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REMARKS ON MODULES OF THE ORTHO-SYMPLECTIC LIE SUPERALGEBRAS

BY

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Abstract

We examine in detail the Jacobi-Trudi characters over the ortho-symplectic Lie superalgebras $\mathfrak{spo}(2|2m+1)$ and $\mathfrak{spo}(2n|3)$. We furthermore relate them to Serganova's notion of Euler characters.

1. Introduction

The representation theory of Lie superalgebras over $\mathbb{C}$ and that of modular representations of algebraic groups share similarities in their lack of complete reducibility and difficulties of finding irreducible characters. There has been significant progress in representation theory of Lie superalgebras over $\mathbb{C}$ in recent years, thanks largely to the works of Brundan, Serganova, and others, mostly for the so-called type I classical Lie superalgebras in the classification of Kac [7] and the queer Lie superalgebras.

However, the representation theory of the Lie superalgebras $\mathfrak{spo}(2n|\ell)$ with $n \geq 1$ and $\ell \geq 3$ (which are of type II classical in the sense of [7]) turns out to be much more challenging. In her fundamental paper [13], Serganova announced in 1998 an algorithm of finding the irreducible characters of $\mathfrak{spo}(2n|\ell)$. In the simplest yet already rather nontrivial case of $\mathfrak{spo}(2|3)$, the irreducible characters have been calculated by Van der Jeugt

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[17] and Germoni [4] using different approaches (also see Gruson [6] and Santos [16] for related developments).

This Note arises from our attempt to understand Serganova's work [13], by examining mainly various aspects of the representation theory of $\mathfrak{spo}(2n|3)$ and $\mathfrak{spo}(2|2m+1)$. The notion of Euler characters, which are alternating sums of certain sheaf cohomology groups, plays a key role in Serganova's theory. In Section 2, we introduce the Jacobi-Trudi characters, imitating the determinant character formula for classical Lie algebras of type B, C, D (cf., e.g., [1, 2, 3, 10]). In general, the Euler characters and the Jacobi-Trudi characters are only virtual $\mathfrak{spo}$-characters. We show that while being different in general, the Euler characters coincide with the Jacobi-Trudi characters for a large class of highest weights for $\mathfrak{spo}(2n|3)$, and hence demystify somewhat the notion of Euler characters. Also, some of our Euler character calculations look surprising in light of [13].

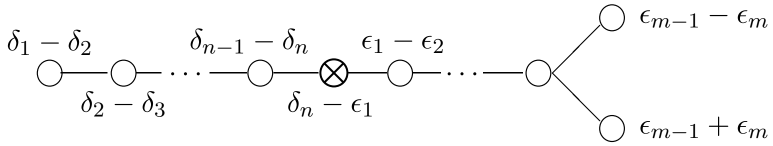
In Section 3, we determine in a number of cases when the kernel of a Laplacian on an exterior tensor power of the natural module is irreducible or reducible. By computations, we also establish some explicit tensor product decomposition formula for $\mathfrak{spo}(2|2m+1)$-modules.

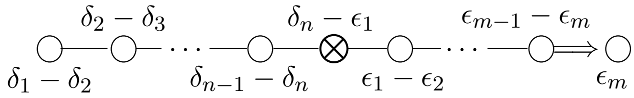
In the case of $\mathfrak{spo}(2|3)$, we provide and compare the explicit formulas for the "composition factors" and the virtual dimensions of the Euler characters, Kac characters, and the Jacobi-Trudi characters, respectively. We also write down the composition factors of the tensor products of any irreducible $\mathfrak{spo}(2|3)$-module with the natural module. We end the Note with a conjecture on a relation betwen the Euler characters of $\mathfrak{spo}(2n|2m+1)$ with respect to the parabolic sublagebra whose Levi subalgebra is $\mathfrak{gl}(n|m)$ and the Kac characters of $\mathfrak{gl}(n|m)$. All of these are treated in Section 4.

2. The Euler and Jacobi-Trudi Characters

2.1. Preliminaries

Throughout this paper we shall denote by $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ the orthosymplectic Lie superalgebra $\mathfrak{spo}(2n|\ell)$ for $n \geq 1$ and $\ell \geq 3$, whose respective standard Dynkin diagrams and simple roots are as follows (where we write $\ell = 2m+1$ or $2m$):





Here and further we use the standard notation for the simple roots of the ortho-symplectic Lie superalgebra, i.e. $\{\epsilon_1, \ldots, \epsilon_m, \delta_1, \ldots, \delta_n\}$ denotes the standard dual basis of the standard Cartan subalgebra $\mathfrak{h}$ equipped with a symmetric bilinear form determined by $(\epsilon_i, \epsilon_j) = -\delta_{ij}$, $(\delta_i, \epsilon_j) = 0$, and $(\delta_i, \delta_j) = \delta_{ij}$. Recall that an odd root α is called *isotropic* if $(\alpha, \alpha) = 0$. Note that $\mathfrak{spo}(2n|2m)_{\bar{0}} = \mathfrak{sp}(2n) \oplus \mathfrak{so}(2m)$ and $\mathfrak{spo}(2n|2m+1)_{\bar{0}} = \mathfrak{sp}(2n) \oplus \mathfrak{so}(2m+1)$.

In concrete matrix form, the Lie superalgebra $\mathfrak{spo}(2n|2m+1)$ consists of the $(2n+2m+1) \times (2n+2m+1)$ matrices in the following $(n|n|m|m|1)$-block form

$$g = \begin{bmatrix} d & e & y_1^t & x_1^t & z_1^t \\ f & -d^t & -y^t & -x^t & -z^t \\ x & x_1 & a & b & -v^t \\ y & y_1 & c & -a^t & -u^t \\ z & z_1 & u & v & 0 \end{bmatrix}, \tag{2.1}$$

where b, c are skew-symmetric, and e, f are symmetric matrices. The remaining $a, d, x, y, x_1, y_1, z, z_1, u, v$ are arbitrary matrices of respective sizes. Similarly, the Lie superalgebra $\mathfrak{spo}(2n|2m)$ consists of the $(2n+2m) \times (2n+2m)$ matrices that are obtained from g of the form (2.1) with the last row and column deleted. The natural $\mathfrak{g}$-module will be denoted by V.

Let $\lambda = \sum_{i=1}^n a_i \delta_i + \sum_{j=1}^m b_j \epsilon_j \in \mathfrak{h}^*$ with $a_i, b_j \in \mathbb{Z}$. Denote by $L(\lambda)$ the highest weight irreducible $\mathfrak{g}$-module of highest weight λ. The following goes back to [7].

Proposition 2.1. *Let* $\lambda = \sum_{i=1}^n a_i \delta_i + \sum_{j=1}^m b_j \epsilon_j$ *be as above.*

(i) *Suppose that* $\mathfrak{g} = \mathfrak{spo}(2n|2m)$. *Then* $L(\lambda)$ *is finite-dimensional if and only if* $a_1 \geq \cdots \geq a_n \geq 0$, $b_1 \geq \cdots b_{m-1} \geq |b_m|$, *and* $a_n < m$ *implies that* $b_{a_n+1} = \cdots = b_m = 0$.

(ii) *Suppose that* $\mathfrak{g} = \mathfrak{spo}(2n|2m+1)$. *Then* $L(\lambda)$ *is finite-dimensional if and only if* $a_1 \geq \cdots \geq a_n \geq 0$, $b_1 \geq \cdots b_{m-1} \geq b_m \geq 0$, *and* $a_n < m$ *implies that* $b_{a_n+1} = \cdots = b_m = 0$.

A weight $\lambda \in \mathfrak{h}^*$ satisfying the condition in Proposition 2.1 will be called a *dominant weight* for $\mathfrak{g}$.

Remark 2.2. It can be seen that the set of dominant weights for $\mathfrak{spo}(2n|2m+1)$ is in 1-1 correspondence with the set of partitions λ with $\lambda_{n+1} \leq m$ as follows. For such λ let $\lambda^\sharp = (\lambda_1, \ldots, \lambda_n, \langle \lambda_1' - n\rangle, \ldots, \langle \lambda_m' - n\rangle)$, where $\langle \ell \rangle = \ell$, if $\ell \geq 0$, and zero otherwise. We may regard $\lambda^\sharp$ as the weight $\sum_{i=1}^n \lambda_i^\sharp \delta_i + \sum_{j=1}^m \lambda_j^\sharp \epsilon_j$. The map $\lambda \to \lambda^\sharp$ is a bijection. Similarly, the set of such partitions parameterizes the set of $\mathfrak{spo}(2n|2m)$-dominant weights with $b_m \geq 0$.

The graded half sum of positive roots ρ is given by

$$\rho = \sum_{i=1}^n (i-m)\delta_{n-i+1} + \sum_{j=1}^m (m-j)\epsilon_j, \quad \text{if } \mathfrak{g} = \mathfrak{spo}(2n|2m),$$
$$\rho = \sum_{i=1}^n (i-m-\frac{1}{2})\delta_{n-i+1} + \sum_{j=1}^m (m-j+\frac{1}{2})\epsilon_j, \quad \text{if } \mathfrak{g} = \mathfrak{spo}(2n|2m+1).$$

Let $\lambda \in \mathfrak{h}^*$. The action of the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ on $L(\lambda)$ defines an algebra homomorphism (called the *central character*) $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$. The following was first stated in [8], and proved in [15] (also cf. [5]).

Proposition 2.3. *Let* $\lambda, \mu \in \mathfrak{h}^*$ *and let* W *be the Weyl group of* $\mathfrak{g}$. *We have* $\chi_\lambda = \chi_\mu$ *if and only if there exists a sequence of isotropic roots* $\alpha_1, \ldots, \alpha_l$ *and* $w \in W$ *such that* $\mu = w(\lambda + \rho + \alpha_1 + \cdots + \alpha_l) - \rho$, *and* $(\lambda + \rho + \alpha_1 + \cdots + \alpha_{s-1}, \alpha_s) = 0$, *for all* $s = 1, \ldots, l$.

Recall [7] that λ is called *typical* if $(\lambda + \rho, \alpha) \neq 0$, for all isotropic root α. Otherwise λ is called *atypical*.

2.2. The Euler characters

Let $\mathfrak{p}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi subalgebra $\mathfrak{l}$ and nilradical $\mathfrak{u}$. Write $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{u} \oplus \tilde{\mathfrak{u}}$. Let M be a finite-dimensional irreducible $\mathfrak{l}$-module extended trivially to an irreducible $\mathfrak{p}$-module. A key role in Serganova's algorithm of finding the irreducible finite-dimensional characters of $\mathfrak{g}$ was played by the notion of Euler characters [13, (1.2)]. By definition, the Euler characters are alternating sums of certain sheaf cohomology groups that themselves are finite-dimensional $\mathfrak{g}$-modules, and hence are in general not $\mathfrak{g}$-characters but only virtual $\mathfrak{g}$-characters. According to [13], a formula for the Euler character is given by

$$E^{\mathfrak{p}}(M) = D \sum_{w \in W} (-1)^{l(w)} w \left(\frac{e^{\rho} \mathrm{ch} M}{\prod_{\alpha \in \Delta^1_{\mathfrak{l},+}} (1 + e^{-\alpha})} \right),$$

where $\mathrm{ch}M$ denotes the character of an $\mathfrak{l}$-module M, and

$$D = D_1/D_0, \quad D_1 = \prod_{\alpha \in \Delta^1_+} (e^{\frac{\alpha}{2}} + e^{\frac{-\alpha}{2}}), \quad D_0 = \prod_{\alpha \in \Delta^0_+} (e^{\frac{\alpha}{2}} - e^{\frac{-\alpha}{2}}).$$

For our purpose, we take this as the definition of some distinguished virtual $\mathfrak{g}$-characters. Furthermore $\Delta^1_{\mathfrak{l},+}$ (respectively $\Delta^0_{\mathfrak{l},+}$) denotes the set of positive odd (respectively even) roots in $\mathfrak{l}$. The simple $\mathfrak{l}$-module of highest weight λ will be denoted by $L^0(\lambda)$.

Let $\mathfrak{b}$ be the standard Borel subalgebra of $\mathfrak{g}$ and let $\mathfrak{h}$ be the corresponding Cartan subalgebra. For a finite-dimensional highest weight $\lambda \in \mathfrak{h}^*$ let $\mathbb{C}_\lambda$ be the one-dimensional $\mathfrak{b}$-module on which $\mathfrak{h}$ transforms by λ. We call $K(\lambda) := E^{\mathfrak{b}}(\mathbb{C}_\lambda)$ the *Kac (virtual) character* (of "highest weight λ"). When λ is a typical dominant weight, indeed $K(\lambda)$ is the character of the simple $\mathfrak{g}$-module $L(\lambda)$ [9].

Remark 2.4. In [16, (12)] Santos defined the functor $\mathcal{L}_0$ that may be regarded as a super analogue of the Bernstein-Zuckerman functor (see e.g. [11]). If we denote the ith derived functor by $\mathcal{L}_i$, then it can be shown that

$$\sum_{i \geq 0} (-1)^i \mathrm{ch} \mathcal{L}_i (\mathrm{Ind}^{\mathfrak{g}}_{\mathfrak{p}} M) = E^{\mathfrak{p}}(M).$$

Since all the irreducible composition factors of the finite-dimensional $\mathfrak{g}$-module $\mathcal{L}_i(\mathrm{Ind}_{\mathfrak{p}}^{\mathfrak{g}} M)$ have the same central character by [16, Proposition 4.5], it follows that all irreducible composition factors of $E^{\mathfrak{p}}(M)$ have the same central character.

Example 2.5. Consider $\mathfrak{spo}(2|4)$ with the maximal parabolic subalgebra $\mathfrak{p}$ obtained by removing the simple root $\epsilon_1+\epsilon_2$. The Levi subalgebra $\mathfrak{l}$ is isomorphic to $\mathfrak{l}=\mathfrak{gl}(1|2)$. Let $\mathbb{C}$ be the trivial module, and let $\mathbb{C}^{1|2}$ be the natural $\mathfrak{gl}(1|2)$-module of highest weight δ_1. One can show that

$$E^{\mathfrak{p}}(\mathbb{C}) = 2\left[\mathrm{ch}\left(\mathbb{C}\right)\right],\ E^{\mathfrak{p}}(\mathbb{C}^{1|2}) = \mathrm{ch}\left(\mathbb{C}^{2|4}\right),\ E^{\mathfrak{p}}(L^0(\delta_1+\epsilon_1)) = \mathrm{ch}L(\delta_1+\epsilon_1).$$

The factor of 2 in the formula of the first example is inconsistent with [13, Theorem 3.3], which in this case predicts the irreducibility of the Euler character with non-vanishing cohomology appearing only in degree zero. Also [13, Proposition 3.4] in this case further seems to imply that $\mathbb{C}$ should only appear with multiplicity one. On the other hand, the last two examples are consistent with [13].*

Example 2.6. Consider $\mathfrak{spo}(2|6)$ with the maximal parabolic subalgebra $\mathfrak{p}$ obtained by removing the simple root $\epsilon_2+\epsilon_3$. The Levi subalgebra $\mathfrak{l}$ is isomorphic to $\mathfrak{l}=\mathfrak{gl}(1|3)$. Let $\mathbb{C}$ be the trivial module and let $\mathbb{C}^{1|3}$ be the standard $\mathfrak{gl}(1|3)$-module of highest weight δ_1. One has

$$E^{\mathfrak{p}}(\mathbb{C})=2\left[\mathrm{ch}\left(\mathbb{C}\right)\right],\ E^{\mathfrak{p}}(\mathbb{C}^{1|3})=2\left[\mathrm{ch}\left(\mathbb{C}^{2|6}\right)\right],\ E^{\mathfrak{p}}((S^2(\mathbb{C}^{1|3}))) = \mathrm{ch}S^2(\mathbb{C}^{2|6}).$$

2.3. The Jacobi-Trudi characters

Let λ be a partition with $\lambda_{n+1}\le m$ and let k be the length of λ. We identify $\lambda^{\natural}$ with $\sum_{i=1}^n \lambda_i^{\natural}\delta_i + \sum_{j=1}^m \lambda_j^{\natural}\epsilon_j$. The *Jacobi-Trudi character* $D(\lambda^{\natural})$ is defined to be the determinant of the following matrix

$$\left(p_{\lambda^*}, p_{\lambda^*+(1^k)} + p_{\lambda^*-(1^k)}, \dots, p_{\lambda^*+(k-1)(1^k)} + p_{\lambda^*-(k-1)(1^k)}\right).$$

*In a private e-mail communication Serganova has informed us that she was aware of these inconsistencies. We were further told that she has found methods to amend these problems. We thank her for kindly sharing this information with us.

Here for a partition $\mu = (\mu_1, \ldots, \mu_k)$ we let $\mu^* = (\mu_1, \mu_2 - 1, \ldots, \mu_k - k + 1)$, and for a k-tuple of integers $a = (a_1, \ldots, a_k)$, p_a stands for the column vector $(p_{a_1}, \ldots, p_{a_k})^t$, where p_{a_j} is the character of $S^{a_j}(V)$. In general, the Jacobi-Trudi character is not a $\mathfrak{g}$-character, but only a virtual $\mathfrak{g}$-character. See [2] for some related discussions.

Lemma 2.7. *Let λ be a partition and suppose that $\mu = \lambda'$ has length ℓ. Then $D(\lambda^\sharp)$ is equal to the determinant of the matrix*

$$\left(e_{\mu^*} - e_{\mu^*-2(1^\ell)}, e_{\mu^*+(1^\ell)} - e_{\mu^*-3(1^\ell)}, \ldots, e_{\mu^*+(\ell-1)(1^\ell)} - e_{\mu^*-(\ell+1)(1^\ell)}\right),$$

where for an ℓ-tuple of integers $a = (a_1, \ldots, a_\ell)$, e_a stands for the column vector $(e_{a_1}, \ldots, e_{a_\ell})^t$, and e_{a_j} is the character of $\Lambda^{a_j}(V)$.

Proof. Denote by $|a_{ik}|$ the determinant of a square matrix $[a_{ik}]$. Recall the classical identity (see e.g. [10, Proposition 2.3.3])

$$\begin{aligned}&\left|p_{\lambda^*}, p_{\lambda^*+(1^k)} + p_{\lambda^*-(1^k)}, \ldots, p_{\lambda^*+(k-1)(1^k)} + p_{\lambda^*-(k-1)(1^k)}\right|\\ &\quad = \left|e_{\mu^*} - e_{\mu^*-2(1^\ell)}, e_{\mu^*+(1^\ell)} - e_{\mu^*-3(1^\ell)}, \ldots, e_{\mu^*+(\ell-1)(1^\ell)} - e_{\mu^*-(\ell+1)(1^\ell)}\right|,\end{aligned}$$

where p_t and e_t, $t \in \mathbb{Z}_+$, are the complete symmetric and elementary symmetric functions in the variables $y_1, y_2, \cdots$, respectively. We regard this identity as a symmetric function identity in the variables $y_{2n+1}, y_{2n+2}, \cdots$ and apply the involution that interchanges the complete and the elementary symmetric functions (see e.g. [12]). Now in the resulting identity we set $y_{2n+2m+1} = y_{2n+2m+2} = \cdots = 0$ in the case of $\mathfrak{spo}(2n|2m)$, and $1 - y_{2n+2m+1} = y_{2n+2m+2} = y_{2n+2m+3} = \cdots = 0$ in the case of $\mathfrak{spo}(2n|2m+1)$. Next we set $y_j^{-1} = y_{n+j}$, for $j = 1, 2, \cdots, n$, and $y_{2n+j}^{-1} = y_{2n+m+j}$, $j = 1, \ldots, m$. Finally, setting $e^{\delta_i} = y_i$, and $e^{\epsilon_j} = y_{2m+j}$, $i = 1, \ldots, n$, and $j = 1, \ldots, m$ we obtain the desired identity. $\square$

2.4. Euler versus Jacobi-Trudi characters

Define

$$\varphi_0(z) = \prod_{k=1}^{n}(1 - u_k z)(1 - u_k^{-1} z),$$

$$\varphi_1(z) = (1+yz)(1+y^{-1}z)(1+z).$$

Note we have the following interpretation

$$\frac{\varphi_1(z)}{\varphi_0(z)} = \sum_l \mathrm{ch}(S^l(\mathbb{C}^{2n|3}))z^l. \tag{2.2}$$

Lemma 2.8. *The following identities hold:*

$$\prod_i \varphi_0(z_i) \cdot \left| \frac{1}{(1-z_iu_k)(1-z_iu_k^{-1})} \right| = \prod_i z_i^{n-1}$$
$$\times \left|u^{n-1}+u^{-n+1}, u^{n-2}+u^{-n+2}, \dots, 1\right| \left|1, z^{-1}+z, \dots, z^{n-1}+z^{-n+1}\right|, \tag{2.3}$$
$$|u^n - u^{-n}, u^{n-1} - u^{-n+1}, \dots, u - u^{-1}|$$
$$= \prod_i (u_i - u_i^{-1}) \left|u^{n-1}+u^{-n+1}, u^{n-2}+u^{-n+2}, \dots, 1\right|, \tag{2.4}$$
$$| z^{-1} - z, \dots, z^{-n} - z^n|$$
$$= (z_n^{-1} - z_n) \prod_{i=1}^{n-1} z_i^{-1}(1-z_iz_n)(1-z_iz_n^{-1}) \left|z - z^{-1}, \dots, z^{n-1} - z^{-n+1}\right|, \tag{2.5}$$

$$\sum_{l \geq 0} (u^{l+\frac{1}{2}} - u^{-l-\frac{1}{2}})z^{l-1} = (1+z^{-1})(u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \frac{1}{(1-uz)(1-u^{-1}z)}. \tag{2.6}$$

Proof. Recall Cauchy's formula (cf., [18])

$$\prod_{i,k} (x_i - y_k) \cdot \left| \frac{1}{x_i - y_k} \right| = \left|1, x, \dots, x^{n-1}\right| \cdot \left|1, y, \dots, y^{n-1}\right|,$$

where x^j $(0 \leq j \leq n-1)$ denotes the column vector $(x_1^j, \dots, x_n^j)^t$ and so on; similar notations apply below. Putting $x_i = z_i - z_i^{-1}$ and $y_k = u_k - u_k^{-1}$ in the Cauchy's formula gives us (2.3). The identity (2.4) is obtained by taking out the common factors $(u_i - u_i^{-1})$ of the determinant on its left-hand side, and applying elementary column operations. The identity (2.5) is proved by using twice the Weyl denominator formula for $\mathfrak{sp}(2n)$:

$$\left|z^{-1} - z, \dots, z^{-n} - z^n\right| = (-1)^n \left|z - z^{-1}, \dots, z^n - z^{-n}\right|$$
$$= (-1)^n \prod_{1 \leq i < j \leq n} (z_j + z_j^{-1} - z_i - z_i^{-1}) \prod_{1 \leq i \leq n} (z_i - z_i^{-1})$$

$$= (z_n^{-1} - z_n) \prod_{i=1}^{n-1} (z_i + z_i^{-1} - z_n - z_n^{-1}) \left| z - z^{-1}, \ldots, z^{n-1} - z^{-n+1} \right|$$
$$= (z_n^{-1} - z_n) \prod_{i=1}^{n-1} z_i^{-1}(1 - z_i z_n)(1 - z_i z_n^{-1}) \left| z - z^{-1}, \ldots, z^{n-1} - z^{-n+1} \right|.$$

The (2.6) follows by the geometric series expansion of its right-hand side. □

Theorem 2.9. *Let $\mathfrak{g} = \mathfrak{spo}(2n|3)$ and $\mathfrak{p}$ be the parabolic subalgebra obtained by removing the simple roots $\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n$ so that $\mathfrak{l} = \mathfrak{gl}(1)^{n-1} \oplus \mathfrak{spo}(2|3)$. Let $\lambda = \sum_{i=1}^{n-1} \lambda_i \delta_i$ be dominant integral. Then the Euler characters coincide with the Jacobi-Trudi characters, that is, $E^{\mathfrak{p}}(L^0(\lambda)) = D(\lambda)$.*

Proof. Denote by D_0^{sp} the Weyl denominator for $\mathfrak{sp}(2n)$. We have

$$\begin{aligned}
&E^{\mathfrak{p}}(L^0(\lambda)) \\
&= \frac{D_1}{D_0} \sum_{w \in W} (-1)^{l(w)} w \left(\frac{e^{\lambda+\rho}}{(1+e^{-\delta_n+\epsilon_1})(1+e^{-\delta_n-\epsilon_1})(1+e^{-\delta_n})} \right) \\
&= \frac{D_1}{D_0^{sp}(e^{\frac{\epsilon_1}{2}} - e^{-\frac{\epsilon_1}{2}})} \sum_{w \in W_{sp}} (-1)^{l(w)} w \left(\frac{e^{\sum_{i=1}^{n}(\lambda_i+n-i-\frac{1}{2})\delta_i}(e^{\epsilon_1/2} - e^{-\epsilon_1/2})}{(1+e^{-\delta_n+\epsilon_1})(1+e^{-\delta_n-\epsilon_1})(1+e^{-\delta_n})} \right) \\
&= \frac{D_1}{D_0^{sp}} \sum_{s \in S_n} (-1)^{l(s)} s \left(\sum_{w \in \mathbb{Z}_2^n} w \left(\frac{e^{\sum_{i=1}^{n-1}(\lambda_i+n-i-\frac{1}{2})\delta_i - \frac{1}{2}\delta_n}}{(1+e^{-\delta_n+\epsilon_1})(1+e^{-\delta_n-\epsilon_1})(1+e^{-\delta_n})} \right) \right) \\
&= \frac{D_1}{D_0^{sp}} \sum_{s \in S_n} (-1)^{l(s)} s \Bigg(\prod_{i=1}^{n-1} (e^{(\lambda_i+n-i-\frac{1}{2})\delta_i} - e^{-(\lambda_i+n-i-\frac{1}{2})\delta_i}) \\
&\qquad\qquad \times \frac{e^{\frac{1}{2}}\delta_n - e^{-\frac{1}{2}\delta_n}}{e_n^{\delta} + e^{-\delta_n} + e^{\epsilon_1} + e^{-\epsilon_1}} \Bigg).
\end{aligned}$$

Setting $\lambda_i + n - i - \frac{1}{2} = \mu_i$, for $i = 1, \ldots, n-1$, $u_i = e^{\delta_i}$, and $y = e^{\epsilon_1}$, we get

$$E^{\mathfrak{p}}(L^0(\lambda)) = \frac{D_1}{D_0^{sp}} \left| u^{\mu_1} - u^{-\mu_1}, \ldots, u^{\mu_{n-1}} - u^{-\mu_{n-1}}, \frac{u^{\frac{1}{2}} - u^{-\frac{1}{2}}}{u + u^{-1} + y + y^{-1}} \right|. \tag{2.7}$$

We write $D_1 = D_1' \cdot \prod_k (u_k^{\frac{1}{2}} + u_k^{-\frac{1}{2}})$, where

$$D_1' = y^{-n}\varphi_0(-y) = \prod_k (u_k^{\frac{1}{2}} y^{\frac{1}{2}} + u_k^{-\frac{1}{2}} y^{-\frac{1}{2}})(u_k^{\frac{1}{2}} y^{-\frac{1}{2}} + u_k^{-\frac{1}{2}} y^{\frac{1}{2}}). \tag{2.8}$$

It is convenient to set $z_n = -y$ and denote by $[z_1^{a_1} \dots z_k^{a_k}]f$ the coefficient of $z_1^{a_1} \cdots z_k^{a_k}$ of f below. Noting that

$$\frac{1}{u_k + u_k^{-1} + y + y^{-1}} = \frac{y}{(1 - u_k z_n)(1 - u_k^{-1} z_n)}$$

and using (2.6), we rewrite (2.7) as

$$\begin{aligned}
&E^{\mathfrak{p}}(L^0(\lambda)) \\
&= [z_1^{\mu_1 - \frac{3}{2}} \cdots z_{n-1}^{\mu_{n-1} - \frac{3}{2}}] \frac{D_1}{D_0^{sp}} \left| \frac{(1 + z_i^{-1})(u_k^{\frac{1}{2}} - u_k^{-\frac{1}{2}})}{(1 - u_k z_i)(1 - u_k^{-1} z_i)}; \frac{y(u_k^{\frac{1}{2}} - u_k^{-\frac{1}{2}})}{(1 - u_k z_n)(1 - u_k^{-1} z_n)} \right| \\
&= [z_1^{\mu_1 - \frac{3}{2}} \cdots z_{n-1}^{\mu_{n-1} - \frac{3}{2}}] \frac{D_1' \prod (u_k^{\frac{1}{2}} + u_k^{-\frac{1}{2}})}{D_0^{sp}} \\
&\qquad \times y \prod (u_k^{\frac{1}{2}} - u_k^{-\frac{1}{2}}) \prod (1 + z_i)^{-1} \left| \frac{1}{(1 - u_k z_i)(1 - u_k^{-1} z_i)} \right| \\
&= [z_1^{\mu_1 - \frac{3}{2}} \cdots z_{n-1}^{\mu_{n-1} - \frac{3}{2}}] \frac{D_1'}{D_0^{sp}} \prod (u_k - u_k^{-1}) \frac{y \prod (z_i^{-1} - z_i)}{(y - y^{-1}) \prod (1 - z_i)} \\
&\qquad \times \frac{\prod_{i=1}^n z_i^{n-1}}{\prod_{i=1}^n \varphi_0(z_i)} \left| u^{n-1} + u^{-n+1}, \dots, 1 \right| \left| 1, z^{-1} + z, \dots, z^{n-1} + z^{-n+1} \right| \\
&= \Big[\prod_{i=1}^{n-1} z_i^{\lambda_i - i - 1} \Big] \frac{D_1'}{D_0^{sp}} \frac{y^n (-1)^{n-1}}{\prod_{i=1}^{n-1} (1 - z_i)(y - y^{-1}) \varphi_0(-y) \prod_{i=1}^{n-1} \varphi_0(z_i)} \\
&\qquad \left| u^n - u^{-n}, \dots, u - u^{-1} \right| \left| z^{-1} - z, \dots, z^{-n} - z^n \right|,
\end{aligned}$$

where (2.4) was used in the last equation. By (2.5), (2.8) and the Weyl denominator formula $D_0^{sp} = \left| u^n - u^{-n}, \dots, u - u^{-1} \right|$, we rewrite the above expression for $E^{\mathfrak{p}}(L^0(\lambda))$ as

$$= \Big[\prod_{i=1}^{n-1} z_i^{\lambda_i - i - 1} \Big] \frac{(-1)^{n-1}}{(y - y^{-1}) \prod_{i=1}^{n-1} \varphi(z_i)(1 - z_i)} \left| z^{-1} - z, \dots, z^{-n} - z^n \right|$$

$$= \Big[\prod_{i=1}^{n-1} z_i^{\lambda_i - i - 1}\Big] \frac{\prod_{i=1}^{n-1} z_i^{-2}(1+z_i)(1+yz_i)(1+y^{-1}z_i)}{\prod_{i=1}^{n-1} \varphi_0(z_i)(z_i - z_i^{-1})} \left|z - z^{-1}, \dots, z^{n-1} - z^{-n+1}\right|$$
$$= \Big[\prod_{i=1}^{n-1} z_i^{\lambda_i - i + 1}\Big] \frac{\varphi_1(z_i)}{\varphi_0(z_i)} \left|z - z^{-1}, \dots, z^{n-1} - z^{-n+1}\right|,$$

which coincides with the Jacobi-Trudi character $D(\lambda)$ by (2.2). □

Remark 2.10. Let $\mathfrak{q}$ be the parabolic with $\mathfrak{l} = \mathfrak{gl}(1)^{n-1} \oplus \mathfrak{gl}(1|1)$. Then one can show similarly that $E^{\mathfrak{q}}(L^0(\lambda)) = 2E^{\mathfrak{p}}(L^0(\lambda))$.

Proposition 2.11. *Assume $n \geq 2$ and let $\mathfrak{g} = \mathfrak{spo}(2n|3)$. Let $\mathfrak{p}$ be the parabolic subalgebra obtained by removing the simple roots $\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n$ and let $\lambda = \sum_{i=1}^{n-1} \lambda_i$ be a dominant integral weight of $\mathfrak{spo}(2n|3)$ with $\lambda_{n-1} > 0$. Set $\tilde{\lambda} = \lambda + \delta_n$. Then we have*

$$E^{\mathfrak{p}}(L^0(\tilde{\lambda})) = K(\tilde{\lambda}) + E^{\mathfrak{p}}(L^0(\lambda)).$$

(Note that $\tilde{\lambda}$ and λ have the same central character.)

Proof. We have

$$\begin{aligned} &E^{\mathfrak{p}}(L^0(\tilde{\lambda})) \\ &\quad = \frac{D_1}{D_0} \sum_{w \in W} (-1)^{l(w)} w \left(\frac{e^{\lambda+\rho}(e^{\delta_n} + e^{-\delta_n} + 1 + e^{\epsilon_1} + e^{-\epsilon_1})}{(1+e^{-\delta_n+\epsilon_1})(1+e^{-\delta_n-\epsilon_1})(1+e^{-\delta_n})} \right) \\ &\quad = \frac{D_1}{D_0^{sp}} \left| u^{\mu_1} - u^{-\mu_1}, \dots, u^{\mu_{n-1}} - u^{-\mu_{n-1}}, (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) + \frac{u^{\frac{1}{2}} - u^{-\frac{1}{2}}}{u + u^{-1} + y + y^{-1}} \right|, \end{aligned}$$

where $\tilde{\lambda}_i + n - i - \frac{1}{2} = \mu_i$, for $i = 1, \dots, n$, $u_i = e^{\delta_i}$, and $y = e^{\epsilon_1}$. The last identity above was derived in a way similar to (2.7). Thus by comparing with (2.7) we have

$$E^{\mathfrak{p}}(L^0(\tilde{\lambda})) = E^{\mathfrak{p}}(L^0(\lambda)) + \left| u^{\mu_1} - u^{-\mu_1}, \dots, u^{\mu_{n-1}} - u^{-\mu_{n-1}}, (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \right|.$$

Now similarly to the derivation of (2.7), we can again show that

$$K(\tilde{\lambda}) = \frac{D_1}{D_0} \sum_{w \in W} (-1)^{l(w)} w \left(e^{\tilde{\lambda}+\rho} \right)$$

$$= \left| u^{\mu_1} - u^{-\mu_1}, \ldots, u^{\mu_{n-1}} - u^{-\mu_{n-1}}, (u^{\frac{1}{2}} - u^{-\frac{1}{2}}) \right|,$$

which completes the proof. □

3. Tensors of $\mathfrak{g}$-Modules

3.1. Exterior tensors of the natural module

Let $\xi_j, \bar{\xi}_j$, $j = 1, \ldots, n$, be the standard basis for $\mathbb{C}^{2n|0}$, and let $x_i, \bar{x}_i, x_0$, $i = 1, \ldots, m$, be the standard basis for $\mathbb{C}^{0|2m+1}$, so that $\Lambda(\mathbb{C}^{2n|2m+1}) \cong \mathbb{C}[x_i, \bar{x}_i, x_0] \otimes \Lambda(\xi_j, \bar{\xi}_j)$. We consider the Laplacian for $\mathfrak{spo}(2n|2m+1)$

$$\Delta = \sum_{j=1}^{n} \frac{\partial}{\partial \xi_j} \frac{\partial}{\partial \bar{\xi}_j} - \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \frac{\partial}{\partial \bar{x}_i} - \frac{1}{2} \frac{\partial^2}{\partial x_0^2}.$$

Dropping the last term in x_0 gives us the Laplacian for $\mathfrak{spo}(2n|2m)$.

Proposition 3.1.

(i) *Let $\mathfrak{g} = \mathfrak{spo}(2|2m+1)$ with $m \geq 1$, and let $k \geq 1$. Then the kernel of $\Delta : \Lambda^k(\mathbb{C}^{2|2m+1}) \to \Lambda^{k-2}(\mathbb{C}^{2|2m+1})$ as a $\mathfrak{g}$-module is irreducible and isomorphic to $L_{\delta_1+(k-1)\epsilon_1}$.*

(ii) *Let $\mathfrak{g} = \mathfrak{spo}(2n|3)$, with $n \geq 1$, and let $1 \leq k \leq n-1$. Then the kernel of $\Delta : \Lambda^k(\mathbb{C}^{2n|3}) \to \Lambda^{k-2}(\mathbb{C}^{2n|3})$ as a $\mathfrak{g}$-module is irreducible.*

(iii) *Let $n \geq 2$. The $\mathfrak{spo}(2n|2n)$-module* $\ker \left[\Delta : \Lambda^2(\mathbb{C}^{2n|2n}) \to \mathbb{C}\right]$ *is not irreducible. It contains a unique submodule which is the trivial module.*

(iv) *Let $n \geq 2$. The $\mathfrak{spo}(2n|2n)$-module* $\ker \left[\Delta : \Lambda^k(\mathbb{C}^{2n|2n}) \to \Lambda^{k-2}(\mathbb{C}^{2n|2n})\right]$ *is irreducible, for $k = 3, 4$.*

Proof. (i). The case of $k = 1$ is clear, so we may assume that $k \geq 2$. We decompose $\Lambda^k(\mathbb{C}^{2|2m+1})$ as an $\mathfrak{sp}(2) \oplus \mathfrak{so}(2m+1)$-module. By a direct computation, as a $\mathfrak{sp}(2) \oplus \mathfrak{so}(2m+1)$-module, the kernel of Δ in $\Lambda^k(\mathbb{C}^{2|2m+1})$ is a direct sum of three irreducibles with respective highest weight vectors x_1^k, $x_1^{k-1}\xi_1$ and $v := x_1^{k-2}\left((k+m-\frac{3}{2})\xi_1\bar{\xi}_1 - \sum_{i=1}^m x_i\bar{x}_i - \frac{1}{2}x_0^2\right)$, which have distinct weights. Among these three vectors, only the weight of $x_1^{k-1}\xi_1$ can be a finite-dimensional $\mathfrak{spo}(2|2m+1)$-highest weight.

From this it follows that the kernel of Δ on $\Lambda^k(\mathbb{C}^{2|2m+1})$ is an irreducible $\mathfrak{spo}(2|2m+1)$-module.

(ii). Under our assumption, the kernel of Δ in $\Lambda^k(\mathbb{C}^{2n|3})$ is isomorphic to $D(\lambda)$ with $\lambda = \delta_1 + \ldots + \delta_k$ by Lemma 2.7. Also by Theorem 2.9 and Remark 2.4 we see that all composition factors of $D(\lambda)$ lie in the same block. Suppose on the contrary that $D(\lambda)$ were not irreducible. As an $\mathfrak{sp}(2n) \oplus \mathfrak{so}(3)$-module, ker $\left[\Delta : \Lambda^k(\mathbb{C}^{2n|3}) \to \Lambda^{k-2}(\mathbb{C}^{2n|3})\right]$ decomposes into a direct sum of the irreducibles of highest weights of the form $\sum_{i=1}^{k-j} \delta_i + s\epsilon_1$, among which only weights of the form $\sum_{i=1}^{k-j} \delta_i$ can possibly be finite-dimensional $\mathfrak{spo}(2n|3)$-highest weights by Proposition 2.1. Hence, $D(\lambda)$ must have an $\mathfrak{spo}(2n|3)$-singular vector with weight $\sum_{i=1}^{k-j} \delta_i$, for $j > 0$. However, a calculation using Proposition 2.3 shows that $\sum_{i=1}^{k-j} \delta_i$ have different central characters for distinct j, which is a contradiction.

(iii). One shows that as an $\mathfrak{sp}(2n) \oplus \mathfrak{so}(2n)$-module, ker$\Delta$ has x_1^2, $x_1\xi_1$, $\xi_1\xi_2$, and $\phi := \sum_{i=1}^n \xi_i\bar{\xi}_i - \sum_{i=1}^n x_i\bar{x}_i$ as a complete set of highest weight vectors. Among the weights of these vectors only $\delta_1 + \delta_2$ and 0 are finite-dimensional $\mathfrak{spo}(2n|2n)$-weights, which implies that kerΔ has at most two composition factors. However, by a direct computation, ϕ is $\mathfrak{spo}(2n|2n)$-invariant (i.e. $\mathbb{C}\phi$ is a trivial module), and thus ker$\Delta/\mathbb{C}\phi$ is irreducible.

(iv). First we write down explict formulas for

$$e_0 = \xi_n \frac{\partial}{\partial x_1} - \bar{x}_1 \frac{\partial}{\partial \bar{\xi}_n}, \quad f_0 = x_1 \frac{\partial}{\partial \xi_n} + \bar{\xi}_n \frac{\partial}{\partial \bar{x}_1},$$

that are the odd simple positive and negative root vectors, respectively.

We will give a proof only in the most involved case of $k = 4$ and $n \geq 4$. We decompose kerΔ as an $\mathfrak{sp}(2n) \oplus \mathfrak{s}o(2n)$-modules and search among those highest weights the ones that are finite-dimensional $\mathfrak{spo}(2n|2n)$-highest weights. The only possibilities are $\sum_{i=1}^4 \delta_i$, $\delta_1 + \delta_2$, and 0, each appearing with multiplicity one. Now it is not difficult to write down the corresponding $\mathfrak{sp}(2n) \oplus \mathfrak{s}o(2n)$-highest weight vectors of these weights, namely explicitly they are

$$\xi_1\xi_2\xi_3\xi_4, \quad \xi_1\xi_2(\phi_1 - \frac{n}{n-2}\phi_0), \quad (\phi_1 - \phi_0)^2 + \frac{1}{n-1}\phi_0^2.$$

Next we observe that the central character of $\sum_{i=1}^4 \delta_i$ is different from that of $\delta_1 + \delta_2$ (which is the same as that of 0). Thus if kerΔ were not irreducible, then among those three vectors at least two would be killed by e_0. Applying

e_0 to these vectors we find that only $\xi_1\xi_2\xi_3\xi_4$ is killed by e_0. This concludes the proof. $\square$

Remark 3.2. Using a similar argument as the one in the proof of Proposition 3.1 (iv) one can show that ker $\left[\Delta : \Lambda^n(\mathbb{C}^{2n|3}) \to \Lambda^{n-2}(\mathbb{C}^{2n|3})\right]$ is irreducible over $\mathfrak{spo}(2n|3)$. Also Proposition 3.1 (iii) shows that the Jacobi-Trudi character is in general not irreducible.

3.2. A tensor product decomposition

Proposition 3.3. *Let* $\mathfrak{g} = \mathfrak{spo}(2|2m+1)$*, and let* $k \geq 2$*. Then, the following* $\mathfrak{g}$*-module decompositions hold:*

$$\begin{aligned} L_{\delta_1+(k-1)\epsilon_1} \otimes \mathbb{C}^{2|2m+1} &\cong L_{2\delta_1+(k-1)\epsilon_1} \oplus L_{\delta_1+k\epsilon_1} \oplus L_{\delta_1+(k-2)\epsilon_1} \\ \mathbb{C}^{2|2m+1} \otimes \mathbb{C}^{2|2m+1} &\cong L_{2\delta_1} \oplus L_{\delta_1+\epsilon_1} \oplus \mathbb{C}. \end{aligned}$$

Proof. We will first consider the tensor product $L_{\delta_1+(k-1)\epsilon_1} \otimes \mathbb{C}^{2|2m+1}$. It follows from the proof of Proposition 3.1(i) that, as a $\mathfrak{g}_{\bar{0}} = \mathfrak{sp}(2) \oplus \mathfrak{so}(2m+1)$-module, $L(\delta_1 + (k-1)\epsilon_1)$ (which is identified with the kernel of Δ on $\Lambda^k(\mathbb{C}^{2|2m+1})$) is a direct sum of three irreducible modules with highest weight vectors v, x_1^k and $x_1^{k-1}\xi_1$. One then checks by an elementary but tedious calculation, that $L(\delta_1 + (k-1)\epsilon_1) \otimes \mathbb{C}^{2|2m+1}$ as a $\mathfrak{g}_{\bar{0}}$-module is a direct sum of 13 irreducible modules with respective highest weight vectors given as follows:

$$x_1^k \otimes x_1, \quad x_1^{k-1}\left(x_1 \otimes x_2 - x_2 \otimes x_1\right),$$

$$x_1^{k-1}\Big(x_0 \otimes x_0 + \sum_{i=1}^m (x_i \otimes \bar{x}_i + \bar{x}_i \otimes x_i)\Big) - \frac{k-1}{k+m-\frac{3}{2}} x_1^{k-2}\big(\frac{1}{2}x_0^2 + \sum_{i=1}^m x_i\bar{x}_i\big) \otimes x_1,$$

$$x_1^k \otimes \xi, \quad x_1^{k-1}\xi \otimes x_1, \quad x_1^{k-2}\xi\left(x_1 \otimes x_2 - x_2 \otimes x_2\right),$$

$$\gamma_0 := \xi x_1^{k-2}\Big(x_0 \otimes x_0 + \sum_{i=1}^m (x_i \otimes \bar{x}_i + \bar{x}_i \otimes x_i)\Big) - \frac{k-2}{k+m-\frac{5}{2}} \xi x_1^{k-3}\Big(\frac{1}{2}x_0^2 + \sum_{i=1}^m x_i\bar{x}_i\Big) \otimes x_1,$$

$$x_1^{k-1}\xi \otimes \xi, \ x_1^{k-1}(\xi \otimes \bar{\xi} - \bar{\xi} \otimes \xi), \ \psi_0 x_1^{k-2} \otimes x_1, \ \psi_0 x_1^{k-3}(x_1 \otimes x_2 - x_2 \otimes x_1),$$

$$\psi_0\Big(x_1^{k-3}(x_0\otimes x_0+\sum_{i=1}^m x_i\otimes\bar{x}_i+\bar{x}_i\otimes x_i)-\frac{k-3}{k+m-\frac{7}{2}}x_1^{k-4}(\frac{1}{2}x_0^2+\sum_{i=1}^m x_i\bar{x}_i)\otimes x_1\Big),$$
$$\psi_0 x_1^{k-2}\otimes\xi,$$

where $\psi_0=(k+n-\frac{3}{2})\xi_1\bar{\xi}_1-\sum_{i=1}^m x_i\bar{x}_i-\frac{1}{2}x_0^2$.

We observe that among the weights of these vectors only five of them can possibly be finite-dimensional $\mathfrak{spo}(2|2m+1)$-highest weights, namely $2\delta_1+(k-1)\epsilon_1$, $\delta_1+k\epsilon_1$, and $\delta_1+(k-2)\epsilon_1$, where the first appears with multiplicity one and the latter two each appears with multiplicity two. Note that these three weights give rise to distinct central characters and hence the tensor product must be complete reducible, with each irreducible component generated by a singular vector. Now both $e_0.\gamma_0$ and $e_0.(x_1^{k-2}\psi_0\otimes\xi)$ are nonzero and proportional to each other. Thus there is exactly one $\mathfrak{spo}(2|2m+1)$-highest weight vector of highest weight $\delta_1+(k-2)\epsilon_1$. Furthermore, both $e_0.x_1^k\otimes\xi$ and $e_0.x_1^{k-1}\xi\otimes x_1$ are nonzero and proportional to each other, and hence there is exactly one $\mathfrak{spo}(2|2m+1)$-highest weight vector of highest weight $\delta_1+k\epsilon_1$. Clearly $L(2\delta_1+(k-1)\epsilon_1)$ appears in the tensor product decomposition with multiplicity one as $2\delta_1+(k-1)\epsilon_1$ is the unique highest weight of multiplicity one. This proves the first identity.

The decomposition for $\mathbb{C}^{2|2m+1}\otimes\mathbb{C}^{2|2m+1}$ can be proved in the same way. Indeed, here the situation is simpler, since as an $\mathfrak{g}_{\bar{0}}$-module $\mathbb{C}^{2|2m+1}\otimes\mathbb{C}^{2|2m+1}$ is a direct sum of only eight irreducibles. We skip the details. □

Remark 3.4. Let $\mathfrak{g}=\mathfrak{spo}(4|5)$. We can show similarly as for Proposition 3.3 that

$$[L_{\delta_1+\delta_2}\otimes\mathbb{C}^{4|5}]=[L_{2\delta_1+\delta_2}+L_{\delta_1}]\oplus[L_{\delta_1+\delta_2+\epsilon_1}]\oplus[L_{\delta_1}],$$

where $[L_{2\delta_1+\delta_2}+L_{\delta_1}]$ denotes a non-trivial extension of modules.

4. Examples and a Conjecture

4.1. Examples of $\mathfrak{spo}(2|3)$

Throughout this Section 4.1, we let $\mathfrak{g}=\mathfrak{spo}(2|3)$. For $\lambda=a\delta_1+b\epsilon_1$ write $L(a|b)=L(\lambda)$, $K(a|b)=K(\lambda)$, and $E^{\mathfrak{p}}(a|b)=E^{\mathfrak{p}}(L^0(\lambda))$. The complete list of atypical weights are: $\lambda=a\delta_1+b\epsilon_1$, where $(a|b)=(\ell|\ell-1)$ for $\ell\geq 1$, or

$(0|0)$. If λ is typical, then the character of $L(\lambda)$ is equal to $K(\lambda)$. On the other hand, the character of $L(\ell|\ell-1)$ was computed in [4] (also see [6]). In particular, $\dim L(\ell|\ell-1) = 2(4\ell^2-1)$ for $\ell \geq 2$, and $\dim L(1|0) = 5$. All the examples in this section are computed from the definitions and the formulas of these irreducible characters. We skip the details.

4.1.1. "Composition factors" of the Euler characters

By abuse of notation, we will regard the Euler characters and the Kac virtual characters as elements in the Grothendieck group of the finite-dimensional $\mathfrak{g}$-modules. We shall denote by $[L(\lambda)]$ the element in the Grothendieck group corresponding to the module $L(\lambda)$.

Example 4.1. Consider the maximal parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{spo}(2|3)$ obtained by removing the simple root ϵ_1 so that $\mathfrak{l} = \mathfrak{gl}(1|1)$. If λ is typical, then $E^{\mathfrak{p}}(\lambda) = K(\lambda) = [L(\lambda)]$. For atypical weights, the following identities of characters hold:

$$\begin{aligned}
E^{\mathfrak{p}}(\ell+1|\ell) &= K(\ell+1|\ell) = [L(\ell+1|\ell)] + [L(\ell|\ell-1)], \quad \ell \geq 2,\\
E^{\mathfrak{p}}(2|1) &= K(2|1) = [L(2|1)] + [L(1|0)] + [L(0|0)],\\
E^{\mathfrak{p}}(1|0) &= K(1|0) = [L(1|0)] - [L(0|0)],\\
E^{\mathfrak{p}}(0|0) &= 2[L(0|0)].
\end{aligned}$$

Example 4.2. Consider $\mathfrak{spo}(2|3)$ with $\mathfrak{p}$ obtained by removing the simple root $\delta_1 - \epsilon_1$ so that $\mathfrak{l} \cong \mathfrak{gl}(1) \oplus \mathfrak{so}(3)$. For atypical weights, the following identities of characters hold:

$$\begin{aligned}
E^{\mathfrak{p}}(\ell+1|\ell) &= K(\ell+1|\ell) = [L(\ell+1|\ell)] + [L(\ell|\ell-1)], \quad \ell \geq 2,\\
E^{\mathfrak{p}}(2|1) &= K(2|1) = [L(2|1)] + [L(1|0)] + [L(0|0)],\\
E^{\mathfrak{p}}(1|0) &= K(1|0) = [L(1|0)] - [L(0|0)],\\
E^{\mathfrak{p}}(0|0) &= K(0|0) = [L(0|0)] - [L(1|0)].
\end{aligned}$$

These identities also follow from [4, Lemma 2.2.1] and our Remark 2.4.

4.1.2. Some virtual dimension formulas

Since each virtual character $D(\lambda)$ or $K(\lambda)$ can be written as the difference of two honest characters, there is a well-defined notion of *virtual* *dimension*, vdim, of these virtual characters as the difference of the degrees of the two honest characters.

Example 4.3. The following virtual dimension formulas hold:

(1) $\mathrm{vdim}K(\lambda) = 2^{|\Delta_1^+|}\prod_{\alpha\in\Delta_0^+}\frac{(\alpha,\lambda+\rho)}{(\alpha,\lambda+\rho_0)}$.

(2) $\mathrm{vdim}K(1|0) = 4$.

(3) $\mathrm{vdim}K(0|0) = -4$.

(4) $\mathrm{vdim}K(\ell|\ell-1) = 4(2\ell-1)^2 = \mathrm{dim}L(\ell|\ell-1) + \mathrm{dim}L(\ell-1|\ell-2)$, $\ell \geq 3$.

(5) $\mathrm{vdim}D(\ell|\ell-1) = \mathrm{vdim}K(\ell|\ell-1) - (-1)^\ell$, $\ell \geq 3$.

(6) $\mathrm{vdim}D(2|1) = 35 = \mathrm{dim}L(2|1) + \mathrm{dim}L(1|0)$.

(7) $\mathrm{vdim}D(1|0) = \mathrm{dim}L(1|0)$; actually, $D(1|0) = L(1|0)$.

(8) $\mathrm{vdim}D(1|k-1) = \mathrm{dim}L(1|k-1)$, for $k \geq 1$.

4.1.3. Tensor products of the simples with the natural module

Below we give explicit formulas for $L(a|b) \otimes L(1|0)$, where we recall $L(1|0) = \mathbb{C}^{2|3}$, the natural $\mathfrak{spo}(2|3)$-module.

Example 4.4.

(1) $\lambda = (a|b)$ atypical.

$$[L(1|0) \otimes L(1|0)] = [L(2|0)] + [L(1|1)] + [L(0|0)],$$
$$[L(l|l-1) \otimes L(1|0)] = [L(l+1|l-1)] + [L(l|l)] + [L(l|l-1)], \quad l \geq 2.$$

(2) $\lambda = (a|b)$ typical with $0 \leq b \leq 1$ or $0 \leq a \leq 1$.

$$\begin{aligned}
&[L(2|0) \otimes L(1|0)] = [L(3|0)] + [L(2|1)] + 2[L(1|0)],\\
&[L(l|0) \otimes L(1|0)] = [L(l+1|0)] + [L(l|1)] + [L(l-1|0)], \quad l \geq 3.\\
&[L(1|1) \otimes L(1|0)] = [L(2|1)] + [L(1|2)] + 2[L(1|0)],\\
&[L(3|1) \otimes L(1|0)] = [L(4|1)] + [L(3|1)] + [L(3|0)]\\
&\qquad\qquad + [L(3|2)] + 2[L(2|1)] + 2[L(1|0)],
\end{aligned}$$

$$\begin{aligned}[L(l|1)\otimes L(1|0)] &= [L(l|1)]+[L(l+1|1)]\\ &\qquad +[L(l-1|1)]+[L(l|2)]+[L(l|0)], \quad l\geq 4.\\ [L(1|l)\otimes L(1|0)] &= [L(2|l)]+[L(1|l+1)]+[L(1|l-1)], \quad l\geq 2.\end{aligned}$$

(3) $\lambda=(l|l)$ typical with $l\geq 2$.

$$\begin{aligned}[L(2|2)\otimes L(1|0)] &= [L(3|2)]+2[L(2|1)]+[L(2|2)]\\ &\quad +[L(1|2)]+[L(2|3)]+[L(1|0)]+[L(0|0)].\\ [L(l|l)\otimes L(1|0)] &= [L(l+1|l)]+2[L(l|l-1)]+[L(l|l)]\\ &\quad +[L(l-1|l)]+[L(l|l+1)]+[L(l-1|l-2)].\end{aligned}$$

(4) $\lambda=(l+2|l)$ typical with $l\geq 0$.

$$\begin{aligned}[L(l+2|l)\otimes L(1|0)] &= [L(l+3|l)]+2[L(l+1|l)]+[L(l+2|l)]\\ &\quad +[L(l|l-1)]+[L(l+2|l+1)]+[L(l+2|l-1)].\end{aligned}$$

(5) $\lambda=(a|b)$ typical and $a,b\geq 2$, $a\neq b+2$, $a\neq b$.

$$\begin{aligned}[L(a|b)\otimes L(1|0)] &= [L(a+1|b)]+[L(a|b)]+[L(a-1|b)]\\ &\quad +[L(a|b+1)]+[L(a|b-1)].\end{aligned}$$

4.2. A conjecture

It is known that the parametrization set of highest weights of the irreducible polynomial representations of $\mathfrak{gl}(n|m)$ (Sergeev [14]) is the same as that of the finite-dimensional simple $\mathfrak{g}=\mathfrak{spo}(2n|2m+1)$-modules (see Remark 2.2).

Conjecture 4.5. Consider the maximal parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}=\mathfrak{spo}(2n|2m+1)$ obtained by removing the simple root ϵ_m so that $\mathfrak{l}=\mathfrak{gl}(n|m)$. Then

(1) The $E^{\mathfrak{p}}(L^0(\lambda^\sharp))$, with λ running over all partitions with $\lambda_{n+1}\leq m$, form a basis for the complexified Grothendieck group of the category of finite-dimensional $\mathfrak{g}$-modules.

(2) When λ is "not close" to the zero weight, then the composition factors for the Euler character $E^{\mathfrak{p}}(L^0(\lambda^\sharp))$ is the same as that for the Kac module $K(\lambda^\sharp)$ for $\mathfrak{gl}(n|m)$.

The comparison of Example 4.1 with some well-known facts for $\mathfrak{gl}(1|1)$ shows that the Conjecture is true for $\mathfrak{spo}(2|3)$.

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