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CHARACTERIZATION OF WEAKLY PRIME SUBTRACTIVE IDEALS IN SEMIRINGS

BY

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Abstract

In the paper we extend some results of [1] to non commutative semirings with $1 \neq 0$. We prove the following Theorem: (1) Let I be a subtractive ideal of a semiring R. Then I is a weakly prime ideal of R if and only if for left ideals A and B of R, $0 \neq AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$. (2) Let R be a semiring in which all nilpotent elements are central and let I be a weakly prime subtractive ideal of R which is not a prime ideal of R. Then $I\sqrt{0} = 0$.

For the definition of a semiring we refer the readers to [2]. Z^+ will denote the set of all non negative integers. All (left, right) ideals are proper. An ideal I of a semiring R is called subtractive if a, $a + b \in I$, $b \in R$, then $b \in I$. An ideal I of a semiring R is called prime if $aRb \subseteq I$ where a, $b \in R$, then $a \in I$ or $b \in I$. Easily, an ideal I of a semiring R is prime if and only if for left ideals A and B of R, $AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$. A semiring R is called prime if 0 is a prime ideal of R. A semiring R is called semi prime if it has no non zero nilpotent ideals. An ideal I of a semiring R is called weakly prime if $0 \neq aRb \subseteq I$ where a, $b \in R$, then $a \in I$ or $b \in I$. Clearly every prime ideal is a weakly prime ideal. If R is a semiring, then 0 is a weakly prime ideal of R. Let $R = Z_6$. Then 0 in $R_{n\times n}$ is a weakly prime ideal but not a prime ideal. If R is a prime semiring and I is a weakly prime ideal of R, then I is a prime ideal of R.

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Easily, I is a prime ideal of a semiring R if and only if $I_{n\times n}$ is a prime ideal of the semiring $R_{n\times n}$.

Now we have the following proposition.

Proposition 1. *Let R be a semiring and let I be an ideal of R. If $I_{n\times n}$ is a weakly prime ideal of $R_{n\times n}$, then I is a weakly prime ideal of R.*

Proof. Let $0 \neq aRb \subseteq I$ where $a, b \in R$. Then $0 \neq aE_{11}R_{n\times n}bE_{11} \subseteq I_{n\times n}$. Hence $aE_{11} \in I_{n\times n}$ or $bE_{11} \in I_{n\times n}$. Now $a \in I$ or $b \in I$. □

Lemma 2. *Let I be a weakly prime subtractive ideal of a semiring R. Then $I^2 = 0$ or I is a prime ideal of R.*

Proof. Suppose $I^2 \neq 0$. Let $aRb \subseteq I$ where $a, b \in R$. If $aRb \neq 0$, then $a \in I$ or $b \in I$. So assume that $aRb = 0$. If $aI \neq 0$, then there exists $x \in I$ such that $ax \neq 0$. Now $0 \neq aRx = aR(b+x) \subseteq I$. Hence $a \in I$ or $b+x \in I$. So $a \in I$ or $b \in I$. Now assume that $aI = 0$. If $Ib \neq 0$, then there exists $u \in I$ such that $ub \neq 0$. Now $0 \neq uRb = (u+a)Rb \subseteq I$. So $a \in I$ or $b \in I$. Hence we can assume that $Ib = 0$. Since $I^2 \neq 0$, there exist $w, z \in I$ such that $wz \neq 0$. Then $0 \neq wRz = (a+w)R(b+z) \subseteq I$. So $a \in I$ or $b \in I$. □

Corollary 3. *Let R be a semiprime semiring. Then a subtractive ideal I of R is weakly prime if and only if $I = 0$ or I is a prime ideal.*

Let I be an ideal of a semiring R and let

$$\begin{aligned}\sqrt{I} &= \{x \in R : \text{ every } m\text{-system containing } x \text{ meets } I\}\\ &\subseteq \{x \in R : x^n \in I \text{ for some } n \geq 1\}.\end{aligned}$$

If R is a commutative semiring, then equality holds. If $I = 0$ and all nilpotent elements are central, then equality also holds.

The following lemma can be proved easily.

Lemma 4. *Let I be an ideal of a semiring R. Then $\sqrt{I}$ is the intersection of all prime ideals of R containing I. So $\sqrt{I}$ is an ideal of R.*

Proposition 5. *Let I be a weakly prime subtractive ideal of a semiring R. Then*

(1) *$I \subseteq \sqrt{0}$ or $\sqrt{0} \subseteq I$.*

(2) *If $I \subsetneq \sqrt{0}$, then I is not a prime ideal of R.*

(3) *If $\sqrt{0} \subsetneq I$, then I is a prime ideal of R.*

Proof. (1) If I is a prime ideal of R, then by Lemma 4, $\sqrt{0} \subseteq I$. If I is not a prime ideal of R, then by Lemma 2, $I^2 = 0 \subseteq \sqrt{0}$. So $I \subseteq \sqrt{0}$. (2) and (3) are obvious. $\square$

The following Lemma can be proved easily.

Lemma 6. *Let I and J be subtractive ideals of a semiring R. Then $I \cup J$ is a subtractive ideal of R if and only if $I \cup J = I$ or $I \cup J = J$.*

We use the following notations: If $x \in R$, then $\langle x\rangle_\ell = Rx$, $\langle x\rangle_r = xR$ and $\langle x\rangle = RxR$. Let I be an ideal of a semiring R and let $x \notin I$. We define

$$(I : \langle x\rangle_\ell)_\ell \;=\; \{y \in R : y\langle x\rangle_\ell \subseteq I\}$$

and

$$(I : \langle x\rangle_r)_r \;=\; \{y \in R : \langle x\rangle_r y \subseteq I\}\,.$$

They form ideals of R containing I.

The following lemma can be proved easily.

Lemma 7. *Let R be a semiring. Let I be an ideal of R and let $x \notin I$.*

(1) *If I is subtractive, then $(I : \langle x\rangle_\ell)_\ell$ is also subtractive.*
(2) *If I is a prime ideal of R and $(I : \langle x\rangle_\ell)_\ell$ is subtractive, then I is also subtractive.*

Lemma 8. *Let I be a subtractive ideal of a semiring R. Then the following statements are equivalent.*

(1) *I is a weakly prime ideal of R.*
(2) *If $x \notin I$, then $(I : \langle x\rangle_\ell)_\ell = I \cup (0 : \langle x\rangle_\ell)_\ell$.*
(3) *If $x \notin I$, then $(I : \langle x\rangle_\ell)_\ell = I$ or $(I : \langle x\rangle_\ell)_\ell = (0 : \langle x\rangle_\ell)_\ell$.*

Proof. (1) $\Rightarrow$ (2). Let $y \in (I : \langle x\rangle_\ell)_\ell$. Then $yRx \subseteq I$. If $yRx = 0$, then $y \in (0 : \langle x\rangle_\ell)_\ell$. If $yRx \neq 0$, then $y \in I$. Now $(I : \langle x\rangle_\ell)_\ell \subseteq I \cup (0 : \langle x\rangle_\ell)_\ell$. Otherway, let $z \in I \cup (0 : \langle x\rangle_\ell)_\ell$. Then $z\langle x\rangle_\ell \subseteq I$. Now $z \in (I : \langle x\rangle_\ell)_\ell$.
(2) $\Rightarrow$ (3). It follows by Lemmas 6 and 7.
(3) $\Rightarrow$ (1). Let $yRx \subseteq I$ and $yRx \neq 0$. We have $(I : \langle x\rangle_\ell)_\ell = I$ or $(I : \langle x\rangle_\ell)_\ell = (0 : \langle x\rangle_\ell)_\ell$. Suppose $(I : \langle x\rangle_\ell)_\ell = (0 : \langle x\rangle_\ell)_\ell$. Since $y \in (I : \langle x\rangle_\ell)_\ell$, we get $y\langle x\rangle_\ell = 0$, a contradiction. Hence $(I : \langle x\rangle_\ell)_\ell = I$. So $y \in I$. $\square$

Similarly the right analogues can be established.

Theorem 9. *Let I be a subtractive ideal of a semiring R. Then the following statements are equivalent*

(1) *I is weakly prime ideal of R.*
(2) *For left ideals A and B of R, $0 \neq AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.*
(3) *For ideals A and B of R, $0 \neq AB \subseteq I$ implies that $A \subseteq I$ or $B \subseteq I$.*

Proof. (1) $\Rightarrow$ (2). Suppose I is a weakly prime ideal of R. If I is a prime ideal of R, then the result holds trivially. So assume that I is not a prime ideal of R. Let A and B be left ideals of R such that $0 \neq AB \subseteq I$. Suppose $A \nsubseteq I$ and $B \nsubseteq I$. Let $a \in A - I$. Now $\langle a\rangle_r B \subseteq I$. Hence $B \subseteq (I : \langle a\rangle_r)_r$. Since $B \nsubseteq I$, by right analogue of Lemma 8, $\langle a\rangle_r B = 0$. So $aB = 0$. Next let $a \in A \cap I$. Let $b \in B$. If $b \in I$, then $ab \in I^2 = 0$ by Lemma 2. If $b \in B - I$, then $A\langle b\rangle_\ell \subseteq I$. Hence $A \subseteq (I : \langle b\rangle_\ell)_\ell$. Since $A \nsubseteq I$, by Lemma 8, $A\langle b\rangle_\ell = 0$. So $ab = 0$. Now $AB = 0$, a contradiction.
(2) $\Rightarrow$ (3). It is obvious.
(3) $\Rightarrow$ (1). Let $0 \neq aRb \subseteq I$ where $a, b \in R$. Then $0 \neq \langle a\rangle\langle b\rangle \subseteq I$. Hence $\langle a\rangle \subseteq I$ or $\langle b\rangle \subseteq I$. So $a \in I$ or $b \in I$. $\square$

Theorem 10. *Let R be semiring in which all nilpotent elements are central. Let I be a weakly prime subtractive ideal of R which is not a prime ideal of R. Then $I\sqrt{0} = 0$.*

Proof. Suppose I is not a prime ideal of R. Let $0 \neq x \in \sqrt{0}$. Then $x \in I$ or $x \notin I$. If $x \in I$, then $Ix \subseteq I^2 = 0$ by Lemma 2. If $x \notin I$, then by Lemma 8, $(I : \langle x\rangle_\ell)_\ell = I$ or $(I : \langle x\rangle_\ell)_\ell = (0 : \langle x\rangle_\ell)_\ell$. Suppose $(I : \langle x\rangle_\ell)_\ell = I$. Since $x \in \sqrt{0}$, we get $x^n = 0$ for some $n > 1$ where $x^{n-1} \neq 0$. Since x is central, we get $x^{n-1}\langle x\rangle_\ell = 0 \subseteq I$. Hence $x^{n-1} \in (I : \langle x\rangle_\ell)_\ell = I$. So $x^{n-2} \in (I : \langle x\rangle_\ell)_\ell = I$. Continue in this way, we get $x \in I$, a contradiction.

Hence $(I:\langle x\rangle_\ell)_\ell = (0:\langle x\rangle_\ell)_\ell$. Since $I \subseteq (I:\langle x\rangle_\ell)_\ell$, we get $I\langle x\rangle_\ell = 0$. Now $I\sqrt{0} = 0$. $\square$

The converse of the Theorem 10 is not true.

Example 11. Let $R = Z_{12}$ and $I = \{0,4,8\}$. Then I is not a weakly prime ideal of R. So it is not a prime ideal of R. It is a subtractive ideal. We have $I\sqrt{0} = 0$ where $\sqrt{0} = \{0,6\}$.

The condition that I is not a prime ideal is essential.

Example 12. Let $R = Z_{12}$ and let $I = \{0,3,6,9\}$. Then I is a subtractive prime ideal of R. So it is a weakly prime ideal of R. But $I\sqrt{0} \neq 0$ where $\sqrt{0} = \{0,6\}$.

Corollary 13. *Let R be a semiring in which all nilpotent elements are central. Let I and J be the weakly prime subtractive ideals of R that are not prime ideals of R. Then $IJ = 0$.*

Proof. By Lemma 2, $J^2 = 0 \subseteq \sqrt{0}$. Hence $J \subseteq \sqrt{0}$. Now $IJ \subseteq I\sqrt{0} = 0$. $\square$

Example 14. Let $R = (Z^+, +, \cdot)$.

Consider the semiring

$$S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,b,c \in R \right\}.$$

Let I be an ideal if R. Then $T = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a,b,c \in I \right\}$ is an ideal of S. Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ be in S.

(i) If $I = \langle 2,3\rangle$, then $0 \neq ASB \subseteq T$ but $A, B \notin T$. Hence T is not a weakly prime ideal of S. It is not subtractive. We have $T^2 \neq 0$.

(ii) If $I = \langle 2\rangle$, then $0 \neq ASB \subseteq T$ but $A, B \notin T$. Hence T is not a weakly prime ideal of S. It is subtractive. We have $T^2 \neq 0$.

Example 15. Let $R = Z_{12}$ and let $I = \langle 6 \rangle$: Let $A = 3E_{11}$, $B = 2E_{11}$. Then $0 \neq AR_{2\times 2}B \subseteq I_{2\times 2}$. But $A, B \notin I_{2\times 2}$. Hence $I_{2\times 2}$ is not a weakly prime ideal of $R_{2\times 2}$. It is subtractive. We have $I^2_{2\times 2} = 0$.

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