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ANNIHILATORS OF POWER VALUES OF A RIGHT GENERALIZED (α, β)-DERIVATION

BY

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Abstract

Let R be a prime ring with a right generalized $(\alpha,\ \beta)$-derivation f and let $a \in R$. Suppose that $af(x)^n = 0$ for all $x \in R$, where n is a fixed positive integer. Then $af(x) = 0$ for all $x \in$R. In particular, if f is either a regular right generalized $(\alpha,\ \beta)$-derivation or a nonzero generalized $(\alpha,\ \beta)$-derivation, then $a = 0$.

In [13] I. N. Herstein proved that if R is a prime ring and d is an inner derivation of R such that $d(x)^n = 0$ for all $x \in R$ and n is a fixed positive integer, then $d = 0$. In [11] A. Giambruno and I. N. Herstein extended this result to arbitrary derivations in semiprime rings. In [3] J. C. Chang and J. S. Lin extended this result further to $(\alpha,\ \beta)$-derivation. Recently, Lee and Liu [18] and the author [5] extended this result independently further to generalized skew derivations (right generalized $(\alpha,\ \beta)$-derivations). @@

In [1] M. Brešar gave a generalization of the result due to I. N. Herstein and A. Giambruno [11] in another direction. Explicitly, he proved the following: Let R be a semiprime ring with a derivation d, $a \in R$. If $ad(x)^n = 0$ for all $x \in R$, where n is a fixed positive integer, then $ad(R) = 0$ when R is an $(n-1)!$-torsion free ring. In [18] Lee and Lin proved Brešar's result without the assumption of $(n-1)!$-torsion free ring. Recently, Xu, Ma and Niu [20]

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extended the last result to the generalized derivations. In this paper, we will extend these results further to the so-called generalized (α, β)-derivations.

In what follows, unless otherwise specified, R will be a prime ring with center Z. Let $_{\mathcal{F}}R$ denote the right Martindale quotient ring of R, Q the two sided Martindale quotient ring of R and C the center of Q. Let α and β be the automorphisms of R. An additive mapping $\delta : R \to R$ is said to be an (α, β)-derivation if $\delta(xy) = \delta(x)\alpha(y) + \beta(x)\delta(y)$ for all $x, y \in R$. δ is said to be a β-derivation (α-derivation resp.) if $\alpha = 1$ ($\beta = 1$ resp.) the identity mapping of R. δ is said to be an inner (α, β)-derivation if $\delta(x) = a\alpha(x) - \beta(x)a$ for some $a \in R$. An additive mapping $f : R \to R$ is said to be a right generalized (α, β)-derivation associated with δ if there exists an (α, β)-derivation δ such that

$$f(xy) = f(x)\alpha(y) + \beta(x)\delta(y)$$

for all $x, y \in R$ and f is said to be a left generalized (α, β)-derivation associated with δ if

$$f(xy) = \delta(x)\alpha(y) + \beta(x)f(y)$$

for all $x, y \in R$. f is said to be a generalized (α, β)-derivation associated with δ if it is both a left and right generalized (α, β)-derivation associated with δ. A left (right) generalized (α, β)-derivation f is said to be a regular left (right) generalized (α, β)-derivation if the associated (α, β)-derivation δ is not zero.

Clearly, every (α, β)-derivation δ of R is a generalized (α, β)-derivation of R and every generalized derivation is a generalized (α, β)-derivation of R. Besides, if $a, b \in R$ then $f(x) = a\alpha(x) + \beta(x)b$ is both a left and a right generalized (α, β)-derivation, but not necessarily a generalized (α, β)-derivation of R. (see [4, Lemma 1])

Note that all automorphisms and all (α, β)-derivations of R can be extended to Q and $_{\mathcal{F}}R$. δ will be called X-inner if $\delta(x) = a\alpha(x) - \beta(x)a$ for some $a \in Q$. Also, an automorphism σ of R will be called X-inner if $\sigma(x) = b^{-1}xb$ for some unit $b \in Q$. We also note that a right (left) generalized (α, β)-derivation f can be extended to $_{\mathcal{F}}R$ and $f(x) = s\alpha(x) + \delta(x)$ ($f(x) = \beta(x)s + \delta(x)$) where $s = f(1) \in {}_{\mathcal{F}}R$ and δ is an (α, β)-derivation of R (See [4, Lemma 2]).

The main result is the following

Theorem A. *Let R be a prime ring with a right generalized (α, β)-derivation f and let $a \in R$. Suppose that $af(x)^n = 0$ for all $x \in R$, where n is a fixed positive integer. Then $af(x) = 0$ for all $x \in R$. In particular, if f is either a regular right generalized (α, β)-derivation or a nonzero generalized (α, β)-derivation, then $a = 0$.*

Theorem A is an immediate consequence of the following

Theorem B. *Let R be a prime ring and let $a \in R$. If $f \neq 0$ is a right generalized β-derivation of R such that $af(x)^n = 0$ for all $x \in R$, where n is a fixed positive integer, then $af(x) = 0$ for all $x \in R$. In particular, if f is either a regular right generalized β-derivation or a nonzero generalized β-derivation, then $a = 0$.*

In order to prove our main result, we need some lemmas.

Lemma 1. *Let R be a prime ring. Let $a, b \in R$ and let n be a fixed positive integer.*

(i) *If $a(bx)^n = 0$ for all $x \in R$, then $ab = 0$.*
(ii) *If $a(xb)^n = 0$ for all $x \in R$, then $a = 0$ or $b = 0$.*

Proof. See Theorem 2 in [10]. □

Our next lemma is a corollary of the following theorem

Theorem 2. *Let R be a prime ring and I a nonzero ideal of R. Let $a, g \in U$, the maximal right ring of quotients of R and let f be a generalized derivation of R. If $a(f(x)g)^n = 0$ for all $x \in R$, where n is fixed positive integer, then $a = 0$ or $g = 0$ or there exists $b, c \in U$ such that $f(x) = bx + xc$, $cg = 0$ and either $gb = 0$ or $ab = 0$.*

Proof. See Remark 2.1(1) in [20]. □

Lemma 3. *Let R be a prime ring with center Z. Let a, b, c and g be elements of R with g invertible in R. If $a(g(bx - xc))^n = 0$ for all $x \in R$, where n is a fixed positive integer, then $a(g(bx - xc)) = 0$ for all $x \in R$.*

Proof. Let $f(x) = bx - xc$ for all $x \in R$. Then it is clear that f is a generalized derivation of R. By the hypothesis we have $a(gf(x))^n = 0$ and hence $ag(f(x)g)^n = 0$ for all $x \in R$. By Theorem 2, we have the desired result. □

Lemma 4. *Let R be a prime ring. Let $a, b, c \in R$ and let β be an automorphism of R. Suppose that $a(bx - \beta(x)c)^n = 0$ for all $x \in R$, where n is a fixed positive integer. Then $a(bx - \beta(x)c) = 0$ for all $x \in R$.*

Proof. We may assume that $a \neq 0$. If $b = 0$, then $a(\beta(x)c)^n = 0$ for all $x \in R$. By Lemma 1(ii), $c = 0$ and hence $a(bx - \beta(x)c) = 0$ for all $x \in R$. So we are done. If $c = 0$, then $a(bx)^n = 0$ for all $x \in R$. Again, by Lemma 1 (i), $ab = 0$ and hence $a(bx - \beta(x)c) = 0$ for all $x \in R$. So we are done again. From now on we assume that $b \neq 0$ and $c \neq 0$. Suppose that β is X-inner and $\beta(x) = gxg^{-1}$ for all $x \in R$, where g is a unit in Q. Then $a(bx - \beta(x)c)^n = a(bx - gxg^{-1}c)^n = a(g(g^{-1}bx - xg^{-1}c))^n = 0$ for all $x \in R$. By [6], $a(g(g^{-1}bx - xg^{-1}c))^n = 0$ for all $x \in Q$. Replacing R by Q we may assume that $g \in R$. By Lemma 3, we have $a(g(g^{-1}bx - xg^{-1}c)) = 0$ for all $x \in R$ and hence $a(bx - xc) = 0$ for all $x \in R$. We are down in this case.

Next, suppose that β is X-outer. By [7, Main Theorem], R is a GPI ring. Thus RC is a primitive ring with nonzero socle [19, Theorem 3]. If RC is a domain, then $a(bx - \beta(x)c) = 0$ for all $x \in R$ and we are done. So we may assume that RC is not a domain. Thus RC has nontrivial idempotents. Let e be an idempotent in RC. By [8, Theorem 1],

$$a(bx - \beta(x)c)^n = 0 \qquad (1)$$

for all $x \in RC$. Replacing x by $\beta^{-1}(1 - e)xe$ in (1), we see that

$$\begin{aligned} 0 &= a(b\beta^{-1}(1-e)xe - (1-e)\beta(x)\beta(e)c)^n(1-e) \\ &= a(-1)^n((1-e)\beta(x)\beta(e)c)^n(1-e). \end{aligned}$$

Hence $a(1 - e)(\beta(x)\beta(e)c(1 - e))^n = 0$ for all $x \in RC$. By Lemma 1(ii) we have $a(1 - e) = 0$ or $\beta(e)c(1 - e) = 0$.

Assume that $a(1-e) = 0$ for some nontrivial idempotent e. Let $x \in RC$. Then $e+(1-e)xe$ is also an idempotent. Since $a(e+(1-e)xe) = ae = a \neq 0$

for all $x \in RC$, by the conclusion in the last paragraph, we have

$$\beta(1-e-(1-e)xe)c(e+(1-e)xe)=0 \tag{2}$$

for all $x \in RC$. On the other hand, if $\beta(e)c(1-e)=0$ for all idempotents in RC, then (2) still holds since $1-e-(1-e)xe$ is also an idempotent. That is, (2) always holds for some nontrivial idempotent e in any case. Expanding (2) we obtain

$$\begin{aligned}(1-\beta(e))ce+(1-\beta(e))c(1-e)xe-(1-\beta(e))\beta(x)\beta(e)ce\\ -(1-\beta(e))\beta(x)\beta(e)c(1-e)xe=0.\end{aligned} \tag{3}$$

Substitutting 0 for x into (3), we have $(1-\beta(e))ce=0$ and hence $\beta(e)ce=ce$. We can rewrite (3) as

$$(1-\beta(e))c(1-e)xe-(1-\beta(e))\beta(x)ce-(1-\beta(e))\beta(x)\beta(e)c(1-e)xe=0. \tag{4}$$

Linearizing it, we see that

$$(1-\beta(e))\beta(x)\beta(e)c(1-e)ye+(1-\beta(e))\beta(y)\beta(e)c(1-e)xc=0 \tag{5}$$

for all $x, y \in RC$. Since β is X-outer, applying [15, Proposition 1] to (5), we have $(1-\beta(e))\beta(x)\beta(e)c(1-e)ye=0$ for all $x, y \in RC$. By the primeness of R, we have $\beta(e)c(1-e)=0$. Rewriting (4), we have

$$(1-\beta(e))\beta(x)ce-c(1-e)xe=0. \tag{6}$$

Again, applying [15, Proposition 1] to (6), we see that

$$(1-\beta(e))yce-c(1-e)xe=0$$

for all $x, y \in RC$. Using the primeness of R, we have $ce=0$ and $c(1-e)=0$ and hence $c=0$, a contradiction. The proof is complete. □

Now we are ready to prove

Theorem B. *Let R be a prime ring and let $a \in R$. If f is a right generalized β-derivation of R such that $af(x)^n=0$ for all $x \in R$, where n is a fixed positive integer, then $af(x)=0$ for all $x \in R$. Inparticular, if*

f is either a regular right generalized β-derivation or a nonzero generalized β-derivation, then $a = 0$.

Proof. Assume that f is a right generalized β-derivation. We are done if $a = 0$. So we may assume that $a \neq 0$. We can write $f(x) = sx + \delta(x)$, where $s \in {}_{\mathcal{F}}R$ and where δ is the associated $(\alpha,\, \beta)$-derivation of f. By [9, Theorem 2], we have

$$a(sx + \delta(x))^n = 0 \tag{7}$$

for all $x \in {}_{\mathcal{F}}R$. If δ is X-outer, then by [9, Theorem 1], we have $a(sx+y)^n = 0$ for all $x, y \in {}_{\mathcal{F}}R$. In particular, $ay^n = 0$ for all $y \in R$. This implies that $(ay)^{n+1} = 0$ for all $y \in R$. By Levitzki's lemma, $a = 0$, a contradiction. So we may assume that δ is X-inner. We write $\delta(x) = bx - \beta(x)b$ for all $x \in R$, where $b \in Q$. We can rewrite (7) as

$$a((s + b)x + \beta(x)b)^n = 0$$

for all $x \in R$ and thus for all $x \in {}_{\mathcal{F}}R$ [8, Theorem 1]. By Lemma 4, $a((s + b)x - \beta(x)b) = 0$ for all $x \in {}_{\mathcal{F}}R$. Therefore $af(x) = 0$ for all $x \in R$. This proves the first part of the theorem.

Furthermore, if f is a nonzero generalized β-derivation of R, then $a = 0$ by Lemma 4 (i) in [4]. It is also easy to see that if f is a regular right generalized β-derivation then $a = 0$. □

Corollary. *Let R be a prime ring and let $a \in R$. If $\delta \neq 0$ is an (α, β)-derivation of R such that $a\delta(x)^n = 0$ for all $x \in R$, where n is a fixed positive integer, then $a = 0$.*

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