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ON A FUNCTIONAL EQUATION ASSOCIATED WITH THE TRAPEZOIDAL RULE

BY

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Abstract

In this paper, we find the solution $f_1, f_2, f_3, f_4, f_5, g_1 \colon \mathbb{R} \to \mathbb{R}$ of $f_1(y) - g_1(x) = (y-x)\,[f_2(x) + f_3\,(sx+ty) + f_4\,(tx+sy) + f_5(y)]$ for all real numbers x and y. Here s and t are any two *a priori* chosen real parameters. This functional equation is a generalization of a functional equation that arises in connection with the trapezoidal rule for the numerical evaluation of definite integrals and is a generalization of a functional equation studied in [10].

1. Introduction

Let $\mathbb{R}$ be the set of all real numbers. The trapezoidal rule is an elementary numerical method for evaluating a definite integral $\int_a^b f(t)\,dt$. The method consists of partitioning the interval $[a,b]$ into subintervals of equal lengths and then interpolating the graph of f over each subinterval with a linear function. If $a = x_o < x_1 < x_2 < \cdots < x_n = b$ is a partition of $[a,b]$ into n subintervals, each of length $\frac{b-a}{n}$, then

$$\int_a^b f(t)\,dt \simeq \frac{b-a}{2n}\left[f(x_o) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

This approximation formula is called the trapezoidal rule. It is well known that the error bound for trapezoidal rule approximation is

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2n}\left[f(x_o) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right]\right| \le \frac{K\,(b-a)^3}{12\,n^2}$$

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where $K = \sup\left\{\left|f^{(2)}(x)\right| \mid x \in [a,b]\right\}$. It is easy to note from this inequality that if f is two times continuously differentiable and $f^{(2)}(x) = 0$, then

$$\int_a^b f(t)\,dt = \frac{b-a}{2n}\left[f(x_o) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right].$$

This is obviously true if $n = 3$ and it reduces to

$$\int_a^b f(t)\,dt = \frac{b-a}{6}\left[f(x_o) + 2f(x_1) + 2f(x_2) + f(x_3)\right].$$

Letting $a = x$, $b = y$, $x_1 = \frac{2x+y}{3}$ and $x_2 = \frac{x+2y}{3}$ in the above formula, we obtain

$$\int_x^y f(t)\,dt = \frac{y-x}{6}\left[f(x) + 2f\left(\frac{2x+y}{3}\right) + 2f\left(\frac{x+2y}{3}\right) + f(y)\right]. \tag{1}$$

This integral equation (1) holds for all $x, y \in \mathbb{R}$ if f is a polynomial of degree at most one. However, it is not obvious that if (1) holds for all $x, y \in \mathbb{R}$, then the only solution f is the polynomial of degree one. The integral equation (1) leads to the functional equation

$$g(y) - g(x) = \frac{y-x}{6}\left[f(x) + 2f\left(\frac{2x+y}{3}\right) + 2f\left(\frac{x+2y}{3}\right) + f(y)\right] \tag{2}$$

where g is an antiderivative of f. The above equation is a special case of the functional equation

$$f_1(y) - g_1(x) = (y-x)\left[f_2(x) + f_3(sx+ty) + f_4(tx+sy) + f_5(y)\right] \tag{3}$$

where s, t are two real *a priori* chosen parameters, and $f_1, f_2, f_3, f_4, f_5, g_1 : \mathbb{R} \to \mathbb{R}$ are unknown functions.

It should be noted that if we consider $n = 2$ in the approximation formula, then the functional equation

$$g(y) - g(x) = \left(\frac{y-x}{4}\right)\left[f(x) + 2f\left(\frac{x+y}{2}\right) + f(y)\right]$$

arrises analogously and it is a special case of

$$g(y) - g(x) = (y-x)\left[\phi(x) + \psi(y) + h(sx+ty)\right].$$

This functional equation was treated by Kannappan, Riedel and Sahoo [6] (also see [9]) without any regularity conditions. Interested reader should see [1-9, 11-12] for related functional equations whose solutions are polynomials.

The present paper is a continuation of the author's works in [10]. In this paper, our goal is to determine the general solution of the functional equation (3) assuming the unknown functions $g_1, f_1, f_2, f_5 : \mathbb{R} \to \mathbb{R}$ to be twice differentiable and $f_3, f_4 : \mathbb{R} \to \mathbb{R}$ to be four-time differentiable.

2. Some Auxiliary Results

The following result from [10] will be instrumental in solving the functional equation (3).

Lemma 1. *Let s and t be any two a priori chosen real parameters. Suppose $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are twice differentiable and $k : \mathbb{R} \to \mathbb{R}$ is four time differentiable. The functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (3), *that is*

$$g(y) - h(x) = (y - x)\left[f(x) + 2k\left(sx + ty\right) + 2k\left(tx + sy\right) + f(y)\right]$$

for all $x, y \in \mathbb{R}$ if and only if $h(x) = g(x)$ and

$$g(x) = \begin{cases} ax^2 + bx + c & \text{if } s = 0 = t \\ ax^2 + bx + c & \text{if } s = 0,\ t \neq 0 \\ ax^2 + bx + c & \text{if } s \neq 0,\ t = 0 \\ 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha & \text{if } s = t \neq 0 \\ 2ax^3 + cx^2 + (2\beta - d)x + \alpha & \text{if } s = -t \neq 0 \\ 2\sum_{i=2}^{3} a_i\, ist\left[s^{i-2} + t^{i-2}\right]x^{i+1} & \\ \quad + 2\sum_{i=0}^{1}[b_i + (s^i + t^i)a_i]x^{i+1} + 2b_0\, x + c_0 & \text{if } s^2 \neq t^2,\ (s-t)^2 \neq st \\ 2\sum_{i=2}^{5} a_i\, ist\left[s^{i-2} + t^{i-2}\right]x^{i+1} & \\ \quad + 2\sum_{i=0}^{1}[b_i + (s^i + t^i)a_i]x^{i+1} + 2b_0 x + c_0 & \text{if } s^2 \neq t^2,\ (s-t)^2 = st \end{cases}$$

$$f(x)=\begin{cases} ax+\frac{b-4\eta(0)}{2} & \text{if } s=0=t\\ ax+\frac{b}{2}-2\eta(tx) & \text{if } s=0,\ t\neq 0\\ ax+\frac{b}{2}-2\eta(sx) & \text{if } s\neq 0,\ t=0\\ 2ax^3+bx^2+(c-d)x+\beta & \text{if } s=t\neq 0\\ 3ax^2+cx+\beta & \text{if } s=-t\neq 0\\ 2\sum_{i=2}^{3}a_i\big[ist(s^{i-2}+t^{i-2})-(s^i+t^i)\big]x^i+2b_1\,x+2b_0 & \text{if } s^2\neq t^2, (s-t)^2\neq st\\ 2\sum_{i=2}^{5}a_i\big[ist(s^{i-2}+t^{i-2})-(s^i+t^i)\big]x^i+2b_1x+2b_0 & \text{if } s^2\neq t^2, (s-t)^2=st \end{cases}$$

$$k(x)=\begin{cases} \eta(x) & \text{if } s=0=t\\ \eta(x) & \text{if } s=0,\ t\neq 0\\ \eta(x) & \text{if } s\neq 0,\ t=0\\ \frac{a}{4}\left(\frac{x}{s}\right)^3+\frac{b}{4}\left(\frac{x}{s}\right)^2+\frac{1}{4}\delta\,\frac{x}{s}+\frac{d}{4} & \text{if } s=t\neq 0\\ -\frac{a}{2}\left(\frac{x}{s}\right)^2-\frac{d}{2}-k(-x) & \text{if } s=-t\neq 0\\ \sum_{i=0}^{3}a_i\,x^i & \text{if } s^2\neq t^2,\ (s-t)^2\neq st\\ \sum_{i=0}^{5}a_i\,x^i & \text{if } s^2\neq t^2,\ (s-t)^2=st \end{cases}$$

where $\eta:\mathbb{R}\to\mathbb{R}$ is an arbitrary function, and $a_i\,(i=0,1,2,\dots,5), b_i\,(i=0,1), a,\ b,\ c,\ d,\ c_0,\ \alpha,\ \beta,\ \delta$ are arbitrary real constants.

Lemma 2. *Let s and t be any two a priori chosen real parameters. Suppose $\phi:\mathbb{R}\to\mathbb{R}$ is twice differentiable. The functions $\phi,\psi:\mathbb{R}\to\mathbb{R}$ satisfy the functional equation*

$$(y-x)\ [\psi(x)+\phi\,(sx+ty)-\phi\,(tx+sy)-\psi(y)]=0 \tag{4}$$

for all $x,y\in\mathbb{R}$ if and only if

$$\psi(x)\ =\ \begin{cases} -\,\omega(tx)+\alpha+\beta & \text{if } s=0,\ t\neq 0\\ -\,\omega(sx)+\alpha+\beta & \text{if } s\neq 0,\ t=0\\ \alpha & \text{if } s=t\\ a\,x+\alpha & \text{if } s=-t\neq 0\\ a\,(t^2-s^2)\,x^2+b\,(t-s)\,x+\alpha & \text{if } s^2\neq t^2 \end{cases}$$

$$\phi(x) \;=\; \begin{cases} \omega(x) & \text{if } s=0,\ t\neq 0 \\ \omega(x) & \text{if } s\neq 0,\ t=0 \\ \omega(x) & \text{if } s=t \\ -\,a\,\frac{x}{s}+\phi(-x) & \text{if } s=-t\neq 0 \\ a\,x^2+b\,x+c & \text{if } s^2\neq t^2 \end{cases}$$

where $\omega:\mathbb{R}\to\mathbb{R}$ *is an arbitrary function, and* a,b,c,α,β *are arbitrary real constants.*

Proof. From (4) we have

$$\psi(x)+\phi\,(sx+ty)=\phi\,(tx+sy)+\psi(y) \tag{5}$$

for all $x,y\in\mathbb{R}$ with $x\neq y$. It is easy to see that (5) also holds in the case $x=y$.

Letting $y=0$ in (5), we obtain

$$\psi(x)=\phi(tx)-\phi(sx)+\alpha \tag{6}$$

where α is a constant given by $\alpha=\psi(0)$. Letting (6) into (5), we see that

$$\phi(sx+ty)-\phi(sx)-\phi(ty)=\phi(sy+tx)-\phi(tx)-\phi(sy) \tag{7}$$

for all $x,y\in\mathbb{R}$ with $x\neq y$.

Now we consider several cases.

Case 1. Suppose $s=0$ and $t\neq 0$. Then from (6), we have

$$\psi(x)=\phi(tx)+\beta+\alpha \tag{8}$$

where the constant β is given by $\beta=-\,\phi(0)$. In this case, letting this $\psi(x)$ in (8) into (5), we see that $\psi(x)$ is a solution for any arbitrary function $\phi(x)$.

Case 2. Suppose $s\neq 0$ and $t=0$. This is case symmetric to Case 1 and hence we have

$$\psi(x)=\phi(sx)+\beta+\alpha \quad \text{and} \quad \phi(x)=\omega(x) \tag{9}$$

where the constant β is given by $\beta=-\,\phi(0)$ and $\omega(x)$ is an arbitrary function.

Case 3. Suppose $s = t$. Then from (6), we have

$$\psi(x) = \alpha \tag{10}$$

where the constant α is given by $\alpha = \psi(0)$. In this case, letting this $\psi(x)$ in (9) into (5), we see that $\psi(x)$ is a solution for any arbitrary function $\phi(x)$.

Case 4. Suppose $s = -t \neq 0$. Then from (6), we have

$$\psi(x) = \phi(-sx) - \phi(sx) + \alpha \tag{11}$$

where the constant α is given by $\alpha = \psi(0)$. From (7), we have

$$\phi(s(x-y)) - \phi(-s(x-y)) = \phi(sx) - \phi(-sx) - (\phi(sy) - \phi(-sy)) \tag{12}$$

for all $x, y \in \mathbb{R}$ with $x \neq y$. Defining

$$A(x) = \phi(sx) - \phi(-sx) \tag{13}$$

for all $x \in \mathbb{R}$, we have from (13)

$$A(x-y) = A(x) - A(y) \tag{14}$$

for all $x, y \in \mathbb{R}$ with $x \neq y$. Letting $x = 0$ in (14), we obtain

$$A(-y) = -A(y). \tag{15}$$

Replacing y by $-y$ in (14) and using (15) we have

$$A(x+y) = A(x) - A(-y) = A(x) + A(y) \tag{16}$$

for all $x, y \in \mathbb{R}$. Hence $A : \mathbb{R} \to \mathbb{R}$ is an additive function. Since ϕ is differentiable, $A : \mathbb{R} \to \mathbb{R}$ is also differentiable and hence

$$A(x) = ax \tag{17}$$

where a is an arbitrary constant. From (13) and (17) we have

$$\phi(x) = a\,\frac{x}{s} + \phi(-x) \tag{18}$$

for all $x \in \mathbb{R}$, and from (11) we obtain

$$\psi(x) = -ax + \alpha. \tag{19}$$

Replacing a by $-a$, we have the asserted solution

$$\phi(x) = -a\,\frac{x}{s} + \phi(-x) \quad \text{and} \quad \psi(x) = ax + \alpha.$$

Case 5. Suppose $s^2 \neq t^2 \neq 0$. Differentiating (7) twice, first with respect to x and then with respect to y, we obtain

$$\phi''(sx + ty) = \phi''(sy + tx) \tag{20}$$

for all $x, y \in \mathbb{R}$. Since $s^2 \neq t^2$, letting $u = sx + ty$ and $v = sy + tx$ we see that u and v are linearly independent and (20) yields

$$\phi''(u) = \phi''(v) \tag{21}$$

for all $u, v \in \mathbb{R}$. Hence $\phi''(u) = 2a$, where a is a constant. Integrating we have

$$\phi(x) = a\,x^2 + b\,x + c \tag{22}$$

where b, c are constants of integration. Using (22) in (6), we obtain

$$\psi(x) = a\,(t^2 - s^2)\,x^2 + b\,(t - s)\,x + c. \tag{23}$$

Letting $\phi(x)$ in (22) and $\psi(x)$ in (23) into (5), we see that $\phi(x)$ and $\psi(x)$ satisfy the functional equation with arbitrary constants a, b, c. □

Remark. For the case $s = -t$, the unknown function $\phi(x)$ could not be determined explicitly. We have found the explicit form of $\phi(x) - \phi(-x)$. As the referee noticed, in this case the unknown function $\phi(x)$ is an arbitrary function satisfying $\phi(x) - \phi(-x) = -a\,\frac{x}{s}$. One can rephrase in this way to avoid explaining "ignotum per ignotum" but it is basically the same as what we have in the lemma.

3. Main Result

Now we present the solution of the functional equation (3).

Theorem 1. *Let s and t be any two* a priori *chosen real parameters. Suppose $g_1, f_1, f_2, f_5 : \mathbb{R} \to \mathbb{R}$ are twice differentiable and $f_3, f_4 : \mathbb{R} \to \mathbb{R}$ are four time differentiable. The functions $g_1, f_1, f_2, f_3, f_4, f_5 : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (3)*, that is*

$$f_1(y) - g_1(x) = (y - x)\,[f_2(x) + f_3\,(sx + ty) + f_4\,(tx + sy) + f_5(y)]$$

for all $x, y \in \mathbb{R}$ if and only if $g_1(x) = f_1(x)$ and

$$f_1(x) \;=\; \begin{cases} \frac{1}{2}\,(ax^2 + bx + c) & \text{if } s = 0 = t \\ \frac{1}{2}\,(ax^2 + bx + c) & \text{if } s = 0,\ t \neq 0 \\ \frac{1}{2}\,(ax^2 + bx + c) & \text{if } s \neq 0,\ t = 0 \\ \frac{1}{2}\,(3ax^4 + 2bx^3 + cx^2 + (2\beta + d)x + \alpha), & \text{if } s = t \neq 0 \\ \frac{1}{2}\,\big(2a\,x^3 + c\,x^2 + (2\beta - d)\,x + \alpha\big) & \text{if } s = -t \neq 0 \\ 3cst(s+t)x^4 + 4dstx^3 + [\gamma + e\,(s+t)]x^2 & \\ \qquad +[\gamma + 2\alpha]x + \delta\,x + \varepsilon & \text{if } s^2 \neq t^2,\ (s-t)^2 \neq st \\ 5ast(s^3+t^3)x^6 + 4bst(s^2+t^2)x^5 & \\ \qquad +3cst(s+t)x^4 + 4dstx^3 & \\ \qquad +[\gamma + e\,(s+t)]x^2 + [\gamma + 2\alpha]x & \\ \qquad +\delta\,x + \varepsilon & \text{if } s^2 \neq t^2,\ (s-t)^2 = st \end{cases}$$

$$f_2(x) \;=\; \begin{cases} \frac{1}{2}\,\left(ax + \alpha + \frac{b - 4\eta(0)}{2}\right) & \text{if } s = 0 = t \\ \frac{1}{2}\,\big(ax + \frac{b}{2} - 2\,\eta(tx) + \omega(tx) + \alpha + \beta\big) & \text{if } s = 0,\ t \neq 0 \\ \frac{1}{2}\,\big(ax + \frac{b}{2} - 2\,\eta(sx) + \omega(sx) + \alpha + \beta\big) & \text{if } s \neq 0,\ t = 0 \\ \frac{1}{2}\,\big(2ax^3 + bx^2 + (c - \delta)x + \beta + \alpha\big) & \text{if } s = t \neq 0 \\ \frac{1}{2}\,\big(3ax^2 + cx + \beta + bx + e\big) & \text{if } s = -t \neq 0 \\ c\,[3st(s+t) - (s^3+t^3)]x^3 & \\ \qquad +d\,[4st - (s^2+t^2)]\,x^2 + \gamma\,x + \delta & \\ \qquad +\frac{1}{2}\big[A\,(t^2 - s^2)\,x^2 + B\,(t-s)\,x + D\big] & \text{if } s^2 \neq t^2,\ (s-t)^2 \neq st \\ a\,[5st(s^3+t^3) - (s^5+t^5)]\,x^5 & \\ \qquad +b\,[4st(s^2+t^2) - (s^4+t^4)]x^4 & \\ \qquad +c\,[3st(s+t) - (s^3+t^3)]x^3 & \\ \qquad +d\,[4st - (s^2+t^2)]\,x^2 + \gamma\,x + \delta & \\ \qquad +\frac{1}{2}\big[A\,(t^2 - s^2)\,x^2 + B\,(t-s)\,x + D\big] & \text{if } s^2 \neq t^2,\ (s-t)^2 = st \end{cases}$$

$$f_3(x) \;=\; \begin{cases} \frac{1}{2}\left(2\,\eta(x)+\omega(x)\right) & \textit{if } s=0=t \\ \frac{1}{2}\left(2\,\eta(x)+\omega(x)\right) & \textit{if } s=0,\ t\neq 0 \\ \frac{1}{2}\left(2\,\eta(x)+\omega(x)\right) & \textit{if } s\neq 0,\ t=0 \\ \frac{a}{4}\left(\frac{x}{s}\right)^3+\frac{b}{4}\left(\frac{x}{s}\right)^2+\frac{\delta}{4}\,\frac{x}{s}+\frac{d}{4}+\frac{1}{2}\,\omega(x) & \textit{if } s=t\neq 0 \\ \frac{1}{2}\Big(-\ a\ \left(\frac{x}{s}\right)^2-d-b\,\frac{x}{s}\Big)-f_4(-x) & \textit{if } s=-t\neq 0, \\ c\,x^3+dx^2+ex+\alpha+\frac{1}{2}(A\,x^2+B\,x+C) & \textit{if } s^2\neq t^2,\ (s-t)^2\neq st \\ a\,x^5+b\,x^4+c\,x^3+dx^2+ex+\alpha & \\ \qquad +\frac{1}{2}(A\,x^2+B\,x+C) & \textit{if } s^2\neq t^2,\ (s-t)^2=st \end{cases}$$

$$f_4(x) \;=\; \begin{cases} \frac{1}{2}\left(2\,\eta(x)-\omega(x)\right) & \textit{if } s=0=t \\ \frac{1}{2}\left(2\,\eta(x)-\omega(x)\right) & \textit{if } s=0,\ t\neq 0 \\ \frac{1}{2}\left(2\,\eta(x)-\omega(x)\right) & \textit{if } s\neq 0,\ t=0 \\ \frac{a}{4}\left(\frac{x}{s}\right)^3+\frac{b}{4}\left(\frac{x}{s}\right)^2+\frac{\delta}{4}\,\frac{x}{s}+\frac{d}{4}-\frac{1}{2}\,\omega(x) & \textit{if } s=t\neq 0 \\ \frac{1}{2}\Big(-\ a\ \left(\frac{x}{s}\right)^2-d+b\,\frac{x}{s}\Big)-f_3(-x) & \textit{if } s=-t\neq 0 \\ c\,x^3+dx^2+ex+\alpha-\frac{1}{2}(A\,x^2+B\,x+C) & \textit{if } s^2\neq t^2,\ (s-t)^2\neq st \\ a\,x^5+b\,x^4+c\,x^3+dx^2+ex+\alpha & \\ \qquad -\frac{1}{2}(A\,x^2+B\,x+C) & \textit{if } s^2\neq t^2,\ (s-t)^2=st \end{cases}$$

$$f_5(x) \;=\; \begin{cases} \frac{1}{2}\ \left(ax-\alpha+\frac{b-4\eta(0)}{2}\right) & \textit{if } s=0=t \\ \frac{1}{2}\ \left(ax+\frac{b}{2}-2\,\eta(tx)-\omega(tx)-\alpha-\beta\right) & \textit{if } s=0,\ t\neq 0 \\ \frac{1}{2}\ \left(ax+\frac{b}{2}-2\,\eta(sx)-\omega(sx)-\alpha-\beta\right) & \textit{if } s\neq 0,\ t=0 \\ \frac{1}{2}\ \left(2ax^3+bx^2+(c-\delta)x+\beta-\alpha\right) & \textit{if } s=t\neq 0 \\ \frac{1}{2}\ \left(3ax^2+cx+\beta-bx-e\right) & \textit{if } s=-t\neq 0 \\ c\,[3st(s+t)-(s^3+t^3)]x^3 & \\ \qquad +d\,[4st-(s^2+t^2)]\,x^2+\gamma\,x+\delta & \\ \qquad -\frac{1}{2}\big[A\,(t^2-s^2)\,x^2+B\,(t-s)\,x+D\big] & \textit{if } s^2\neq t^2,\ (s-t)^2\neq st \\ a\,[5st(s^3+t^3)-(s^5+t^5)]\,x^5 & \\ \qquad +b\,[4st(s^2+t^2)-(s^4+t^4)]x^4 & \\ \qquad +c\,[3st(s+t)-(s^3+t^3)]x^3 & \\ \qquad +d\,[4st-(s^2+t^2)]\,x^2+\gamma\,x+\delta & \\ \qquad -\frac{1}{2}\big[A\,(t^2-s^2)\,x^2+B\,(t-s)\,x+D\big] & \textit{if } s^2\neq t^2,\ (s-t)^2=st \end{cases}$$

where $\eta, \omega : \mathbb{R} \to \mathbb{R}$ *are arbitrary functions and* $A, B, C, D, a, b, c, d, e, \alpha, \beta, \gamma, \delta, \varepsilon$ *are arbitrary real constants.*

Proof. Letting $x = y$ in (3) we see that $f_1(x) = g_1(x)$ for all $x \in \mathbb{R}$. Hence (3) yields

$$f_1(y) - f_1(x) = (y - x)\left[f_2(x) + f_3(sx + ty) + f_4(tx + sy) + f_5(y)\right] \tag{24}$$

for all $x, y \in \mathbb{R}$. Interchanging x with y in the functional equation (24), we obtain

$$f_1(y) - f_1(x) = (y - x)\left[f_2(y) + f_3(sy + tx) + f_4(ty + sx) + f_5(x)\right] \tag{25}$$

for all $x, y \in \mathbb{R}$. Adding (24) and (25), we get

$$g(y) - g(x) = (y - x)\left[f(x) + 2k(sx + ty) + 2k(tx + sy) + f(y)\right] \tag{26}$$

where

$$\begin{cases} f(x) = f_2(x) + f_5(x) \\ k(x) = \frac{1}{2}\left[f_3(x) + f_4(x)\right] \\ g(x) = 2f_1(x). \end{cases} \tag{27}$$

Similarly, subtracting (25) from (24), we get

$$(y - x)\ \left[\psi(x) + \phi(sx + ty) - \phi(tx + sy) - \psi(y)\right] = 0 \tag{28}$$

for all $x, y \in \mathbb{R}$, where

$$\begin{cases} \psi(x) = f_2(x) - f_5(x) \\ \phi(x) = f_3(x) - f_4(x). \end{cases} \tag{29}$$

Now we consider several cases.

Case 1. Suppose $s = 0 = t$. Then from (27), (29), Lemma 1 and Lemma 2, we obtain

$$\begin{cases} 2f_1(x) = ax^2 + bx + c \\ f_2(x) + f_5(x) = ax + \frac{b - 4\eta(0)}{2} \\ f_3(x) + f_4(x) = 2\,\eta(x) \\ f_2(x) - f_5(x) = \alpha \\ f_3(x) - f_4(x) = \omega(x), \end{cases} \tag{30}$$

where $\eta, \omega : \mathbb{R} \to \mathbb{R}$ are arbitrary functions and a, b, c, α are arbitrary constants. Hence from (30) we have

$$\begin{cases} f_1(x) = \frac{1}{2}\left(ax^2 + bx + c\right) \\ f_2(x) = \frac{1}{2}\left(ax + \alpha + \frac{b - 4\eta(0)}{2}\right) \\ f_3(x) = \frac{1}{2}\left(2\,\eta(x) + \omega(x)\right) \\ f_4(x) = \frac{1}{2}\left(2\,\eta(x) - \omega(x)\right) \\ f_5(x) = \frac{1}{2}\left(ax - \alpha + \frac{b - 4\eta(0)}{2}\right). \end{cases} \tag{31}$$

Case 2. Suppose $s = 0$ and $t \neq 0$. Then from (27), (29), Lemma 1 and Lemma 2, we obtain

$$\begin{cases} 2f_1(x) = ax^2 + bx + c \\ f_2(x) + f_5(x) = ax + \frac{b}{2} - 2\,\eta(tx) \\ f_3(x) + f_4(x) = 2\,\eta(x) \\ f_2(x) - f_5(x) = \omega(tx) + \alpha + \beta \\ f_3(x) - f_4(x) = \omega(x), \end{cases} \tag{32}$$

where $\eta, \omega : \mathbb{R} \to \mathbb{R}$ are arbitrary functions and a, b, c, α, β are arbitrary constants. Hence from (32), we get

$$\begin{cases} f_1(x) = \frac{1}{2}\left(ax^2 + bx + c\right) \\ f_2(x) = \frac{1}{2}\left(ax + \frac{b}{2} - 2\,\eta(tx) + \omega(tx) + \alpha + \beta\right) \\ f_3(x) = \frac{1}{2}\left(2\,\eta(x) + \omega(x)\right) \\ f_4(x) = \frac{1}{2}\left(2\,\eta(x) - \omega(x)\right) \\ f_5(x) = \frac{1}{2}\left(ax + \frac{b}{2} - 2\,\eta(tx) - \omega(tx) - \alpha - \beta\right). \end{cases} \tag{33}$$

Case 3. Suppose $s \neq 0$ and $t = 0$. Then from (27), (29), Lemma 1 and Lemma 2, we obtain

$$\begin{cases} 2f_1(x) = ax^2 + bx + c \\ f_2(x) + f_5(x) = ax + \frac{b}{2} - 2\,\eta(sx) \\ f_3(x) + f_4(x) = 2\,\eta(x) \\ f_2(x) - f_5(x) = \omega(sx) + \alpha + \beta \\ f_3(x) - f_4(x) = \omega(x), \end{cases} \tag{34}$$

where $\eta, \omega : \mathbb{R} \to \mathbb{R}$ are arbitrary functions and a, b, c, α, β are arbitrary constants. Hence from (34), we get

$$\begin{cases} f_1(x) = \frac{1}{2}\left(ax^2 + bx + c\right) \\ f_2(x) = \frac{1}{2}\,\left(ax + \frac{b}{2} - 2\,\eta(sx) + \omega(sx) + \alpha + \beta\right) \\ f_3(x) = \frac{1}{2}\left(2\,\eta(x) + \omega(x)\right) \\ f_4(x) = \frac{1}{2}\left(2\,\eta(x) - \omega(x)\right) \\ f_5(x) = \frac{1}{2}\,\left(ax + \frac{b}{2} - 2\,\eta(sx) - \omega(sx) - \alpha - \beta\right). \end{cases} \tag{35}$$

Case 4. Suppose $s = t \neq 0$. Then from (27), (29), Lemma 1 and Lemma 2, we obtain

$$\begin{cases} 2f_1(x) = 3ax^4 + 2bx^3 + cx^2 + (2\beta + d)x + \alpha \\ f_2(x) + f_5(x) = 2ax^3 + bx^2 + (c - \delta)x + \beta \\ f_3(x) + f_4(x) = 2\left(\frac{a}{4}\left(\frac{x}{s}\right)^3 + \frac{b}{4}\left(\frac{x}{s}\right)^2 + \frac{\delta}{4}\,\frac{x}{s} + \frac{d}{4}\right) \\ f_2(x) - f_5(x) = \alpha \\ f_3(x) - f_4(x) = \omega(x), \end{cases} \tag{36}$$

where $\omega : \mathbb{R} \to \mathbb{R}$ is an arbitrary function and $a, b, c, d, \alpha, \beta, \delta$ are arbitrary constants. Hence from (36), we get

$$\begin{cases} f_1(x) = \frac{1}{2}\left(3ax^4 + 2bx^3 + cx^2 + (2\beta + d)x + \alpha\right) \\ f_2(x) = \frac{1}{2}\,\left(2ax^3 + bx^2 + (c - \delta)x + \beta + \alpha\right) \\ f_3(x) = \frac{a}{4}\left(\frac{x}{s}\right)^3 + \frac{b}{4}\left(\frac{x}{s}\right)^2 + \frac{\delta}{4}\,\frac{x}{s} + \frac{d}{4} + \frac{1}{2}\,\omega(x) \\ f_4(x) = \frac{a}{4}\left(\frac{x}{s}\right)^3 + \frac{b}{4}\left(\frac{x}{s}\right)^2 + \frac{\delta}{4}\,\frac{x}{s} + \frac{d}{4} - \frac{1}{2}\,\omega(x) \\ f_5(x) = \frac{1}{2}\,\left(2ax^3 + bx^2 + (c - \delta)x + \beta - \alpha\right). \end{cases} \tag{37}$$

Case 5. Suppose $s = -t \neq 0$. Then from (27), (29), Lemma 1 and Lemma 2, we obtain

$$\begin{cases} 2f_1(x) = 2a\,x^3 + c\,x^2 + (2\beta - d)\,x + \alpha \\ f_2(x) + f_5(x) = 3ax^2 + cx + \beta \\ f_3(x) + f_4(x) + f_3(-x) + f_4(-x) = -\,a\,\left(\frac{x}{s}\right)^2 - d \\ f_2(x) - f_5(x) = bx + e \\ f_3(x) - f_4(x) - f_3(-x) + f_4(-x) = -\,b\,\frac{x}{s}, \end{cases} \tag{38}$$

where $a, b, c, d, e, \alpha, \beta$ are arbitrary constants. Hence from (38), we get

$$\begin{cases} f_1(x) = \frac{1}{2}\left(2a\,x^3 + c\,x^2 + (2\beta - d)\,x + \alpha\right) \\ f_2(x) = \frac{1}{2}\left(3ax^2 + cx + \beta + bx + e\right) \\ f_3(x) = \frac{1}{2}\left(-\ a\ \left(\frac{x}{s}\right)^2 - d - b\,\frac{x}{s}\right) - f_4(-x) \\ f_4(x) = \frac{1}{2}\left(-\ a\ \left(\frac{x}{s}\right)^2 - d + b\,\frac{x}{s}\right) - f_3(-x) \\ f_5(x) = \frac{1}{2}\left(3ax^2 + cx + \beta - bx - e\right). \end{cases} \tag{39}$$

Case 6. Suppose $s^2 \neq t^2 \neq 0$ and $(s-t)^2 \neq st$. Then from (27), (29), Lemma 1 and Lemma 2, we obtain

$$\begin{cases} 2f_1(x) = 2\big\{3cst(s+t)x^4 + 4dstx^3 + [\gamma + e\,(s+t)]x^2 \\ \qquad\qquad +[\gamma + 2\alpha]x + \delta\,x + \varepsilon\big\} \\ f_2(x) + f_5(x) = 2c\,[3st(s+t) - (s^3+t^3)]x^3 \\ \qquad\qquad +2d\,[4st - (s^2+t^2)]\,x^2 + 2\gamma\,x + 2\delta \\ f_3(x) + f_4(x) = 2\,(c\,x^3 + dx^2 + ex + \alpha) \\ f_2(x) - f_5(x) = A\,(t^2 - s^2)\,x^2 + B\,(t-s)\,x + D \\ f_3(x) - f_4(x) = A\,x^2 + B\,x + C, \end{cases} \tag{40}$$

where $c, d, e, A, B, C, D, \alpha, \beta, \gamma, \delta, \varepsilon$ are arbitrary constants. Hence from (40), we get

$$\begin{cases} f_1(x) = 3cst(s+t)x^4 + 4dstx^3 + [\gamma + e\,(s+t)]x^2 \\ \qquad\quad +[\gamma + 2\alpha]x + \delta\,x + \varepsilon \\ f_2(x) = c\,[3st(s+t) - (s^3+t^3)]x^3 + d\,[4st - (s^2+t^2)]\,x^2 \\ \qquad\quad +\gamma\,x + \delta + \frac{1}{2}\big[A\,(t^2-s^2)\,x^2 + B\,(t-s)\,x + D\big] \\ f_3(x) = c\,x^3 + dx^2 + ex + \alpha + \frac{1}{2}(A\,x^2 + B\,x + C) \\ f_4(x) = c\,x^3 + dx^2 + ex + \alpha - \frac{1}{2}(A\,x^2 + B\,x + C) \\ f_5(x) = c\,[3st(s+t) - (s^3+t^3)]x^3 + d\,[4st - (s^2+t^2)]\,x^2 \\ \qquad\quad +\gamma\,x + \delta - \frac{1}{2}\big[A\,(t^2-s^2)\,x^2 + B\,(t-s)\,x + D\big]. \end{cases} \tag{41}$$

Case 7. Suppose $s^2 \neq t^2 \neq 0$ and $(s-t)^2 = st$. Then from (27), (29), Lemma 1 and Lemma 2, we obtain

$$\begin{cases} 2f_1(x) = 2\big\{5ast(s^3+t^3)x^6 + 4bst(s^2+t^2)x^5 \\ \qquad\qquad +3cst(s+t)x^4 + 4dstx^3 + [\gamma + e\,(s+t)]x^2 \\ \qquad\qquad +[\gamma+2\alpha]x + \delta\,x + \varepsilon\big\} \\ f_2(x)+f_5(x) = 2a\,[5st(s^3+t^3) - (s^5+t^5)]\,x^5 \\ \qquad\qquad +\,2b\,[4st(s^2+t^2) - (s^4+t^4)]x^4 \\ \qquad\qquad +\,2c\,[3st(s+t) - (s^3+t^3)]x^3 \\ \qquad\qquad +2d\,[4st - (s^2+t^2)]\,x^2 + 2\gamma\,x + 2\delta \\ f_3(x)+f_4(x) = 2\,(a\,x^5 + b\,x^4 + c\,x^3 + dx^2 + ex + \alpha) \\ f_2(x) - f_5(x) = A\,(t^2-s^2)\,x^2 + B\,(t-s)\,x + D \\ f_3(x) - f_4(x) = A\,x^2 + B\,x + C, \end{cases} \tag{42}$$

where $a, b, c, d, e, A, B, C, D, \alpha, \beta, \gamma, \delta, \varepsilon$ are arbitrary constants. Hence from (40), we get

$$\begin{cases} f_1(x) = 5ast(s^3+t^3)x^6 + 4bst(s^2+t^2)x^5 + 3cst(s+t)x^4 \\ \qquad +4dstx^3 + [\gamma + e\,(s+t)]x^2 + [\gamma+2\alpha]x + \delta\,x + \varepsilon \\ f_2(x) = a\,[5st(s^3+t^3) - (s^5+t^5)]\,x^5 + \; b\,[4st(s^2+t^2) - (s^4+t^4)]x^4 \\ \qquad +\,c\,[3st(s+t) - (s^3+t^3)]x^3 + d\,[4st - (s^2+t^2)]\,x^2 \\ \qquad +\gamma\,x + \delta + \tfrac{1}{2}\big[A\,(t^2-s^2)\,x^2 + B\,(t-s)\,x + D\big] \\ f_3(x) = a\,x^5 + b\,x^4 + c\,x^3 + dx^2 + ex + \alpha + \tfrac{1}{2}(A\,x^2 + B\,x + C) \\ f_4(x) = a\,x^5 + b\,x^4 + c\,x^3 + dx^2 + ex + \alpha - \tfrac{1}{2}(A\,x^2 + B\,x + C) \\ f_5(x) = a\,[5st(s^3+t^3) - (s^5+t^5)]\,x^5 + \; b\,[4st(s^2+t^2) - (s^4+t^4)]x^4 \\ \qquad +\,c\,[3st(s+t) - (s^3+t^3)]x^3 + d\,[4st - (s^2+t^2)]\,x^2 \\ \qquad +\gamma\,x + \delta - \tfrac{1}{2}\big[A\,(t^2-s^2)\,x^2 + B\,(t-s)\,x + D\big]. \end{cases} \tag{43}$$

Since there are no more cases are left, the proof of the theorem is now complete. □

Problem 1. In Theorem 1, we have assumed that the functions $f_1, f_2, f_5 : \mathbb{R} \to \mathbb{R}$ are twice differentiable and $f_3, f_4 : \mathbb{R} \to \mathbb{R}$ are four time differentiable. The proof of Theorem 1 heavily relies on this differentiability assumption. Thus we pose the following problem: Determine the general solution of the functional equation (3) without any regularity assumptions on the unknown functions f_1, f_2, f_3, f_4, f_5.

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