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AN ELEMENTARY INEQUALITY FOR PSI FUNCTION

BY

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Abstract

For $x > 0$, let $\Gamma(x)$ be the Euler's gamma function, and $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$ be the psi function. In this paper, we prove that $(b-L(a,b))\Psi(b)+(L(a,b)-a)\Psi(a) > (b-a)\Psi(\sqrt{ab})$ for $b > a \geq 2$, where $L(a,b) = \frac{b-a}{\log b - \log a}$.

1. Introduction

For real and positive values of x, the Euler's gamma function Γ and its logarithmic derivative Ψ, the so-called psi function, are defined as

$$\Gamma(x) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \quad \Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}, \tag{1.1}$$

respectively. For extensions of these functions to complex variables and for basic properties see [1]. There is a vast literature on these functions and a good reference to this can be found, for example, in the recent papers [2-12].

The ratio $\frac{\Gamma(s)}{\Gamma(r)}(s > r > 0)$ has been researched by many authors. For $0 < s < 1, n = 1, 2, 3, \cdots$, W. Gautschi [13] first showed that

$$n^{1-s} < \frac{\Gamma(n+1)}{\Gamma(n+s)} < \exp((1-s)\Psi(n+1)). \tag{1.2}$$

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A strengthened upper bound was given by T. Erber in [14]

$$\frac{\Gamma(n+1)}{\Gamma(n+s)} < \frac{4(n+s)(n+1)^{1-s}}{4n+(s+1)^2}. \tag{1.3}$$

J. D. Kečkić and P. M. Vasić obtained the following inequalities in [15]

$$\frac{b^{b-1}}{a^{a-1}}e^{a-b} < \frac{\Gamma(b)}{\Gamma(a)} < \frac{b^{b-(1/2)}}{a^{a-(1/2)}}e^{a-b}, \quad 0<a<b. \tag{1.4}$$

For $b>a>0$, the generalized logarithmic mean $L_p(a,b)$ of a and b is defined as

$$L_p(a,b) = \begin{cases} \left(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & p\neq -1,0, \\ \frac{b-a}{\log b-\log a}, & p=-1, \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}, & p=0. \end{cases} \tag{1.5}$$

K.B.Stolarsky [16] proved that $L_p(a,b)$ is an increasing function on p for fixed a and b. Denote $A(a,b)=L_1(a,b)=\frac{b+a}{2}$, $I(a,b)=L_0(a,b)=\frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}}$, $L(a,b)=L_{-1}(a,b)=\frac{b-a}{\log b-\log a}$ and $G(a,b)=L_{-2}(a,b)=\sqrt{ab}$ are the arithmetic mean, identric mean, logarithmic mean and geometric mean of a and b, respectively.

For $b>a>0$, D. Kershaw [17] proved

$$\log\frac{\Gamma(b)}{\Gamma(a)} > (b-a)\Psi(\sqrt{ab}) \tag{1.6}$$

and X. M. Zhang and Y. M. Chu [18] obtained

$$\log\frac{\Gamma(b)}{\Gamma(a)} > (b-L(a,b))\Psi(b)+(L(a,b)-a)\Psi(a), \tag{1.7}$$

but Zhang and Chu can't compare the lower bound of $\log\frac{\Gamma(b)}{\Gamma(a)}$ in (1.6) with that in (1.7).

The main purpose of this paper is to compare $(b-a)\Psi(\sqrt{ab})$ with $(b-L(a,b))\Psi(b)+(L(a,b)-a)\Psi(a)$ and prove the following result.

Theorem 1.1. *If $b > a \geq 2$, then*

$$(b - L(a,b))\Psi(b) + (L(a,b) - a)\Psi(a) > (b-a)\Psi(\sqrt{ab}). \tag{1.8}$$

Although we can not prove Theorem 1.1 is also true for all $b > a > 0$, the elementary computation by a computer and the Mathematica software suggests the following conjecture.

Conjecture 1.1. *Whenever $b > a > 0$ we have*

$$(b - L(a,b))\Psi(b) + (L(a,b) - a)\Psi(a) > (b-a)\Psi(\sqrt{ab}).$$

2. Proof of Theorem 1.1

For the convenience of the reader, first we shall introduce and establish the following nine lemmas; they will be used in proof of our Theorem 1.1.

Lemma 2.1.(see [19]) *If $x > 0$, then*

$$2\Psi'(x) + x\Psi''(x) < \frac{1}{x}. \tag{2.1}$$

Lemma 2.2.(see [20]) *If $x > 0$, then*

$$\Psi'(x) > \frac{1}{x} + \frac{1}{2x^2}. \tag{2.2}$$

Lemma 2.3.(see [21]) *If $x > 0$, then*

$$(\Psi'(x))^2 + \Psi''(x) > 0. \tag{2.3}$$

Lemma 2.4.(see [22]) *If $x > 0$, then*

$$\Psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \sum_{n=1}^{m}(-1)^{n-1}\frac{B_n}{x^{2n+1}} + (-1)^m\theta\frac{B_{m+1}}{x^{2m+3}}, \tag{2.4}$$

where $m = 1, 2, 3, \cdots$, $0 < \theta < 1$, $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}, \cdots$.

Lemma 2.5. *If $x > 0$, then*

$$\Psi'(x) < \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}. \tag{2.5}$$

Proof. Taking $m = 1$ in (2.4), we have

$$\begin{aligned}\Psi'(x) &= \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{\theta}{30x^5}\\ &< \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}.\end{aligned}$$

Lemma 2.6. *If $x \geq 2$, then*

$$\frac{6x^2}{3x+1} > \log x + 3 - \log 2. \tag{2.6}$$

Proof. Let $f(x) = \frac{6x^2}{3x+1} - \log x - 3 + \log 2$, then

$$f(2) = \frac{3}{7} > 0 \tag{2.7}$$

and

$$f'(x) = \frac{18x^3 + 3x^2 - 6x - 1}{x(3x+1)^2} > 0 \tag{2.8}$$

for $x \geq 2$. (2.6) follows from (2.7) and (2.8).

Lemma 2.7. *If $b > a > 0$, then*

$$\frac{b - L(a,b)}{L(a,b) - a} > \frac{\log b - \log a + 3}{3}. \tag{2.9}$$

Proof. For $b > a > 0$ and $x \in [a, b]$, let

$$f(x) = (3x + a\log x + 3a - a\log a)(\log x - \log a) - (\log x + 6 - \log a)(x - a) \tag{2.10}$$

and

$$g(x) = xf'(x). \tag{2.11}$$

Then the elementary computation reveals

$$f(a) = 0, \tag{2.12}$$

$$g(x) = (3x+a)(\log x-\log a)+a\log x+x\log a-x\log x-a\log a +4a-4x, \tag{2.13}$$

$$g(a) = 0, \tag{2.14}$$

$$g'(x) = 2(\log x - \log a) + 2(\frac{a}{x} - 1), \tag{2.15}$$

$$g'(a) = 0, \tag{2.16}$$

$$g''(x) = \frac{2}{x^2}(x-a), \tag{2.17}$$

$$g''(a) = 0 \tag{2.18}$$

and

$$g''(x) > 0, \quad x \in (a,b]. \tag{2.19}$$

(2.10)-(2.19) imply that $f(b) > f(a) = 0$, this leads to

$$\frac{b - L(a,b)}{L(a,b) - a} > \frac{\log b - \log a + 3}{3}.$$

Lemma 2.8. *If* $b > a \geq 2$*, then*

$$b(b - L(a,b))\Psi'(b) > a(L(a,b) - a)\Psi'(a). \tag{2.20}$$

Proof. For $x \in [2, \infty)$, Lemma 2.6 implies

$$\log x + (4 - \log 2) - x(\log x + 3 - \log 2)(\frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3}) > 0. \tag{2.21}$$

Lemma 2.5 and (2.21) lead to

$$\log x + (4 - \log 2) - x(\log x + 3 - \log 2)\Psi'(x) > 0. \tag{2.22}$$

Lemma 2.2 and (2.22) yield

$$(\log x + 4 - \log 2)\Psi'(x) - x(\log x + 3 - \log 2)(\Psi'(x))^2 > 0. \tag{2.23}$$

From Lemma 2.3 and (2.23) we get

$$(\log x + 4 - \log 2)\Psi'(x) + x(\log x + 3 - \log 2)\Psi''(x) > 0,$$

this leads to

$$\big[x(\log x + 3 - \log 2)\Psi'(x)\big]' > 0.$$

Hence $x(\log x + 3 - \log 2)\Psi'(x)$ is strictly increasing on $[2, \infty)$. For $b > a \geq 2$, we have

$$b\Psi'(b)\frac{\log b + 3 - \log 2}{\log a + 3 - \log 2} > a\Psi'(a) \tag{2.24}$$

and

$$\frac{\log b + 3 - \log 2}{\log a + 3 - \log 2} \leq \frac{\log b + 3 - \log a}{\log a + 3 - \log a} = \frac{\log b - \log a + 3}{3}. \tag{2.25}$$

Lemma 2.8 follows from Lemma 2.7, (2.24) and (2.25).

Lemma 2.9. *If $x > 0$, then*

$$\Psi'(x) + x\Psi''(x) < 0. \tag{2.26}$$

Proof. By Lemma 2.1 and Lemma 2.2 we have

$$\Psi'(x) + x\Psi''(x) < \frac{1}{x} - \Psi'(x) < -\frac{1}{2x^2} < 0.$$

Proof of theorem 1.1. If $b > a \geq 2$, then Lemma 2.8 yields

$$b(b - L(a, b))(\log b - \log \sqrt{ab})\Psi'(b) > a(L(a, b) - a)(\log \sqrt{ab} - \log a)\Psi'(a). \tag{2.27}$$

Making use of Lemma 2.9 and the Cauchy mean value theorem we can get

$$\frac{\Psi(b) - \Psi(\sqrt{ab})}{\log b - \log \sqrt{ab}} > b\Psi'(b) \tag{2.28}$$

and

$$\frac{\Psi(\sqrt{ab}) - \Psi(a)}{\log \sqrt{ab} - \log a} < a\Psi'(a). \tag{2.29}$$

Theorem 1.1 follows from (2.27), (2.28) and (2.29).

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