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EXACT STRONG LAWS FOR SIMULTANEOUS ST. PETERSBURG GAMES

BY

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This paper is written in memory of Professor Sándor Csörgő who passed away on February 14, 2008. Sándor was both a mentor and a friend. I will miss him.

Abstract

Herein we show a simple way to make the St. Petersburg Game fair for both Peter and Paul. By just considering the minimum of two simultaneous games we establish a way that could easily have been understood three centuries ago. Similarly, we show how to play the St. Petersburg Game by observing the maximum of two concurrent games. After showing how to make this fair for both Peter and Paul by playing two games at a time we extend this so that they can play m simultaneous St. Petersburg Games.

1. Introduction

We let X denote the winnings of Paul in a generalized St. Petersburg Game. By a generalized St. Petersburg Game it is understood that the coin may be biased where p is the probability that heads appears on any one toss. Thus $P\{X = q^{-n}\} = pq^{n-1}$ with $0 < q = 1 - p < 1$. We play this game repeatedly and the goal has always been to figure a way to make this game fair for both Peter (the house) and Paul (the player). The problem is that Paul's expectation is infinite. This created the paradox three hundred years

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ago, see Csorgo and Simons (2006). It was clear that if a game had a finite expectation then the player should pay the house that amount for each play of that game. But once the expectation became infinite a fair entrance fee was no longer possible. What was not understood three hundred years ago was that the first moment of the random variable associated with the St. Petersburg Game was barely infinite. Which meant that a weighted strong law could be established, making it fair for both Peter and Paul, see Adler and Rosalsky (1989).

By fair we want to find a cumulative entrance fee, say b_N, that Paul would pay by the N^{th} play so that the sum of his winnings, $\sum_{n=1}^{N} X_n$, would balance in the sense that $\sum_{n=1}^{N} X_n/b_N$ converges to one in some way. In Feller (1968), page 252, there is a solution in the weak sense, i.e., convergence in probability. But that isn't an acceptable way to play this game since the almost sure upper limit is infinite while the almost sure lower limit is finite. It turns out that these almost sure lower limits are the same as the weak law limits, see Adler (1990). But from this weak law and Adler and Wittmann (1994) we know that there is a weighted strong law that will make this game fair for everyone, see Adler (2000). These strong laws only apply to these type of random variables, those barely with or without finite means, see Klass and Teicher (1977), of which our random variable X just happens to be one of them. Similarly the weights must be of the order n^{-1}. A strong law cannot exist whenever we observed sums of i.i.d. random variables whenever the common mean is zero or infinite, see Chow and Robbins (1961). So, in those cases we are forced to look at weighted sums of our winnings. In the weighted case, Paul's winnings after N plays is $\sum_{n=1}^{N} a_n X_n$ while his entrance fee is still b_N.

Now we turn to a different way to solve this game. The motivation comes from Csorgo and Simons (2005). In that paper they looked at order statistics from the St. Petersburg distribution. In this paper we look at simultaneous games and repeat that process. With just two simultaneous games we see how the Bernoulli's could easily have played this game making both Peter and Paul happy, see Theorem 1.

The key here is to examine the minimum of two simultaneous games. Let X_1 and X_2 be i.i.d. copies of X. Then set $W_{(2)} = X_1 \wedge X_2$. We do this repeatedly obtaining the sequence $W_{(2)n}$. Similarly, we will establish strong laws with the max of our simultaneous games, $M_{(2)} = X_1 \vee X_2$. But that will require our unusual weighted strong laws, since the distribution of the max is similar to that of the underlying distribution, see Adler (2003). However, with the minimum there is no need to use our exact strong laws.

The distribution of $W_{(2)}$ is

$$\begin{aligned}
P\{W_{(2)}=q^{-n}\} &= 2P\{X_1=q^{-n}\}P\{X_2>q^{-n}\}+P\{X_1=q^{-n}\}P\{X_2=q^{-n}\}\\
&= 2pq^{n-1}\Big[1-P\{X_2\le q^{-n}\}\Big]+[pq^{n-1}]^2\\
&= 2pq^{n-1}\left[1-\sum_{j=1}^{n}pq^{j-1}\right]+[pq^{n-1}]^2\\
&= 2pq^{n-1}\Big[1-(1-q^n)\Big]+[pq^{n-1}]^2\\
&= 2pq^{n-1}\big[q^n\big]+[pq^{n-1}]^2\\
&= pq^{2n-2}(p+2q)
\end{aligned}$$

and the distribution of $M_{(2)}$ is

$$\begin{aligned}
P\{M_{(2)}=q^{-n}\} &= 2P\{X_1=q^{-n}\}P\{X_2<q^{-n}\}+P\{X_1=q^{-n}\}P\{X_2=q^{-n}\}\\
&= 2pq^{n-1}\sum_{j=1}^{n-1}pq^{j-1}+[pq^{n-1}]^2\\
&= 2pq^{n-1}(1-q^{n-1})+p^2q^{2n-2}\\
&= pq^{n-1}\big[2-(2-p)q^{n-1}\big].
\end{aligned}$$

Now, the mean of $W_{(2)}$ is

$$\begin{aligned}
EW_{(2)} &= \sum_{n=1}^{\infty}q^{-n}pq^{2n-2}(p+2q)=(p+2q)p\sum_{n=1}^{\infty}q^{n-2}\\
&= \frac{(p+2q)p}{q}\sum_{n=0}^{\infty}q^n=\frac{p+2q}{q}.
\end{aligned}$$

Hence our first result could have easily been established three hundred years ago.

Theorem 1. $\lim_{N\to\infty}\frac{\sum_{n=1}^{N}W_{(2)n}}{N}=\frac{p+2q}{q}$ *almost surely.*

Thus by playing two games at the same time we can find a simple way that both Peter and Paul will be interested in playing. As for the max of two simultaneous games, in order to make that fair for both Peter and Paul we are required to use our work on exact strong laws. Before we proceed, two comments about our notation are in order. The first is that we set $\lg x = \log(\max\{e, x\})$ so that we avoid dividing by zero. The second is that the constant C will be used as a generic upper bound that is not necessarily the same in each appearance.

Theorem 2. *For all* $\beta > 0$ *we have*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{(\lg n)^{\beta-2}}{n} M_{(2)n}}{(\lg N)^{\beta}} = \frac{2p}{q(\lg q^{-1})\beta} \quad \textit{almost surely.}$$

Proof. Let $a_n = (\lg n)^{\beta-2}/n$, $b_n = (\lg n)^{\beta}$ and $c_n = b_n/a_n = n(\lg n)^2$. We also need to define a sequence of integers, k_n, such that

$$(q^{-1})^{k_n} \le c_n < (q^{-1})^{k_n+1}.$$

Next, we partition our random variables in the usual way

$$\begin{aligned}
&\frac{1}{b_N}\sum_{n=1}^{N} a_n M_{(2)n} \\
&= \frac{1}{b_N}\sum_{n=1}^{N} a_n \big[M_{(2)n} I(1 \le M_{(2)n} \le c_n) - EM_{(2)n} I(1 \le M_{(2)n} \le c_n)\big] \\
&\quad + \frac{1}{b_N}\sum_{n=1}^{N} a_n M_{(2)n} I(M_{(2)n} > c_n) \\
&\quad + \frac{1}{b_N}\sum_{n=1}^{N} a_n EM_{(2)n} I(1 \le M_{(2)n} \le c_n).
\end{aligned}$$

The truncated second moment of the maximum of two St. Petersburg games is

$$\begin{aligned}
EM_{(2)}^2 I(1 \le M_{(2)} \le c_n) &= \sum_{k=1}^{k_n} (q^{-1})^{2k} p q^{k-1} \big[2 - (2-p) q^{k-1}\big] \\
&\le 2p \sum_{k=1}^{k_n} q^{-k-1} \le C q^{-k_n}.
\end{aligned}$$

Hence the first term vanishes almost surely by the Khintchine-Kolmogorov

Convergence Theorem, see page 113 of Chow and Teicher (1997), and Kronecker's lemma since

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{c_n^2}EM_{(2)n}^2I(1\le M_{(2)n}\le c_n) &\le C\sum_{n=1}^{\infty}\frac{q^{-k_n}}{c_n^2}\\ \le\ C\sum_{n=1}^{\infty}\frac{1}{c_n} &= C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.\end{aligned}$$

As for the second term we have

$$\begin{aligned}P\{M_{(2)n}>c_n\} &\le P\{M_{(2)}>q^{-k_n}\}=1-P\{M_{(2)}\le q^{-k_n}\}\\ &= 1-\sum_{k=1}^{k_n}pq^{k-1}\big[2-(2-p)q^{k-1}\big]\\ &= 1-2p\sum_{k=1}^{k_n}q^{k-1}+(2-p)p\sum_{k=1}^{k_n}q^{2(k-1)}\\ &= 1-2(1-q^{k_n})+(1-q^{2k_n})=2q^{k_n}-q^{2k_n}\end{aligned}$$

thus the second term vanishes, with probability one, by the Borel-Cantelli lemma since

$$\sum_{n=1}^{\infty}P\{M_{(2)n}>c_n\}\le C\sum_{n=1}^{\infty}q^{k_n}\le C\sum_{n=1}^{\infty}\frac{1}{c_n}<\infty.$$

Now, for the expectation of the truncated first moment

$$\begin{aligned}EM_{(2)n}I(1\le M_{(2)n}\le c_n) &= \sum_{k=1}^{k_n}\big(q^{-1}\big)^kpq^{k-1}\big[2-(2-p)q^{k-1}\big]\\ &= \sum_{k=1}^{k_n}2pq^{-1}-(2-p)p\sum_{k=1}^{k_n}q^{k-2}\\ &= 2pq^{-1}k_n-(2-p)pq^{-1}\sum_{k=1}^{k_n}q^{k-1}\\ &= 2pq^{-1}k_n-(2-p)q^{-1}\big(1-q^{k_n}\big)\sim 2pq^{-1}k_n.\end{aligned}$$

Since $\lg c_n\sim k_n\lg(q^{-1})$ and $\lg c_n\sim\lg n$ we have

$$EM_{(2)n}I(1\le M_{(2)n}\le c_n)\sim\frac{2p\lg n}{q\lg(q^{-1})}.$$

Therefore

$$\frac{\sum_{n=1}^{N} a_n EM_{(2)n} I(1 \le M_{(2)n} \le c_n)}{b_N} \sim \left(\frac{2p}{q\lg(q^{-1})}\right) \frac{\sum_{n=1}^{N} \frac{(\lg n)^{\beta-1}}{n}}{(\lg N)^{\beta}} \to \frac{2p}{q\lg(q^{-1})\beta}$$

which completes the proof. □

We next extend this last result to m games. Here we are playing m games repeatedly and looking at the maximum times it takes to see a head in these independent games. Thus the distribution of $M_{(m)}$ is

$$P\{M_{(m)} \le q^{-n}\} = (1-q^n)^m$$

since $M_{(m)} = X_1 \vee X_2 \vee \cdots \vee X_m$ with each X_i being i.i.d. copies of X and

$$P\{X \le q^{-n}\} = \sum_{i=1}^{n} pq^{i-1} = p\sum_{i=0}^{n-1} q^i = 1-q^n.$$

Using that, we have

$$\begin{aligned} P\{M_{(m)} = q^{-n}\} &= P\{M_{(m)} \le q^{-n}\} - P\{M_{(m)} \le q^{-(n-1)}\} \\ &= (1-q^n)^m - (1-q^{n-1})^m \\ &= \sum_{j=0}^{m} \binom{m}{j}(-q^n)^j - \sum_{j=0}^{m} \binom{m}{j}(-q^{n-1})^j \\ &= \sum_{j=1}^{m} \binom{m}{j}(-1)^j \left[q^{nj} - q^{nj-j}\right] \\ &= \sum_{j=1}^{m} \binom{m}{j}(-1)^j q^{nj}\left[1-q^{-j}\right]. \end{aligned}$$

Next, we establish a generalization of Theorem 2 to m simultaneous games.

Theorem 3. *For all $\beta > 0$ we have*

$$\lim_{N\to\infty} \frac{\sum_{n=1}^{N} \frac{(\lg n)^{\beta-2}}{n} M_{(m)n}}{(\lg N)^{\beta}} = \frac{mp}{q(\lg q^{-1})\beta} \quad \text{almost surely.}$$

Proof. As in the last proof, let $a_n = (\lg n)^{\beta-2}/n$, $b_n = (\lg n)^{\beta}$, $c_n = b_n/a_n = n(\lg n)^2$ and define k_n such that $(q^{-1})^{k_n} \le c_n < (q^{-1})^{k_n+1}$.

The usual partition of $M_{(m)n}$ suffices

$$\begin{aligned}
&\frac{1}{b_N}\sum_{n=1}^{N} a_n M_{(m)n}\\
&= \frac{1}{b_N}\sum_{n=1}^{N} a_n\big[M_{(m)n}I(1\le M_{(m)n}\le c_n) - EM_{(m)n}I(1\le M_{(m)n}\le c_n)\big]\\
&\quad+\frac{1}{b_N}\sum_{n=1}^{N} a_n M_{(m)n}I(M_{(m)n}>c_n)\\
&\quad+\frac{1}{b_N}\sum_{n=1}^{N} a_n EM_{(m)n}I(1\le M_{(m)n}\le c_n).
\end{aligned}$$

The first term almost surely disappears since

$$\begin{aligned}
EM_{(m)}^2 I(1\le M_{(m)}\le c_n) &= \sum_{k=1}^{k_n}\big(q^{-1}\big)^{2k}\sum_{j=1}^{m}\binom{m}{j}(-1)^j q^{kj}(1-q^{-j})\\
&= \sum_{j=1}^{m}\binom{m}{j}(-1)^j(1-q^{-j})\sum_{k=1}^{k_n}\big(q^{j-2}\big)^k\\
&\le \sum_{j=1}^{m}\binom{m}{j}(-1)^{j+1}(q^{-j}-1)\sum_{k=1}^{k_n}q^{-k}\\
&\le \sum_{j=1}^{m}\binom{m}{j}q^{-j}\sum_{k=1}^{k_n}q^{-k}\le C\sum_{j=1}^{m}\binom{m}{j}q^{-j}q^{-k_n}\\
&\le Cq^{-k_n}
\end{aligned}$$

allowing

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{1}{c_n^2}EM_{(m)n}^2 I(1\le M_{(m)n}\le c_n) &\le C\sum_{n=1}^{\infty}\frac{q^{-k_n}}{c_n^2}\le C\sum_{n=1}^{\infty}\frac{1}{c_n}\\
&= C\sum_{n=1}^{\infty}\frac{1}{n(\lg n)^2}<\infty.
\end{aligned}$$

As for the second term, we have

$$\begin{aligned}
P\{M_{(m)n} > c_n\} &\leq P\{M_{(m)n} > q^{-k_n}\} = 1 - P\{M_{(m)n} \leq q^{-k_n}\} \\
&= 1 - \left[P\{X \leq q^{-k_n}\}\right]^m = 1 - (1 - q^{k_n})^m \\
&= 1 - \sum_{j=0}^{m} \binom{m}{j} (-1)^j q^{k_n j} = \sum_{j=1}^{m} \binom{m}{j} (-1)^{j+1} q^{k_n j} \\
&\leq \sum_{j=1}^{m} \binom{m}{j} q^{k_n j}
\end{aligned}$$

hence

$$\begin{aligned}
\sum_{n=1}^{\infty} P\{M_{(m)n} > c_n\} &\leq \sum_{n=1}^{\infty} \sum_{j=1}^{m} \binom{m}{j} q^{k_n j} = \sum_{j=1}^{m} \binom{m}{j} q^{-j} \sum_{n=1}^{\infty} q^{(k_n+1)j} \\
&\leq \sum_{j=1}^{m} \binom{m}{j} q^{-j} \sum_{n=1}^{\infty} \left[\frac{1}{n(\lg n)^2}\right]^j < \infty.
\end{aligned}$$

Finally, our truncate first moment is

$$\begin{aligned}
EM_{(m)n} I(1 \leq M_{(m)n} \leq c_n) &= \sum_{k=1}^{k_n} (q^{-1})^k \sum_{j=1}^{m} \binom{m}{j} (-1)^j q^{kj} (1 - q^{-j}) \\
&= \sum_{j=1}^{m} \binom{m}{j} (-1)^j (1 - q^{-j}) \sum_{k=1}^{k_n} q^{-k} q^{kj} \\
&= \frac{mp}{q} k_n + \sum_{j=2}^{m} \binom{m}{j} (-1)^j (1 - q^{-j}) \sum_{k=1}^{k_n} (q^{j-1})^k \\
&\sim \frac{mp}{q} k_n \sim \frac{mp}{q(\lg q^{-1})} \lg n
\end{aligned}$$

since

$$\begin{aligned}
\left| \sum_{j=2}^{m} \binom{m}{j} (-1)^j (1 - q^{-j}) \sum_{k=1}^{k_n} (q^{j-1})^k \right| &\leq C \sum_{j=2}^{m} \binom{m}{j} \sum_{k=1}^{k_n} (q^{j-1})^k \\
&\leq C \sum_{j=2}^{m} \binom{m}{j} \sum_{k=0}^{\infty} q^k \\
&\leq C \sum_{j=2}^{m} \binom{m}{j} = O(1).
\end{aligned}$$

Putting this all together we have

$$EM_{(m)n}I(1 \le M_{(m)n} \le c_n) \sim \frac{mp\lg n}{q\lg(q^{-1})}$$

thus

$$\begin{aligned}\frac{\sum_{n=1}^{N} a_n EM_{(m)n}I(1 \le M_{(m)n} \le c_n)}{b_N} &\sim \left(\frac{mp}{q\lg(q^{-1})}\right)\frac{\sum_{n=1}^{N}\frac{(\lg n)^{\beta-1}}{n}}{(\lg N)^{\beta}} \\ &\to \frac{mp}{q\lg(q^{-1})\beta}\end{aligned}$$

which completes the proof. □

Naturally one can play m concurrent games and observe the minimum of those games. But as it was shown in Theorem 1, all we need is two games to make the St. Petersburg Game fair for both Peter and Paul when using the minimum as a criteria.

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