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COHOMOGENEITY TWO ACTIONS ON $\mathbb{R}^m, m \geq 3$

BY

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Abstract

We suppose that a connected and closed Lie group G of isometries of $\mathbb{R}^m, m \geq 3$, acts by cohomogeneity two on $\mathbb{R}^m$. Then we show that under some conditions, the orbit space is homeomorphic to $\mathbb{R}^2$ or $[0, +\infty) \times \mathbb{R}$.

1. Introduction

Let G be a connected and closed Lie group of isometries of a Riemannian manifold M. For each point $x \in M$, we denote the orbit containing x by:

$$G(x) = \{gx : g \in G\}.$$

We say that G acts by "Cohomogeneity K" on M, if

$$\dim M = K + \max\{\dim G(x) : x \in M\}.$$

If $K = 0$, then for each point $x \in M$, we have $M = G(x)$ and M is called homogeneous G-manifold. Homogeneous and cohomogeneity one manifolds are studied by several authors (see [1], [2], [7], [10], [11]). Study of cohomogeneity two Riemannian manifolds is still wide open. In [3] the authors studied cohomogenity two Riemannian manifolds from a algebraic view point. In [8] it is considered that M is flat and G has fixed point in M. Then the orbits and orbit space are characterized. In this paper we consider cohomogeneity two actions on $\mathbb{R}^m, m \geq 3$. In Theorem 3.6 we suppose that G is a compact

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connected subgroup of $\text{Isom}(R^m)$, which acts by cohomogeneity two on R^m. Then we show that the orbit space is homeomorphic to $[0, +\infty) \times R$. In Theorem 3.8 we suppose that G(compact or noncompact) has an irreducible orbit. Then we show that the orbit space is homeomorphic to $[0, +\infty) \times R$ or R^2.

2. Preliminaries

In this paper, when two spaces X and Y are homeomorphic we denote this by $X \sim Y$. Now, we mention some facts which we will use in the sequel. Let G be a connected and closed Lie subgroup of isometries of M. We denote by $\frac{M}{G}$ the set of orbits of this action and we equip $\frac{M}{G}$ with the quotient topology relative to the canonical projection $M \to \frac{M}{G}, x \to G(x)$.

Defnition 2.1. Let G and H be closed and connected subgroups of $Isom(M)$. We say that G and H are orbit-equivalent on M, if the set of orbits of G-action on M is equal to the set of orbits of H-action on M.

$$\{G(x) : x \in M\} = \{H(x) : x \in M\}$$

The following fact is clear.

Fact 2.2. *If G and H are orbit-equivalent on M, then $\frac{M}{G} = \frac{M}{H}$.*

Fact 2.3. *Let $\widetilde{M}$ and $\widetilde{G}$ be the universal covering manifolds of M and G, with covering maps $\pi : \widetilde{M} \to M$ and $\kappa : \widetilde{G} \to G$. It is well known* (*see* [4] *pages* 62-63) *that $\widetilde{G}$ acts on $\widetilde{M}$, such that for each $\widetilde{x} \in \widetilde{M}$ and $\widetilde{g} \in \widetilde{G}$ we have:*

$$\pi(\widetilde{g}\widetilde{x}) = \kappa(\widetilde{g})\pi(\widetilde{x})$$

If M is simply connected then G and $\widetilde{G}$ both act on M orbit equivalently and the map $\kappa : \widetilde{G} \to G$ is a representation of $\widetilde{G}$ as isometries of M (the action of $\widetilde{G}$ on M may be not effective).

Definition 2.4. Let G be a connected and closed subgroup of isometries of R^m. In Fact 2.3 if we let $M = R^m$ then we have $\widetilde{M} = R^m$ and π is the identity map. So for each $\widetilde{x} \in R^m$ and $\widetilde{g} \in \widetilde{G}$ we have

$$\widetilde{g}\widetilde{x} = \kappa(\widetilde{g})\widetilde{x}$$

The covering map $\kappa : \widetilde{G} \to G$ is a representation of $\widetilde{G}$ on $G \subset Isom(R^m)$. This representation of $\widetilde{G}$ is called "induced representation".

By Fact 2.3 we have the following fact.

Fact 2.5. *If G is a closed and connected Lie subgroup of isometries of R^m, then the group $\tilde{G}$ the universal covering group of G acts on R^m by induced representation, orbit-equivalently to G, and we have:*

$$\frac{R^m}{G} = \frac{R^m}{\tilde{G}}.$$

Fact 2.6. (See [2], [7] and [10]) *Let G act by cohomogeneity one on M, then*

(a) *The orbit space $\frac{M}{G}$ is homeomorphic to one of the following spaces*

$$R;\ [0,+\infty);\ S^1;\ [-1,1].$$

(b) *If M is simply connected, then $\frac{M}{G} \nsim S^1$.*
(c) *If M is compact, then $\frac{M}{G} \sim S^1$ or $\frac{M}{G} \sim [-1,1]$.*

The isometry group of R^n is in the form $O(n) \times R^n$, where the action of $(A,b) \in O(n) \times R^n$ on R^n is as follows:

$$(A,b)(x) = A(x) + b.$$

The isometry (I,b) is called an ordinary translation.

$$(I,b)(x) = x + b.$$

Fact 2.7. *If R^n is of cohomogeneity one under the action of a closed and connected Lie subgroup G of isometries of R^n, then*

(a) $\frac{R^n}{G} \sim R$ or $\frac{R^n}{G} \sim [0,+\infty)$.
(b) *If G contains ordinary translations only, then $\frac{R^n}{G} \sim R$.*

Proof. (a). By Theorem 2.8 in [7], $\frac{R^n}{G} \not\approx [-1,1]$ and by Fact 2.6(b), we have $\frac{R^n}{G} \not\approx S^1$. Therefore $\frac{R^n}{G} \sim R$ or $\frac{R^n}{G} \sim [0,+\infty)$.

(b) If G contains ordinary translations only, then for each two points $x, y \in R^n$ we have

$$G(x) = \{x + b : b \in G\},\ G(y) = \{y + b : b \in G\}.$$

So all orbits are diffeomorphic to each other and there is not any singular orbit(see[7] proof of Theorem 3.1). Thus by part (a), we have $\frac{R^n}{G} \sim R$. □

Definition 2.8. Let M be a submanifold of R^m, we say that M is reducible, if M is isometric to $M_1 \times M_2$, where M_1, M_2 are submanifolds of R^m and $dimM_i \geq 1$.

3. Results

Before stating our results we give a definition and lemma in general topology.

Definition 3.1. Let $I = [0,+\infty)$, $X = \bigcup_{t\in I} X_t$, where X is a topological space and for each t, X_t is a subspace of X and the union is disjoint. We say that X is a continuous motion of X_1on I, if there exist a continuous map $\psi : X_1 \times I \longrightarrow X$ such that

(1) $\psi(x,t) \in X_t$.

(2) $\psi(x,1) = x$.

(3) The collection B containing all of the sets in the form $\psi(U \times (a,b))$ and $\psi(X_1 \times [0,b))$ is a basis for the topology of X, (where $(a,b) \subset I$ and U is open in X_1).

The map ψ is called motion map.

Example 3.2. Let $X = R^2$, $X_t = S^1(t) = \{(x_1,x_2) \in R^2 : x_1^2 + x_2^2 = t^2\}$ and let $\psi : S^1(1) \times I \longrightarrow R^2$, $\psi(x,t) = tx$. It is easy to see that X is a continuous motion of X_1.

Lemma 3.3. *Let $X = \bigcup_t X_t$, $Y = \bigcup_t Y_t$ be two spaces which are continuous motions of X_1, Y_1 under the motions $\psi : X_1 \times I \longrightarrow X$ and*

$\phi : Y_1 \times I \longrightarrow Y$. *Also let for each* t *in* I, *there is a homeomorphism* $F_t : X_t \longrightarrow Y_t$, *such that*

$$F_t o \psi_t = \phi_t o F_1 (*)$$

where $\psi_t(x) = \psi(x,t)$, $\phi_t(x) = \phi(x,t)$. *Then* X *is homeomorphic to* Y.

Proof. Consider the map F as :

$$F : X \longrightarrow Y, F(x) = F_t(x), x \in X_t.$$

By definition, the collection $B = \{\phi(V \times (a,b)), \phi(Y_1 \times [0,b)) : V$ open in Y_1, $(a,b) \subset I\}$ is a basis for topology of Y. F_1 is a homeomorphism. So if V is open in Y_1 then $U = F_1^{-1}(V)$ is open in X_1. By using (*), we have:

$$\begin{aligned} F^{-1}\{\phi(V \times (a,b))\} &= \bigcup_{t \in (a,b)} F_t^{-1}\{\phi(x,t) : x \in V\} = \bigcup_t F_t^{-1}\{\phi_t(x) : x \in V\} \\ &= \bigcup_t \{\psi_t o F_1^{-1}(x) : x \in V\} = \bigcup_t \{\psi_t(y) : y \in U\} \\ &= \psi(U \times (a,b)). \end{aligned}$$

In similar way we can show that:

$$F^{-1}(\phi(Y_1 \times [0,b)) = \psi(X_1 \times [0,b)).$$

So for each open set W in Y, $F^{-1}(W)$ is open in X. This means that F is continuous. In the similar way we can show that F^{-1} is continuous. Therefore F is a homeomorphism between X and Y. □

Theorem 3.4.([5, p.56]) *Let* $M = G(x)$ *be a homogeneous irreducible submanifold of* R^n, *where* G *is a connected Lie subgroup of isometries of* R^n. *Then the universal covering group* $\tilde{G}$ *of* G *is isomorphic to the direct product* $K \times R^d$, *where* K *is a simply connected Lie group. Moreover, the induced representation of* $\tilde{G}$ *is equivalent to* $P_1 \bigoplus P_2$ *where* P_1 *is a representation of* $\tilde{G}$ *in to* $SO(d)$ *and* P_2 *is linear map from* R^d *to* R^e, $n = d + e$ *regarding* R^e *as ordinary translations.*

From Theorem 3.4 and its proof (in [5] pages 56, 57) we get the following corollary.

Corollary 3.5. *If $M = G(x)$ is a homogeneous irreducible submanifold of R^n, then $\tilde{G}$, the universal covering group of G, is orbit equivalent to a subgroup H of the group $\{(A,b) : A \in SO(d), b \in R^e\}$, where H acts on R^n, as follows*

$$(A,b)(x,y) = (Ax, y+b);\ (x,y) \in R^d \times R^e = R^n.$$

Theorem 3.6. *If $G \subset ISo(R^m)$ is compact and connected and acts by cohomogeneity two on $R^m, m \geq 3$, then*

$$\frac{R^m}{G} \sim [0,+\infty) \times R.$$

Proof. Since G is compact,by Cartan's theorem(see [6] vol II page 111) it has at least one fixed point in R^m. Without loss of generality, we assume that the origin is a fixed point of G. Let $S^{m-1}(r)$ be the standard sphere of radius r in R^m.

$$S^{m-1}(r) = \{(x_1, \cdots, x_m) \in R^m : \sum_{i=1}^{m} x_i^2 = r^2\}.$$

Since each $g \in G$ fixes the origin of R^m invariant, for any $x \in S^{m-1}(r)$ we have $g(x) \in S^{m-1}(r)$. So we can consider G as a subgroup of isometries of $S^{m-1}(r)$ (i.e $G \subset O(m)$). Let $r_2 > r_1 > 0$ and consider the following map

$$\begin{cases} \phi_{r_1r_2} : S^{m-1}(r_1) \to S^{m-1}(r_2), \\ \phi_{r_1r_2}(x) = \frac{r_2}{r_1}x. \end{cases}$$

Each $g \in G$ is a linear map on R^m. So we have:

$$\phi_{r_1r_2}(gx) = \frac{r_2}{r_1}(gx) = g(\frac{r_2}{r_1}x) = g\phi_{r_1r_2}(x).$$

Therefore $\phi_{r_1r_2}$ maps each orbit of the G-action on $S^{m-1}(r_1)$ diffeomorphically on to an orbit of G-action on $S^{m-1}(r_2)$. So topologically, the orbit foliation of $S^{m-1}(r_1)$ is alike the orbit foliation of $S^{m-1}(r_2)$. Since R^m is of cohomogeneity two under the action of G, then for each $r > 0, S^{m-1}(r)$ is of cohomogeneity one. Consider the sphere $S^{m-1}(1)$. By Fact 2.6(b,c), $\frac{S^{m-1}(1)}{G}$

is homeomorphic to $[-1,1]$. Let P be this homeomorphism.

$$P : \frac{S^{m-1}(1)}{G} \to [-1,1].$$

We have $R^m = \bigcup_{t \in I} S^{m-1}(t)$, where $I = [0,+\infty)$. So $\frac{R^m}{G} = \bigcup_t \frac{S^{m-1}(t)}{G}$. Let $X_t = \frac{S^{m-1}(t)}{G}$, $X = \frac{R^m}{G}$, it is easy to see that X is a continuous motion of X_1 under the motion map ψ defined by:

$$\psi : X_1 \times I \longrightarrow X; \psi(G(x),t) = G(tx), x \in S^{m-1}(1).$$

Let Y be the subset of R^2 defined by:

$$Y = \bigcup_{t \in I} \{t\} \times [-t,t].$$

and let

$$Y_t = \{t\} \times [-t,t], t \in I.$$

Y is a continuous motion of $Y_1 = \{1\} \times [-1,1]$ under the map ϕ defined by:

$$\phi : Y_1 \times I \longrightarrow Y, \phi((1,a),t) = (t,ta).$$

Now for each t in I define the map $F_t : X_t \longrightarrow Y_t$ as follows

$$\begin{cases} F_t(G(x)) = (t, tP(G(\frac{x}{|x|}))), & |t| \neq 0, \\ F_0(o) = (0,0), & |t| = 0. \end{cases}$$

Note that $X_0 = \{o\}, Y_0 = \{(0,0)\}$. For each t in I, F_t is homeomorphism and the conditions of Lemma 3.3 are valid. Thus X is homeomorphic to Y. But easily we can show that Y is homeomorphic to $[0,+\infty) \times R$. Therefore X is homeomorphic to $[0,+\infty) \times R$. □

Lemma 3.7. *Let H be a closed and connected subgroup of $SO(d) \times R^e$ which acts by cohomogeneity two on $R^d \times R^e = R^m$ and let*

$$S = \{A : (A,b) \in H \text{ for some } b \in R^e\},$$
$$T = \{b : (A,b) \in H \text{ for some } A \in SO(d)\}.$$

Then

(1) *One of the following is true.*
   (a) *The cohomogeneity of S-action on R^d is 1 and the cohomogeneity of T-action on R^e is 1 or 0.*
   (b) *The cohomogeneity of S-action on R^d is 2 and the cohomogeneity of T-action on R^e is 0.*

(2) *For $r > 0$, let $M_r = S^{d-1}(r) \times R^e \subseteq R^d \times R^e$, where $S^{d-1}(r)$ is the standard sphere in R^d with radius r. Then for each $r > 0, H$ acts by cohomogeneity one on M_r and for each $r_1, r_2 > 0$ we have $\frac{M_{r_1}}{H} \sim \frac{M_{r_2}}{H}$.*

(3) *In (2), if for one $r > 0$, $\frac{M_r}{H}$ is compact,then $\frac{M_0}{H}$ is a one point space.*

(4) *$\frac{R^m}{H}$ is homeomorphic to $[0, +\infty) \times R$ or R^2.*

Proof. (1) Since $H \subset S \times T$, we have:

$$\begin{aligned} 2 \; = \; & \text{cohomogeneity of } H \text{ action on } R^m \geq \text{cohomogeneity of} \\ & + S\text{-action on } R^d \text{ cohomogeneity of } T\text{-action on } R^e. \end{aligned}$$

Since S is compact, it has fixed point in R^d. Thus the cohomogeneity of S-action on R^d is ≥ 1. These yield to (a) or (b).

(2) Consider $(x, y) \in M_r, x \in S^{d-1}(r), y \in R^e$, we have:

$$H(x, y) \subseteq (S \times T)(x, y) = S(x) \times T(y) \subseteq S^{d-1}(r) \times R^e = M_r.$$

So H maps M_r on to itself and we can consider H as a subgroup of isometries of M_r. For $r_1, r_2 > 0$, the map $\varphi_{r_1 r_2} : M_{r_1} \to M_{r_2}; (x, y) \to (\frac{r_2 x}{r_1}, y)$ induces a homeomorphism between $\frac{M_{r_1}}{H}$ and $\frac{M_{r_2}}{H}$. Since $\dim M_r = m - 1$ and the action of H on R^m is of cohomogeneity two, the action of H on M_r is of cohomogeneity one.

(3) Consider the map: $\phi_r : M_r \to M_0$ defined by $:\phi_r(x, y) = y$. ϕ_r induces a continuous and on to map: $\overline{\phi_r} : \frac{M_r}{H} \longrightarrow \frac{M_0}{H}$. So $\frac{M_0}{H}$ must be compact. But it is easy to see that $\frac{M_0}{H} = \frac{R^e}{T}$. By part (1) of Lemma and Fact 2.7, we have $\frac{R^e}{T} = \{0\}$ or R. Since $\frac{M_0}{H} \sim \frac{R^e}{T}$ is compact, we get that $\frac{M_0}{H} \sim \frac{R^e}{T} = \{0\}$.

(4) We have $R^m = \bigcup_{t \in I} M_t$, where $I = [0, +\infty)$. So

$$\frac{R^m}{H} = \bigcup_t \frac{M_t}{H}.$$

Let

$$X = \frac{R^m}{H}, X_t = \frac{M_t}{H}.$$

X is a continuous motion of X_1 under the motion map ψ defined by

$$\psi : X_1 \times I \longrightarrow X, \psi(H(x,y),t) = H(tx,y); (x,y) \in M_1 = S^{d-1}(1) \times R^e$$

By Fact 2.6(a) and part (2) of Lemma, for all $r > 0$, $\frac{M_r}{H}$ is homomorphic to one of the following spaces.

(I) $S^1(r)$ (II) $[-r,r]$ (III) $[0,+\infty)$ (IV) R.

We study each case separately

(I) $\frac{M_r}{H} \sim S^1(r), r > 0$.
Let

$$Y = R^2, Y_t = S^1(t), t \in [0,+\infty)$$

Y is a continuous motion of Y_1,under the map:

$$\phi : Y_1 \times I \longrightarrow Y, \phi(a,t) = ta.$$

Let P be the homeomorphism between $X_1 = \frac{M_1}{H}$ and $Y_1 = S^1(1)$. For each t in I,define the map $F_t : X_t \longrightarrow Y_t$ as follows:

$$\begin{cases} F_t(H(x,y)) = tP(H(\frac{x}{|x|},y)), & t \neq 0, \\ F_0(o) = (0,0), & t = 0. \end{cases}$$

Note that $Y_0 = (0,0)$ and by part (3) of Lemma, we have $X_0 = \{o\}$. For each $t \in I$, F_t is homeomorphism and the conditions of Lemma 3.3 are valid. So X is homeomorphic to $Y = R^2$.

(II) $\frac{M_r}{H} = [-r,r],\ r > 0$.
In this case we let

$$Y = \bigcup_t Y_t$$

where

$$Y_t = t \times [-t,t],$$

$$\phi : Y_1 \times I \longrightarrow Y, \quad \phi((1,a),t) = (t, ta).$$

As like as the proof of Theorem 3.6, we can show that X is homeomorphic to Y. Since Y is homeomorphic to $[0,\infty) \times R$, we get that X is homeomorphic to $[0,\infty) \times R$.

(III) $\frac{M_r}{H} \sim [0,+\infty),\ r > 0.$

We show that this case can not occur .Consider the continuous and onto map

$$\begin{cases} \phi_r : M_r \to R^e \\ \phi_r(x,y) = y \end{cases}$$

ϕ_r induces continuous and onto map $\overline{\phi_r}$ between orbit spaces

$$\overline{\phi_r} : \frac{M_r}{H} \sim [0,+\infty) \to \frac{R^e}{T}.$$

By part (1) of Lemma and Fact 2.7, $\frac{R^e}{T}$ is homeomorphic to $\{0\}$ or R. If $\frac{R^e}{T} \sim R$ then $\overline{\phi}$ is a continuous and onto map as follows:

$$\overline{\phi_r} : [0,+\infty) \to R.$$

So the following map is continuous and onto

$$\overline{\phi_r} : (0,+\infty) \to R - \{\overline{\phi_r}(0)\},$$

which is a contradiction (because $R - \{\overline{\phi_r}(0)\}$ is not connected.)

If $\frac{R^e}{T} = 0$, then T acts transitively on R^e. So for each $(x,y) \in M_r$ there exists $(A,b) \in H$ such that $(A,b)(x,y) = (x_1, 0)$ for some $x_1 \in S^{d-1}(r)$. Thus each H-orbit of M_r intersects the set $S^{d-1}(r) \times \{0\} \subset S^{d-1}(r) \times R^e = M_r$. Let $\kappa : M_r \to \frac{M_r}{H}$ be the projection on the orbit space and consider the map $\eta : M_r \to S^{d-1}(r) \times \{0\}, \eta(x,y) = (x,0)$ and let κ_1 be the restriction of κ on $S^{d-1}(r) \times \{0\}$. Easily we see that the following diagram is commutative

$$\begin{cases} \eta : M_r \to S^{d-1} \times \{0\} \\ \quad \kappa \searrow \quad \swarrow \kappa_1 \\ \qquad\quad \frac{M_r}{H} \end{cases}$$

Since $S^{d-1}(r) \times \{0\}$ is compact, $\frac{M_r}{H}$ must be compact, which is a contradiction. Therefore the case III can not occur.

(IV) $\frac{M_r}{H} \sim R,\ r > 0$.
If $\frac{R^e}{T} = 0$. As like as the case III we get a contradiction. Let $\frac{R^e}{T} = R$, we have:$R^m = (\{0\} \times R^e) \cup (\cup_{r>0} M_r)$ and $\frac{\{0\}\times R^e}{H} = \frac{R^e}{T} = R$. Let

$$Y = [0, +\infty) \times R, Y_t = \{t\} \times R$$

Y is a continuous motion of Y_1, by the map $\phi : Y_1 \times I \longrightarrow Y$ defined by

$$\phi((1, a), t) = (t, a).$$

As like as before by suitable choice of the maps $F_t : X_t \longrightarrow Y_t$, we can show that X is homeomorphic to $Y = [0, +\infty) \times R$. □

Theorem 3.8. *Let $R^m, m > 3$, be of cohomogeneity two, under the action of a connected and closed Lie subgroup G of $Isom(R^m)$, and suppose that there exists an irreducible orbit $G(x)$ for some x in R^m, then $\frac{R^m}{G}$ is homeomorphic to one of the following spaces:*

$$[0, +\infty) \times R;\ R^2$$

Proof. Let $G(x)$ be an irreducible orbit of this action. By Corollary 3.5, $\tilde{G}$ the universal covering Lie group of G acts on R^m, orbit-equivalent to a subgroup of the group $\{(A, b) : A \in SO(d), b \in R^e\} = SO(d) \times R^e$, $d + e = m$, which we denote it by H. By Fact 2.5 and Corollary 3.5, we get that:

$$\frac{R^m}{G} \sim \frac{R^m}{H}$$

So we get the result by Lemma 3.7(4). □

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