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A LOCALIZED LUSIN THEOREM AND A RADEMACHER TYPE THEOREM

BY

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Abstract

For each positive number a slightly more general function space than that in [6] is introduced and a corresponding version of Lusin theorem is established with a view to cover an application to a Rademacher type theorem for classes of functions introduced by Calderón and Zygmund.

1. Introduction

The Rademacher phenomenon is coined to refer to a general phenomenon which came to light through a well-known theorem of Rademacher [10] which affirms that a Lipschitz function is differentiable almost everywhere. This theorem has been generalized by Stepanoff [12] to the following result:

Theorem 1.1. *Let u be a measurable function defined on an open set $D \subset R^n$, then u is differentiable almost everywhere on D if and only if*

$$\limsup_{y \to x} \frac{|u(y) - u(x)|}{|y - x|} < +\infty$$

for almost all $x \in D$.

A certain property of functions, the so-called Lusin property, is later found to be closely related to the Rademacher phenomenon. Unless explicitly stated otherwise a function defined on a measurable subset D of R^n will be assumed to be real measurable and finite almost everywhere on D. A function u defined on D is said to have the *Lusin property of order k* if for any

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$\epsilon > 0$ there is a C^k-function g on R^n such that $|\{x \in D : u(x) \neq g(x)\}| < \epsilon$, where we use $|A|$ to denote the Lebesgue measure of a set A in R^n. A classical theorem of Lusin states that measurable functions which are finite almost everywhere has the Lusin property of order zero. It is observed first by Federer that a Lipschitz function u defined on a domain D in R^n has the Lusin property of order 1. This is then strengthened by Whitney [14] in the following theorem:

Theorem 1.2. *A measurable function u defined on a measurable set D has the Lusin property of order 1 if and only if it is approximately differentiable almost everywhere on D.*

We recall that a measurable function u defined on a measurable set $D \subset R^n$ is approximately differenriable at $x \in D$ if there is $d \in R^n$ such that

$$\operatorname{ap}\lim_{y \to x} \frac{|u(y) - u(x) - d \cdot (y - x)|}{|y - x|} = 0,$$

where by $\operatorname{ap}\lim_{y \to x} v(y) = l$ it is meant that the set $\{y \in D : |v(y) - l| \leq \epsilon\}$ has density 1 at x for all $\epsilon > 0$. Eventually Federer closes this sequence of results in [4, 3.1.16] with the theorem below:

Theorem 1.3. *A measurable function u defined on a measurable set $D \subset R^n$ has the Lusin property of order 1 if and only if*

$$\operatorname{ap}\limsup_{y \to x} \frac{|u(y) - u(x)|}{|y - x|} < +\infty$$

for almost all $x \in D$.

In the theorem above $\operatorname{ap}\limsup_{y \to x} v(y)$ is the infimum of all those $\lambda \in R$ such that the set $\{y \in D : v(y) > \lambda\}$ has density zero at x.

While the content of Theorem 1.1 is usually refered to as the Rademacher phenomenon, Theorem 1.3 expresses also the Rademacher phenomenon with "limit" replaced by "approximate limit". These two types of the Rademacher phenomenon are later generalized to situations of differentiability and approximate differentiability of higher order in [9] as we shall now describe. A function u defined on D is said to admit a $(k-1)$-*Taylor polynomial* at x if there is a polynomial $P_x(\cdot)$ of degree at most $k-1$ such that

$$\limsup_{y\to x} \frac{|u(y)-P_x(y)|}{|y-x|^k} < +\infty; \tag{1.1}$$

while u will be said to be *Taylor-differentiable of order* k at x if there is a polynomial $Q_x(\cdot)$ of degree at most k such that

$$\lim_{y\to x} \frac{|u(y)-Q_x(y)|}{|y-x|^k} = 0. \tag{1.2}$$

Weakening "limit" to "approximate limit", we introduce also the following definitions. A function u defined on D is said to admit approximately a $(k-1)$-*Taylor polynomial* at x if there is a polynomial $P_x(\cdot)$ of degree at most $k-1$ such that

$$\operatorname{ap}\limsup_{y\to x} \frac{|u(y)-P_x(y)|}{|y-x|^k} < +\infty; \tag{1.3}$$

while u will be said to be *approximately Taylor-differentiable of order* k at x if there is a polynomial $Q_x(\cdot)$ of degree at most k such that

$$\operatorname{ap}\lim_{y\to x} \frac{|u(y)-Q_x(y)|}{|y-x|^k} = 0. \tag{1.4}$$

It is easily verified that the polynomials $P_x(\cdot)$ and $Q_x(\cdot)$ in (1.3) and (1.4) respectively are uniquely determined if x is a point of density of D and so are those in (1.1) and (1.2).

We note that Taylor-differentiability and approximate Taylor-differentiability of order k at a point are simply called in [9] differentiability and approximate differentiability of order k which might lead to confusion with differentiability and approximate differentiability in ordinary sense when $k \geq 2$. It is also to be noted that Taylor-differentiability of order k at a point is first introduced by de la Vallé-Poussin in [13] for functions of a real variable and that Taylor-differentiability is different from differentiability in ordinary sense when $k \geq 2$ has been observed by Denjoy [4] for functions of a real variable.

It is shown in [9] that the following three statements are equivalent:

(1) u admits approximately a $(k-1)$-Taylor polynomial at almost every point of D;
(2) u is approximately Taylor-differentiable of order k at almost every point of D;
(3) u has the Lusin property of order k.

The equivalence of (1) and (2) is the Rademacher phenomenon of general order in terms of approximate differentiability, generalizing Theorem 1.3. As a consequence of the equivalence of (1), (2), and (3), we also show in [9] the following generalization of Theorem 1.1:

Theorem 1.4. *A function u defined on a measurable set D is Taylor-differentiable of order k at almost all points of D if and only if it admits a (k-1)-Taylor polynomial at almost all points of D.*

It is worthwhile at this point to mention the following interesting variant of Theorem 4 in [9]:

Theorem 1.5. *Suppose that u is a measurable function defined on a neighborhood of a measurable set D in R^n and assume that both*

$$ap \limsup_{y \to x} \frac{|u(y)-u(x)|}{|y-x|} < +\infty$$

and

$$\limsup_{|y| \to 0} \frac{|u(x+y)+u(x-y)-2u(x)|}{|y|} < \infty$$

hold almost everywhere on D. Then u is differentiable almost everywhere on D.

This theorem could be considered as a Rademacher theorem of mixed type.

On the other hand, we may also consider the Rademacher phenomenon in terms of point-wise differentiability in p-mean as introduced by Calderón and Zygmund in [1]. For this purpose, relevant classes of functions are defined as follows. Let Ω be an open set in R^n. For a positive integer k,

$1 \leq p < \infty$, and $x \in \Omega$, let $T_k^p(x)$ be the class of all those functions $u \in L_b^p(\Omega)$ with the property that there exists a polynomial P_x of degree $< k$ such that

$$\sup_{0<\rho\leq 1} \rho^{-k}\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y) - P_x(y)|^p dy\}^{1/p} < +\infty;$$

while $t_k^p(x)$ will denote the class of those functions $u \in T_k^p(x)$ with the property that there exists a polynomial Q_x of degree $\leq k$ such that

$$\lim_{\rho\to 0} \rho^{-k}\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y) - Q_x(y)|^p dy\}^{1/p} = 0.$$

In the definitions above, $L_b^p(\Omega)$ is the space of all such measurable function u on Ω which is in $L^p(B)$ for all bounded measurable subset $B \subset \Omega$ and $\Omega(x,\rho) = B_\rho(x) \cap \Omega$ with $B_\rho(x)$ being the ball centered at x and with radius ρ. These definitions are slight variations of those in [1] where Ω is taken to be R^n and the supremum is taken over all $\rho > 0$.

We shall establish the following Rademacher type theorem:

Theorem 1.6. *Suppose that $S \subset \Omega$ is measurable. Then $u \in t_k^p(x)$ for almost all $x \in S$ if and only if $u \in T_k^p(x)$ for almost all $x \in S$.*

When $p > 1$, this theorem has been proved by Ziemer in [15]. The proof in [15] for the case $p > 1$ as well as our proof of the general case $p \geq 1$ depend essentially on the fact that if $u \in T_k^p(x)$ for almost all $x \in S$, then u has Lusin property of order k. This fact is established in [15] with quite delicate tools when $p > 1$. We shall demonstrate this fact in the next section through a localized version of a Lusin type theorem proved in [7]. Theorem 1.6 will then be proved in the last section.

2. A Lusin Type Theorem

Suggested by the classes of functions introduced in [1] by Calderón and Zygmund and those in [2] and [3] by Campanato, we introduce in [7] a class of functions for each $\gamma > 0$ on which a basic operation in [1] for the case $p = 1$ can be applied to define a maximal mean estimate of Taylor remainder for functions in the class, and for which a form of Lusin property similar to that in [8] is established. A slightly more general situation will be considered here.

Let Ω be an open set in R^n and a constant $A > 0$ be given. Ω is said to satisfy A-condition at $x \in R^n$ if $|\Omega(x,\rho)| \geq A\rho^n$ for all $0 < \rho \leq 1$, where $\Omega(x,\rho) = B_\rho(x) \cap \Omega$. Ω is said to satisfy A-condition on a set $S \subset R^n$ if it satisfies A-condition at every point $x \in S$. If Ω satisfies A-condition on a measurable set $S \subset \Omega$, then the pair $\{S, \Omega\}$ is called an A-pair. When Ω satisfies A-condition on itself, Ω is simply said to satisfy A-condition. When Ω is bounded, A-condition is introduced by Campanato in [3] for the case $S = \Omega$ and with $0 < \rho \leq \mathrm{diam}\Omega$. For convenience of some of our later statements, for a measurable set S in R^n, $M_p(S)$, $p \geq 0$, will be used to denote the class of all measurable functions u such that

$$\lim_{\lambda\to\infty} \lambda^p |\{x \in S : |u(x)| \geq \lambda\}| = 0.$$

For $\gamma \in R$, denote by $\bar\gamma$ the largest integer strictly less than γ and write $\gamma = \bar\gamma + \mu$, then $0 < \mu \leq 1$. Let $u \in L^1_b(\Omega)$ and $x \in \Omega$, if there is a polynomial $P_x(\cdot)$ with degree $\leq \bar\gamma$ satisfying

$$[u]_\gamma(x) := \sup_{0<\rho\leq 1} \rho^{-\gamma} \frac{1}{|\Omega(x,\rho)|} \int_{\Omega(x,\rho)} |u(y) - P_x(y)| dy < +\infty,$$

then P_x is uniquely determined (see preliminary remarks below) and hence $[u]_\gamma(x)$ is well-defined by u and x. If there exists no such a polynomial, let $[u]_\gamma(x) = \infty$. Now we are ready to define the classes of functions alluded. In the following, Ω is a fixed open subset of R^n and S is a measurable subset of Ω.

Definition 1. For $\gamma > 0$, let $\mathcal{L}^\gamma(S;\Omega)$ be the class of all those functions $u \in L^1_b(\Omega)$ such that

(i) for almost all $x \in S$,

$$[u]_\gamma(x) < +\infty;$$

(ii) if we set

$$\sigma_u(x) = [u]_\gamma(x) + \int_{\Omega(x,1)} |u(y)| dy,$$

then σ_u is in $M_0(S)$.

It is worthwhile to remark at this point that if S is of finite measure, then (ii) is a consequence of (i).

When Ω is clearly implied, $\mathcal{L}^\gamma(S;\Omega)$ will be usually abbreviated as $\mathcal{L}^\gamma(S)$.

Some preliminary remarks are now in order. In the following, we refer to [15] for notations involving multi-indices. First, at each point $x \in \Omega$ for which $[u]_\gamma(x)$ takes finite value, the polynomial $P_x(\cdot)$ is uniquely determined. Secondly, if we write

$$P_x(y) = \sum_{|\alpha| \le \bar{\gamma}} \frac{u_\alpha(x)}{\alpha!}(y-x)^\alpha,$$

then each u_α is a measurable function and so is $[u]_\gamma$. For these facts we refer to [7].

On p. 89 of [7] we have actually shown the following

Basic Proposition. *Let (S,Ω) be an A-pair, then there is $C > 0$ depending only on n, r and A such that for $u \in \mathcal{L}^r(S;\Omega)$ and x, y in S we have for $|\alpha| < r$*

(i) $|D^\alpha P_x(x)| \le C\sigma_u(x)$; *and*
(ii) $|D^\alpha P_y(y) - D^\alpha P_x(y)| \le C|x-y|^{r-|\alpha|} \max\{[u]_r(x), [u]_r(y)\}$.

Remark. Earlier hints for the Basic Proposition appear in [6] and [8]; actually the Basic Proposition is a result of efforts to make clear and simplify the main argument in [6].

We recall that a measurable function u defined on S is said to be in $L^p_w(S)$ if

$$|\{x \in S : |u(x)| \ge \lambda\}| \le \frac{C}{\lambda^p}$$

for some $C \ge 0$ and for all $\lambda > 0$. The smallest such a C will be denoted by $N_p(u)^p$. We note that if $u \in BV(R^n)$, then $u \in \mathcal{L}^1(R^n)$ with $N_1([u]_\gamma)$ being less than or equal to the total variation of u and that if $u \in W^k_p(R^n)$, then $u \in \mathcal{L}^k(R^n)$ with $\sigma_u \in L^p_w(R^n) \cap M_p(R^n)$. For these facts we refer to [8]. Now let

$$\mathcal{L}^\gamma_p(S;\Omega) = \{u \in \mathcal{L}^\gamma(S;\Omega) : \sigma_u \in L^p_w(S)\}$$

and

$$\mathcal{L}^\gamma_{p,0}(S;\Omega) = \{u \in \mathcal{L}^\gamma_p(S;\Omega) : \sigma_u \in M_p(S)\}.$$

$\mathcal{L}_p^\gamma(S;\Omega)$ and $\mathcal{L}_{p,0}^\gamma(S;\Omega)$ will be abbreviated respectively as $\mathcal{L}_p^\gamma(S)$ and $\mathcal{L}_{p,0}^\gamma(S)$ if Ω is clearly implied. Then $BV(\Omega) \subset \mathcal{L}_1^1(\Omega)$ and $W_p^k(\Omega) \subset \mathcal{L}_{p,0}^k(\Omega)$, if Ω is minimally smooth in the sense as defined in [11].

If f is a measurable function defined on a measurable set S, set

$$\mu_f(\lambda) = |\{x \in S : |f(x)| > \lambda\}|, \quad \lambda \geq 0.$$

The nonincreasing rearrangement f^* of f is defined as

$$f^*(t) = \sup\{\lambda : \mu_f(\lambda) > t\}.$$

We note that in terms of nonincreasing rearrangement $M_0(S)$ consists exactly of those functions f for which $f^*(t) < \infty$ for all $t > 0$. Also, it is not hard to see that for $f \in M_p(S)$ we have

$$\lim_{t\to 0} t[f^*(t)]^p = 0. \tag{2.1}$$

From the Basic Proposition, by an obvious modification of the proof for Theorem 1.2 in [7] we can establish the following theorem:

Theorem 2.1. *Let $\{S,\Omega\}$ be an A-pair. There exists a constant $C > 0$ depending only on n, A, and γ such that for $u \in \mathcal{L}^\gamma(S)$, and $t > 0$, there is a closed set $F_t \subset S$ and $u_t \in C^{\bar{\gamma},\mu}(R^n)$ such that*

1. $|S \backslash F_t| < 2t$,

2. $u_t = u$ *on* F_t, *and* $\|u_t\|_{C^{\bar{\gamma},\mu}(R^n)} \leq C\sigma_u^*(t)$.

The merit of this theorem is that it implies many forms of Lusin properties for functions depending on σ_u^*, for example, one may refer to [7] for situations $u \in \mathcal{L}_p^\gamma(\Omega)$ and $u \in \mathcal{L}_{p,0}^\gamma(\Omega)$. Our proof of Theorem 1.6 depends on the following consequence of Theorem 2.1 under the weakest possible conditions:

Theorem 2.2. *Suppose that $S \subset \Omega$ is measurable and $\gamma > 0$. Let $u \in L_b^1(\Omega)$ be such that for almost all $x \in S$,*

$$[u]_\gamma(x) < +\infty.$$

Then for almost all $x \in S$ we have

$$\operatorname{ap}\limsup_{y\to 0} \frac{|u(y) - P_x(y)|}{|y-x|^\gamma} < +\infty.$$

Proof. We may assume that S has positive measure. Let K be a compact subset of S with positive measure, $\{K, \Omega\}$ is an A-pair for some A, then, from the remark following Definition 1, $u \in \mathcal{L}^\gamma(K, \Omega)$. From Theorem 2.1, for any given $t > 0$, there is a closed set $F_t \subset K$ and $u_t \in C^{\bar{\gamma},\mu}(R^n)$ such that $|K\backslash F_t| < t$ and $u_t = u$ on F_t. Since $D^\alpha u_t(x) = u_\alpha(x)$ for almost all $x \in F_t$,

$$\limsup_{y\to 0} \frac{|u_t(y) - P_x(y)|}{|y-x|^\gamma} < +\infty,$$

for almost all $x \in F_t$. For such an x which is also a point of density of F_t we have

$$\operatorname{ap}\limsup_{y\to 0} \frac{|u(y) - P_x(y)|}{|y-x|^\gamma} < +\infty.$$

Since $t > 0$ is arbitrary, the previous inequality holds for almost all $x \in K$. As K is any compact subset of S with positive measure, this proves our theorem. □

3. Proof of Theorem 1.6

We precede the proof of Theorem 1.6 by stating a theorem of Calderón and Zygmund in [1]:

Theorem 3.1. *Let $u \in L^p(R^n)$, $1 \le p < \infty$, and let $a > 0$. If*

$$\left\{\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} |u(y)|^p dy\right\}^{1/p} = O(\rho^a)$$

as $\rho \to 0$ for almost all x in a measurable set $S \subset R^n$, then

$$\left\{\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} |u(y)|^p dy\right\}^{1/p} = o(\rho^a)$$

as $\rho \to 0$.

Proof of Theorem 1.6. We only have to prove that $u \in t_k^p(x)$ for almost all $x \in S$ if $u \in T_k^p(x)$ for almost all $x \in S$. From Hölder inequality we have

$$[u]_k(x) \le \sup_{0<\rho\le 1} \rho^{-k}\left\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y)-P_x(y)|^p dy\right\}^{1/p},$$

hence, if $u \in T_k^p(x)$ for almost all $x \in S$, then by Theorem 2.2 with $\gamma = k$, u admits approximately a $(k-1)$-Taylor polynomial at almost every point of S. As we have stated in the first section, u has the Lusin property of order k on S. Therefore for any $\epsilon > 0$ there is a C^k-function g on R^n such that if we let $D = \{x \in S : u(x) = g(x), D^\alpha P_x(x) = D^\alpha g(x), |\alpha| \le k-1\}$, then $|S \backslash D| < \epsilon$. Then for $x \in D$, we have

$$\begin{aligned}&\left\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y)-g(y)|^p dy\right\}^{1/p}\\ &\le \left\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y)-P_x(y)|^p dy\right\}^{1/p} + \left\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |g(y)-P_x(y)|^p dy\right\}^{1/p}\\ &= \mathrm{O}(\rho^k),\end{aligned}$$

because P_x is also the Taylor polynomial of degree $k-1$ of g at x. Hence by Theorem 3.1, it follows that

$$\left\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y)-g(y)|^p dy\right\}^{1/p} = \mathrm{o}(\rho^k) \,\mathrm{as}\, \rho \to 0$$

for almost all $x \in D$. For such an x, if we let Q_x be the Taylor polynomial of degree k of g at x, then

$$\begin{aligned}&\left\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y)-Q_x(y)|^p dy\right\}^{1/p}\\ &\le \left\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |u(y)-g(y)|^p dy\right\}^{1/p} + \left\{\frac{1}{|\Omega(x,\rho)|}\int_{\Omega(x,\rho)} |g(y)-Q_x(y)|^p dy\right\}^{1/p}\\ &= \mathrm{o}(\rho^k) \quad \text{as} \quad \rho \to 0.\end{aligned}$$

Thus $u \in t_k^p(x)$, i.e. $u \in t_k^p(x)$ for almost all $x \in D$. As $|S \backslash D| < \epsilon$ and ϵ is arbitrary, this proves the theorem.

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