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ON THE PERIODIC SOLUTIONS OF A CERTAIN NONLINEAR VECTOR DIFFERENTIAL EQUATION OF FOURTH-ORDER

BY

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Abstract

Our aim is essentially to establish a theorem which contains sufficient conditions that guarantee the non-existence of non-trivial periodic solutions of a certain nonlinear vector differential equation of fourth order.

1. Introduction

With respect to our observations in the literature, so far, periodic behaviors of solutions, in particular nonexistence of nontrivial periodic solutions, for various nonlinear ordinary scalar and vector differential equations of fourth order were investigated only by a few authors. The papers were carried out on the nonexistence of nontrivial periodic solutions of nonlinear differential equations of fourth order can be summarized as follows: First, in 1974, Tejumola [6] proved a result on the existence of periodic solutions for certain fourth order scalar nonlinear differential equations of the form

$$x^{(4)} + a_1\dddot{x} + g(\dot{x})\ddot{x} + a_3\dot{x} + f(x, \dot{x}, \ddot{x}, \dddot{x}) = p(t),$$

in which a_1 and a_3 are constants; g, f and p are continuous functions for the variables displayed explicitly and p is also ω-periodic in t. Later, in 1989 and 1990, Tiryaki ([7], [8], [9]) discussed some similar problems on the same

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subject for the following scalar differential equation and its different scalar versions

$$x^{(4)} + f_1(\ddot{x})\dddot{x} + f_2(\dot{x})\ddot{x} + f_3(\dot{x}) + f_4(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

where $f_1,\ f_2,\ f_3,\ f_4$ and p are continuous functions for the variables displayed explicitly and p is also ω-periodic in t. Besides, in 1979, Ezeilo and Tejumola [3] also established a result which includes sufficient conditions on the existence of an ω-periodic solution of the ordinary differential equation

$$x^{(4)} + a_1\dddot{x} + f(x, \dot{x}, \ddot{x}, \dddot{x})\ddot{x} + g(x)\dot{x} + h(x) = p(t, x, \dot{x}, \ddot{x}, \dddot{x}),$$

in which a_1 is a constant; $f,\ g,\ h$ and p are continuous functions for the variables displayed explicitly and p is also ω-periodic in t. Finally, in 2005, E. Tunç [11] demonstrated an alike problem on the subject for the nonlinear vector differential equations of the form

$$X^{(4)} + \Phi(\ddot{X})\dddot{X} + \Psi(\dot{X})\ddot{X} + F(\dot{X}) + G(X) = 0,$$

by imposing certain assumptions on the functions $\Phi,\ \Psi,\ F$ and G that appeared in the above equation.

In this paper, we discuss a similar problem, that is, the existence of periodic solutions for the fourth order nonlinear vector differential equation

$$X^{(4)} + \Psi(\ddot{X})X + G(X, \dot{X}, \ddot{X}, \dddot{X})\ddot{X} + \Theta(\dot{X}) + F(X) = 0, \tag{1}$$

in which $X \in \mathcal{R}^n$; Ψ and G are continuous $n \times n$-symmetric matrix functions for the variables displayed explicitly; $\Theta : \mathcal{R}^n \to \mathcal{R}^n$, $F : \mathcal{R}^n \to \mathcal{R}^n$ and $\Theta(0) = F(0) = 0$. It is also supposed that the functions Θ and F are continuous for all $X,\ \dot{X} \in \mathcal{R}^n$, respectively. The equation (1) can be rewritten as

$$\begin{cases} \dot{X} = Y,\ \dot{Y} = Z,\ \dot{Z} = W, \\ \dot{W} = -\ \Psi(Z)W - G(X, Y, Z, W)Z - \Theta(Y) - F(X), \end{cases} \tag{2}$$

which was obtained by taking $\dot{X} = Y$, $\ddot{X} = Z$, $X = W$ in the equation (1). Let $J_F(X)$, $J_\Phi(Y)$ and $J_\Psi(Z)$, respectively, denote the linear operators from the vectors $F(X)$, $\Theta(Y)$ and the matrix $\Psi(Z)$ to the matrices

$$J_F(X)=\left(\frac{\partial f_i}{\partial x_j}\right),\ J_\Theta(Y)=\left(\frac{\partial \theta_i}{\partial y_j}\right),\ J_\Psi(Z)=\left(\frac{\partial \psi_{ik}}{\partial z_j}\right), (i,j,k=1,2,\ldots,n),$$

where $(x_1,x_2,\ldots,x_n)$, $(y_1,y_2,\ldots,y_n)$, $(z_1,z_2,\ldots,z_n)$, $(f_1,f_2,\ldots,f_n)$, $(\theta_1,\theta_2,\ldots,\theta_n)$ and (ψ_{ik}) are components of X, Y, Z, F, Θ and Ψ, respectively. Furthermore, it is also assumed that $F(X)$ and $\Theta(Y)$ are gradient vector fields, that is, there are scalar functions f and θ such that $F=\nabla f$ and $\Theta=\nabla\theta$, and the matrices $J_F(X)$, $J_\Theta(Y)$ and $J_\Psi(Z)$ exist and are continuous.

It is worth mentioning that, to the best of our knowledge in the literature, no paper was found on the existence of periodic solutions of nonlinear vector differential equations of the form (1). Meanwhile, two problems on the instability of solutions of the differential equation (1) and its scalar case, that is,

$$x^{(4)}+\psi(\ddot{x})\dddot{x}+g(x,\dot{x},\ddot{x},\dddot{x})\ddot{x}+\theta(\dot{x})+f(x)=0$$

were studied by Ezeilo [2] and Tunç [10], respectively. One can also refer to the book of Reissig et al. [5] as a survey for some related papers published on the subject. It should also be clarified that through all of the papers mentioned above, the Lyapunov's [4] second (or direct) method has been used as a basic tool to prove the results established there. However, the Leray-Schauder principle can also be used to prove the existence of nontrivial periodic solutions of the above mentioned equations. Next, in the literature, there also exist some results on the subject that were verified by using the Leray-Schauder principle. But, here, we will only take into consideration the Lyapunov's [4] second (or direct) method to verify our forthcoming main result.

Throughout this paper, the symbol $\langle X,Y\rangle$ is used to denote the usual scalar product in $\mathcal{R}^n$ for given any X, Y in $\mathcal{R}^n$, that is, $\langle X,Y\rangle=\sum_{i=1}^n x_iy_i$, thus $\|X\|^2=\langle X,X\rangle$, and $\lambda_i(A)$, $(i=1,2,\ldots,n)$, will represent the eigenvalues of the $n\times n$-matrix A. It is also well-known that a real symmetric matrix $A=(a_{ij}),(i,\ j=1,\ 2,\ldots,\ n)$ is said to be positive definite if and only if the quadratic form X^TAX is positive definite, where $X\in\mathcal{R}^n$ and X^T denotes the transpose of X.

2. Main Result

We indeed prove the following theorem:

Theorem. *We assume that the function G and the functions Ψ, Θ and F that appeared in the system* (2)*, respectively, are continuous and continuously differentiable, and there exists a constant a_2 such that the following conditions are satisfied: There are scalar functions f and θ such that $F = \nabla f$ and $\Theta = \nabla\theta$;*

Ψ, J_F and G are symmetric such that

$$\frac{\partial\psi_{ij}}{\partial z_k} = \frac{\partial\psi_{ik}}{\partial z_j}, \quad (i, j, k = 1, 2, \ldots, n);$$
$$F(X) \neq 0 \quad \textit{if} \quad X \neq 0, \quad \lambda_i(G(X, Y, Z, W)) \leq a_2$$

and

$$\lambda_i(J_F(X)) > \frac{1}{4}a_2^2 \quad , \quad (i = 1,\ 2, \ldots,\ n),$$

for all $X,\ Y,\ Z,\ W \in \mathcal{R}^n$, where $\lambda_i(J_F(X))$ and $\lambda_i(G(X,Y,Z,W))$, $(i = 1, 2, \ldots, n)$, are the eigenvalues of $G(X,Y,Z,W)$ and $J_F(X)$. Then the equation (1) *has no periodic solution other than $X = 0$ for arbitrary θ.*

Remark. It should be clarified that there is no restriction on the eigenvalues of the matrices $\Psi(Z)$ and $J_\Theta(Y)$, which is obtained from $\Theta(Y)$.

Now, we will state a lemma, which will be needed in the proof of the main result.

Lemma. *Let A be a real symmetric $n \times n$-matrix and*

$$a' \geq \lambda_i(A) \geq a > 0, \quad (i = 1, 2, \ldots, n),$$

where a', a are constants. Then

$$a' \langle X, X\rangle \geq \langle AX, X\rangle \geq a \langle X, X\rangle$$

and

$$a'^2 \langle X, X\rangle \geq \langle AX, AX\rangle \geq a^2 \langle X, X\rangle .$$

Proof. See (Bellman [1]). □

Proof of the theorem. Now, let $(X,Y,Z,W) = (X(t),Y(t),Z(t),W(t))$ be an arbitrary $\alpha-(\alpha>0)$ periodic solution of the system (2), that is, $(X(t),Y(t),Z(t),W(t)) = (X(t+\alpha),Y(t+\alpha),Z(t+\alpha),W(t+\alpha))$. Corresponding to this solution, we consider the function $V=V(t)=V(X(t),Y(t),Z(t),W(t))$, which is defined by:

$$\begin{aligned} V &= \langle Y, a_2IZ\rangle + \langle W,Z\rangle + \langle F(X),Y\rangle + \int_0^1 \langle \Theta(\sigma Y),Y\rangle\, d\sigma \\ &\quad + \int_0^1 \langle \sigma\Psi(\sigma Z)Z,Z\rangle\, d\sigma. \end{aligned} \tag{3}$$

Differentiating the function (3) through the system (2), we obtain

$$\begin{aligned} \dot{V} &= \frac{d}{dt}V(X,Y,Z,W) \\ &= \langle Z,a_2IZ\rangle + \langle W,W\rangle + \langle Y,a_2IW\rangle - \langle G(X,Y,Z,W)Z,Z\rangle \\ &\quad + \langle J_F(X)Y,Y\rangle - \langle \Psi(Z)W,Z\rangle - \langle\Theta(Y),Z\rangle \\ &\quad + \frac{d}{dt}\int_0^1 \langle\Theta(\sigma Y),Y\rangle\, d\sigma + \frac{d}{dt}\int_0^1 \langle\sigma\Psi(\sigma Z)Z,Z\rangle\, d\sigma. \end{aligned} \tag{4}$$

Recall that

$$\begin{aligned} \frac{d}{dt}\int_0^1 \langle\Theta(\sigma Y),Y\rangle\, d\sigma &= \int_0^1 \sigma\langle J_\Theta(\sigma Y)Z,Y\rangle d\sigma + \int_0^1 \langle\Theta(\sigma Y),Z\rangle\, d\sigma \\ &= \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle\Theta(\sigma Y),Z\rangle\, d\sigma + \int_0^1 \langle\Theta(\sigma Y),Z\rangle\, d\sigma \\ &= \sigma\langle\Theta(\sigma Y),Z\rangle\,\big|_0^1 = \langle\Theta(Y),Z\rangle \end{aligned} \tag{5}$$

and

$$\begin{aligned} &\frac{d}{dt}\int_0^1 \langle\sigma\Psi(\sigma Z)Z,Z\rangle\, d\sigma \\ &= \int_0^1 \langle\sigma\Psi(\sigma Z)W,Z\rangle\, d\sigma + \int_0^1 \sigma^2\langle J_\Psi(\sigma Z)ZW,Z\rangle\, d\sigma + \int_0^1 \sigma\langle\Psi(\sigma Z)Z,W\rangle\, d\sigma \\ &= \int_0^1 \langle\sigma\Psi(\sigma Z)W,Z\rangle\, d\sigma + \int_0^1 \sigma\frac{\partial}{\partial\sigma}\langle\sigma\Psi(\sigma Z)W,Z\rangle\, d\sigma \\ &= \sigma^2\langle\Psi(\sigma Z)W,Z\rangle\,\big|_0^1 = \langle\Psi(Z)W,Z\rangle \end{aligned} \tag{6}$$

Substituting the estimates (5) and (6) in (4), we get

$$\dot{V} = \langle Z, a_2 I Z\rangle + \langle W, W\rangle + \langle Y, a_2 I W\rangle - \langle G(X,Y,Z,W)Z, Z\rangle + \langle J_F(X)Y, Y\rangle . \tag{7}$$

Now, consider the expression

$$\langle Z, a_2 I Z\rangle - \langle G(X,Y,Z,W)Z, Z\rangle ,$$

which is contained in (7). Taking into account the assumption

$$\lambda_i(G(X,Y,Z,W) \leq a_2$$

of the theorem and the lemma , we get

$$\langle Z, a_2 I Z\rangle - \langle G(X,Y,Z,W)Z, Z\rangle \geq a_2 \langle Z, Z\rangle - a_2 \langle Z, Z\rangle = 0. \tag{8}$$

Next, making the use of the inequality (8), it follows from (7) that

$$\begin{aligned} \dot{V} &\geq \langle W, W\rangle + \langle Y, a_2 I W\rangle + \langle J_F(X)Y, Y\rangle \\ &\geq \langle W, W\rangle - |a_2| \, \|Y\| \, \|W\| + \langle J_F(X)Y, Y\rangle . \end{aligned} \tag{9}$$

Hence

$$\begin{aligned} \dot{V} &\geq \left\| W \pm \frac{1}{2} a_2 Y \right\|^2 + \langle J_F(X)Y, Y\rangle - \frac{1}{4} a_2^2 \langle Y, Y\rangle \\ &\geq \langle J_F(X)Y, Y\rangle - \frac{1}{4} a_2^2 \langle Y, Y\rangle . \end{aligned} \tag{10}$$

If we assume the assumption $\lambda_i(J_F(X)) > \frac{1}{4}a_2^2$ holds, then, $\dot{V}(t) \geq 0$ for all t. This case implies that $V(t)$ is monotone in t. Further, since V is continuous and $(X,\ Y,\ Z,\ W)$ is a periodic solution in t, $V(t)$ is bounded. Thus we have that

$$\lim_{t\to\infty} V(t) = V_0(constant). \tag{11}$$

Using (11) and the fact that $V(t) = V(t + m\alpha)$ for any fixed t and for arbitrary integer m, it can be shown that

$$V(t) \equiv V_0 \ \text{ for all } \ t. \tag{12}$$

Hence, clearly, we have from (12) that

$$\dot{V}(t) = 0 \text{ for all } t. \tag{13}$$

Now, in view of (10) and (13) together, it follows that $Y = 0$ for all $t \geq 0$, and hence also that $X = \xi$ (a constant vector). By turns, it follows that

$$X = \xi, \quad Y = \dot{X}, \quad Z = \dot{Y} = 0, \quad W = \dot{Z} = 0, \quad \text{for all } t \geq 0. \tag{14}$$

Replacing the estimate (14) in the system (2) pioneering to the result $F(\xi) = 0$, which necessarily implies (only) that $\xi = 0$ because of the assumption of $F(X) \neq 0$ if $X \neq 0$. Thus, the above discussion on $\dot{V} = 0 \quad (t \geq 0)$ gives that

$$X = Y = Z = W = 0 \text{ for all } t \geq 0. \tag{15}$$

This fact completes the proof of the theorem. □

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