

MONOSTABLE-TYPE TRAVELING WAVES OF BISTABLE REACTION-DIFFUSION EQUATIONS IN THE MULTI-DIMENSIONAL SPACE

BY

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Dedicated to Professor Mimura on the occasion of his 65th birthday

Abstract

We are dealing with a reaction-diffusion equation $u_t = \Delta u + u_{yy} + f(u)$ in \mathbb{R}^{n+1} , where $(x, y) = (x_1, \dots, x_n, y) \in \mathbb{R}^{n+1}$ and Δ is the Laplacian in \mathbb{R}^n . Suppose that the equation has a bistable nonlinearity, namely it has two stable constant solutions $u = 0, 1$ and an unstable one between those. With the unbalanced condition $\int_0^1 f(u) du > 0$ the equation allows planar traveling waves connecting two constant solutions and an unstable standing solution $v(x) > 0$ of $\Delta v + f(v) = 0$ with $\lim_{|x| \rightarrow \infty} v(x) = 0$. Then we show that there are a family of traveling waves $u = U(x, z)$, $z = y - ct$ connecting $u = 1$ (or $u = 0$) at $z = -\infty$ to $u = v(x)$ at $z = \infty$ with speeds belonging to a half infinite interval. The proof is carried out by using the comparison principle and constructing a subsolution and a supersolution appropriately. The existence theorem can be extended to a more general reaction-diffusion equation.

1. Introduction

There are variety of wave propagation phenomena in the field of chemical reaction, population biology, morphogenesis etc. Those phenomena are well studied by mathematical models. Among other things models of reaction-

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diffusion equations are well accepted for the mathematical understanding of the phenomena.

In this article we are dealing with the following scalar reaction-diffusion equation in the multi-dimensional space \mathbb{R}^{n+1} :

$$u_t = \Delta u + u_{yy} + f(u), \quad (1.1)$$

where $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, $y \in \mathbb{R}$, $u_t = \partial u / \partial t$, $\Delta = \sum_{k=1}^n \partial^2 / \partial x_k^2$ and $u_{yy} = \partial^2 u / \partial y^2$. We assume that $f(u)$ is C^2 on an open interval containing $[0, 1]$ and that $f(u)$ satisfies

$$f(0) = f(a) = f(1) = 0, \quad f'(0) < 0, \quad f'(a) > 0, \quad f'(1) < 0, \quad (1.2)$$

together with

$$f(u) \neq 0 \quad \text{for } u \in (0, a) \cup (a, 1). \quad (1.3)$$

The condition (1.2) implies that constant solutions $u = 0, 1$ are nondegenerate stable equilibria of the diffusion free equation of (1.1) while $u = a$ is an unstable one. The equation (1.1) satisfying (1.2) and (1.3) is called a bistable reaction-diffusion equation. This equation is used as a simple model describing propagation of species in population biology or propagation of nerve excitation. In fact, (1.1) with (1.2)-(1.3) allows a planar traveling wave with a monotone profile, that is, there is a solution $u = \psi(y - ct)$ satisfying

$$\begin{cases} \psi_{zz} + c\psi_z + f(\psi) = 0, & \psi(z) > 0 & (z \in \mathbb{R}), \\ \lim_{z \rightarrow -\infty} \psi(z) = 1, & \lim_{z \rightarrow \infty} \psi(z) = 0. \end{cases}$$

where $z = x - ct$. Since $\psi'(z) < 0$ holds for the solution $\psi(z)$, we may call it a monotone planar traveling wave (solution). Henceforth we assume

$$\int_0^1 f(u) du > 0,$$

which implies that the speed c is positive.

One of attracting problems in the study for traveling waves in the multi-dimensional space is to determine all the types of traveling waves. Besides the planar traveling wave, the equation possesses traveling wave solution with other profiles when $n \geq 1$. Hamel-Monneau-Roquejoffre [7, 8] and

Ninomiya-Taniguchi [13, 14] proved the existence of such a traveling wave in different ways. Their studies revealed that for $n + 1 = 2$, a contour of the traveling wave has two asymptotic lines as it goes to infinity and the inclination of the asymptotic line determines its speed, which is greater than the above speed c [13]. On the other hand for the case of $n + 1 \geq 2$, the traveling wave solutions with a conical profile are constructed in [8] and those with pyramidal shapes are constructed in [18, 19].

We also remark that Hamel-Nadirashvili [9, 10] study traveling waves and entire solutions of (1.1) for the monostable nonlinearity instead of (1.2). Such solutions also arise in the bistable case since $f(u)$ restricted on $[0, a]$ or $[a, 1]$ possesses a monostable nonlinearity.

The purpose of this article is to show the existence of a traveling wave solution (1.1) equipped with a different type of profile from the previous ones in the multi-dimensional space. To explain the profile, let $v(x)$ be a stationary solution of (1.1), that is, a standing wave solution of

$$\Delta v + f(v) = 0 \quad (x \in \mathbb{R}^n), \quad v(x) > 0, \quad \lim_{|x| \rightarrow \infty} v(x) = 0.$$

It is known that there is a one hump solution $v(x)$ for $n = 1$ or a radially symmetric solution $v(x) = \Phi(r)$ ($r = |x|$)

$$\begin{cases} \Phi_{rr} + \frac{n-1}{r}\Phi_r + f(\Phi) = 0, & \Phi(r) > 0 \quad (0 < r < \infty), \\ \Phi_r(0) = 0, & \lim_{r \rightarrow \infty} \Phi(r) = 0 \end{cases} \quad (1.4)$$

for $n \geq 2$. This standing wave solution $v(x)$ is unique up to translation and unstable ([3, 15]). In fact the linearized eigenvalue problem

$$\Delta \phi + f'(v(x))\phi = \mu \phi, \quad (x \in \mathbb{R}^n) \quad (1.5)$$

allows a positive eigenvalue $\mu > 0$ and a corresponding eigenfunction $\phi(x)$ satisfying

$$\lim_{|x| \rightarrow \infty} \phi(x) = 0, \quad \phi(x) > 0 \quad (x \in \mathbb{R}^n).$$

Therefore we may suppose that there exists a traveling wave connecting $u = 1$ (at $y = -\infty$) with $u = v(x)$ (at $y = \infty$) or connecting $u = v(x)$ (at

$y = -\infty$) with $u = 0$ (at $y = \infty$). In the sequel we look for a traveling wave solution $u = U(x, y - ct)$, $z = y - ct$ satisfying

$$\begin{cases} \Delta U + U_{zz} + cU_z + f(U) = 0 & ((x, z) \in \mathbb{R}^{n+1}), \\ \lim_{z \rightarrow -\infty} U(x, z) = 1, & \lim_{z \rightarrow \infty} U(x, z) = v(x), \end{cases} \quad (1.6)$$

or $u = V(x, y - ct)$ satisfying

$$\begin{cases} \Delta V + V_{zz} + cV_z + f(V) = 0 & ((x, z) \in \mathbb{R}^{n+1}), \\ \lim_{z \rightarrow -\infty} V(x, z) = v(x), & \lim_{z \rightarrow \infty} V(x, z) = 0. \end{cases} \quad (1.7)$$

We have the following theorem.

Theorem 1.1. *Let μ_+ be a positive eigenvalue of (1.5). Then for each $c \geq 2\sqrt{\kappa}$, there exist solutions $U(x, z)$ and $V(x, z)$ to (1.6) and (1.7) respectively, satisfying*

$$U_z(x, z) \leq 0, \quad V_z(x, z) \leq 0 \quad ((x, z) \in \mathbb{R}^{n+1}),$$

where

$$\kappa := \max \left\{ - \min_{0 \leq u \leq 1} f'(u), \mu_+ \right\}.$$

We prove this theorem in the remaining of the present article by the comparison principle with an appropriate subsolution and a supersolution.

We remark that

$$\mu_+ \leq \max_{0 \leq u \leq 1} f'(u)$$

holds. Indeed, setting the x value which attains the maximum of the positive eigenfunction $\phi(x)$ in (1.5), we easily obtain the inequality. Hence, we have a sufficient condition

$$c \geq 2\sqrt{\max_{0 \leq u \leq 1} |f'(u)|}$$

for $c \geq 2\sqrt{\kappa}$.

From the statement of the above theorem we see that there are a family of traveling wave solutions for $c \geq 2\sqrt{\kappa}$. This reminds us of the traveling

waves of the Fisher-KPP equation. In fact, in our proof a similar monostable situation arises, though we treat the bistable reaction-diffusion equation. We, however, emphasize that unlike the conventional Fisher-KPP equation, we have to handle a sign changing nonlinearity. Although there are many results for traveling waves in the Fisher-KPP equation ([12, 1, 2, 11], et al.), in particular, [2, 5, 10] for the multi-dimensional space, those cannot directly apply to the present case. In fact, the bistable nonlinearity affects the argument for the comparison principle, that is, the construction of an appropriate subsolution or supersolution. In the sequel, we need a new existence theorem for nonplanar traveling waves in the multi-dimensional space.

In related to the above existence theorem we also refer to nice papers by Berestycki-Nirenberg [4] and Vega [20] for a multi-dimensional cylindrical domain, $\mathbb{R} \times \Omega$, where Ω is a bounded domain of \mathbb{R}^n . We, however, note that their theorems are not applicable to our case in direct since the boundedness of the cross section Ω is crucial in their arguments.

In the next section we treat a more general equation and establish the existence theorem for traveling wave solutions of the equation (see Theorem 2.1). Since the condition for the equation is rather weak, this existence theorem covers not only the above bistable case but also a monostable equation with an inhomogeneous coefficient, for instance,

$$u_t = \Delta u + u_{yy} + (a(x) - u)u, \quad (1.8)$$

where $a(x) \geq 0$ and $a(|x|) \rightarrow 0$ ($|x| \rightarrow \infty$). Those applications are stated in Sec. 3.2.

We finally give two remarks on the main result. Firstly, by a technical reason, we do not show the uniqueness of the traveling wave of (1.6) and (1.7) for each $c \geq 2\sqrt{\kappa}$ in Theorem 1.1, which would be a future study. Secondly, the condition $c \geq 2\sqrt{\kappa}$ is a technical condition in our argument. It is an interesting problem to determine the minimum speed of the traveling waves.

Throughout the next section we establish the existence theorem for nonplanar traveling wave $u = U(x, z)$, $z = y - ct$ connecting two equilibrium solutions which are uniform in z direction. Then Sec. 3 we apply the result to (1.1) and (1.8).

2. A Theorem for the Existence of Nonplanar Traveling Waves

We treat a more general equation than (1.1). Consider the following equation:

$$u_t = \Delta u + u_{yy} + g(x, u), \quad (2.1)$$

where g and its derivatives in u up to the second order are continuous and bounded in $\{(x, u) : x \in \mathbb{R}^n, |u| \leq K\}$ for a large constant $K > 0$. Hereafter we consider the solution u satisfying $|u(x, t)| \leq K$. We assume that the equation (2.1) has two stationary solutions v_- and v_+ , namely,

$$\Delta v_- + g(x, v_-(x)) = 0, \quad \Delta v_+ + g(x, v_+(x)) = 0. \quad (2.2)$$

Let $\mu = \mu_{\pm}$ be the first eigenvalues and let $\phi = \phi_{\pm}(x)$ be corresponding eigenfunctions of

$$\begin{aligned} \Delta \phi + g_u(x, v_{\pm}(x))\phi &= \mu\phi, & x \in \mathbb{R}^n, \\ \lim_{|x| \rightarrow \infty} \phi(x) &= 0, & \phi(x) > 0 \ (x \in \mathbb{R}^n). \end{aligned} \quad (2.3)$$

We assume one of the following four conditions:

- (1) $\mu_+ > 0$ and $v_-(x) \geq v_+(x)$ for $x \in \mathbb{R}^n$,
- (2) $\mu_+ > 0$ and $v_-(x) \leq v_+(x)$ for $x \in \mathbb{R}^n$,
- (3) $\mu_- > 0$ and $v_-(x) \geq v_+(x)$ for $x \in \mathbb{R}^n$,
- (4) $\mu_- > 0$ and $v_-(x) \leq v_+(x)$ for $x \in \mathbb{R}^n$.

We look for a traveling wave solution of (2.1) connecting $v_-(x)$ at $y = -\infty$ to $v_+(x)$ at $y = \infty$. Putting $u = U(x, y - ct)$, $z = y - ct$ into the equation, we obtain a nonlinear elliptic equation

$$\begin{cases} \Delta U + U_{zz} + cU_z + g(x, U) = 0, & ((x, z) \in \mathbb{R}^{n+1}), \\ \lim_{z \rightarrow -\infty} U(x, z) = v_-(x), & \lim_{z \rightarrow \infty} U(x, z) = v_+(x). \end{cases} \quad (2.4)$$

By the change of variables $z \mapsto -z$ and $c \mapsto -c$ in the equation (2.4), the conditions (3) and (4) are identified with (2) and (1) respectively. It is

therefore suffices to consider the cases (1) and (2). The following theorem ensures the existence of a solution to (2.4):

Theorem 2.1. *Let $v_{\pm}(x)$ be stationary solutions of (2.1) with the eigenvalues μ_{\pm} and the corresponding eigenfunction ϕ_{\pm} of (2.3).*

- (i) *Case (1): Suppose that there are no other stationary solutions $v(x)$ sandwiched by $v_{\pm}(x)$ as $v_{+}(x) \leq v(x) \leq v_{-}(x)$ ($x \in \mathbb{R}^n$). In addition, assume $v_{+}(x) + \varepsilon\phi_{+}(x) \leq v_{-}(x)$ ($x \in \mathbb{R}^n$) for a small positive constant ε . Then for each $c \geq 2\sqrt{\kappa_g}$, there exists a solution $U(x, z)$ to (2.4) satisfying*

$$U_z(x, z) \leq 0,$$

where

$$\kappa_g := \max \left\{ - \min_{x \in \mathbb{R}^n, v_{-}(x) \leq u \leq v_{+}(x)} g_u(x, u), \mu_{+} \right\}.$$

- (ii) *Case (2): Suppose that there are no other stationary solutions $v(x)$ sandwiched by $v_{\pm}(x)$ and that $v_{-}(x) + \varepsilon\phi_{-}(x) \leq v_{+}(x)$ ($x \in \mathbb{R}^n$) holds for a constant $\varepsilon > 0$. Then for each $c \geq 2\sqrt{\kappa_g}$, there exists a solution $V(x, z)$ to (2.4) satisfying*

$$V_z(x, z) \geq 0.$$

Using the comparison principle, we prove the above theorem in the rest of this section. First we consider the case (1).

2.1. A subsolution

We let $w(z)$ be a solution to

$$\begin{cases} w_{zz} + cw_z + \mu_{+}w - w^2 = 0, & w(z) > 0, \quad w_z(z) < 0 \quad (z \in \mathbb{R}), \\ \lim_{z \rightarrow -\infty} w(z) = \mu_{+}, & \lim_{z \rightarrow \infty} w(z) = 0, \end{cases} \tag{2.5}$$

where $c \geq 2\sqrt{\mu_{+}}$. It is known that for each $c \geq 2\sqrt{\mu_{+}}$ there is a unique solution $w(z)$ of (2.5) up to translation (for instance see [1]). Set

$$W(z) := \sigma w(z).$$

We normalize $\phi_{\pm}(x)$ of (2.3) as

$$\max_{x \in \mathbb{R}} \phi_{\pm}(x) = 1.$$

Define

$$\mathcal{F}[U] := -\Delta U - U_{zz} - cU_z - g(x, U).$$

and put

$$\underline{U}(x, z) := v_+(x) + \phi_+(x)W(z).$$

Then we compute

$$\begin{aligned} \mathcal{F}[\underline{U}] &= -\Delta v_+ - \Delta \phi_+ W - \phi_+ W_{zz} - c\phi_+ W_z - g(x, v_+ + \phi_+ W) \\ &= -\phi_+ \{W_{zz} + cW_z + \mu_+ W\} \\ &\quad -g(x, v_+ + \phi_+ W) + g(x, v_+) + g_u(x, v_+) \phi_+ W \\ &\leq -\phi_+ \{W_{zz} + cW_z + \mu_+ W\} + M_g \phi_+^2 W^2 \end{aligned}$$

where

$$M_g := \min_{v_+(x) \leq u \leq v_-(x), x \in \mathbb{R}} g_{uu}(x, u).$$

Since $W = \sigma w$, we have

$$\mathcal{F}[\underline{U}] \leq -\phi_+ \sigma \{w_{zz} + cw_z + \mu_+ w - \sigma M_g w^2\}.$$

This implies that if $\sigma \leq 1/M_g$, then \underline{U} is a subsolution to (2.4).

2.2. A supersolution

We define

$$\tilde{g}^{\delta}(x, u) := g(x, u) + \delta$$

for a small positive δ and consider

$$\Delta v + \tilde{g}^{\delta}(x, v) = 0. \tag{2.6}$$

In what follows this equation will work as an approximate equation of (2.2). With the aid of the assumption $\mu_+ \neq 0$ we can show that there is a solution

$v = \tilde{v}_+^\delta(x)$ to (2.6) such that

$$\lim_{\delta \rightarrow 0} \tilde{v}_+^\delta(x) = v_+(x).$$

Moreover, we can check

$$0 < v_+(x) + \frac{\delta}{\kappa_g} \leq \tilde{v}_+^\delta(x) \quad (x \in \mathbb{R}^n). \tag{2.7}$$

Indeed,

$$\begin{aligned} & -\Delta(v_+(x) + \delta/\kappa_g) - \tilde{g}^\delta(x, v_+(x) + \delta/\kappa_g) \\ &= g(x, v_+(x)) - g(x, v_+(x) + \delta/\kappa_g) - \delta \\ &\leq -\left(\min_{x \in \mathbb{R}^n, v_+(x) \leq u \leq v_+(x) + \delta/\kappa_g} g_u(x, u) + \kappa_g \right) \frac{\delta}{\kappa_g}. \end{aligned}$$

Hence, the condition $\kappa_g + \min g_u(x, u) \geq 0$ implies that $v_+(x) + \delta/\kappa_g$ is a subsolution to (2.6), which yields (2.7).

Now put

$$U^+(x, z) := \tilde{v}_+^\delta(x) + Q(z)$$

where

$$Q(z) = \alpha \exp(\lambda z), \quad \lambda := -\frac{c - \sqrt{c^2 - 4\kappa_g}}{2}, \quad \alpha > 0.$$

We note that $Q(z)$ is a solution to

$$Q_{zz} + cQ_z + \kappa_g Q(z) = 0.$$

Then

$$\begin{aligned} \mathcal{F}[U^+] &= -\Delta \tilde{v}_+^\delta - Q_{zz} - cQ_z - g(x, \tilde{v}_+^\delta + Q) \\ &= \tilde{g}^\delta(x, \tilde{v}_+^\delta) + \kappa_g Q - g(x, \tilde{v}_+^\delta + Q) \\ &= g(x, \tilde{v}_+^\delta) + \delta - g(x, \tilde{v}_+^\delta + Q) + \kappa_g Q \\ &= [\kappa_g - g_u(x, \tilde{v}_+^\delta + \theta Q)]Q + \delta \\ &\geq \delta > 0 \end{aligned}$$

where $\theta \in (0, 1)$. Thus

$$\bar{U}(x, z) := \min_{(x,z) \in \mathbb{R}^2} \{ \tilde{v}_+^\delta(x) + Q(z), v_-(x) \}$$

is a supersolution to (2.4).

2.3. Existence and monotonicity

We prove the existence of a solution to (2.4) together with the monotonicity of the solution in z direction.

First we compare \underline{U} and \bar{U} . Let $\bar{U} = \tilde{v}_+^\delta(x) + Q(z)$. Then

$$\begin{aligned} \bar{U}(x, z) - \underline{U}(x, z) &= \tilde{v}_+^\delta(x) + Q(z) - \{v_+(x) + \phi_+(x)W(z)\} \\ &\geq \tilde{v}_+^\delta(x) - v_+(x) + Q(z) - W(z) \\ &\geq \tilde{v}_+^\delta(x) - v_+(x) - \sigma w(z) \\ &\geq \frac{\delta}{\kappa_g} - \sigma \mu_+. \end{aligned}$$

Thus given δ , we take $\sigma \leq \delta/(\kappa_g \mu_+)$ so that

$$\underline{U}(x, z) \leq \bar{U}(x, z) \tag{2.8}$$

holds. On the other hand, when $\bar{U} = v_-(x)$, we take $\sigma \leq \varepsilon/\mu_+$ to realize (2.8); indeed, we have

$$v_+(x) + \sigma \phi_+(x)w(z) \leq v_+(x) + \sigma \mu_+ \phi_+(x).$$

By the assumption $v_+(x) + \varepsilon \phi_+(x) \leq v_-(x)$ we obtain (2.8) for $0 < \sigma \leq \min\{\delta/\kappa_g \mu_+, \varepsilon/\mu_+\}$.

Next define

$$\mathcal{L}[u] := -\Delta u - u_{zz} - cu_z + \gamma u$$

where we fix a positive constant γ satisfying

$$\gamma > \max \left\{ \frac{c^2}{4}, - \min_{v_+(x) \leq u \leq v_-(x), x \in \mathbb{R}^n} g_u(x, u) \right\}.$$

Let $\{u_n(x)\}_{n=0,1,2,\dots}$ be a sequence given by

$$\begin{aligned}\mathcal{L}[u_n] &= g(x, u_{n-1}) + \gamma u_{n-1}, \\ u_0 &= \underline{U}.\end{aligned}\tag{2.9}$$

Then we see the following property:

Lemma 2.2. The sequence $\{u_n\}$ satisfies

$$\begin{aligned}u_0 &< u_1 < \dots < u_n < u_{n+1} < \dots < \bar{U}, \\ \frac{\partial u_n}{\partial z} &< 0 \quad (n = 0, 1, 2, \dots).\end{aligned}$$

Proof. The argument below is essentially found in Sattinger [17]. We have

$$\begin{aligned}\mathcal{L}[u_1 - u_0] &= g(x, u_0) + \gamma u_0 - (-\Delta u_0 - u_{0zz} - cu_{0z} + \gamma u_0) \\ &= \Delta \underline{U} + \underline{U}_{zz} + c\underline{U}_z + g(x, \underline{U}) \geq 0, \\ \mathcal{L}[u_{n+1} - u_n] &= \{g_u(x, \theta u_n + (1 - \theta)u_{n-1}) + \gamma\} (u_n - u_{n-1})\end{aligned}$$

for some θ . Applying the strong maximum principle to these inequalities yields the former part of the lemma.

Similarly, we have

$$\begin{aligned}(u_0)_z &= \phi_+(x)W'(z) < 0, \\ \mathcal{L}[(u_{n+1})_z] &= (g_u(x, u_n) + \gamma)(u_n)_z.\end{aligned}$$

Applying the inductive argument to this equation with the inequality for $n = 0$ leads to the latter part of the lemma. \square

By (2.8) and Lemma 2.2 we assert that u_n converges as $n \rightarrow \infty$. We denote the limit by U , i.e.,

$$U(x, z) := \lim_{n \rightarrow \infty} u_n(x, z),$$

which satisfying

$$u_n(x, z) \leq U(x, z) \leq \bar{U}(x, z).$$

By the Schauder estimate we can assert that the limiting function $U(x, z)$ is smooth and satisfies

$$\Delta U + U_{zz} + cU_z + g(x, U) = 0. \quad (2.10)$$

This implies that $U(x, z)$ is a solution.

2.4. Completion of the proof for Theorem 2.1 (i)

Given $\delta > 0$, we take σ small so that

$$\underline{U}(x, z) \leq U(x, z) \leq \overline{U}(x, z)$$

Then we easily see

$$\phi_+(x)W(z) \leq U(x, z) - v_+(x) \leq \tilde{v}_+^\delta(x) - v_+(x) + Q(z)$$

for any large z . Thus

$$0 \leq \limsup_{z \rightarrow \infty} \{U(x, z) - v_+(x)\} = O(\delta).$$

Since U is independent of δ , δ is taken arbitrarily small. Thus there is a solution $U(x, z)$ satisfying

$$\lim_{z \rightarrow \infty} \sup_{x \in \mathbb{R}} (U(x, z) - v_+(x)) = 0.$$

The monotonicity of U in z immediately follows from Lemma 2.2.

Next we show the behavior of U in $z \rightarrow -\infty$. The monotonicity in z implies that there exist functions $\varphi(x)$ such that

$$\varphi(x) := \lim_{z \rightarrow -\infty} U(x, z).$$

By $v_+(x) < U(x, z) \leq \overline{U}(x, z) \leq v_-(x)$ and $U_z \leq 0$, we have

$$\lim_{z \rightarrow -\infty} U_z = 0.$$

We note that U is bounded in $C^3(\mathbb{R}^{n+1})$. It follows from the above facts and

the Schauder estimate that

$$\lim_{z \rightarrow -\infty} U_{zz} = 0.$$

Thus we have

$$\Delta\varphi + g(x, \varphi) = 0$$

by taking the limit of (2.10) as $z \rightarrow -\infty$. Noticing

$$v_+(x) < U(x, 0) \leq \varphi(x) \leq v_-(x).$$

We conclude $\varphi \equiv v_-$ by the assumption of Theorem 2.1. □

2.5. Proof for Theorem 2.1 (ii)

Since the proof is similar to the case (1), we only construct a subsolution and a supersolution.

Set

$$\begin{aligned} \underline{U}(x, z) &:= \max_{(x, z) \in \mathbb{R}^2} \{\hat{v}_+^\delta(x) - \alpha \exp(\lambda z), v_-(x)\}, \\ \overline{U}(x, z) &:= v_+(x) - \phi_+(x)W(z), \end{aligned}$$

where $\hat{v} = \hat{v}_+^\delta(x)$ is a solution to

$$\Delta\hat{v} + g(x, \hat{v}) - \delta = 0.$$

Let

$$\hat{g}^\delta(x, u) := g(x, u) - \delta.$$

From

$$\begin{aligned} &-\Delta(v_+(x) - \delta/\kappa_g) - \hat{g}^\delta(x, v_+(x) - \delta/\kappa_g) \\ &= \{g_u(x, v_+(x) - \delta/\kappa_g) + \kappa_g\} \frac{\delta}{\kappa_g} \geq 0, \end{aligned}$$

it follows that

$$\lim_{\delta \rightarrow 0} \hat{v}_+^\delta(x) = v_+(x),$$

$$\hat{v}_+^\delta(x) < v_+(x) - \frac{\delta}{\kappa_g} \quad (x \in \mathbb{R}).$$

Putting

$$Q(z) := \alpha \exp(\lambda z), \quad \lambda := -\frac{c - \sqrt{c^2 - 4\kappa_g}}{2},$$

we have

$$\begin{aligned} \mathcal{F}[\hat{v}_+^\delta(x) - Q(z)] &= -\Delta \hat{v}_+^\delta - (-Q)_{zz} - c(-Q)_z - g(x, \hat{v}_+^\delta - Q) \\ &= \hat{g}^\delta(x, \hat{v}_+^\delta) - \kappa_g Q - g(x, \hat{v}_+^\delta - Q) \\ &= g(x, \hat{v}_+^\delta) - \delta - g(x, \hat{v}_+^\delta - Q) - \kappa_g Q \\ &= -(\kappa_g - g_u(x, \hat{v}_+^\delta - \theta Q))Q + \delta \\ &\leq -\delta < 0. \end{aligned}$$

Thus $\underline{U} = \max\{\hat{v}_+^\delta(x) - Q(z), v_-(x)\}$ is a subsolution.

Now recall that $W(z) = \sigma w(z)$ and w satisfies (2.5). Putting $\overline{U} = v_+(x) - \phi_+(x)W(z)$, we obtain

$$\begin{aligned} \mathcal{F}[\overline{U}] &= -\Delta v_+ + \Delta \phi_+ W + \phi_+ W_{zz} + c\phi_+ W_z - g(x, v_+ - \phi_+ W) \\ &= \phi_+ \{W_{zz} + cW_z + \mu_+ W\} - g(x, v_+ + \phi_+ W) + g(x, v_+) - g_u(x, v_+) \phi_+ W \\ &\geq \phi_+ \{W_{zz} + cW_z + \mu_+ W\} - M_g \phi_+^2 W^2 \\ &\geq \phi_+ \sigma \{w_{zz} + cw_z + \mu_+ w - \sigma M_g w^2\}, \end{aligned}$$

which implies that \overline{U} is a supersolution if $\sigma \leq 1/M_g$.

A solution $U(x, z)$ is obtain by the limit of the $\{u_n\}$ as in (2.9) starting from \overline{U} . Consequently, we can prove Theorem 2.1. This completes the proof of the theorem. \square

3. Applications

3.1. Proof of Theorem 1.1

We go back to the equation (1.1) and apply Theorem 2.1 to (1.6) and (1.7). As seen in Sec. 1, there is a radially symmetric stationary solution $v(x) = \Phi(|x|)$. Since the linearized eigenvalue problem for $v(x)$ of (1.5) allows the zero eigenvalue and corresponding eigenfunctions $\partial\Phi/\partial x_j$ ($j =$

$1, \dots, n$), we see that there are a positive eigenvalue μ and a corresponding eigenfunction $\phi(x) = \tilde{\phi}(|x|)$. Moreover, since $\Phi(r)$ and $\tilde{\phi}(r)$ asymptotically satisfy

$$\Phi_{rr} + \frac{n-1}{r}\Phi_r + f'(0)\Phi = 0$$

and

$$\tilde{\phi}_{rr} + \frac{n-1}{r}\tilde{\phi}_r + (f'(0) - \mu)\tilde{\phi} = 0$$

respectively as $r \rightarrow \infty$, we notice

$$\lim_{r \rightarrow \infty} \frac{\tilde{\phi}(r)}{\Phi(r)} = 0.$$

In addition by the uniqueness of the solution of (1.4), any radially symmetric solution is given by a translation of $v(x)$ in space ([15]).

Now we put

$$v_-(x) = 1, \quad v_+(x) = v(x) \quad (x \in \mathbb{R}^n)$$

for the case (1) in Sec. 2, while

$$v_-(x) = v(x), \quad v_+(x) = 0 \quad (x \in \mathbb{R}^n)$$

for the case (3). Recall that the case (3) is equivalent to the case (2) by the transformation $(z, c) \mapsto (-z, -c)$. Then we can assert that the conditions of Theorem 1.1 are met. This concludes the proof of Theorem 1.1. \square

3.2. The inhomogeneous case

We consider the equation

$$u_t = \Delta u + u_{yy} + f(x, u), \quad f(x, u) := (a(x) - u)u.$$

Let $a(x)$ be a non-constant function such as

$$a(x) \geq 0, \quad \lim_{|x| \rightarrow \infty} a(x) = 0. \tag{3.1}$$

In this subsection we prove the existence of a traveling wave by applying Theorem 2.1.

First we show that there is a positive solution to

$$\Delta v + (a(x) - v)v = 0 \quad (x \in \mathbb{R}^n). \quad (3.2)$$

Consider the spectrum of the linearized operator $-\Delta - a(x)$ in $L^2(\mathbb{R}^n)$. Since (3.1) holds, the set of essential spectrum is the half line $[0, \infty)$ and there are a negative eigenvalue $-\mu$ and a corresponding positive eigenfunction $\phi(x)$ (see [16]). Then we can easily prove that $\underline{v} := \alpha\phi(x)$ is a subsolution of (3.2) for a sufficiently small $\alpha > 0$. Then we can construct a sequence starting from $v = \alpha\phi$ which converges a positive solution in the similar argument in Sec. 2.1. We denote this solution by $v = \tilde{v}(x)$.

Now setting $v_- = \tilde{v}$ and $v_+ = 0$, we can directly apply Theorem 2.1 (i) to obtain a traveling wave $u = U(x, z)$, $z = y - ct$ to (1.6) for each $c \geq 2\sqrt{a_\infty}$ where

$$a_\infty := \max_{x \in \mathbb{R}} a(x).$$

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References

1. D. G. Aronson and H. F. Weinberger, Nonlinear diffusion in population genetics, combustion, and nerve pulse propagation, *Partial Differential Equations and Related Topics*, ed. J. A. Goldstein, *Lecture Notes in Math.* **446**(1975), 5-49, Springer.
2. D. G. Aronson and H. F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv. Math.*, **30**(1978), 33-76.
3. P. W. Bates and J. Shi, Existence and instability of spike layer solutions to singular perturbation problems, *J. Funct. Anal.*, **196**(2002), 211-264.

4. H. Berestycki and L. Nirenberg, Travelling fronts in cylinders, *Ann. Inst. Henri Poincaré*, **9** (1992), 497-572.
5. A. Bonnet and F. Hamel, Existence of nonplanar solutions of a simple model of premixed Bunsen flames, *SIAM J. Math. Anal.*, **31**(1999), 80-118.
6. E. N. Dancer, New solutions of equations on \mathbb{R}^n , *Ann. Scuola Norm. Sup. Pisa. Cl. Sci.* (4), **30**(2001), 535-563.
7. F. Hamel, R. Monneau, and J.-M. Roquejoffre, Existence and qualitative properties of multidimensional conical bistable fronts, *Disc. Cont. Dyn. Systems* **13** (2005), 1069-1096.
8. F. Hamel, R. Monneau, and J.-M. Roquejoffre, Asymptotic properties and classification of bistable fronts with Lipschitz level sets, *Disc. Cont. Dyn. Systems* **14** (2006), 75-92.
9. F. Hamel and N. Nadirashvili, Entire solutions of the KPP equation, *Comm. Pure Appl. Math.*, **52**(1999), 1255-1276.
10. F. Hamel and N. Nadirashvili, Travelling fronts and entire solutions of the Fisher-KPP equation in \mathbb{R}^N , *Arch. Ration. Mech. Anal.*, **157** (2001), 91-163.
11. Ya. I. Kanel, The behavior of solutions of the Cauchy problem when the time tends to infinity, in the case of quasilinear equations arising in the theory of combustion, *Soviet Math. Dokl.*, **1**(1960), 533-536.
12. A. Kolmogorov, I. Petrovsky, and N. Piskunov, Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique, *Bjul. Moskovskogo Gos. Univ. Ser. Internat. Sec. A*, **1**(1937), 1-26.
13. H. Ninomiya and M. Taniguchi, Existence and global stability of traveling curved fronts in the Allen-Cahn equations, *J. Differential Equations*, **213** (2005), 204-233.
14. H. Ninomiya and M. Taniguchi, Global stability of traveling curved fronts in the Allen-Cahn equations, submitted to *Disc. Cont. Dyn. Systems*, **15** (2006), 819-832.
15. T. Ouyang and J. Shi, Exact multiplicity of positive solutions for a class of semilinear problem, II, *J. Differential Equations*, **158** (1999), 94-151.
16. M. Reed and B. Simon, *Methods of Modern Mathematical Physics IV*, Academic Press, 1979.
17. D. H. Sattinger, On the stability of waves of nonlinear parabolic systems, *Adv. Math.* **22** (1976), 312-355.
18. M. Taniguchi, Traveling fronts of pyramidal shapes in the Allen-Cahn equations, *SIAM J. Math. Anal.*, **39** (2007), 319-344.
19. M. Taniguchi, The uniqueness and asymptotic stability of pyramidal traveling fronts in the Allen-Cahn equations *J. Differential Equations*, in press.
20. J. M. Vega, The asymptotic behavior of the solutions of some semilinear elliptic equations in cylindrical domains, *J. Differential Equations*, **102** (1993), 119-152.

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