ON METRIC SPACES IN WHICH METRIC SEGMENTS HAVE UNIQUE PROLONGATIONS

BY

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Abstract

An M-space is a metric space (X, d) having the property that for each pair of points $p, q \in X$ with $d(p, q) = \lambda$ and for each real number $\alpha \in [0, \lambda]$, there is a unique $r_\alpha \in X$ such that $d(p, r_\alpha) = \alpha$ and $d(r_\alpha, q) = \lambda - \alpha$. In an M-space (X, d), we say that metric segments have unique prolongations if points p, q, r, s satisfy $d(p, q) + d(q, r) = d(p, r)$, $d(p, q) + d(q, s) = d(p, s)$ and $d(q, r) = d(q, s)$ then $r = s$.

This paper mainly deals with some results on best approximation in metric spaces for which metric segments have unique prolongations.

Rotundity or strict convexity has been studied extensively in Banach spaces (see e.g., [6]). It is well known that metric lines are unique in a Banach space B if and only if B is strictly convex ([1], [2], [3], [10]). This result is not valid in the metric space setting. There are complete convex, externally convex metric spaces (see [4], [5], [7]) in which the concepts of strict convexity and unique metric lines are not equivalent. However, Freese and Murphy [4], Freese, Murphy and Andalafte [5] and Khalil [7] have shown that unique metric lines (metric segments have unique prolongations) and strict convexity (redefined in purely metric terms) are equivalent in a larger class

of spaces. This paper mainly deals with some results on best approximation in metric spaces for which metric segments have unique prolongations.

Following Khalil [7], we call a metric space (X, d) an *M-space* provided for every two points $p, q \in X$ with $d(p, q) = \lambda$ and every real number $\alpha \in [0, \lambda]$, there is a unique point $r_\alpha \in X$ such that $d(p, r_\alpha) = \alpha$ and $d(r_\alpha, q) = (\lambda - \alpha)$. It is easy to show that M-spaces have unique between points and hence unique metric segments, but do not necessarily have unique metric lines or metric lines at all (see [7]).

A metric space (X, d) is called *strictly convex* [5] or has *strictly convex spheres* [4] provided if $p, q, r \in X$ with $d(p, q) = d(p, r)$, then for every point s metrically between q and r, $d(p, s) < d(p, q)$.

An M-space (X, d) is *externally convex* [5] provided for each two distinct points $p, q \in X$, there exists a point $r \in X$, $r \neq q$ such that $d(p, q) + d(q, r) = d(p, r)$. The space X is called *strongly externally convex* [5] provided for all distinct points $p, q \in X$ such that $d(p, q) = \lambda$ and for $k > \lambda$, there exists a unique point $r \in X$ such that $d(p, q) + d(q, r) = d(p, r) = k$.

If (X, d) is an M-space, we say that metric segments in X have *unique prolongations* [5] if points p, q, r, s satisfy $d(p, q) + d(q, r) = d(p, r)$, $d(p, q) + d(q, s) = d(p, s)$ and $d(q, r) = d(q, s)$ then $r = s$.

A subset V of a metric space (X, d) is said to be *proximinal* provided for each $p \in X$ there is at least one point ν in V, called a *foot* of p on V (or a *best approximation* of p in V), such that $d(p, \nu) = \inf\{d(p, q) : q \in V\}$. If this point ν is also unique for each $p \in X$ then V is called a *Chebyshev set*. The set of all such $\nu \in V$ is denoted by $P_V(p)$. The mapping P_V which takes each element of X into its sets of best approximations in V is called *metric projection*.

A proximinal set V is said to be a *semi-sun* [8] if for each $x \in X \setminus V$ and $r > 0$ there exists $z \in X$ and $\nu \in P_V(z)$ such that $d(z, x) = r$ and $x \in G[\nu, z] \equiv \{u \in X : d(\nu, u) + d(u, z) = d(\nu, z)\}$, the metric segment joining ν and z.

A proximinal set V is called a *sun* [8] if for all $x \in X \setminus V$, there is $\nu \in P_V(x)$, such that $\nu \in P_V(z)$ for all $z \in G(\nu, x, -) \equiv$ the largest line segment joining $G[x, \nu]$ for which x is an extreme point i.e., the ray starting from ν and passing through x.

A normed linear space X is called *strictly convex* (see [6]) if $x, y \in X$, $x \neq 0$, $y \neq 0$ and $\|x + y\| = \|x\| + \|y\|$ imply $x = \lambda y$, $\lambda > 0$. Equivalently, whenever $x, y \in X$, $\|x\| = \|y\| = 1$ and $x \neq y$, then $\left\| \dfrac{x + y}{2} \right\| < 1$.

The following two propositions show that in a strictly convex normed linear space, metric segments have unique prolongations.

Proposition 1. *A strictly convex normed linear space is an M-space.*

Proof. Let X be a strictly convex normed linear space and $z_1, z_2 \in B[x, r] \cap B[y, \lambda - r]$, where $\lambda = d(x, y)$. This gives $\|x - z_1\| \leq r$, $\|x - z_2\| \leq r$, $\|y - z_1\| \leq \lambda - r$, $\|y - z_2\| \leq \lambda - r$. Consider $\|x - y\| \leq \|x - z_1\| + \|z_1 - y\| \leq r + \lambda - r = \lambda = \|x - y\|$. Therefore equality holds throughout and so $\|x - y\| = \|x - z_1\| + \|z_1 - y\|$. By the strict convexity, $z_1 - y = t(x - z_1)$ for some $t > 0$ (see [9], p.4) i.e., $z_1 = \dfrac{1}{1 + t} y + \dfrac{t}{1 + t} x$ i.e., z_1 lies in between x and y. Similarly, z_2 lies in between x and y. Since $\|x - y\| = \|x - z_1\| + \|z_1 - y\|$, $\|x - z_1\| = r$ and $\|y - z_1\| = \lambda - r$. Similarly, $\|x - z_2\| = r$ and $\|y - z_2\| = \lambda - r$. Consider

$$\|z_1 - z_2\| = \Big| \|x - z_2\| - \|x - z_1\| \Big| = |r - r| = 0$$

and so $z_1 = z_2$. $\qquad \square$

Proposition 2. *If X is a strictly convex normed linear space then metric segments in X have unique prolongations.*

Proof. Let $p, q, r, s \in X$ be such that

$$\|p - q\| + \|q - r\| = \|p - r\| \tag{1}$$
$$\|p - q\| + \|q - s\| = \|p - s\| \tag{2}$$

and $\|q - r\| = \|q - s\|$.

Since X is strictly convex, (1) gives $p - q = t(q - r)$ and (2) gives $p - q = \lambda(q - s)$ (see [9], p.4) i.e., $q = \dfrac{1}{1 + t} p + \dfrac{t}{1 + t} r$ and $q = \dfrac{1}{1 + \lambda} p + \dfrac{\lambda}{1 + \lambda} s$ i.e., q lies on the line segment joining p and r and q lies on the line segment joining p and s.

Now $\|r - s\| = \Big| \|q - r\| - \|q - s\| \Big| = 0$ gives $r = s$. $\qquad \square$

Remarks 1. A strictly convex metric space is an M-space [5].

2. In a strictly convex metric space, metric segments need not have unique prolongations. (see [5])

There are some metric spaces in which metric segments have unique prolongations.

Example 1. Let X be the union of two distinct half lines of the Euclidean plane with common origin O. The metric d of x, y is defined to be their Euclidean distance if they lie on the same half line but their distance is defined to be the sun of their respective distances from O if they lie on different half lines.
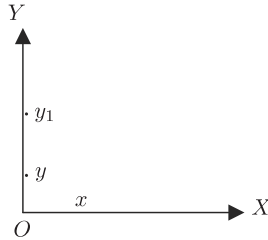

Take any two points x and y on different half lines then $d(x, y) = d(x, 0) + d(0, y)$. Suppose y_1 is such that $d(x, y_1) = d(x, 0) + d(0, y_1)$ together with $d(0, y) = d(0, y_1)$ then y must be equal to y_1.

Other cases are trivial. Hence metric segments in (X, d) have unique prolongations.

The following examples show that in an externally convex metric space, metric segments need not have unique prolongations:

Example 2. Let X be the union of two lines in the cartesian plane whose equations are $y = 1$ and $y = 2$. Let the distance $d(A, B)$ for $A = (x_1, y_1)$, $B = (x_2, y_2)$ be given by $|x_1 - x_2|$ if $y_1 = y_2$ and $1 + |x_1| + |x_2|$ if $y_1 \neq y_2$.
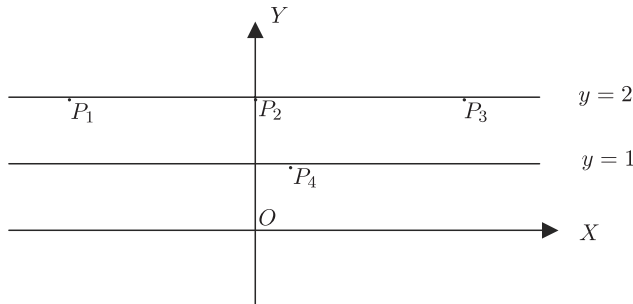
Consider points $P_1 = (-3, 2)$, $P_2 = (0, 2)$, $P_3 = (3, 2)$, $P_4 = (1, 1)$. Then $d(P_1, P_4) = 5 = d(P_1, P_2) + d(P_2, P_4)$, $d(P_3, P_4) = 5 = d(P_3, P_2) + d(P_2, P_4)$, $d(P_1, P_2) = d(P_3, P_2)$ but $P_1 \neq P_3$.

Example 3. Let X be the union of three distinct half lines given by $x = y$, $x = -2y$, $x = 2y$ of the Euclidean plane with common origin O. The metric d of x, y is defined to be their Euclidean distance if they lie on the same half line but their distance is defined to be the sun of their respective distances from O if they lie on different half lines.

Consider points $P_1 = (1, 1)$, $P_2 = (-2, 1)$, $P_3 = (-2, -1)$, $O = (0, 0)$. Then

$$d(P_1, P_2) = d(P_1, O) + d(O, P_2),$$
$$d(P_1, P_3) = d(P_1, O) + d(O, P_3),$$
$$d(O, P_2) = d(O, P_3), \quad \text{but} \quad P_2 \neq P_3.$$

However, we have the following proposition giving another class of metric spaces in which metric segments have unique prolongations:

Proposition 3. *If (X, d) is a metric space with strong external convexity then metric segments in X have unique prolongations.*

Proof. Let $p, q, r, s \in X$ be such that

$$d(p, q) + d(q, r) = d(p, r), \quad d(p, q) + d(q, s) = d(p, s) \qquad (*)$$

and $d(q, r) = d(q, s)$. This gives $d(p, r) = d(p, s) = k$ (say) and $d(p, r) > d(p, q)$ i.e., $k > d(p, q) \equiv \lambda$. Therefore by the strong external convexity of X, there exists unique element satisfying $(*)$ and so $r = s$. $\square$

Next proposition deals with best approximation in metric spaces in which metric segments have unique prolongations.

Proposition 4. *Let (X, d) be a convex metric space, $V \subseteq X$, $x \in X$, $\nu_0 \in P_V(x)$ and $x_\lambda \in G[\nu_0, x]$ then $\nu_0 \in P_V(x_\lambda)$. If (X, d) is an M-space in which metric segments have unique prolongations then $P_V(x_\lambda)$ is a singleton.*

Proof. Consider

$$
\begin{aligned}
d(x_\lambda, \nu_0) &= d(x, \nu_0) - d(x, x_\lambda) \\
&\leq d(x, \nu) - d(x, x_\lambda) \ \text{ for all } \ \nu \in V \\
&\leq d(x, x_\lambda) + d(x_\lambda, \nu) - d(x, x_\lambda) \ \text{ for all } \ \nu \in V \\
&= d(x_\lambda, \nu) \ \text{ for all } \ \nu \in V.
\end{aligned}
$$

This gives $\nu_0 \in P_V(x_\lambda)$.

Now suppose metric segments in X have unique prolongations. Let $\nu_0, \nu_1 \in P_V(x_\lambda)$. Consider

$$
\begin{aligned}
d(x, \nu_1) &\leq d(x, x_\lambda) + d(x_\lambda, \nu_1) \\
&= d(x, x_\lambda) + d(x_\lambda, \nu_0) = d(x, \nu_0) = d(x, \nu_1).
\end{aligned}
$$

Therefore equality holds throughout and so

$$
\begin{aligned}
d(x, \nu_1) &= d(x, x_\lambda) + d(x_\lambda, \nu_1), \\
d(x, \nu_0) &= d(x, x_\lambda) + d(x_\lambda, \nu_0).
\end{aligned}
$$

Also $d(x_\lambda, \nu_1) = d(x_\lambda, \nu_0)$. By unique prolongations, we have $\nu_0 = \nu_1$. $\square$

Corollary 1. *Let* (X, d) *be a strongly externally convex metric space,* $V \subseteq X$, $x \in X$, $\nu_0 \in P_V(x)$ *and* $x_\lambda \in G[\nu_0, x]$ *then* $P_V(x_\lambda) = \{\nu_0\}$.

Corollary 2.([8], Lemma 3, p.371) *Let* V *be a subset of a normed linear space* X *and* $x \in X \setminus \overline{V}$. *If* $\nu_0 \in P_V(x)$ *and* $x_\lambda = \nu_0 + \lambda(x - \nu_0)$ *for some* $\lambda \in [0, 1[$ *is a point of* $]x, \nu_0]$, *then* $\nu_0 \in P_V(x_\lambda)$. *Moreover, if* X *is strictly convex and* $x_\lambda \in]x, \nu_0]$ *then* $P_V(x_\lambda)$ *is the singleton* $\{\nu_0\}$.

Clearly every sun is a semi-sun. However, it is not known whether every semi-sun is a sun. The following proposition gives conditions under which a semi-sun is a sun:

Proposition 5. *If* (X, d) *is an M-space in which metric segments have unique prolongations,* V *is a semi-sun and* P_V *is compact valued (i.e.,* $P_V(x)$ *is compact for each* $x \in X$*) then* V *is a sun.*

Proof. Let $x \in X \setminus V$. Since V is a semi-sun, for each $n \in N$, we can pick $z_n \in X$ and $\nu_n \in P_V(z_n)$ such that $x \in G[\nu_n, z_n]$ and $d(z_n, x) = nd(x, V)$. Then $\nu_n \in P_V(x)$ for all n i.e., $\{\nu_n\} \subseteq P_V(x)$ for all n. Since $P_V(x)$ is compact, there is a subsequence $\{v_{n_k}\}$ converging to ν in $P_V(x)$. Let $z \in G(\nu, x, -)$ be such that $d(z, x) = \lambda d(x, \nu) = \lambda d(x, V)$. For V to be a sun, we show that $\nu \in P_V(z)$.

Let $w_n \in G(\nu_n, x, -)$ be such that $d(w_n, x) = \lambda d(x, \nu_n) = \lambda d(x, V)$. Now $z_n, w_n \in G(\nu_n, x, -)$ are such that $d(z_n, x) = nd(x, V)$ and $d(w_n, x) = \lambda d(x, V)$. For $n \geq \lambda$, $w_n \in G[\nu_n, z_n]$. Now

$$d(w_n, \nu_n) = d(\nu_n, x) + d(x, w_n)$$

implies

$$d(\lim w_{n_k}, \nu) = d(\nu, x) + d(x, \lim w_{n_k})$$

and so, $\lim w_{n_k} \in G(\nu, x, -)$. Also $z \in G(v, x, -)$ and $d(\lim w_{n_k}, x) = \lambda d(x, V) = d(z, x)$. By unique prolongations, we get $\lim w_{n_k} = z$ and $n_k \geq \lambda$ eventually. As $w_{n_k} \in G[\nu_{n_k}, z_{n_k}]$, $\nu_{n_k} \in P_V(w_{n_k})$ eventually i.e., $d(w_{n_k}, \nu_{n_k}) = d(w_{n_k}, V)$. On taking limit, this gives $d(z, \nu) = d(z, V)$ i.e., $\nu \in P_V(z)$ and hence V is a sun. □

Note. For Banach spaces, Proposition 5 is given in [8], p.471.

The following theorem gives conditions under which a sun is a Chebyshev set:

Theorem 1. *If (X, d) is an M-space in which metric segments have unique prolongations then every sun in X is a Chebyshev set.*

Proof. Let V be a sun in X and $x \in X$. Suppose $u_0, u_1 \in P_M(x)$ i.e., $d(x, u_o) = d(x, u_1) = d(x, V)$. Let $x_t \in G(u_0, x, -)$ then $u_0 \in P_V(x_t)$. Consider

$$\begin{aligned}
d(x_t, u_1) &\leq d(x_t, x) + d(x, u_1) \\
&= d(x_t, x) + d(x, u_0) = d(x_t, u_0) \leq d(x_t, u_1)
\end{aligned}$$

and so equality holds throughout. Consequently, $d(x_t, u_1) = d(x_t, u_0)$. Also $d(x_t, u_1) = d(x_t, x) + d(x, u_1)$, $d(x_t, u_0) = d(x_t, x) + d(x, u_0)$. Since metric

segments in X have unique prolongations, $u_0 = u_1$ and hence V is Chebyshev. □

Corollary.([8], p.470) *If X is a strictly convex normed linear space then every sun in X is Chebyshev.*

Combining Proposition 5 and Theorem 1, we get

Theorem 2. *If (X, d) is an M-space in which metric segments have unique prolongations, V is a semi-sun and P_V is compact valued then V is Chebyshev.*

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