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RIGHT CHAIN PO-Γ-SEMIGROUPS

BY

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Abstract

In this paper, we prove, among others, that in right chains, the proper ideals are completely prime if and only if they are completely semiprime and that the proper right ideals are prime if and only if they are semiprime. Moreover, the proper non-zero idempotent ideals are completely prime.

1. Preliminaries

The concept of po-Γ-semigroup was introduced by Y. I. Kwon and S. K. Lee in 1996, and it has been studied by several authors [1, 8].

Let M and Γ be any two non-empty sets. M is called a Γ-semigroup if there exists a mapping $M \times \Gamma \times M \to M$, written as $(a,\ \gamma,\ b) \to a\gamma b$, satisfying the following identities $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in M$ and $\gamma,\ \mu \in \Gamma$.

A po-Γ-semigroup (: partially ordered Γ-semigroup) is an ordered set M which is a Γ-semigroup such that

$a \leq b \Rightarrow a\gamma c \leq b\gamma c$ and $c\gamma a \leq c\gamma b$ for all $a, b, c \in M$ and $\gamma \in \Gamma$[1].

From [1], an element $a \in M$ is called a zero element of M if $x\gamma a = a\gamma x = a$ and $a \leq x$ for all $x \in M$ and $\gamma \in \Gamma$ and it is denoted by 0. An element $e \in M$ is called an identity element of M if $x\gamma e = e\gamma x = x$ and $x \leq e$ for all $x \in M, \gamma \in \Gamma$ and it is denoted by e.

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Example 1.1. Let $M = \{\{a,b,c\}, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$ and $\Gamma = \{\{a,b,c\}, \phi, \{a\}\}$. If $ABC = A \cap B \cap C$ and $A \leq C \Leftrightarrow A \subseteq C$ for all $A, C \in M$ and $B \in \Gamma$, then M is a po-Γ-semigroup with zero element.

In general, let $P(X)$ be the power set of any non-empty set X and τ a topology on X. If we define $ABC = A \cap B \cap C$ and $A \leq C \Leftrightarrow A \subseteq C$ for all $A, C \in P(X)$ and $B \in \tau$, then $P(X)$ is a po-τ- semigroup with zero element.

Example 1.2. Let $\Gamma = \{\{a,b,c\}\}$, and M is as in the Example 1.1. If $ABC = A \cap B \cap C$ and $A \leq C \Leftrightarrow A \subseteq C$ for all $A, C \in M$ and $B \in \Gamma$, then M is a po-Γ-semigroup with zero and identity element.

In this paper, M stands for a nonzero po-Γ-semigroup with identity and zero element.

A po-Γ-semigroup M is called a chain if for any $a, b \in M$, either $a \leq b$ or $b \leq a$.

For a subset A of M, we denote $(A] = \{t \in M \ : \ t \leq a \text{ for some } a \in A\}$. We see that $A \subseteq (A]$, $((A]] = (A]$, $(A]\Gamma(B] \subseteq (A\Gamma B]$ for all $A, B \subseteq M$ and $A \subseteq (B]$ for $A \subseteq B \subseteq M$([6] and [1]).

A non-empty subset L of M is called a left (resp. right) ideal of M if $M\Gamma L \subseteq L$ (resp. $L\Gamma M \subseteq L$) and for all $a \in L,\ b \in M,\ b \leq a$ implies $b \in L$. A non-empty subset I of M is called an ideal if I is a both left and right ideal of M (cf. [5] and [10]) .

We denote by $I(a)$ (resp. $R(a),\ L(a)$) the ideal (resp. right ideal, left ideal) of M generated by a. We can easily prove that $I(a) = (M\Gamma a\Gamma M];\ R(a) = (a\Gamma M];\ L(a) = (M\Gamma a]$.

For any one sided ideal A of M, we have $(A] = A$. Also, if A and B are ideals of M, then $(A\Gamma B]$ and $A \cup B$ are also ideals of M. Clearly if A is any one sided ideal of M, then $B \subseteq A$ if and only if $(B] \subseteq A$ for any subset B of M.

A po-Γ-semigroup M is called a right chain if

i) For any $a, b \in M$, there exists $c \in M$ and $\gamma \in \Gamma$ with $a \leq b\gamma c$ or $b \leq a\gamma c$.

ii) If $a\gamma b \leq a\gamma_1 c$ with $a\gamma b \neq 0$ for any $a, b, c \in M$ and $\gamma, \gamma_1 \in \Gamma$, then $b \leq c\gamma_2 t$ for some $\gamma_2 \in \Gamma$ and $t \in M$.

Observe that every chain is a right chain, and the condition (i) implies that for any two right ideals $I_1,\ I_2$ of M, either $I_1 \subseteq I_2$ or $I_2 \subseteq I_1$.

Example 1.3. Let $M = \{\{a,b,c\}, \phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}\}$ and $\Gamma = \{\phi\}$. If $ABC = A \cup B \cup C$ and $A \leq C \Leftrightarrow A \subseteq C$ for all $A, C \in M$ and $B \in \Gamma$, then M is a po-Γ-semigroup with zero and identity element. It is a right chain, but it is not chain.

An element $u \in M$ is called a unit of M if there exists an element $v \in M$ with $u\gamma v = v\gamma u = e$ for all $\gamma \in \Gamma$. A right ideal (resp. ideal) A of M is said to be an idempotent if $(A\Gamma A] = A$[6]. A right ideal (resp. ideal) R of M is called proper if $R \neq M$ ([4] and [10]). A proper right ideal (resp. ideal) P of M is said to be prime if for any right ideals (resp. ideals) $A,\ B$ of M such that $A\Gamma B \subseteq P$, then $A \subseteq P$ or $B \subseteq P$. Clearly a right ideal P is prime if and only if $a\Gamma M\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$. An ideal P is called completely prime if $a\Gamma b \subseteq P$ implies $a \in P$ or $b \in P$[5]. The union of all proper right ideals of M is denoted by J. Clearly J is a proper right ideal of M. An ideal Q is called exceptional prime if Q is a prime ideal which is not completely prime ideal. We set P as the intersection of all completely prime ideals containing Q.

For any subset S of M and for any $\gamma \in \Gamma$, let $S^n = S\ \overbrace{\Gamma S ... \Gamma S}^{n-1\ term\ \Gamma S}$ and $S^n_\gamma = S\ \overbrace{\gamma S ... \gamma S}^{n-1\ term\ \gamma S}$ for any $n \in \mathbb{N}$.

We say that an element $a \in M$ is $Q-$nilpotent (resp. nilpotent) if $a^n \in Q$ (resp. $a^n = 0$) for some $n \in \mathbb{N}$.

2. Main Results

Lemma 2.1. *Let M be a right chain. Then*

(1) *J is an ideal of M*

(2) *All one-sided units in M are units.*

Proof. (1) Let $m \in M,\ \gamma \in \Gamma,\ j \in J$. If $m\gamma j \notin J$, then $m \neq 0$ and $m \leq m\gamma j\gamma_1 m_1$ for some $\gamma_1 \in \Gamma$ and $m_1 \in M$ which implies $e \leq j\gamma_1 m_1 \gamma_3 m_2 \in J$ for some $m_2 \in M$ and some $\gamma_3 \in \Gamma$, a contradiction.

(2) Suppose $u \in M$ is a right unit. Then there exists $v \in M$ such that $u\gamma v = e$ for all $\gamma \in \Gamma$.

If $(v\Gamma M] = M$, then $e = v\gamma_1 u_1$ for some $u_1 \in M$ and $\gamma_1 \in \Gamma$. Now $u\gamma_2 e = u\gamma_2 v\gamma_1 u_1$ for all $\gamma_2 \in \Gamma$. Thus $u = u_1$. Hence $v\gamma u = e$. Otherwise $(v\Gamma M] \subseteq J$. Then $v \in J$, is a contradiction. $\square$

Theorem 2.2. *Let M be a right chain. Then*

(1) *An ideal $I \neq M$ of M is completely prime if and only if I is completely semiprime.*

(2) *A right ideal $I \neq M$ of M is prime if and only if I is semiprime*

Proof. (1) Suppose $x\Gamma y \subseteq I$ for some $x, y \in M$. If $x \leq y\gamma c$ for some $\gamma \in \Gamma$ and $c \in M$, then $x\gamma_1 x \leq x\gamma_1 y\gamma c \in I$ for all $\gamma_1 \in \Gamma$ which implies $x\gamma_1 x \in I$. Thus $x\Gamma x \subseteq I$ and hence $x \in I$.

Otherwise $y \leq x\gamma c$ for some $\gamma \in \Gamma$ and $c \in M$. Now $(y\gamma_1 x)\gamma_2(y\gamma_1 x) = y\gamma_1(x\gamma_2 y)\gamma_1 x \in I$ for all $\gamma_1, \gamma_2 \in \Gamma$. Then $y\gamma_1 x \in I$ for all $\gamma_1 \in \Gamma$ which implies $y\gamma_3 y \leq y\gamma_3 x\gamma c = (y\gamma_3 x)\gamma c \in I$ for all $\gamma_3 \in \Gamma$, and so $y \in I$.

(2) Suppose $a\Gamma M\Gamma b \subseteq I$. If $a \leq b\gamma c$ for some $\gamma \in \Gamma$ and $c \in M$, then $a\gamma_1 m\gamma_2 a \leq a\gamma_1 m\gamma_2 b\gamma c \in a\Gamma M\Gamma b\Gamma c \subseteq I$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $m \in M$ which implies $a\Gamma M\Gamma a \subseteq I$. So $a \in I$.

Otherwise $b \leq a\gamma c$ for some $\gamma \in \Gamma$ and $c \in M$. Then $b\gamma_1 m\gamma_2 b \leq a\gamma c\gamma_1 m\gamma_2 b \in a\Gamma M\Gamma M\Gamma b \subseteq a\Gamma M\Gamma b \subseteq I$ for any $\gamma_1, \gamma_2 \in \Gamma$ and $m \in M$ which implies $b\Gamma M\Gamma b \subseteq I$, and so $b \in I$. $\square$

Lemma 2.3. *Let M be a right chain. Then J is a completely prime ideal of M.*

Proof. From Lemma 2.1, J is an ideal of M. Let $x \notin J$ for some $x \in M$. Then $(x\Gamma M] = M$ and $e = x\gamma m_1$ for some $\gamma \in \Gamma$ and $m_1 \in M$. Suppose $x\Gamma x \subseteq J$. Then $x\gamma_1 e = x\gamma_1 x\gamma m_1 \in J$ for all $\gamma_1 \in \Gamma$. Thus $x \in J$, is a contradiction. Therefore J is a completely prime ideal by Theorem 2.2. $\square$

Theorem 2.4. *Let M be a right chain and $I \neq M$ be an ideal of M.*

(1) *If $I^n \neq \{0\}$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} (I^n] = P$ is a completely prime ideal of M.*

(2) *Idempotent ideals $\neq M, \{0\}$ are completely prime.*

(3) *Let* $t \in J$. *If* $t^n \Gamma M \neq \{0\}$ *for all* $n \in \mathbb{N}$, *then* $\bigcap_{n\in\mathbb{N}} (t^n \Gamma M] = P$ *is a prime right ideal.*

If P *is, in addition, an ideal and* M *is a chain, then* $\bigcap_{n\in\mathbb{N}} (t^n \Gamma M] = P$ *is completely prime.*

Proof. **(1)** Assume that $I^n \neq \{0\}$ for any $n \in \mathbb{N}$.

Case(a) If $I = (I^2]$, then $P = I$. Indeed: Since $I = (I\Gamma I] = ((I\Gamma I]\Gamma I] \subseteq (I\Gamma I\Gamma I] \subseteq I$, we have $(I\Gamma I\Gamma I] = I$. In a similar way we get $(I^n] = I$ for any $n \in \mathbb{N}$. Let $x \notin I$ for some $x \in M$. Then $I \subset (x\Gamma M]$. Now $I = (I\Gamma I] \subseteq ((x\Gamma M]\Gamma I] \subseteq (x\Gamma M\Gamma I] \subseteq (x\Gamma I]$. Thus $x\Gamma I \subseteq x\Gamma(x\Gamma I]$. Also, $I = (I\Gamma I] \subseteq ((x\Gamma M]\Gamma I] \subseteq (x\Gamma M\Gamma I] \subseteq (x\Gamma I] \subseteq (x\Gamma(x\Gamma I]] \subseteq (x\Gamma x\Gamma I]$, thus there exists $\gamma_1 \in \Gamma$ such that $x\gamma_1 x \neq 0$.

Suppose $x\Gamma x \subseteq I$, then $I = (x\Gamma I]$. Since $x\gamma_1 x \in I = (x\Gamma I]$, we have $x\gamma_1 x \leq x\gamma_2 i$ for some $i \in I$ *and* $\gamma_2 \in \Gamma$ which implies $x \leq i\gamma_3 c \in I$ for some $\gamma_3 \in \Gamma$ and $c \in M$. Thus $x \in I$, a contradiction.

Case (b) If $P = (I^n]$ for some $n = 2, 3, \ldots$, then $(I^n] = P \subseteq (I^n \Gamma I^n] \subseteq ((I^n]\Gamma(I^n]] \subseteq (I^n]$. Thus $((I^n]\Gamma(I^n]] = (I^n]$. By using the previous argument, we have that P is completely prime.

Case (c) Let $P \subset (I^n]$ for all $n \in \mathbb{N}$ and $x \notin P$ for some $x \in M$. Then there exists $n \in \mathbb{N}$ with $(I^n] \subset (x\Gamma M]$. Now $P \subset (I^{2n}] = (I^n \Gamma I^n] \subseteq ((I^n]\Gamma I^n] \subseteq ((x\Gamma M]\Gamma I^n] \subseteq (x\Gamma M\Gamma I^n] \subseteq (x\Gamma I^n] \subseteq (x\Gamma(I^n]] \subseteq (x\Gamma(x\Gamma M]] \subseteq (x\Gamma x\Gamma M]$. Hence there exists $\gamma_1 \in \Gamma$ such that $x\gamma_1 x \neq 0$.

Suppose $x\Gamma x \subseteq P$, then $P = (x\Gamma I^n]$. Since $x\gamma_1 x \in P$, we have $x\gamma_1 x \leq x\gamma_2 i$ for some $i \in I^n$ and $\gamma_2 \in \Gamma$. Then $x \leq i\gamma_3 c \in I^n$ for some $\gamma_3 \in \Gamma$ *and* $c \in M$, thus $x \in (I^n]$, a contradiction.

(2) It follows directly from (1).

(3) Let $t \in J$ and $t^n \Gamma M \neq 0$ for all $n \in \mathbb{N}$. Then $t\gamma_1 t \ldots \gamma_{n-1} t \neq 0$ for some $\gamma_1, \gamma_2, \ldots \gamma_{n-1} \in \Gamma$. Suppose $(t^{n+1}\Gamma M] = (t^n \Gamma M]$. Then $t\gamma_1 t \ldots \gamma_{n-1} t\gamma e \leq t\gamma_1^{'} t \ldots \gamma_n^{'} t\gamma_2 m_2$ for all $\gamma \in \Gamma$ and for some $\gamma_1^{'}, \gamma_2^{'}, \ldots, \gamma_{n-1}^{'}, \gamma_2 \in \Gamma,\ m_2 \in M$ which imply $e \leq t\gamma^{'} m_1 \in J$ for some $\gamma^{'} \in \Gamma$ and $m_1 \in M$, a contradiction. Hence $(t^{n+1}\Gamma M] \subset (t^n \Gamma M]$ and $P \subset (t^n \Gamma M]$ for all $n \in \mathbb{N}$.

Let $x \notin P$ for some $x \in M$ and suppose $x\Gamma M\Gamma x \subseteq P$. Then $(t^n \Gamma M] \subset (x\Gamma M]$ for some $n \in \mathbb{N}$. Now $(t^{2n}\Gamma M] = (t^n \Gamma t^n \Gamma M] = (t^n \Gamma e \Gamma t^n \Gamma M] \subseteq$

$(t^n\Gamma M\Gamma t^n\Gamma M] \subseteq ((x\Gamma M]\Gamma(x\Gamma M]] \subseteq (x\Gamma M\Gamma x\Gamma M] \subseteq P$. Thus $(t^{2n}\Gamma M] \subseteq P$, a contradiction. Hence $x\Gamma M\Gamma x \nsubseteq P$.

Suppose M is a chain and P is an ideal. Let $x \notin P$ for some $x \in M$ and suppose $x\Gamma x \subseteq P$. Then $x \notin (t^n\Gamma M]$ for some $n \in \mathbb{N}$. Since M is a chain, we have $t^n = x$. Then $t^n\Gamma t^n \subseteq x\Gamma x \subseteq P$. Since P is a right ideal, we have $t^{2n} \in P$. Thus $t^{2n}\Gamma M \subseteq P$ and hence $(t^{2n}\Gamma M] \subseteq P$, a contradiction. So P is a completely prime ideal. $\square$

Theomrem 2.5. *Let M be a right chain and I a proper non-zero ideal of M.*

(1) *Let $\gamma \in \Gamma$. If $I^n_\gamma \neq \{0\}$ for all $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} (I^n_\gamma] = P$ is a completely prime ideal of M.*
(2) *If $(I^2_\gamma] = I$ for all $\gamma \in \Gamma$, then I is a completely prime ideal.*
(3) *Let $t \in J$ and $\gamma \in \Gamma$. If $t^n_\gamma\gamma M \neq \{0\}$ for any $n \in \mathbb{N}$, then $\bigcap_{n\in\mathbb{N}} (t^n_\gamma\gamma M] = P$ is a prime right ideal.*

If P is, in addition, an ideal and M is a chain, then $\bigcap_{n\in\mathbb{N}} (t^n_\gamma\gamma M] = P$ is a completely prime ideal.

Proof. The proof is similar to the proof of Theorem 2.4. $\square$

Theorem 2.6. *Let M be a right chain and Q be an exceptional prime ideal. Then $Q = \{0\}$ or $Q\Gamma Q \subset Q$ holds. Furthermore, there exists a completely prime ideal P minimal over Q. Also, P is an idempotent and there is no ideal between P and Q.*

Proof. Suppose $Q \neq \{0\}$. If $Q\Gamma Q = Q$, then $(Q\Gamma Q] = Q$. By Theorem 2.4(2), Q is a completely prime ideal, a contradiction. Hence $Q\Gamma Q \subset Q$.

From Lemma 2.3, there exists a completely prime ideal minimal over Q.

We show that P is an idempotent ideal. Clearly $P^n \neq 0$ for all $n \in \mathbb{N}$ and $Q \subset P^n$ for all $n \in \mathbb{N}$.

Consider $P^n \neq 0$ and $Q \subset P^n$ for all $n \in \mathbb{N}$. By Theorem 2.4(1), $\bigcap_{n\in\mathbb{N}} (P^n]$ is a completely prime ideal containing Q. Thus $Q \subseteq \bigcap_{n\in\mathbb{N}} (P^n] \subseteq P$. Since P is a minimal completely prime ideal containing Q, we have $P = \bigcap_{n\in\mathbb{N}} (P^n]$. Thus $P \subseteq (P^2] = (P\Gamma P]$. Hence P is idempotent.

Suppose I is an ideal of M such that $Q \subseteq I \subseteq P$. Suppose $I^n = 0$ or $I^n \subseteq Q$ or $P \subseteq I^n$, then $I = Q$ or $I = P$. So let us assume that $I^n \neq \{0\}$, $Q \subseteq I^n$ and $I^n \subseteq P$ for all $n \in \mathbb{N}$.

By Theorem 2.4(1), $\bigcap\limits_{n\in\mathbb{N}} (I^n]$ is a completely prime ideal containing Q. Since P is a minimal completely prime ideal containing Q, we have $P = \bigcap\limits_{n\in\mathbb{N}} (I^n]$. Thus $P = I$. □

Lemma 2.7. *Let M be a right chain. If P is an ideal and Q is an ideal with $Q \subseteq P$ and $a\Gamma a \subseteq Q$ for all $a \in P$, then $(P\Gamma P\Gamma P] \subseteq Q$.*

Proof. Let $x \in (P\Gamma P\Gamma P]$. Then $x \leq a\gamma_1 b\gamma_2 c$ for some $a, b, c \in P$ and $\gamma_1, \gamma_2 \in \Gamma$. If $b \leq a\gamma_3 x$ for some $\gamma_3 \in \Gamma$ and $x \in M$, then $a\gamma_1 b\gamma_2 c \leq a\gamma_1 a\gamma_3 x = (a\gamma_1 a)\gamma_3 x \in Q$, and so $x \in Q$. Otherwise $a \leq b\gamma_3 x$. Then $a\gamma_1 b\gamma_2 c \leq b\gamma_3 x\gamma_1 b\gamma_2 c$.

If $b\gamma_2 c \leq b\gamma_3 x\gamma_4 t$ for some $\gamma_4 \in \Gamma$ and $t \in M$, then $a\gamma_1 b\gamma_2 c \leq b\gamma_3 x\gamma_1(b\gamma_2 c)$ $\leq (b\gamma_3 x)\gamma_1(b\gamma_3 x)\gamma_4 t \in Q$. Hence $x \in Q$. Otherwise $b\gamma_3 x \leq b\gamma_2 c\gamma_4 t$. Then $a\gamma_1 b\gamma_2 c \leq b\gamma_2 c\gamma_4 t\gamma_1 b\gamma_2 c$.

Now, if $c\gamma_4 t\gamma_1 b \leq b\gamma_5 m$ for some $\gamma_5 \in \Gamma; m \in M$, then $a\gamma_1 b\gamma_2 c \leq b\gamma_2(c\gamma_4 t\gamma_1 b)\gamma_2 c \leq b\gamma_2 b\gamma_5 m\gamma_2 c \in Q$. Hence $x \in Q$. Otherwise $b \leq c\gamma_4 t\gamma_1 b\gamma_5 m$. Then $b \leq (c\gamma_4 t)\gamma_1(c\gamma_4 t)\gamma_1 b\gamma_5 m\gamma_5 m \in Q$. Hence $x \in Q$. □

Theorem 2.8. *Let M be a right chain, Q is an exceptional prime ideal and P is a completely prime ideal minimal over Q. Then*

(1) *For any $a \in P\backslash Q$, there exists $\gamma \in \Gamma$ such that $a\gamma a \notin Q$.*
(2) *There exists an element in $P\backslash Q$ which is not $Q-$nilpotent.*
(3) *If M is a chain, then there exists $a \in P\backslash Q$ with $Q \subset \bigcap\limits_{n\in\mathbb{N}} (a^n\Gamma M]$.*

Proof. **(1)** Suppose $a \in P\backslash Q$ and $a\gamma a \in Q$ for all $\gamma \in \Gamma$. Then $a\Gamma a \subseteq Q$. By Lemma 2.7, we have $(P\Gamma P\Gamma P] \subseteq Q$ which implies $P = Q$ since Q is exceptional prime ideal, a contradiction. Hence $a\gamma a \notin Q$ for some $\gamma \in \Gamma$.

(2) Suppose all elements in P/Q are $Q-$nilpotent. Let $\gamma \in \Gamma$, and $a \in P/Q$. By (1), there exists $\gamma_3 \in \Gamma$ with $a\gamma_3 a \notin Q$. We show that $M\gamma a\gamma_3 a \subseteq a\Gamma M$. Let $m \in M$. If $m\gamma a\gamma_3 a \leq a\gamma_2 s$ for some $\gamma_2 \in \Gamma$ and $s \in M$, we are done. Otherwise $a \leq m\gamma a\gamma_3 a\gamma_2 s \leq (m\gamma a)\gamma_3(m\gamma a)\cdots\gamma_3(m\gamma a)\gamma_2' s' \in Q$ for some

$\gamma_2^{'} \in \Gamma$ and $s^{'} \in M$. Then $a \in Q$, a contradiction. Thus $M\gamma a\gamma_3 a \subseteq a\Gamma M$ and hence $(M\gamma a\gamma_3 a\Gamma M] \subseteq (a\Gamma M]$. Clearly $Q \subset (M\gamma a\gamma_3 a\Gamma M] \subseteq P$.

If $P = (M\gamma a\gamma_3 a\Gamma M]$, then $P = (a\Gamma M]$. Since $a \in P\backslash Q$, we have $a^n \in Q$ with $a^{n-1} \notin Q$ for some $n \in \mathbb{N}$. Consider $(M\gamma a^{n-1}\Gamma M\Gamma P]$ and Q. Clearly $(M\gamma a^{n-1}\Gamma M\Gamma P]$ is an ideal of M.

If $(M\gamma a^{n-1}\Gamma M\Gamma P] \subseteq Q$, then $(M\gamma a^{n-1}\Gamma M] \subseteq Q$ or $P \subseteq Q$. Thus $a^{n-1} \in Q$ or $P \subseteq Q$, a contradiction. So $Q \subseteq (M\gamma a^{n-1}\Gamma M\Gamma P]$. Then $Q \subseteq (M\gamma a^{n-1}\Gamma M\Gamma P] \subseteq (M\gamma a^{n-1}\Gamma P] = (M\gamma a^{n-1}\Gamma(a\Gamma M]] \subseteq (M\gamma(a^{n-1}\Gamma a)\Gamma M] = (M\gamma a^n\Gamma M] \subseteq Q$. Thus $(M\gamma a^{n-1}\Gamma M\Gamma P] = Q$. Since $(M\gamma a^{n-1}\Gamma M]\Gamma P \subseteq (M\gamma a^{n-1}\Gamma M\Gamma P] = Q$ and Q is a exceptional prime ideal, we have $(M\gamma a^{n-1}\Gamma M] \subseteq Q$ or $P \subseteq Q$, a contradiction. Thus inclusion does not exists in between two ideals $(M\gamma a^{n-1}\Gamma M\Gamma P]$ and Q, a contradiction. Hence $(M\gamma a\gamma_3 a\Gamma M]$ is a proper ideal between P and Q, which is also a contradiction to Theorem 2.6. Hence there exists an element in $P\backslash Q$ which is not a Q-nilpotent.

(3) It follows from (2). □

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