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OSCILLATORY AND ASYMPTOTIC BEHAVIOUR OF A HOMOGENEOUS NEUTRAL DELAY DIFFERENCE EQUATION OF SECOND ORDER

BY

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Abstract

In this paper we find sufficient conditions for every solution of the neutral delay difference equation

$$\Delta(r_n\Delta(y_n - p_n y_{n-m})) + q_n G(y_{n-k}) = 0$$

to oscillate or to tend to zero or $\pm\infty$ as $n \to \infty$, where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n, p_n, q_n$, and r_n are infinite sequences of real numbers with $q_n \geq 0, r_n > 0$. Different ranges of $\{p_n\}$ are considered. This paper improves,generalizes and corrects some recent results of [1, 9, 12, 13, 14].

1. Introduction

In this paper sufficient conditions are found so that every solution of

$$\Delta(r_n\Delta(y_n - p_n y_{n-m})) + q_n G(y_{n-k}) = 0 \tag{E}$$

oscillates or tends to zero or $\pm\infty$ as $n \to \infty$, where Δ is the forward difference operator given by $\Delta x_n = x_{n+1} - x_n, \{p_n\}, \{q_n\}$, and $\{r_n\}$ are assumed to be infinite sequences of real numbers with $q_n \geq 0$, $r_n > 0$. We assume m, k, are non negative integers, and $G \in C(R, R)$. Various ranges of $\{p_n\}$ are

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considered. Further the following conditions are assumed for its use in the sequel.

(H_1) $xG(x) > 0$ for $x \neq 0$ and G is non-decreasing.

(H_2) $\sum_{n=0}^{\infty} q_n = \infty$.

(H_3) $\sum_{n=0}^{\infty} \frac{1}{r_n} = \infty$.

(H_4) $\sum_{n=0}^{\infty} \frac{1}{r_n} < \infty$.

(H_5) $\sum_{n=1}^{\infty} \frac{1}{r_n} \sum_{i=0}^{n-1} q_i = \infty$.

In recent years there have been much interest in studing the oscillation of neutral delay difference equations (NDDE in short). For recent results and references see the monograph by Agarwal[2] and the papers (see [1, 3, 9, 17] and [12]–[15]and the references cited there in). The authors of this paper feel (E) is not yet being given serious attention. In [9] the authors have given sufficient conditions for all solutions of (E) to be oscillatory. In their work p_n is confined to $-1 \leq p_n \leq 0$ only. They restrict r_n with (H_3). Also they impose a super linear condition on G i.e

$$\frac{G(x)}{x} \geq \gamma > 0 \quad \text{for} \quad x \neq 0. \tag{1}$$

When $G(u) = u^{\frac{1}{3}}$,then the above condition is not satisfied. Hence [9] does not cover a class of NDDEs. Again in [1] the authors have some results for (E) where they restricted G with the condition that there exists a non-negative function H such that

$$G(u) - G(v) = H(u, v)(u - v). \tag{2}$$

If $u \to v$ then $H = G'(v)$. Then eq(2) implies G is non-decreasing, which most authors assume while dealing with non-linear equations. In most of the results of [1], the authors have assumed the conditions (H_3), $p_n \equiv p$, which is a constant and m to be an odd positive integer. Not all results of [1] can be compared with our work but certainly their Theorem 8 invites a direct comparison with our Theorem 3.1. With all humbleness we have to say that our Example 2 contradicts the conclusion of that result. The several restrictions on G imposed in [1, 9] are due to the technique employed by the authors in their work. Here in this paper an attempt is made to remove all these restrictions (See our results in section 3,particularly the Theorems 3.1 and 3.3). Here we may observe that when (H_3) holds ,it is possible to prove all solutions of (E) to be oscillatory as in [9], but when (H_4) holds and $-1 \leq -b \leq p_n \leq 0$, then one may look at the following example where the NDDE

$$\Delta(2^{2n}\Delta(y_n + \frac{1}{2}y_{n-1})) + \frac{1}{4}y_{n-2} = 0$$

has a non oscillatory solution $y_n = 2^{-n}$ which approaches to zero as n approaches ∞. Hence it looks justified when in this work we show that (H_3) or (H_5) is a sufficient condition for all solutions of (E) to be oscillatory or tending to zero or $\pm\infty$ as $n \to \infty$ with p_n in different ranges. We allow G to be linear, sublinear or super linear in the entire work unlike [1, 9]. Further this paper extends and generalize [12, 13] to 2nd order NDDEs since our assumption (H_3) allows us to take $r_n \equiv 1$. Whether it is the discrete or continuous case so far as study neutral equation (E) is concerned almost all results (see[1, 4, 16, 9])use (H_3) or (H_4). But we have some results, where neither (H_3) nor (H_4) is required(see Theorems 3.3 and 3.9). We illustrate our results with suitable examples, which not only establishes the significance of our work over [9] but also contradict some of the existing results of [1, 14] (see Examples 1, 2 and 3).

Let $s = \max\{m, k\}$ and n_0 be a fixed nonnegative integer. By a solution of (E) we mean a real sequence $\{y_n\}$ which is defined for all positive integer $n \geq n_0 - s$ and satisfies (E) for $n \geq n_0$. Clearly if the initial condition

$$x_n = a_n \quad \text{for} \quad n_0 - s \leq n \leq n_0 \tag{3}$$

are given, then the equation (E) has a unique solution satisfying the given initial condition (3). A solution $\{y_n\}$ of (E) is said to be oscillatory if for every positive integer $n_0 > 0$, there exists $n \geq n_0$ such that $y_n y_{n+1} \leq 0$, otherwise $\{y_n\}$ is said to be non-oscillatory.

In the sequel for convenience when we write a functional inequality without specifying its domain of validity, we assume that it holds for all sufficiently large positive integer n.

2. Some Lemmas

First we state a lemma which is due to [13].

Lemma 2.1. *Let* $\{f_n\}, \{q_n\}$ *and* $\{p_n\}$ *be sequences of real numbers defined for* $n \geq N_0 \geq 0$ *such that*

$$f_n = q_n - p_n q_{n-m}, n \geq N_0 + m$$

where $m \geq 0$ *is an integer. Suppose that there exist real numbers* b, b_1, b_2 *such that* p_n *satisfies one of the following three conditions*

(i) $-1 < -b \leq p_n \leq 0$,
(ii) $-b_2 \leq p_n \leq -b_1 < -1$,
(iii) $0 \leq p_n \leq b_2$.

If $q_n > 0$ *for* $n \geq N_0, \liminf_{n\to\infty} q_n = 0$ *and* $\lim_{n\to\infty} f_n = L$ *exists then* $L = 0$.

Before we start our main results, for a better understanding of our assumptions, first we would like to present a useful remark.

Remark 1.

(i) Since $r_n > 0$,therefore either (H_3) or (H_4)holds but not both.
(ii) If (H_3) holds, then (H_2) implies (H_5) but not conversely. This is justified from the the example when $r_n = 3^{-n}$ and $q_n = 2^{-n}$.
(iii) If (H_4) holds, then (H_5) implies (H_2) but not conversely. Indeed this can be verified from the example when $r_n = n^3$ and $q_n \equiv 1$
(iv) If (H_2) and (H_5) hold, then nothing can be said about (H_3) and (H_4). This can be seen from the example $r_n = n^2$ and $q_n \equiv 1$. In this case (H_2), (H_5) and (H_4) hold but not (H_3). Next consider the example $r_n \equiv 1$ and $q_n \equiv 1$. Here (H_2), $(H_5)$and (H_3) hold but not (H_4).

Lemma 2.2. *Suppose that* $\{p_n\}$ *lies in one of the following three ranges*

(A_1) : $0 \leq p_n \leq b < 1$,

(A_2) : $-1 < -b \leq p_n \leq 0$,

(A_3) : $-d \leq p_n \leq -c < -1$.

where b, c *and* d *are positive real numbers. If* (H_1), (H_2) *and* (H_3) *hold and* $\{y_n\}$ *be a nonoscillatory solution of* (E) *for* $n \geq n_0$,*then setting*

$$z_n = y_n - p_n y_{n-m} \tag{4}$$

for $n > n_0$*, one may conclude that* $\lim_{n\to\infty} z_n = 0$.

Proof. Let $\{y_n\}$ be an eventually positive solution of (E) for $n \geq n_0$. Then setting

$$r_n \Delta z_n = w_n \tag{5}$$

for $n \geq n_1 > n_0$, we obtain

$$\Delta w_n = -q_n G(y_{n-k}) \leq 0. \tag{6}$$

Then $\{w_n\}$ is monotonic and consequently $\lim_{n\to\infty} w_n = l$ where $-\infty \leq l < \infty$. Consider the first case when $-\infty \leq l < 0$. Then $w_n < 0$ and is decreasing, so we can find a large positive intrger n_2 such that for $n \geq n_2$ implies $w(n) \leq w_{n_2}$. This further implies $\Delta z_n \leq \frac{w_{n_2}}{r_n}$. Then summing it from $n = n_2$ to ∞ and using (H_3), we obtain $z_n \to -\infty$. Then p_n must be in (A_1) and $z_n < 0$ for large n. Hence $y_n < y_{n-m}$ for large n, which implies y_n is bounded. Consequently z_n is bounded, which is a contradiction. Next we consider the second case $l \geq 0$. This implies w_n is eventually positive. We claim

$$\liminf_{n\to\infty} y_n = 0. \tag{7}$$

Otherwise $y_n > \alpha > 0$ for large $n \geq n_2$ which implies

$$\sum_{n=n_2}^{\infty} q_n G(y_{n-k}) > G(\alpha) \sum_{n=n_2}^{\infty} q_n = \infty \tag{8}$$

by (H_2). On the other hand taking summation of Eq(6) from $n = n_2$ to ∞ we obtain

$$\sum_{n=n_2}^{\infty} q_n G(y_{n-k}) < \infty \tag{9}$$

a contradiction to Eq(8). Thus our claim holds. Since $w_n > 0$ for $n \geq n_3 \geq n_2$ then, Eq(5) with $r_n > 0$ yields $\Delta z_n > 0$. Consequently $z_n > 0$ or $z_n < 0$ for large n. Hence $\lim_{n\to\infty} z_n = a$ where $-\infty < a \leq \infty$. If $a = \infty$ then $\Delta z_n > 0$ with $z_n > 0$ and

$$\liminf_{n\to\infty} \frac{y_n}{z_n} = 0. \tag{10}$$

However,

$$\lim_{n\to\infty}\left\{\frac{y_n}{z_n}-\frac{p_n^* y_{n-m}}{z_{n-m}}\right\}=1 \tag{11}$$

where $p_n^*=\frac{p_n z_{n-m}}{z_n}$. Since z_n is increasing then $\frac{z_{n-m}}{z_n}<1$. Hence it is clear that $p_n^*\to p_n$ as $n\to\infty$. If p_n is in (A_1) then $0<p_n^*<p_n$. However,if p_n is in (A_2) or (A_3) then $0>p_n^*>p_n$. Then it follows that p_n^* lies in the required ranges of Lemma 2.1, when p_n is in $(A_1),(A_2)$ or (A_3). Hence from Eq(10) and Lemma 2.1 it follows that

$$\lim_{n\to\infty}\left\{\frac{y_n}{z_n}-\frac{p_n^* y_{n-m}}{z_{n-m}}\right\}=0,$$

which contradicts (11). Thus a is finite and is equal to zero again by lemma 2.1. What is important to note here is that z_n is eventually negative since it is increasing. The proof for the case $y_n<0$, is similar. □

Remark 2. In the above lemma $\{w_n\}$ is positive and decreasing but $\{z_n\}$ is negative and increasing. Further we may note that $a=-\infty$ is not possible even if z_n is decreasing because p_n satisfies (A_1), (A_2) or (A_3).

Lemma 2.3. *Suppose that $\{p_n\}$ is in one of the ranges (A_1), (A_2) or (A_3). If (H_1), (H_4) and (H_5) hold and $\{y_n\}$ be a nonoscillatory solution of (E) for $n\ge n_0$, then setting z_n as in Eq(4) one may conclude* $\lim_{n\to\infty} z_n=0$.

Proof. By Remark 1(iii) one may observe that (H_2) holds. Here we proceed as in the previous lemma and need only prove the first part of the lemma where (H_3) is used. In that case $l<0$ and we proceed as follows. It is clear that z_n is decreasing and $\lim_{n\to\infty} z_n=a$. We claim that Eq(7) holds. Otherwise $y_n>\beta>0$ for large n say $n>n_1$. Then from Eq(6) one may obtain

$$\sum_{n=n_1}^{i-1}\Delta w_n=-\sum_{n=n_1}^{i-1} q_n G(y_{n-k})\quad \text{for } i>n_1+1,$$

that is

$$w_i-w_{n_1}=-\sum_{n=n_1}^{i-1} q_n G(y_{n-k})<-G(\beta)\sum_{n=n_1}^{i-1} q_n.$$

This implies

$$\Delta z_i < -\frac{G(\beta)}{r_i}\sum_{n=n_1}^{i-1} q_n + \frac{w_{n_1}}{r_i} \quad \text{for } i > n_1 + 1.$$

Then one may find $n_2 > n_1$ and $\delta > 0$ such that for $i \geq n_2 + 1$

$$\Delta z_i < -\frac{G(\beta)}{r_i}\sum_{n=n_2}^{i-1} q_n - \frac{\delta}{r_i}.$$

This implies

$$\begin{aligned}\sum_{i=n_2}^{j-1} \Delta z_i &< -G(\beta)\sum_{i=n_2}^{j-1}\frac{1}{r_i}\sum_{n=n_2}^{i-1} q_n - \delta\sum_{i=n_2}^{j-1}\frac{1}{r_i}\\ &< -G(\beta)\sum_{i=n_2}^{j-1}\frac{1}{r_i}\sum_{n=n_2}^{i-1} q_n \to -\infty,\end{aligned}$$

as $j \to \infty$ by (H_5). Thus $z_j \to -\infty$ as $j \to \infty$, which implies $z_n < 0$ for large n. Then p_n must be in (A_1). This implies y_n is bounded. Consequently z_n is bounded, a contradiction. Hence $\liminf_{n\to\infty} y_n = 0$ holds. Then $\lim_{n\to\infty} z_n = 0$, follows from Lemma 2.1. □

Lemma 2.4. *Suppose that* (H_1), (H_2) *and* (H_5) *hold, and let* $\{p_n\}$ *be in one of the ranges* (A_1), (A_2) *or* (A_3). *If* $\{y_n\}$ *is a non-oscillatory solution of* (E) *for* $n \geq n_0$ *then setting* z_n *as in Eq*(4) *one may obtain* $\lim_{n\to\infty} z_n = 0$.

Proof. The proof for the case when $l \geq 0$ follows from Lemma 2.2, and the proof for the case when $l < 0$ follows from Lemma 2.3. □

Lemma 2.5. *Suppose* $\{p_n\}$ *is in the range* (A_4): $1 \leq p_n \leq b$, *where* b *is a real number. If* (H_1), (H_2) *and* (H_3) *hold and* $\{y_n\}$ *be a non oscillatory solution of* (E) *for* $n \geq n_0$ *then setting* z_n *as in Eq*(4), *one may obtain*

$$\lim_{n\to\infty} z_n = 0 \quad \text{or} \quad \lim_{n\to\infty} z_n = -\infty.$$

Proof. Let $\{y_n\}$ be an eventually positive solution of (E) for $n \geq n_0$. Then setting z_n and w_n as in Eq(4) and Eq(5)respectively one may obtain Eq(6). Then w_n is monotonic and cosequently $\lim_{n\to\infty} w_n = l$ where $-\infty \leq$

$l < \infty$. First consider the case when $l < 0$. Then $\Delta z_n < 0$ and hence $\{z_n\}$ is decreasing. Consequently $\lim_{n\to\infty} z_n = a$, where $-\infty \le a < \infty$. Suppose that $a \ne -\infty$. Then using Remark 1(ii) and proceeding as in Lemma 2.3 we prove $\liminf_{n\to\infty} y_n = 0$. Then application of Lemma 2.1 yields $a = 0$. Since $\{z_n\}$ is decreasing then $z_n > 0$. Because of (A_4) we obtain $\liminf_{n\to\infty} y_n > 0$, a contradiction. Next Consider the case when $l \ge 0$ and is finite. Then we proceed as in Lemma 2.2 and prove $\liminf_{n\to\infty} y_n = 0$. From Eq(5) and the fact that $r_n > 0$ it follows that $\{z_n\}$ is monotonic increasing and hence $\lim_{n\to\infty} z_n = a$, where $-\infty < a \le \infty$. If $a > 0$ then because of (A_4) we obtain $y_n > y_{n-m}$. This implies $\liminf_{n\to\infty} y_n > 0$, a contradiction. If $a \le 0$ then since it is finite so application of Lemma 2.1 yields $a = 0$. The proof for the case when $y_n < 0$ is similar. Thus the lemma is proved. □

Remark 3. In the above Lemma if $l \ge 0$ then $a = 0$ and $z_n < 0$. However, if $l < 0$ then $a = -\infty$.

Lemma 2.6. *Suppose that p_n is in the range (A_4). If (H_1), (H_4) and (H_5) hold and $\{y_n\}$ be a non oscillatory solution of (E) for $n \ge n_0$, then setting z_n as in Eq(4), one may obtain*

$$\lim_{n\to\infty} z_n = 0 \quad \text{or} \quad \lim_{n\to\infty} z_n = -\infty.$$

Proof. The proof is omitted because it follows on similar lines as the proof of the previous lemmas. □

3. Sufficient Conditions

Theorem 3.1. *If we assume (H_1), (H_2) and (H_3) then the following holds.*

(i) *If $\{p_n\}$ is in (A_1) then every solution of (E) oscillates or tends to zero as $n \to \infty$.*

(ii) *If $\{p_n\}$ is in (A_2) or (A_3) then every solution of (E) is oscillatory.*

Proof. First consider the proof of (i). Let $y = \{y_n\}$ be a nonoscillatory and eventually positive solution of (E) for $n \ge n_0$. Then setting z_n as in

Eq(4) and applying Lemma 2.2 we obtain $\lim_{n\to\infty} z_n = 0$. If $\{p_n\}$ is in (A_1) then

$$\begin{aligned} 0 &= \lim_{n\to\infty} z_n = \limsup_{n\to\infty}(y_n - p_n y_{n-m}) \\ &\geq \limsup_{n\to\infty} y_n + \liminf_{n\to\infty}(-p_n y_{n-m}) \\ &\geq (1-b)\limsup_{n\to\infty} y_n, \end{aligned}$$

which implies $\limsup_{n\to\infty} y_n = 0$. Hence $\lim_{n\to\infty} y_n = 0$.

Next consider the proof for (ii). If y_n is an eventually positive solution of (E) and if $\{p_n\}$ is in (A_2) or (A_3) then $z_n > 0$ for large n. But from Lemma 2.2 and Remark 2 it follows that $z_n < 0$. Hence we get a contradiction. The proof for the case $y_n < 0$ is similar. □

Theorem 3.2. *Suppose that* (H_1), (H_4) *and* (H_5) *hold. If* $\{p_n\}$ *is in one of the ranges* (A_1), (A_2) *or* (A_3) *then every solution of* (E) *oscillates or tends to zero as* $n \to \infty$.

Proof. The proof is similar to that of Theorem 3.1 and the only difference is that here Lemma 2.3 is to be applied in place of Lemma 2.2. If $\{p_n\}$ satisfies (A_2) or (A_3) then since $y_n \leq z_n$, implies $\lim_{n\to\infty} y_n = 0$. □

Theorem 3.3. *Let* $\{p_n\}$ *be in one of the ranges* (A_1), (A_2), (A_3) *and suppose that* (H_1), (H_2), (H_5) *hold. Then every solution of* (E) *oscillates or tends to zero as* $n \to \infty$.

Proof. It follows from Theorem 3.1 with the application of Lemma 2.4. If $\{p_n\}$ satisfies (A_2) or (A_3) then since $y_n \leq z_n$, implies $\lim_{n\to\infty} y_n = 0$. □

Example 1. Consider the NDDE

$$\Delta^2(y_n - 4y_{n-1}) + 4^{(n+1)/3} y_{n-2}^{1/3} = 0, \quad n \geq 3$$

In the above NDDE $r_n \equiv 1$ is in (H_3). Also $(H_1), (H_2)$ hold. But p_n is in (A_4). It may be verified that $y_n = 2^n$ is a solution of the above NDDE which $\to \infty$ as $n \to \infty$. This example is the motivation behind our next result.

Theorem 3.4. *Suppose that* (H_1), (H_2) *and* (H_3) *hold. If* $\{p_n\}$ *is in*

(A_5) $b_2 \geq p_n \geq b_1 > 1$, *then every bounded solution of* (E) *oscillates or tends to zero as* $n \to \infty$.

Proof. Let $y = \{y_n\}$ be an nonoscillatory and eventually bounded positive solution of (E) for $n \geq n_0$. Then setting z_n as in Eq(4) and applying Lemma 2.5 we obtain $\lim_{n\to\infty} z_n = 0$. Then

$$
\begin{aligned}
0 = \lim_{n\to\infty} z_n &= \liminf_{n\to\infty}(y_n - p_n y_{n-m}) \\
&\leq (1 - b_1) \limsup_{n\to\infty} y_n.
\end{aligned}
$$

Hence $\lim_{n\to\infty} y_n = 0$. The proof for the case $y_n < 0$ is similar. □

Theorem 3.5. *Suppose* $\{p_n\}$ *is in* (A_5). *If* (H_1), (H_4) *and* (H_5) *hold then every bounded solution of* (E) *oscillates or tends to zero as* $n \to \infty$.

Proof. The proof is similar to that of Theorem 3.4 but here Lemma 2.6 is to be applied. □

Example 2. Consider the NDDE

$$
\Delta^2(y_n - 2e^{-1} y_{n-1}) + \frac{(1-e)^2}{e^5} y_{n-3} = 0 \tag{12}
$$

Here $p_n = \frac{2}{e} \in (A_1)$, $r_n \equiv 1$, $q_n = \frac{(1-e)^2}{e^5}$. It is clear that the above NDDE satisfies all the conditions (H_1), (H_2) and (H_3) of Theorem 3.1(i). Since (H_5) too holds then it satisfies all the conditions of Theorem 3.3 also. Hence all solutions of Eq(12) oscillates or tends to zero as $n \to \infty$. As such $y_n = e^{-n}$ is a non-oscillatory solution which $\to 0$ as $n \to \infty$. Further this example contradicts the conclusion of Theorem 8 of [1] which says all solutions of (E) are oscillatory under the conditions (H_2), (H_3), (A_1) and m odd.

Example 3. The NDDE

$$
\Delta^2(y_n + \frac{1}{2} y_{n-1}) + 2y_{n-3}^{\frac{1}{3}} = 0
$$

satisfies all the conditions of Theorem 3.1(ii) and hence all its solutions are oscillatory. As such $y_n = (-1)^n$ is the oscillatory solution of the above NDDE. But the results of [9] cannot be applied to this NDDE as $G(U) = U^{\frac{1}{3}}$ is sublinear and does not satisfy (1).

Example 4. The NDDE

$$\Delta(2^{2n}\Delta(y_n - \frac{1}{2}y_{n-1})) + 4^{n-2}y_{n-2} = 0$$

satisfies all the conditions of Theorems 3.2 and 3.3. Here (H_2), (H_4) and (H_5) hold. The range of p_n is (A_1). Hence all solutions either oscillate or tend to zero, as such $y_n = 2^{-n}$ is the solution which $\to 0$ as $n \to \infty$.

Theorem 3.6. *Suppose that* (H_1) *and* (H_3) *hold. Let* $\{p_n\}$ *be in the range* (A_4) *and if for every sequence* $\{q_{n_j}\}$ *of* $\{q_n\}$

$$(H_6) \qquad \sum_{j=0}^{\infty} q_{n_j} = \infty$$

holds then every solution of (E) *oscillates or tends to zero or tends to* $\pm\infty$ *as* $n \to \infty$.

Proof. First we may note that (H_6) implies (H_2) and we follow the proof of Lemma 2.5 to prove this result. Let $y = \{y_n\}$ be a non-oscillatory and eventually positive solution of (E) for $n \geq n_0$. Then set z_n and w_n as in Eq(4) and Eq(5)respectively. Then let $\lim_{n\to\infty} w_n = l$ and $\lim_{n\to\infty} z_n = a$. Consider the first case $l < 0$. Then by Remark 3 one may obtain $a = -\infty$ Then from (A_4) and Eq(4) it follows that $y_{n-m} \geq -\frac{z_n}{b}$. This implies $\lim_{n\to\infty} y_n = \infty$. Next consider the second case that $l \geq 0$ and is finite. Using Remark 3 we get $a = 0$. Then taking summation of Eq(6) from $n = n_2$ to ∞ we obtain Eq(9). We claim $\{y_n\}$ is bounded. Otherwise if y_n is unbounded then there exists a subsequence $\{y_{n_j}\}$ such that $y_{n_j-k} > \alpha > 0$ for $j > n_1$. Hence

$$\sum_{j=n_1}^{\infty} q_{n_j} G(y_{n_j-k}) > G(\alpha) \sum_{j=n_1}^{\infty} q_{n_j} = \infty \tag{13}$$

by (H_6) which contradicts (9). Hence y_n is bounded. If $\limsup_{n\to\infty} y_n = \gamma > 0$ then there exists a subsequence $\{y_{n_j}\}$ such that $y_{n_j-k} > \alpha > 0$ for $j > n_1$. From this using (H_6) we again obtain Eq(13) which contradicts Eq(9). Hence $\limsup_{n\to\infty} y_n = 0$. Thus $\lim_{n\to\infty} y_n = 0$. The proof for the case when $y_n < 0$ is similar. $\square$

Remark 4. Theorem 2.6 of [14, pp.761] for m even says "Under (A_4) and (H_2) every solution of (E) with $r_n \equiv 1$ is oscillatory." Theorem 2.7 of [14, pp.762] for m even says "Under (A_5) and (H_2) every solution of (E) with $r_n \equiv 1$ oscillates." Our Example 1 contradicts both the theorems. The above two theorems of this paper are the corresponding correct results.

Example 5. Consider the NDDE

$$\Delta(n^{-1}\Delta(y_n - 4y_{n-1}) + \frac{4^{\frac{n+1}{3}}(n-1)}{n(n+1)} y_{n-2}^{\frac{1}{3}} = 0, \quad n \geq 3$$

In the above NDDE $r_n = n^{-1}$ is in $(H_3), q_n = \frac{4^{\frac{n+1}{3}}(n-1)}{n(n+1)}$. Also $(H_1), (H_6)$ hold. Here $p_n = 4$ is in (A_4). It may be veryfied that $y_n = 2^n$ is a solution of the above NDDE which $\to \infty$ as $n \to \infty$. This example illustrates Theorem 3.6

Theorem 3.7. *Let* $\{p_n\}$ *satisfy*

(A_6): $\quad -d \leq p_n \leq 0,$

where d *is a positive real number. Suppose that* (H_1) *and* (H_3) *hold. Further assume*

- (H$_7$) $\sum_{n=s}^{\infty} q_n^* = \infty$, *where* $q_n^* = \min(q_n, q_{n-m})$,
- (H$_8$) *For* $u > 0, v > 0$ *there exists* $\delta > 0$ *such that* $G(u) + G(v) \geq \delta G(u+v)$,
- (H$_9$) $G(u)G(v) \geq G(uv)$ *for* $u > 0, v > 0$,
- (H$_{10}$) $G(-u) = -G(u)$.

Then every solution of (E) *oscillates or tends to zero as* $n \to \infty$.

Proof. Let $y = \{y_n\}$ be an eventually positive solution of (E) for $n \geq n_0 > 0$. Then setting z_n, w_n as in (2.1) and (2.2) we arrive at (2.3). Consequently $\lim_{n\to\infty} w_n = l$, where $-\infty \leq l < \infty$. Consider the first case $l < 0$. Then $w_n < 0$ and consequently $\Delta z_n < 0$. Because of (A_6) we have $\lim_{n\to\infty} z_n = a$ where $a \geq 0$ and is finite. Then as in Lemma 2.2 one may use (H_3) and obtain $z_n \to -\infty$, a contradiction. Next consider the other case $l \geq 0$. Since $r_n > 0$ then $\Delta z_n > 0$. Consequently $\lim_{n\to\infty} z_n = a$ where $-\infty < a \leq \infty$. Due to (A_6), $z_n > 0$. Hence $a \geq 0$. If $a = 0$ then

$\lim_{n\to\infty} y_n = 0$ follows from the inequality $y_n \le z_n$. If $a > 0$ then $z_n > \lambda > 0$ for large $n > n_1$. Then using (H_7) and (H_{10}) we obtain for $n \ge n_1$

$$\begin{aligned} 0 &= \Delta w_n + q_n G(y_{n-k}) + G(-p_{n-k})\{\Delta w_{n-m} + q_{n-m} G(y_{n-k-m})\} \\ &\ge \Delta w_n + G(d)\Delta w_{n-m} + \delta q_n^* G(z_{n-k}) \\ &\ge \Delta w_n + G(d)\Delta w_{n-m} + \delta q_n^* G(\lambda). \end{aligned}$$

If we take sum from $n = n_2$ to $n = i-1$ and take limit $i \to \infty$ and use (H_7) then we obtain the contradiction that $w_i + G(d) w_{i-m} \to -\infty$ as $i \to \infty$. The proof for the case $y_n < 0$ for large n is similar. In this case (H_{10}) is required. Thus the theorem is proved. □

Remark 5.

(i) (H_7) implies (H_2) but not conversely.

(ii) The prototype of G satisfying $(H_1), (H_8), (H_9)$ and (H_{10}) is $G(u) = (\beta + |u|^\mu)|u|^\lambda sgnu$ where $\lambda > 0, \mu > 0, \beta \ge 1$. For proof one may refer [7, page-292].

Theorem 3.8. *Let p_n satisfy (A_6). Suppose that (H_1), (H_4), and $(H_8) - (H_{10})$ hold. Further if*

(H_{11}) $\sum_{n=n_0}^{\infty} \frac{1}{r_n} \sum_{i=s}^{n-1} q_i^* = \infty, \quad$ for $n_0 > s$

holds, then every solution of (E) oscillates or tends to zero as $n \to \infty$.

Proof. It may be noted that (H_4) and (H_{11}) implies (H_7). Then the proof is similar to that of the above theorem except in case when $w_n < 0$ we use (H_{11}) to get a contradiction. □

Theorem 3.9. *Let p_n satisfy (A_6). Suppose that (H_1), and $(H_7)-(H_{11})$ hold. Then every solution of (E) oscillates or tends to zero as $n \to \infty$.*

Proof. The proof is similar to that of the above theorem. When $w_n > 0$ we use (H_7) and proceed as in Theorem 3.7 and in case when $w_n < 0$ we use (H_{11}) as in Theorem 3.8 to get a contradiction. □

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