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OSCILLATION CRITERIA FOR SOME CLASSES OF LINEAR DELAY DIFFERENTIAL EQUATIONS OF FIRST-ORDER

BY

BAŞAK KARPUZ AND ÖZKAN ÖCALAN

Abstract

In this article, we study delay differential equations having forms

$$x'(t)+P(t)x(t-\tau)-Q(t)x(t-\sigma) \;=\; 0$$

$$[x(t)-R(t)x(t-\rho)]'+P(t)x(t-\tau) \;=\; 0$$

and

$$[x(t)-R(t)x(t-\rho)]'+P(t)x(t-\tau)-Q(t)x(t-\sigma) \;=\; 0,$$

where $P,Q,R\in C\left([t_0,\infty),\mathbb{R}^+\right)$ and $\tau,\sigma,\rho\geq 0$. We use recursive methods to obtain new oscillation criterions.

1. Introduction

In the recent years, the oscillation theory of delay differential equations has grown rapidly. It is a relatively new field with interesting applications from the real world. In fact, delay differential equations appear in modeling of the problems as population dynamics and transformation of information. We refer readers to [1]-[41] for theorical studies on this subject.

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In [21], G. Ladas and Y. G. Sficas obtained every solution of

$$x'(t) + px(t-\tau) - qx(t-\sigma) = 0 \tag{1}$$

is oscillatory when

$$\begin{aligned} &\tau \geq \sigma \geq 0, \\ &p > q \geq 0, \\ &q(\tau-\sigma) \leq 1, \\ &(p-q) > \tfrac{1}{e}\left(1 - q(\tau-\sigma)\right). \end{aligned} \tag{2}$$

Also, A. Faiz studied (1) and obtained new results in [11].

First study of the equation (1) with continuous coefficients

$$x'(t) + P(t)x(t-\tau) - Q(t)x(t-\sigma) = 0 \tag{3}$$

was done in [23] by G. Ladas and C. Qian of which solutions are oscillatory under conditions

$$\begin{aligned} &P, Q \in C\left([t_0,\infty), \mathbb{R}^+\right), \\ &\tau \geq \sigma \geq 0, \\ &\bar{P}(t) := P(t) - Q(t-\tau+\sigma) \geq 0, \\ &\textstyle\int_{t-\tau+\sigma}^{t} Q(s)\,ds \leq 1, \end{aligned} \tag{4}$$

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s)\,ds > \frac{1}{e} \tag{5}$$

or

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s)\,ds > 1.$$

Furthermore, J. Shen and X. Tang improved the condition (5) by replacing

$$\liminf_{t\to\infty} P_i(t) > \tfrac{1}{e^i},\ i \in \mathbb{N}, \tag{6}$$

where

$$P_i(t) := \begin{cases} 1, & t \geq t_0,\ i = 0 \\ \displaystyle\int_{t-\tau}^{t} \bar{P}(s)\left(1 + \int_{s-\tau+\sigma}^{s} Q(u-\tau)\,du\right) P_{i-1}(s)\,ds, & t \geq t_0 + i\tau,\ i \in \mathbb{N} \end{cases} \tag{7}$$

in [34].

Oscillatory behavior of

$$[x(t) - rx(t-\rho)]' + px(t-\tau) = 0 \tag{8}$$

is investigated by many authors. We improve the result of [22] by G. Ladas and Y. G. Sficas holding

$$\begin{aligned} &1 \geq r \geq 0,\ p \geq 0, \\ &p(\tau-\rho) > \tfrac{1}{e}(1-r) \end{aligned} \tag{9}$$

conditions for oscillation. Also, the case including continuous functions as coefficients

$$[x(t) - R(t)x(t-\rho)]' + P(t)x(t-\tau) = 0 \tag{10}$$

have been studied by I. Kubiaczyk, S. H. Saker and J. Morchalo in [18], J. S. Yu , M. P. Chen and H. Zhang in [40], K. A. Dib and R. M. Mathsen in [4].

The equation

$$[x(t) - R(t)x(t-\rho)]' + P(t)x(t-\tau) - Q(t)x(t-\sigma) = 0 \tag{11}$$

is also studied by many authors. J. H. Shen and L. Debnath in [35] by getting rid of the known condition

$$\int_{t_0}^{\infty} \bar{P}(s)\,ds = \infty,$$

showed that all solutions of (11) are oscillatory when

$$\begin{aligned} &P, Q, R \in C([t_0,\infty),\ \mathbb{R}^+), \\ &\rho \geq 0,\ \tau \geq \sigma \geq 0, \\ &R(t) \geq 0,\ \bar{P}(t) \geq 0 \end{aligned} \tag{12}$$

and

$$R(t) + \int_{t-\tau+\sigma}^{t} Q(s)\,ds \equiv 1,$$

$$\int_{t_0}^{\infty} \bar{P}(s)\, e^{\frac{1}{\delta}\int_{t_0}^{s} u\bar{P}(u)du} ds = \infty, \quad \delta = \max\{\rho, \tau\}.$$

Also, most of the known papers require

$$R(t)+\int_{t-\tau+\sigma}^{t} Q(s)\,ds \leq 1 \tag{13}$$

and (5). For instance, Z. Luo, J. Shen and X. Liu in [27].

We call a solution of a delay differential equation non-oscillatory if the function satisfying the equation for $t \geq t_0$ has eventually constant sign, otherwise we call the function oscillatory. And when we write an expression, we assume that it holds eventually.

2. Improved Oscillation Criterion for (3)

In this section, we build new oscillation criterion for (3) and without furthermore mentioning, we assume (4) holds.

The following well-known lemma is from [14].

Lemma 1. *Assume that $x(t)$ is an eventually positive solution of* (3) *and* (4) *holds. Then*

$$z(t) := x(t)-\int_{t-\tau+\sigma}^{t} Q(s)\,x(s-\sigma)\,ds \tag{14}$$

satisfies

$$z'(t) \leq 0, z(t) > 0 \tag{15}$$

eventually.

Before stating our results, we need to define some special functions. Assume that (4) holds for $t \geq \bar{t}$ and

$$Q_i(t) := \begin{cases} 1, & t \geq \bar{t},\ i=0 \\ \displaystyle\int_{t-\tau+\sigma}^{t} Q(s)\,Q_{i-1}(s-\sigma)\,ds, & t \geq \bar{t}+i\sigma,\ i \in \mathbb{N}, \end{cases} \tag{16}$$

where $\bar{t} \geq t_0$.

Lemma 2. *Assume that all conditions of Lemma* 1 *are held. Then for*

$n \in \mathbb{N}$, *eventually positive* $z(t)$ *in* (14) *satisfies*

$$z'(t) + \bar{P}(t) \sum_{i=0}^{n} Q_i(t-\tau) z(t-\tau) \leq 0 \tag{17}$$

eventually.

Proof. Assume that $x(t)$ is an eventually positive solution of (3). Then there exists a $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. Set $t_2 := \max\{t_1 + \tau, \bar{t}\}$. From (14) and (15), we have

$$0 < z(t) \leq x(t), \quad t \geq t_2. \tag{18}$$

Rewritting (14) as

$$z(t) + \int_{t-\tau+\sigma}^{t} Q(s) x(s-\sigma) ds = x(t), \quad t \geq t_2,$$

we have

$$\begin{aligned} & z(t) + \int_{t-\tau+\sigma}^{t} Q(s_1) \left(z(s_1 - \sigma) + \int_{s_1-\tau+\sigma}^{s_1} Q(s_2 - \sigma) x(s_2 - 2\sigma) ds_2 \right) ds_1 \\ & = x(t), \quad t \geq t_2 + \sigma. \end{aligned}$$

Since $z'(t) \leq 0$, we have

$$\begin{aligned} x(t) &\geq z(t) + z(t-\sigma) \int_{t-\tau+\sigma}^{t} Q(s) ds \\ &\quad + \int_{t-\tau+\sigma}^{t} Q(s_1) \int_{s_1-\tau+\sigma}^{s_1} Q(s_2-\sigma) x(s_2 - 2\sigma) ds_2 ds_1 \\ &\geq z(t) \left(1 + \int_{t-\tau+\sigma}^{t} Q(s) ds\right) \\ &\quad + \int_{t-\tau+\sigma}^{t} Q(s_1) \int_{s_1-\tau+\sigma}^{s_1} Q(s_2-\sigma) x(s_2 - 2\sigma) ds_2 ds_1 \\ &= z(t)(Q_0(t) + Q_1(t)) \\ &\quad + \int_{t-\tau+\sigma}^{t} Q(s_1) \int_{s_1-\tau+\sigma}^{s_1} Q(s_2-\sigma) x(s_2 - 2\sigma) ds_2 ds_1 \\ &= z(t) \sum_{i=0}^{1} Q_i(t) + \int_{t-\tau+\sigma}^{t} Q(s_1) \int_{s_1-\tau+\sigma}^{s_1} Q(s_2-\sigma) x(s_2 - 2\sigma) ds_2 ds_1 \end{aligned}$$

for $t \geq t_2 + \sigma$. Repeating the above procedure for n times, we have

$$z(t) \sum_{i=0}^{n} Q_i(t) + \int_{t-\tau+\sigma}^{t} Q(s_1) \cdots \int_{s_n-\tau+\sigma}^{s_n} Q(s_{n+1} - n\sigma)\, x(s_{n+1} - (n+1)\sigma)\, ds_{n+1} \cdots ds_1 \leq x(t)$$

or

$$z(t) \sum_{i=0}^{n} Q_i(t) \leq x(t) \tag{19}$$

for $t \geq t_2 + n\sigma$. Since

$$z'(t) + \bar{P}(t)\, x(t-\tau) = 0,$$

we have

$$z'(t) + \bar{P}(t) \sum_{i=0}^{n} Q_i(t-\tau)\, z(t-\tau) \leq 0, \;\; t \geq t_2 + n\sigma + \tau$$

by considering (18) and (19). This is the desired result. □

For the rest of the section, we define

$$\alpha(n) := \liminf_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s) \sum_{i=0}^{n} Q_i(s-\tau)\, ds \tag{20}$$

and

$$\beta(n) := \limsup_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s) \sum_{i=0}^{n} Q_i(s-\tau)\, ds. \tag{21}$$

Remark 3. In the view of (16), (20) and (21), $\alpha(n)$ and $\beta(n)$ are non-decreasing sequences respect to n.

As an immediate consequence of these definitions, we can give the following theorem which improves Theorem 2.6.1 in [14].

Theorem 4. *Assume that all conditions of Lemma* 1 *are held. Further-*

more, assume that there exists $n \in \mathbb{N}$ *such that*

$$\alpha(n) > \frac{1}{e} \tag{22}$$

or

$$\alpha(n) \leq \tfrac{1}{e},\ \beta(n) > 1 - \tfrac{1-\alpha(n)-\sqrt{1-2\alpha(n)-\alpha^2(n)}}{2} \tag{23}$$

holds. Then every solution of (3) *is oscillatory.*

Proof. Assume for contrary that $x(t)$ is an eventually positive solution of (3). Then, in the view of (22) or (23) $z(t)$ in (14) can not be an eventually positive solution of (17).This contradiction completes the proof □

Remark 5. It is easy to see that known results are obtained with $n = 0$ in Theorem 4.

Corollary 6. *Assume that all conditions of Lemma* 1 *are held. Furthermore, assume that there exists* $n \in \mathbb{N}$ *such that*

$$\liminf_{t\to\infty}(n+1)\int_{t-\tau}^{t}\bar{P}(s)\sqrt[n+1]{\prod_{i=0}^{n}Q_i(s-\tau)}ds > \frac{1}{e}$$

holds, then every solution of (3) *is oscillating.*

Proof. Proof is clear by the relation between arithmetic and geometric mean. □

Theorem 7. *Assume that conditions of Lemma* 1 *are satisfied and* $Q(t)$ *is a non-increasing function then if there exists* $n \in \mathbb{N}$ *such that*

$$\liminf_{t\to\infty}\int_{t-\tau}^{t}\bar{P}(s)\sum_{i=0}^{n}\left(Q(s-\tau)(\tau-\sigma)\right)^i ds > \frac{1}{e},$$

then every solution of (3) *is oscillatory.*

Proof. Considering (16), we have,

$$\begin{aligned} Q_0(t) &= 1, \\ Q_1(t) &= \int_{t-\tau+\sigma}^{t} Q(s)\,ds \end{aligned}$$

$$\begin{aligned}
&\geq Q(t)(\tau-\sigma), \\
Q_2(t) &= \int_{t-\tau+\sigma}^{t} Q(s) Q_1(s-\sigma)\, ds \\
&\geq (\tau-\sigma) \int_{t-\tau+\sigma}^{t} Q(s) Q(s-\sigma)\, ds \\
&\geq (Q(t)(\tau-\sigma))^2 .
\end{aligned}$$

It is not hard to see that

$$Q_i(t) \geq (Q(t)(\tau-\sigma))^i, \ \ i \in \mathbb{N}$$

for sufficiently large t. Then, considering (20),

$$\alpha(n) \geq \liminf_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s) \sum_{i=0}^{n} (Q(s)(\tau-\sigma))^i\, ds > \frac{1}{e},$$

the proof is done. □

Theorem 8. *Assume that conditions of Lemma* 1 *are satisfied. If*

$$\alpha(\infty) > \frac{1}{e},$$

then every solution of (3) *is oscillating.*

Proof. Since $\alpha(n)$ is non-decreasing, there exists $n_1 \in \mathbb{N}$ such that

$$\alpha(n_1) \geq \frac{1}{e}$$

and a number $n_2 > n_1$ with

$$\alpha(n_2) > \frac{1}{e}.$$

Thus, every solution of (3) is oscillatory by Theorem 4. □

As metioned in the introduction, the following theorem can be found in [14] as Theorem 2.2.4 considering the autonomous case with the result (2) of which proof is done in a different way by us.

Theorem 9. *Assume that* (2) *holds. Then every solution of* (1) *is* *oscillatory.*

Proof. First of all, we calculate $Q_i(t)$ functions for this case. Clearly, $Q_0(t) \equiv 1$ and

$$
\begin{aligned}
Q_1(t) &= \int_{t-\tau+\sigma}^{t} qQ_0(s-\sigma)\,ds = q(\tau-\sigma),\\
Q_2(t) &= \int_{t-\tau+\sigma}^{t} qQ_1(s-\sigma)\,ds = (q(\tau-\sigma))^2.
\end{aligned}
$$

Then, it is easy to see

$$Q_i(t) = (q(\tau-\sigma))^i,\ \ i \in \mathbb{N}.$$

Now, there are two possible cases.

Case 1. $q(\tau-\sigma) < 1$. In this case,

$$
\begin{aligned}
\alpha(\infty) &= \liminf_{t\to\infty} \int_{t-\tau}^{t} (p-q) \sum_{i=0}^{\infty} (q(\tau-\sigma))^i\,ds\\
&= \tau(p-q)\frac{1}{1-q(\tau-\sigma)}.
\end{aligned}
$$

Thus, by (2)

$$\alpha(\infty) > \frac{1}{e}$$

and all solutions of (1) are oscillatory by Theorem 8.

Case 2. $q(\tau-\sigma) \equiv 1$. In this case,

$$\alpha(\infty) = \infty > \frac{1}{e}.$$

Thus, every solution of (1) is oscillatory by Theorem 8. $\square$

3. Improved Oscillation Criterion for (10)

In this section, we investigate (10) with conditions $P, R \in C([t_0,\infty), \mathbb{R}^+)$

with

$$0 \leq R(t) \leq 1 \tag{24}$$

and $\tau, \rho \geq 0$. The following lemma can be found in [35].

Lemma 10. *Assume that $x(t)$ is an eventually positive solution of* (10) *and* (24) *hold. Then*

$$z(t) := x(t) - R(t)\, x(t-\rho) \tag{25}$$

satisfies

$$z'(t) \leq 0, \ \ 0 < z(t) \tag{26}$$

eventually.

For the rest of the section, we define

$$R_i(t) := \begin{cases} 1, & t \geq t_0, \ i = 0 \\ R(t)\, R_{i-1}(t-\rho), & t \geq t_0 + i\rho, \ i \in \mathbb{N}. \end{cases} \tag{27}$$

As an immediate consequence of preceding results and definitions, we have the following lemma.

Lemma 11. *Assume that assumptions of Lemma 10 are held. Then for $n \in \mathbb{N}$, eventually positive $z(t)$ in* (25) *is a solution of the following inequality*

$$z'(t) + P(t) \sum_{i=0}^{n} R_i(t-\tau)\, z(t-\tau) \leq 0. \tag{28}$$

Proof. Assume that $x(t)$ is an eventually positive solution of (10). Then, there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1 - \tau$. From (25) and (26), we have

$$0 < z(t) \leq x(t), \ \ t \geq t_1. \tag{29}$$

Rewriting (25) as

$$z(t) + R(t)\, x(t-\rho) = x(t), \ \ t \geq t_1,$$

we have

$$z\left(t\right)+R\left(t\right)\left(z\left(t-\rho\right)+R\left(t-\rho\right)x\left(t-2\rho\right)\right)=x\left(t\right),\ \ t\geq t_{1}+\rho$$

and considering non-increasing behavior of $z\left(t\right)$,

$$z\left(t\right)\left(1+R\left(t\right)\right)+R\left(t\right)R\left(t-\rho\right)x\left(t-2\rho\right)\leq x\left(t\right),\ \ t\geq t_{1}+\rho$$

that is

$$z\left(t\right)\sum_{i=0}^{1}R_{i}\left(t\right)+R_{2}\left(t\right)x\left(t-2\rho\right)\leq x\left(t\right),\ \ t\geq t_{1}+\rho.$$

A pattern appears to be emerging and it is natural to assume

$$z\left(t\right)\sum_{i=0}^{n}R_{i}\left(t\right)+R_{n+1}\left(t\right)x\left(t-\left(n+1\right)\rho\right)\leq x\left(t\right),\ \ t\geq t_{1}+n\rho$$

or

$$z\left(t\right)\sum_{i=0}^{n}R_{i}\left(t\right)\leq x\left(t\right),\ \ t\geq t_{1}+n\rho \tag{30}$$

for $n\in\mathbb{N}$. Since

$$z'\left(t\right)+P\left(t\right)x\left(t-\tau\right)=0,$$

we have

$$z'\left(t\right)+P\left(t\right)\sum_{i=0}^{n}R_{i}\left(t-\tau\right)z\left(t-\tau\right)\leq 0,\ \ t\geq t_{1}+n\rho+\tau,\ \ n\in\mathbb{N}$$

from (30). The proof of the lemma is done. $\square$

For the sake of convenience, we set

$$\alpha\left(n\right)\ :=\ \liminf_{t\to\infty}\int_{t-\tau}^{t}P\left(s\right)\sum_{i=0}^{n}R_{i}\left(s-\tau\right)ds \tag{31}$$

and

$$\beta\left(n\right)\ :=\ \limsup_{t\to\infty}\int_{t-\tau}^{t}P\left(s\right)\sum_{i=0}^{n}R_{i}\left(s-\tau\right)ds. \tag{32}$$

Remark 12. Considering definition of $R_{i}\left(t\right)$ functions $\alpha\left(n\right)$ and $\beta\left(n\right)$ are non-decreasing sequences.

The following theorem improves Theorem 3.2.1 in [10] by removing the condition

$$\int_{t_0}^{\infty} P(s)\,ds = \infty.$$

Theorem 13. *Assume all conditions of Lemma* 10 *are held. If there exists* $n \in \mathbb{N}$ *such that*

$$\alpha(n) > \frac{1}{e} \tag{33}$$

or

$$\alpha(n) \leq \tfrac{1}{e},\ \beta(n) > 1 - \tfrac{1-\alpha(n)-\sqrt{1-2\alpha(n)-\alpha^2(n)}}{2}, \tag{34}$$

then every solution of (10) *is oscillating.*

Proof. Proof is trivial. □

Theorem 14. *Assume that conditions of Lemma* 10 *hold. If*

$$\alpha(\infty) > \frac{1}{e},$$

then every solution of (10) *is oscillatory.*

Proof. Proof is similar to the proof of Theorem 8 and omitted. □

The following theorem improves Theorem 6.1.3 in [14] by removing the condition on ρ.

Theorem 15. *Assume that* $0 \leq r \leq 1$ *and* $0 \leq p, \rho, \tau$. *If*

$$\tau p > \frac{1}{e}(1-r) \tag{35}$$

holds, then every solution of (8) *is oscillatory.*

Proof. We need to calculate $R_i(t)$ functions. One can easily show that

$$R_i(t) = r^i,\ t \geq t_0 + i\rho,\ i \in \mathbb{N}.$$

Now, there exists two possible cases.

Case 1. $r<1$. Thus,

$$\alpha(\infty)=\liminf_{t\to\infty}\int_{t-\tau}^{t}p\sum_{i=0}^{\infty}r^{i}ds=\frac{\tau p}{1-r}$$

by (35)

$$\alpha(\infty)=\frac{\tau p}{1-r}>\frac{1}{e}$$

that every solution of (8) is oscillatory by Theorem 14.

Case 2. $r\equiv 1$. In this case,

$$\alpha(\infty)=\infty>\frac{1}{e}.$$

Thus, Theorem 14 can be applied to reveal oscillatory behavior of solutions of (8).

The proof is completed. $\square$

4. Oscillation of (11)

In the following subsections, we give two different oscillation criterions for (11). First of them is adaptation of Section 2 and Section 3 and the other one is improving these results with the key idea of [34]. Throughout this section, we let $\kappa:=\max\{\rho,\sigma\}$ and assume (12) and (13) hold for $t\geq\bar{t}\geq t_0$.

4.1. Oscillation criterion 1

In this section, we combine results of Section 2 and Section 3 to obtain advanced oscillation criterion for the equation (11).

We have the following lemma from [35].

Lemma 16. *Assume that* (12) *and* (13) *hold, and* $x(t)$ *is an eventually positive solution of* (11). *Setting*

$$z(t):=x(t)-R(t)x(t-\rho)-\int_{t-\tau+\sigma}^{t}Q(s)x(s-\sigma)ds, \tag{36}$$

then

$$z'(t) \leq 0, \;\; z(t) > 0$$

eventually.

We set,

$$H_i(t) := \begin{cases} 1, & t \geq \bar{t}, \;\; i = 0 \\ R(t)\, H_{i-1}(t-\rho) + \displaystyle\int_{t-\tau+\sigma}^{t} Q(s)\, H_{i-1}(s-\sigma)\, ds, & t \geq \bar{t}+i\kappa, \;\; i \in \mathbb{N}. \end{cases} \tag{37}$$

Lemma 17. *Assume that all conditions of Lemma* 16 *hold. Then eventually positive* $z(t)$ *in* (36) *eventually satisfies*

$$z'(t) + \bar{P}(t) \sum_{i=0}^{n} H_i(t-\tau)\, z(t-\tau) \leq 0 \tag{38}$$

for every $n \in \mathbb{N}$.

Proof. The proof is very similar to proofs of Lemma 2 and Lemma 11, and is omitted for reasons of space. □

As in preceding sections, we set

$$\alpha(n) := \liminf_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s) \sum_{i=0}^{n} H_i(s-\tau)\, ds \tag{39}$$

and

$$\beta(n) := \limsup_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s) \sum_{i=0}^{n} H_i(s-\tau)\, ds. \tag{40}$$

Remark 18. By the definition in (37), $\alpha(n)$ and $\beta(n)$ are non-decreasing respect to n.

Theorem 19. *Assume all conditions of Lemma* 16 *are held. If there exists* $n \in \mathbb{N}$ *such that*

$$\alpha(n) > \frac{1}{e} \tag{41}$$

or

$$\alpha(n) \le \tfrac{1}{e},\ \beta(n) > 1 - \tfrac{1-\alpha(n)-\sqrt{1-2\alpha(n)-\alpha^2(n)}}{2}, \tag{42}$$

then every solution of (10) *is oscillatory.*

Proof. Proof is trivial. □

Theorem 20. *Assume all conditions of Lemma* 16 *are held, furthermore* $R(t)$ *and* $Q(t)$ *are non-increasing functions. If there exists* $n \in \mathbb{N}$ *such that*

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s) \sum_{i=0}^{n} (R(s-\tau) + Q(s-\tau)(\tau-\sigma))^i \, ds > \frac{1}{e},$$

then every solution of (11) *is oscillating.*

Proof. By the definition in (37),

$$\begin{aligned}
H_0(t) &= 1,\\
H_1(t) &= R(t) + \int_{t-\tau+\sigma}^{t} Q(s)\,ds\\
&\ge R(t) + Q(t)(\tau-\sigma),\\
H_2(t) &= R(t)\,H_1(t-\rho) + \int_{t-\tau+\sigma}^{t} Q(s)\,H_1(s-\sigma)\,ds\\
&\ge R(t)\,(R(t-\rho) + Q(t-\rho)(\tau-\sigma))\\
&\quad + \int_{t-\tau+\sigma}^{t} Q(s)\,(R(s-\sigma) + Q(s-\sigma)(\tau-\sigma))\,ds\\
&\ge R^2(t) + R(t)\,Q(t)(\tau-\sigma) + R(t)\,Q(t)(\tau-\sigma) + (Q(t)(\tau-\sigma))^2\\
&= (R(t) + (Q(t)(\tau-\sigma)))^2.
\end{aligned}$$

Generally, we obtain

$$H_i(t) \ge (R(t) + (Q(t)(\tau-\sigma)))^i$$

for $i \in \mathbb{N}$ and sufficiently large t. Therefore,

$$\alpha(n) \ge \liminf_{t\to\infty} \int_{t-\tau}^{t} \bar{P}(s) \sum_{i=0}^{n} (R(s-\tau) + Q(s-\tau)(\tau-\sigma))^i \, ds > \frac{1}{e}.$$

Application of Theorem 19 completes the proof. □

Theorem 21. *Assume all conditions of Lemma* 16 *are held. If*

$$\alpha(\infty) > \frac{1}{e}, \tag{43}$$

then every solution of (11) *is oscillating.*

Proof. With a similar way to proofs of Theorem 8 and Theorem 14, proof can be done. □

The following theorem considers the equation (11) with autonomous case as

$$[x(t) - rx(t-\rho)]' + px(t-\tau) - qx(t-\sigma) = 0 \tag{44}$$

with

$$\begin{aligned} &p > q, \\ &\tau \geq \sigma, \\ &1 \geq r + q(\tau - \sigma) \geq 0. \end{aligned} \tag{45}$$

Theorem 22. *Assume that* (45) *and*

$$\frac{\tau(p-q)}{1-(r+q(\tau-\sigma))} > \frac{1}{e} \tag{46}$$

are held, then every solution of (44) *is oscillatory.*

Proof. First, calculate the $H_i(t)$ functions. Obviously,

$$H_0(t) = 1, \ t \geq t_1,$$

then

$$H_1(t) = r + q(\tau - \sigma), \ t \geq t_1 + \kappa,$$

and

$$H_2(t) = rH_1(t) + q(\tau-\sigma)H_1(t) = (r + q(\tau-\sigma))^2, \ t \geq t_1 + 2\kappa.$$

By continuation, we obtain

$$H_n(t) = (r + q(\tau - \sigma))^n, \quad t \geq t_1 + n\kappa.$$

Case 1. $r+q\,(\tau-\sigma)<1$. Thus,

$$\begin{aligned}\alpha\,(\infty) &= \liminf_{t\to\infty}\int_{t-\tau}^{t}(p-q)\sum_{i=0}^{\infty}(r+q\,(\tau-\sigma))^{i}\,ds\\ &= \frac{\tau\,(p-q)}{1-(r+q\,(\tau-\sigma))},\end{aligned}$$

which implies by (46) that every solution of (44) is oscillatory by Theorem 21.

Case 2. $r+q\,(\tau-\sigma)\equiv 1$. In this case,

$$\alpha\,(\infty)=\infty>\frac{1}{e}.$$

Thus, Theorem 21 can be applied. The proof is completed. □

4.2. Oscillation criterion 2

In this section, we join our key idea with the key idea in [34] to obtain a new criterion. We define the following functions

$$\bar{H}_{lj}\,(t):=\begin{cases}1, & t\ge\bar{t},\ j=0\\ \int_{t-\tau}^{t}\bar{P}\,(s)\sum\limits_{i=0}^{l}H_i\,(s-\tau)\,\bar{H}_{lj-1}\,(s)\,ds, & t\ge\bar{t}+(l+j)\,\kappa,\ j\in\mathbb{N},\end{cases}$$

where $H_i\,(t)$ functions are defined in (37).

We result our study with the following theorem which improves Theorem 3 in [34].

Theorem 23. *Assume that* (12) *and* (13) *hold. Also, assume that there exists a pair of positive integers* n,m *such that*

$$\liminf_{t\to\infty}\bar{H}_{nm}\,(t)>\frac{1}{e^{m}}$$

holds. Then every solution of (11) *is oscillatory.*

Proof. If these conditions are held, then (38) can not have an eventually positive solution. This implies (11) can not have eventually positive solution. Since the equation is linear multiplying, an eventually negative solution by

-1 forms an eventually positive solution. Thus every solution of (11) is oscillatory. $\square$

5. Applications

This section is dedicated to illustrative examples.

Example 24. Assume $A \in C\left([t_0, \infty), \mathbb{R}^+\right), B > 0$ and $\tau > \sigma \geq 0$. And consider

$$x'(t) + (A(t) + B)\, x(t - \tau) - Bx(t - \sigma) = 0 \tag{47}$$

with

$$0 < m \leq \int_{t-\tau}^{t} A(s)\, ds \leq \tau < \tfrac{1}{2e},\ \ B(\tau - \sigma) \equiv 1.$$

It is obvious that (5) can not be applied. And,

$$\tau < \frac{1}{2e} < \frac{1}{2} < 1 - \frac{1 - m - \sqrt{1 - 2m - m^2}}{2}$$

implies the known result

$$\beta(0) > 1 - \frac{1 - \alpha(0) - \sqrt{1 - 2\alpha(0) - \alpha^2(0)}}{2}$$

can not be applied where $\alpha(n)$ and $\beta(n)$ are defined in (20) and (21) respectively. Either, (6) is not useful, in fact,

$$\begin{aligned} A_0(t) &= 1, \\ A_1(t) &= \int_{t-\tau}^{t} A(s)\left(1 + B(\tau - \sigma)\right) ds = 2\int_{t-\tau}^{t} A(s)\, ds \leq 2\tau < \frac{1}{e} \end{aligned}$$

and

$$A_2(t) = 2\int_{t-\tau}^{t} A_1(s)\, ds \leq 4\tau^2 < \frac{1}{e^2}.$$

So, in general we have

$$A_i(t) \leq (2\tau)^i < \frac{1}{e^i},$$

which implies (6) can not be applied. Clearly, all known results are useless.

Obviously, for sufficiently large t values

$$Q_i(t) = 1, \ \ i \in \mathbb{N}.$$

Denoting greatest integer function with $\lceil . \rceil$, we can see that

$$\begin{aligned}
\alpha\left(\left\lceil \frac{1}{me} \right\rceil\right) &= \liminf_{t\to\infty} \int_{t-\tau}^{t} A(s) \sum_{i=0}^{\lceil \frac{1}{me} \rceil} 1 ds \\
&= \left(\left\lceil \frac{1}{me} \right\rceil + 1\right) \liminf_{t\to\infty} \int_{t-\tau}^{t} A(s)\, ds \\
&\geq \left(\left\lceil \frac{1}{me} \right\rceil + 1\right) m > \frac{1}{e}
\end{aligned}$$

holds. Thus, by Theorem 4, every solution of (47) is oscillatory.

Example 25. Assume $A \in C\left([t_0, \infty), \mathbb{R}^+\right)$, and $\rho, \tau \geq 0$. And consider

$$[x(t) - x(t-\rho)]' + A(t)\, x(t-\tau) = 0 \tag{48}$$

with

$$0 < m \leq \int_{t-\tau}^{t} A(s)\, ds \leq \tau < \frac{1}{2e}. \tag{49}$$

As in Example 24, most of the known results are useless either.
Obviously, for sufficiently large t values

$$R_i(t) \equiv 1, \ \ i \in \mathbb{N}.$$

Since,

$$\alpha\left(\left\lceil \frac{1}{me} \right\rceil\right) > \frac{1}{e}$$

Theorem 10 guaranties that every solution of (48) is oscillatory. $\alpha(n)$ is as defined in (31).

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Department of Mathematics, Faculty of Science and Arts, Kocatepe University, ANS Campus, 3200 Afyonkarahisar, Turkey.
E-mail: bkarpuz@aku.edu.tr

Department of Mathematics, Faculty of Science and Arts, Kocatepe University, ANS Campus, 3200 Afyonkarahisar, Turkey.
E-mail: ozkan@aku.edu.tr