

## ON SPARSE SETS AND DENSITY POINTS DEFINED BY FAMILIES OF SEQUENCES

BY

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### Abstract

In the paper a condition equivalent to sparseness of a set is formulated. This condition is a base for a definition of a density point with respect to a family of sequences ( $\mathcal{C}$ -density). It is shown that  $\mathcal{C}$ -density is a generalization of a density with respect to one sequence ( $\langle a \rangle$ -density).

D. N. Sarkhel and A. K. De in the paper [5] defined some integrals of Peron type. They used notions of a proximal limit, continuity and derivative, which are generalizations of approximate ones. These generalizations are based on a notion of a set sparse at a point. Sparse sets were also investigated in [3] and [4]. In [3] such sets were used to define an analogue of the density point (called in our paper the proximal density point) and an analogue of the density topology.

In our paper we formulate a condition equivalent to the definition of a sparse set (Theorem 1). Replacing in this condition the family of sequences satisfying  $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 0$  by another family  $\mathcal{C}$  of sequences (and taking complements) we get definition of an abstract density point depending on family  $\mathcal{C}$  ( $\mathcal{C}$ -density). We note that  $\mathcal{C}$ -density is a generalization of  $\langle a \rangle$ -density studied in [2] and [1] (Proposition 4). We show that proximal density (so sparseness too) cannot be defined by means of  $\langle a \rangle$ -density (Theorem 2).

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Through out the paper we shall use standard notation:  $\mathbb{R}$  will be the set of real numbers,  $\mathbb{N}$  the set of positive integers,  $\mathcal{L}$  the family of Lebesgue measurable subsets of  $\mathbb{R}$ ,  $|A|$  the Lebesgue measure of a measurable set  $A$  and  $|A|^*$  the outer Lebesgue measure of a set  $A \subset \mathbb{R}$ . We shall also write  $A \sim B$  if  $|A \Delta B| = 0$ . By the density of a measurable set  $A$  we mean

$$d(A, x) = \lim_{h \rightarrow 0^+} \frac{|A \cap (x - h; x + h)|}{2h}.$$

If  $d(A, x) = 1$ , then we say that  $x$  is a *density point* of  $A$ . The set of all density points of  $A$  we denote by  $\Phi(A)$ . Similarly we define *upper densities*

$$\begin{aligned} \bar{d}(A, x) &= \limsup_{h \rightarrow 0^+} \frac{|A \cap (x - h; x + h)|}{2h}, \\ \bar{d}_+(A, x) &= \limsup_{h \rightarrow 0^+} \frac{|A \cap (x; x + h)|}{h}, \quad \bar{d}_-(A, x) = \limsup_{h \rightarrow 0^+} \frac{|A \cap (x - h; x)|}{h} \end{aligned}$$

and sets  $\bar{\Phi}(A)$ ,  $\bar{\Phi}_+(A)$ ,  $\bar{\Phi}_-(A)$  of all points for which the suitable upper density is equal to 1. Sequences of real numbers we shall denote by  $\langle a \rangle$  or  $(a_n)$ .

D. N. Sarkhel and A. K. De have defined in [5] a set sparse at a point. Their definition may be apply to every subset of the real line (not necessarily measurable). We restrict our considerations to measurable sets, because sparseness of a set is equivalent to sparseness of it measurable cover.

A measurable set  $A$  is said to be *sparse at a point  $x$  on the right* if for any  $\varepsilon > 0$  there exists  $h > 0$  such that each interval  $(\alpha; \beta) \subset (x; x + h)$  with  $\alpha - x < h(\beta - x)$  contains at least one point  $y$  such that  $|A \cap (x; y)| < \varepsilon(y - x)$ . *Sparseness on the left* is defined similarly. A set  $A$  is said to be *sparse at  $x$*  if it is sparse at  $x$  on the right and on the left. In [5] it was shown that if  $x$  is a dispersion point of  $A$  (i.e.  $d(A, x) = 0$ ) then  $A$  is sparse at  $x$ , and if  $A$  is sparse at  $x$  on the right, then  $\bar{d}_+(A, x) < 1$  and  $\bar{d}_+(\mathbb{R} \setminus A, x) = 1$ .

**Theorem 1.** *A measurable set  $A$  is sparse at  $x$  on the right if and only if for any  $\varepsilon > 0$  there is a decreasing sequence  $(a_n)$  satisfying  $\lim_{n \rightarrow \infty} a_n = 0$ ,  $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 0$  and such that for each  $n \in \mathbb{N}$*

$$\frac{|A \cap (x; x + a_n)|}{a_n} < \varepsilon.$$

*Proof.* Without loss of generality we can assume that  $x = 0$ .

Suppose that  $A$  is sparse at 0 on the right and fix  $\varepsilon > 0$ . By the assumption, there exists  $h \in (0; 1)$  such that each interval  $(\alpha; \beta) \subset (0; h)$  with  $\alpha < h\beta$  contains a point  $y$  satisfying

$$\frac{|A \cap (0; y)|}{y} < \varepsilon.$$

Let  $c_n = (\frac{h}{2})^n$  for  $n \in \mathbb{N}$ . Since  $c_n < h$  and  $\frac{c_{n+1}}{c_n} < h$ , we can find a sequence  $(y_n)$  such that  $y_n \in (c_{n+1}; c_n)$  and

$$\frac{|A \cap (0; y_n)|}{y_n} < \varepsilon$$

for every  $n$ . Moreover

$$\frac{y_{n+1}}{y_n} > \frac{c_{n+2}}{c_n} = \frac{h}{4} > 0,$$

which completes the proof of necessity.

To prove the inverse implication, set any positive  $\varepsilon$  and find a decreasing sequence  $(a_n)$  satisfying  $\lim_{n \rightarrow \infty} a_n = 0$ ,  $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 0$  and

$$\frac{|A \cap (0; a_n)|}{a_n} < \varepsilon \tag{1}$$

for  $n \in \mathbb{N}$ . We can find  $\delta \in (0; 1)$  such that  $\frac{a_{n+1}}{a_n} > \delta$  for all  $n$ . Set  $h = \min\{\delta, \delta a_1\}$  and consider any interval  $(\alpha; \beta) \subset (0; h)$  with  $\alpha < \beta h$ . Since  $\beta \leq h < a_1$ ,  $\beta \in (a_{p+1}; a_p)$  for some  $p \in \mathbb{N}$ . Thus the inequality

$$\frac{\alpha}{\beta} < h \leq \delta < \frac{a_{p+1}}{a_p}$$

implies  $a_{p+1} \in (\alpha; \beta)$ , and from (1) we conclude that  $A$  is sparse at 0 on the right.  $\square$

Theorem 1 motivates us to define a new kind of density point. A family  $\mathcal{C}$  of decreasing sequences convergent to 0 such that

$$\inf\{a_1; \langle a \rangle \in \mathcal{C}\} = 0$$

we call *acceptable*. We say that  $x$  is a  $\mathcal{C}$ -density point (*right-hand  $\mathcal{C}$ -density point*) of a measurable set  $A$  if for every  $\varepsilon > 0$  there exists  $\langle a \rangle \in \mathcal{C}$  such that for each  $n$

$$\frac{|A \cap (x - a_n; x + a_n)|}{2a_n} > 1 - \varepsilon \left( \frac{|A \cap (x; x + a_n)|}{a_n} > 1 - \varepsilon \right).$$

Similarly we define *left-hand  $\mathcal{C}$ -density points*. The set of all  $\mathcal{C}$ -density points (right-hand  $\mathcal{C}$ -density points, left-hand  $\mathcal{C}$ -density points) we denote by  $\Phi_{\mathcal{C}}(A)$  ( $\Phi_{\mathcal{C}}^+(A)$ ,  $\Phi_{\mathcal{C}}^-(A)$ ).

It is clear that

**Proposition 1.** *For any  $A, B \in \mathcal{L}$  and any acceptable  $\mathcal{C}, \mathcal{C}_1, \mathcal{C}_2$*

- (a)  $\Phi_{\mathcal{C}}(\emptyset) = \emptyset$  and  $\Phi_{\mathcal{C}}(\mathbb{R}) = \mathbb{R}$ .
- (b) If  $A \sim B$  then  $\Phi_{\mathcal{C}}(A) = \Phi_{\mathcal{C}}(B)$ .
- (c) If  $A \subset B$  then  $\Phi_{\mathcal{C}}(A) \subset \Phi_{\mathcal{C}}(B)$ .
- (d) If  $\mathcal{C}_1 \subset \mathcal{C}_2$  then  $\Phi_{\mathcal{C}_1}(A) \subset \Phi_{\mathcal{C}_2}(A)$ .

The following definition enables us to explain connections between sparseness and  $\mathcal{C}$ -density. We say that  $x$  is a proximally density point (*right-hand proximally density point, left-hand proximally density point*) of a set  $A$  if its complement  $\mathbb{R} \setminus A$  is sparse at  $x$  (sparse at  $x$  on the right, sparse at  $x$  on the left). By  $\Phi_{pr}(A)$  we denote the set of all proximally density points of a measurable set  $A$ . Similarly we define  $\Phi_{pr}^+(A)$  and  $\Phi_{pr}^-(A)$ . Clearly  $\Phi_{pr}^+(A) \cap \Phi_{pr}^-(A) = \Phi_{pr}(A)$ . In [3] it was proved that  $\Phi_{pr}$  is a lower density operator.

Let us denote by  $\mathcal{C}_0$  the family of all decreasing sequences convergent to 0 and satisfying  $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 0$ . Theorem 1 implies

**Corollary 1.**  $\Phi_{\mathcal{C}_0}^+ = \Phi_{pr}^+$ .

Let  $\tilde{\mathcal{C}}$  denote the family of all decreasing sequences convergent to 0. Clearly,  $\tilde{\mathcal{C}}$  is acceptable and we have

**Proposition 2.**  $\Phi_{\tilde{\mathcal{C}}} = \overline{\Phi}$ .

**Proposition 3.** *For any  $A \in \mathcal{L}$  and any acceptable  $\mathcal{C}$*

- (a)  $\Phi(A) \subset \Phi_{\mathcal{C}}(A) \subset \overline{\Phi}(A)$ .

- (b)  $\Phi_{\mathcal{C}}(A) \in \mathcal{L}$ .  
 (c)  $\Phi_{\mathcal{C}}(A) \sim A$ .

*Proof.* Condition (a) is evident. Since  $\overline{\Phi}(A) \subset \mathbb{R} \setminus \Phi(\mathbb{R} \setminus \mathbb{A})$ , (a) and the Lebesgue Density Theorem imply

$$|\Phi_{\mathcal{C}}(A) \setminus \Phi(A)|^* \leq |\overline{\Phi}(A) \setminus \Phi(A)|^* \leq |\mathbb{R} \setminus \Phi(\mathbb{R} \setminus \mathbb{A}) \setminus \Phi(A)| = 0.$$

Hence we conclude (b) and (c).  $\square$

Let us notice that Propositions 1, 2 and 3 are still true for one-sided  $\mathcal{C}$ -densities  $\Phi_{\mathcal{C}}^+$  and  $\Phi_{\mathcal{C}}^-$ .

It is obvious that  $\Phi_{\mathcal{C}}(A) \subset \Phi_{\mathcal{C}}^+(A) \cap \Phi_{\mathcal{C}}^-(A)$ . The inverse inclusion need not be true.

**Example 1.** If

$$A = \bigcup_{n=1}^{\infty} \left( \frac{1}{(2n)!}; \frac{1}{(2n-1)!} \right) \cup \bigcup_{n=1}^{\infty} \left( \frac{-1}{(2n)!}; \frac{-1}{(2n+1)!} \right),$$

then  $\overline{d}_+(A, 0) = \overline{d}_-(A, 0) = 1$ , and so  $0 \in \overline{\Phi}_+(A) \cap \overline{\Phi}_-(A) = \Phi_{\mathcal{C}}^+(A) \cap \Phi_{\mathcal{C}}^-(A)$ . But  $\overline{d}(A, 0) = d(A, 0) = \frac{1}{2}$ , and consequently  $0 \notin \overline{\Phi}(A) = \Phi_{\mathcal{C}}(A)$ .

Proposition 1 yields  $\Phi_{\mathcal{C}}(A \cap B) \subset \Phi_{\mathcal{C}}(A) \cap \Phi_{\mathcal{C}}(B)$ . The equality need not hold.

**Example 2.** If

$$A = \bigcup_{n=1}^{\infty} \left( \frac{1}{(2n)!}; \frac{1}{(2n-1)!} \right), \quad B = \bigcup_{n=1}^{\infty} \left( \frac{1}{(2n+1)!}; \frac{1}{(2n)!} \right),$$

then  $\overline{d}_+(A, 0) = \overline{d}_+(B, 0) = 1$ , and so  $0 \in \Phi_{\mathcal{C}}^+(A) \cap \Phi_{\mathcal{C}}^+(B)$ . On the other hand  $\Phi_{\mathcal{C}}^+(A \cap B) = \Phi_{\mathcal{C}}^+(\emptyset) = \emptyset$ . Adding to  $A$  and  $B$  their mirror images, we can get the same conditions for two-sided  $\tilde{\mathcal{C}}$ -density.

In [2] it was defined a notion of a density point with respect to an increasing sequence convergent to infinity. For simplicity of notation, we formulate this definition for decreasing sequences convergent to 0.

Let  $\langle a \rangle$  be a decreasing sequence convergent to 0. We say that  $x$  is an  $\langle a \rangle$ -density point of a measurable set  $A$  if

$$\lim_{n \rightarrow \infty} \frac{|A \cap (x - a_n; x + a_n)|}{2a_n} = 1.$$

The set of all  $\langle a \rangle$ -density points of  $A$  we denote by  $\Phi_{\langle a \rangle}(A)$ . In the same way one can define *one-sided  $\langle a \rangle$ -density points* and sets  $\Phi_{\langle a \rangle}^+(A)$  and  $\Phi_{\langle a \rangle}^-(A)$ . Obviously,  $\Phi_{\langle a \rangle}(A) = \Phi_{\langle a \rangle}^+(A) \cap \Phi_{\langle a \rangle}^-(A)$ . In [2] it was proved that  $\Phi_{\langle a \rangle}(A) \in \mathcal{L}$  for  $A \in \mathcal{L}$  and that  $\Phi_{\langle a \rangle}$  is a lower density operator.

Let  $\langle a \rangle$  be a decreasing sequence convergent to 0. It is clear that the family

$$\mathcal{C}\langle a \rangle = \{(a_k, a_{k+1}, a_{k+2}, \dots); k \in \mathbb{N}\}$$

is acceptable and

**Proposition 4.**  $\Phi_{\mathcal{C}\langle a \rangle} = \Phi_{\langle a \rangle}$ .

The notion of  $\mathcal{C}$ -density is strictly wider than that of  $\langle a \rangle$ -density. From Example 2 it follows that  $\Phi_{\tilde{\mathcal{C}}}$  is not a lower density, and hence  $\Phi_{\tilde{\mathcal{C}}}$  is generated by no sequence  $\langle a \rangle$ .

In [2] it was showed that  $\Phi_{\langle a \rangle} = \Phi$  for any sequence  $\langle a \rangle$  with  $\liminf_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} > 0$ . Thus  $\Phi_{\mathcal{C}\langle a \rangle} = \Phi$  too. It is easy to see that the equality  $\Phi_{\mathcal{C}} = \Phi$  can also be true for families greater than  $\mathcal{C}\langle a \rangle$ .

**Example 3.** Let  $\mathcal{C}_1$  be the family of all decreasing sequences convergent to 0 and satisfying  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$ . We shall show that  $\Phi_{\mathcal{C}_1} = \Phi$ . On the contrary, suppose that  $\Phi_{\mathcal{C}_1}(A) \setminus \Phi(A) \neq \emptyset$  for some measurable set  $A$ . We can assume that  $0 \in \Phi_{\mathcal{C}_1}(A) \setminus \Phi(A)$ . Thus there are a positive number  $\varepsilon_0$ , a decreasing sequence  $(x_n)$  convergent to 0 and a sequence  $(a_n) \in \mathcal{C}_1$  such that

$$\frac{|(0; x_n) \setminus A|}{x_n} > \varepsilon_0 \text{ and } \frac{|(0; a_n) \setminus A|}{a_n} < \frac{\varepsilon_0}{2} \tag{2}$$

for  $n \in \mathbb{N}$ . Choose two positive integers  $n_0$  and  $p$  such that  $a_n < 2a_{n+1}$  for  $n \geq n_0$  and  $x_p \leq a_{n_0}$ . Thus for every  $n \geq p$  there exists  $k_n \geq n_0$  with

$a_{k_n+1} < x_n \leq a_{k_n}$ , and consequently

$$\frac{|(0; x_n) \setminus A|}{x_n} \leq \frac{|(0; a_{k_n}) \setminus A|}{a_{k_n+1}} \leq \frac{|(0; a_{k_n}) \setminus A|}{a_{k_n}} \cdot \frac{a_{k_n}}{a_{k_n+1}} < \varepsilon_0.$$

This contradicts (2) and completes the proof.

We finish the paper proving that proximal density cannot be defined by means of  $\langle a \rangle$ -density.

**Theorem 2.**  $\Phi_{pr}^+ \neq \Phi_{\langle a \rangle}^+$  for each decreasing sequence  $\langle a \rangle$  convergent to 0.

*Proof.* Let  $\langle a \rangle$  be a decreasing sequence convergent to 0 and let  $\langle b \rangle$  be a subsequence of  $\langle a \rangle$  such that

$$\lim_{n \rightarrow \infty} \frac{b_{n+1}}{b_n} = 0. \quad (3)$$

Set

$$A = \bigcup_{n=1}^{\infty} \left( \frac{b_n}{2}; b_n \right).$$

Since

$$\frac{|A \cap (0; b_n)|}{b_n} \geq \frac{b_n - \frac{b_n}{2}}{b_n} = \frac{1}{2},$$

$$0 \notin \Phi_{\langle a \rangle}^+(\mathbb{R} \setminus A).$$

Now we show that  $A$  is sparse at 0 on the right (i.e.  $0 \in \Phi_{pr}^+(\mathbb{R} \setminus A)$ ). Choose any  $\varepsilon > 0$ . We may assume that  $\varepsilon < 1$ . From (3) it follows that there is a positive integer  $n_0$  such that for each  $n \geq n_0$

$$\frac{b_{n+1}}{b_n} < \frac{\varepsilon}{2}.$$

Set  $h = \min\{\frac{b_{n_0}}{2}, \frac{\varepsilon}{4}\}$  and fix any interval  $(\alpha; \beta) \subset (0; h)$  with  $\alpha < \beta h$ . We look for a point  $y \in (\alpha; \beta)$  satisfying  $|A \cap (0; y)| < \varepsilon y$ .

If  $\frac{b_p}{2} \in (\alpha; \beta)$  for some  $p \in \mathbb{N}$ , then  $b_p < 2\beta \leq 2h \leq b_{n_0}$ , and so  $p > n_0$ .

Consequently,

$$\frac{|A \cap (0; \frac{1}{2}b_p)|}{\frac{1}{2}b_p} \leq 2 \frac{b_{p+1}}{b_p} < \varepsilon. \quad (4)$$

If  $\frac{b_n}{2} \notin (\alpha; \beta)$  for each  $n \in \mathbb{N}$ , then there is a positive integer  $p$  such that

$$\frac{1}{2}b_p \leq \alpha < \beta \leq \frac{1}{2}b_{p-1}.$$

Thus for  $y = \frac{\alpha+\beta}{2}$  we have  $y \in (\alpha; \beta)$  and

$$\frac{|A \cap (0; y)|}{y} \leq \frac{b_p}{\frac{1}{2}\beta} \leq \frac{2\alpha}{\frac{1}{2}\beta} = 4 \frac{\alpha}{\beta} < 4h \leq \varepsilon. \quad (5)$$

From (4) and (5) it follows that  $0 \in \Phi_{pr}^+(\mathbb{R} \setminus A)$ , which ends the proof.  $\square$

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