,\ m \in N). \tag{4.14}$$

The estimates in (4.13) *and* (4.14) *are sharp for $m \in N$.*

Proof. (i) It is clear that $z^{-p} \in \Lambda_{p,k}(\alpha, A, B, n)$. According to Theorem 7 and condition (4.12), we have $N_1(z^{-p}) \subset \Lambda_{p,k}(\alpha, A, B, n)$. It follows that $f(z) \in \Lambda_{p,k}(\alpha, A, B, n)$.

(ii) Under the hypothesis in part (ii) of Theorem 9, we can see from (4.12) that

$$\sum_{k=0}^{m-2} |a_k| + d_{m-1} \sum_{k=m-1}^{\infty} |a_k| \leq \sum_{k=0}^{\infty} d_k\, |a_k| < 1, \tag{4.15}$$

by using the hypothesis (4.12).

By setting

$$g_1(z) = d_{m-1}\left\{\frac{f(z)}{s_m(z)} - (1 - \frac{1}{d_{m-1}})\right\} = 1 - \frac{d_{m-1}\sum\limits_{k=m-1}^{\infty} a_k z^{k+p}}{1 - \sum\limits_{k=0}^{m-2} a_k z^{k+p}}, \quad (4.16)$$

and applying (4.15), we find that

$$\left|\frac{g_1(z)-1}{g_1(z)+1}\right| \leq \frac{d_{m-1}\sum\limits_{k=m-1}^{\infty}|a_k|}{2 - 2\sum\limits_{k=0}^{m-2}|a_k| - d_{m-1}\sum\limits_{k=m-1}^{\infty}|a_k|} \leq 1 \qquad (z \in U), \qquad (4.17)$$

which readily yields the assertion (4.13) of Theorem 9. If we take

$$f(z) = z^{-p} - \frac{z^{m-1}}{d_{m-1}}, \qquad (4.18)$$

then

$$\frac{f(z)}{s_m(z)} = 1 - \frac{z^{m+p-1}}{d_{m-1}} \to 1 - \frac{1}{d_{m-1}} \quad as \ \ z \to 1,$$

which shows that the bound in (4.13) is best possible for each $m \in N$.

(iii) Similarly, if we put

$$g_2(z) \ = \ (1 + d_{m-1})(\frac{s_m(z)}{f(z)} - \frac{d_{m-1}}{1 + d_{m-1}}) = 1 + \frac{(1 + d_{m-1})\sum\limits_{k=m-1}^{\infty}|a_k|\, z^{k+p}}{1 - \sum\limits_{k=0}^{\infty}|a_k|\, z^{k+p}},$$

and make use of (4.15), we can deduce that

$$\left|\frac{g_2(z)-1}{g_2(z)+1}\right| \leq \frac{(1 + d_{m-1})\sum\limits_{k=m-1}^{\infty}|a_k|}{2 - 2\sum\limits_{k=0}^{m-2}|a_k| + (1 - d_{m-1})\sum\limits_{k=m-1}^{\infty}|a_k|} \leq 1 \quad (z \in U), \quad (4.19)$$

which yields inequality (4.14) of Theorem 9. The bound in (4.14) is sharp for each $m \in N$ with the extremal function $f(z)$ given by (4.18). The proof of Theorem 9 is completed. □

Remark 3. Choosing A, B, n and p appropriately in the above results in (Sec. 4) we obtain new results for the classes studied by Srivastava et al. [12], Uralegaddi and Ganigi [13] and Aouf and Shammaky [5].

5. Properties involving modified Hadamard Product

Let the functions $f_j(z)(j = 1, 2)$ be defined by

$$f_j(z) = z^{-p} - \sum_{k=p}^{\infty} |a_{k,j}|\, z^k. \tag{5.1}$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^{-p} - \sum_{k=p}^{\infty} |a_{k,1}|\,|a_{k,2}|\, z^k. \tag{5.2}$$

Theorem 10. *Let the functions $f_j(z)$ $(j = 1, 2)$ defined by* (5.1) *be in the $\Lambda^*_{p,k}(\alpha, A, B, n)$. Then $(f_1 * f_2)(z) \in \Lambda^*_{p,k}(\zeta, A, B, n)$, where*

$$\begin{aligned} \zeta &= \zeta(\alpha, A, B, p, n) \\ &= p\left(1 - \frac{2(1+B)(B-A)(p-\alpha)^2}{(2p+1)^n[p(2+A+B)+(B-A)\alpha] + (B-A)^2(p-\alpha)^2}\right). \end{aligned} \tag{5.3}$$

The result is sharp, for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_j(z) = z^{-p} - \frac{(B-A)(p-\alpha)}{(2p+1)^n[p(2+A+B)] + (B-A)\alpha]} z^p \quad (j = 1, 2; p \in N). \tag{5.4}$$

Proof. Employing the technique used earlier by Schild and Silverman [11], we need to find the largest $\zeta = \zeta(\alpha, A, B, p, n)$ such that,

$$\sum_{k=p}^{\infty} (k+p+1)^n \frac{k}{p} \left[\frac{[k(1+B) + p(1+A) + (B-A)\zeta}{(B-A)(p-\zeta)}\right] |a_{k,1}|\,|a_{k,2}| \le 1, \tag{5.5}$$

for $f_j(z) \in \Lambda^*_{p,k}(\alpha, A, B, n)$ $(j = 1, 2)$. Since $f_j(z) \in \Lambda^*_{p,k}(\alpha, A, B, n)$ $(j =$

1, 2), we readly see that

$$\sum_{k=p}^{\infty}(k+p+1)^n\frac{k}{p}\left[\tfrac{[k(1+B)+p(1+A)+(B-A)\alpha}{(B-A)(p-\alpha)}\right]|a_{k,j}\,|\leq 1 \quad (j=1,2). \tag{5.6}$$

Therefore, by the Cauchy - Schwarz inequality, we obtain

$$\sum_{k=p}^{\infty}(k+p+1)^n\frac{k}{p}\left[\tfrac{[k(1+B)+p(1+A)+(B-A)\alpha}{(B-A)(p-\alpha)}\right]\sqrt{|a_{k,1}\,|\,|a_{k,2}|}\leq 1. \tag{5.7}$$

Thus it is sufficient to show that

$$\begin{aligned}&\left[\frac{[k(1+B)+p(1+A)+(B-A)\zeta}{(p-\zeta)}\right]|a_{k,1}\,|\,|a_{k,2}|\\ &\leq \left[\frac{[k(1+B)+p(1+A)+(B-A)\alpha}{(p-\alpha)}\right]\sqrt{|a_{,1}\,|\,|a_{k,2}|} \quad (k\geq p),\end{aligned} \tag{5.8}$$

or, equivaletly, that

$$\sqrt{|a_{k,1}\,|\,|a_{k,2}|}\leq \frac{[k(1+B)+p(1+A)(B-A)\alpha](p-\zeta)}{[(k(1+B)+(1+A)(B-A)\zeta](p-\alpha)} \quad (k\geq p). \tag{5.9}$$

Note that

$$\sqrt{|a_{k,1}\,|\,|a_{k,2}|}\leq \frac{(B-A)(p-\alpha)}{(k+p+1)^n\frac{k}{p}[k(1+B)+p(1+A)+(B-A)\alpha]}. \tag{5.10}$$

Consequently, we need only to prove

$$\begin{aligned}&\frac{(B-A)(p-\alpha)}{(k+p+1)^n\frac{k}{p}[k(1+B)+p(1+A)+(B-A)\alpha]}\\ &\leq \frac{[k(1+,B)+p(1+A)+(B-A)\alpha](p-\zeta)}{[k(1+B)+p(1+A)+(B-A)\zeta](p-\alpha)} \quad (k\geq p).\end{aligned} \tag{5.11}$$

It follows from (5.11) that

$$\zeta\leq p-\frac{(k+p)(1+B)(B-A)(p-\alpha)^2}{(k+p+1)^n\frac{k}{p}[k(1+B)+p(1+A)+(B-A)\alpha]^2+(B-A)^2(p-\alpha)^2} \quad (k\geq p). \tag{5.12}$$

Now, defining the function $\varphi(k)$ by

$$\varphi(k) = p - \frac{(k+p)(1+B)(B-A)(p-\alpha)^2}{(k+p+1)^n \frac{k}{p}[k(1+B)+p(1+A)+(B-A)\alpha]^2+(B-A)^2(p-\alpha)^2} \quad (k \geq p), \quad (5.13)$$

we see that is an increasing function of k $(k \geq p)$. Therefore we coclude that

$$\zeta \leq \varphi(p) = p\left(1 - \frac{2(1+B)(B-A)(p-\alpha)^2}{(2p+1)^n[p(2+A+B)+(B-A)\alpha]^2+(B-A)^2(p-\alpha)^2}\right), \quad (5.14)$$

which complete the proof of Theorem 10. □

Corollary 2. *For $f_1(z)$ and $f_2(z)$ as in Theorem* 10*, the function*

$$h(z) = z^{-p} - \sum_{k=p}^{\infty} \sqrt{|a_{k,1}|\,|a_{k,2}|}\; z^k \quad (5.15)$$

belongs to the class $\Lambda^*_{p,k}(\alpha, A, B, n)$. The result follows from the inequality (5.7)*. It is sharp for the same functions $f_j(z)(j = 1, 2)$ given by* (5.4)*.*

Finally it is easy to prove the following theorems so we will omitte the proofs.

Theorem 11. *Let the function $f_1(z)$ defined by* (5.1) *be in the class $\Lambda^*_{p,k}(\alpha, A, B, n)$ and that function $f_2(z)$ defined by* (5.1) *be in the class $\Lambda^*_{p,k}(\gamma, A, B, n)$. Then $(f_1 * f_2)(z) \in \Lambda^*_{p,k}(\delta, A, B, n)$, where*

$$\begin{aligned}
\delta &= \delta(\alpha, \gamma, A, B, p, n) \\
&= p\left(1 - \frac{2(1+B)(B-A)(p-\alpha)(p-\gamma)}{(2p+1)^n\theta(\alpha)\theta(\gamma) + (B-A)^2(p-\alpha)(p-\gamma)}\right) \\
&\quad (\theta(\lambda) = [p(2+A+B) + (B-A)\lambda]).
\end{aligned} \quad (5.16)$$

The result is sharp for the functions $f_j(z)$ $(j = 1, 2)$ given by

$$f_1(z) = z^{-p} - \frac{(B-A)(p-\alpha)}{(2p+1)^n[p(2+A+B)+(B-A)\alpha]} z^p \quad (p \in N) \quad (5.17)$$

and

$$f_2(z) = z^{-p} - \frac{(B-A)(p-\gamma)}{(2p+1)^n[p(2+A+B)+(B-A)\gamma]} z^p \quad (p \in N). \quad (5.18)$$

Corollary 3. *Let the functions* $f_j(z)(j=1,2,3)$ *defined by* (5.1) *be in the class* $\Lambda^*_{p,k}(\alpha, A, B, n)$. *Then* $(f_1 * f_2 * f_3)(z) \in \Lambda^*_{p,k}(\zeta, A, B, n)$, *where*

$$\begin{aligned} \zeta &= \zeta(\alpha, \gamma, A, B, p, n) \\ &= p\left(1 - \frac{2(1+B)(B-A)^2(p-\alpha)^3}{(2p+1)^{2n}[p(2+A+B)+(B-A)\alpha]^3+(B-A)^3(p-\alpha)^3}\right). \end{aligned} \tag{5.19}$$

The result is sharp for the functions $f_j(z)(j=1,2,3)$ *given by* (5.4).

Theorem 12. *Let the functions* $f_j(z)\,(j=1,2)$ *defined by* (5.1) *be in the class* $\Lambda^*_{p,k}(\alpha, A, B, n)$, *then the function*

$$h(z) = z^{-p} - \sum_{k=p}^{\infty}(|a_{k,1}|^2 + |a_{k,2}|^2)z^k \tag{5.20}$$

belongs to the class $\Lambda^*_{p,k}(\eta, A, B, n)$, *where*

$$\begin{aligned} \eta &= \eta(\alpha, A, B, p, n) \\ &= p\left(1 - \frac{4(1+B)(B-A)(p-\alpha)^2}{(2p+1)^n[p(2+A+B)+(B-A)\alpha]^2+2(B-A)^2(p-\alpha)^2}\right). \end{aligned} \tag{5.21}$$

The result is sharp for the functions $f_j(z)\,(j=1,2)$ *given by* (5.4).

Remark 4. (i) Putting $n=0$, $p=1$, and replacing A by βA and B by βB $(0<\beta\le 1)$ in the above results (Theorems 10, 11 and 12), we obtain the corresponding results obtained by Srivastava et al. ([12, Theorems 8, 9 and 10, respectively]);

(ii) Putting $n=0$, $p=1$, $A=1$ and $B=-1$ in the above results (Theorems 10, 11 and 12), we obtain new results for the class studied by Uralegaddi and Ganigi [13].

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