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ANALYSIS OF THE BOUNDARY LAYER EQUATION IN THE KINETIC THEORY OF GASES

BY

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Dedicated to Professor Yoshio Sone, with admiration

Abstract

The present paper completes an earlier result by S. Ukai, T. Yang, and S.-H. Yu [Commun. Math. Phys. **236** (2003), 373−393] on weakly nonlinear half-space problems for the steady Boltzmann equation with hard-sphere potential.

1. Statement of the Problem & Main Result

Boundary layers are an important class of flows in the kinetic theory of gases. One of the most significant problems in this context is the study of half-space problems for the Boltzmann equation matching the state of the gas on a surface immersed in the flow to some thermodynamic equilibrium far from that surface. A classical example of this situation is the case of a phase transition, where the rarefied gas is the vapor above its condensed phase. From the mathematical viewpoint, these boundary layers explain how boundary conditions for the hydrodynamic equations can be obtained from the boundary data at the mesoscopic level of description corresponding to the kinetic theory of gases.

Henceforth, we consider the steady state of a rarefied gas in a half-space

$$\mathbf{R}^3_+ := \{(x,y,z) \in \mathbf{R}^3 \mid x \geq 0\}.$$

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In kinetic theory, the state of the gas is described by its distribution function F, and we shall assume this state to have slab-symmetry, meaning that F depends on one space variable $x \geq 0$ and three velocity variables $v = (v_1, v_2, v_3) \in \mathbf{R}^3$. Here, $F(x,v)$ is the density of gas molecules with velocity $v \in \mathbf{R}^3$ that are located in the plane

$$\{(x,y,z) \mid y,z \in \mathbf{R}\}.$$

Assuming that the gas is monatomic and viewing the gas molecules as hard spheres, the distribution function F satisfies the Boltzmann equation

$$v_1 \partial_x F = \mathcal{B}(F,F), \quad x > 0, \quad v \in \mathbf{R}^3. \tag{1.1}$$

Here, the notation $\mathcal{B}(F,F)$ designates the Boltzmann collision integral, which acts only on the v-variable in F. In other words,

$$\mathcal{B}(F,F)(x,v) = \mathcal{B}(F(x,\cdot), F(x,\cdot))(v)$$

where, for each $\phi \equiv \phi(v)$ that is continuous and rapidly decaying at infinity, one has

$$\mathcal{B}(\phi,\phi)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (\phi'\phi_1' - \phi\phi_1)|(v-v_1)\cdot\omega| dv_1 d\omega$$

with the notation

$$\phi = \phi(v), \quad \phi_1 = \phi(v_1), \quad \phi' = \phi(v'), \quad \phi_1' = \phi(v_1'). \tag{1.2}$$

Here

$$\begin{aligned} v' = v'(v,v_1,\omega) &= v - (v-v_1)\cdot\omega\omega, \\ v_1' = v_1'(v,v_1,\omega) &= v_1 + (v-v_1)\cdot\omega\omega. \end{aligned} \tag{1.3}$$

In order for the Boltzmann equation (1.1) to define a unique solution F, one must add boundary conditions at $x = 0$ and/or some limiting condition at infinity.

Typically, the density F at infinity must be a thermodynamic equilibrium, i.e. a Maxwellian state defined by its temperature $\theta_\infty > 0$, its bulk velocity $u_\infty \in \mathbf{R}^3$ and its pressure p_∞. In the sequel, this Maxwellian state is denoted by

$$M_{(\rho_\infty, u_\infty, \theta_\infty)}(v) = \frac{\rho_\infty}{(2\pi\theta_\infty)^{3/2}} e^{-\frac{|v-u_\infty|^2}{2\theta_\infty}} \tag{1.4}$$

where $\rho_\infty = p_\infty/\theta_\infty$.

As for the boundary condition at $x=0$, we shall restrict our attention to the case where the density of molecules entering the half-space is given:

$$F(0,v) = F_b(v), \quad v \in \mathbf{R}^3, \quad v_1 > 0, \tag{1.5}$$

where F_b is given.

Of course the boundary condition (1.5) and the condition at infinity

$$F(x,v) \to M_{(\rho_\infty,u_\infty,\theta_\infty)}(v) \text{ as } x \to +\infty \tag{1.6}$$

cannot in general be prescribed independently. In other words, one expects that there should exist (finitely many) relations between the boundary data F_b and the equilibrium state at infinity, i.e. the parameters ρ_∞, u_∞ and θ_∞.

Another way of stating the same problem is as follows:

Given $\rho_\infty > 0$, u_∞ and $\theta_\infty > 0$, to find all boundary data F_b such that the steady Boltzmann equation (1.1) *with boundary condition* (1.5) *has a unique solution F satisfying the condition* (1.6) *at infinity.*

Henceforth, we denote $\mathbf{S}[M_{(\rho_\infty,u_\infty,\theta_\infty)}]$ this set of boundary data — which we can think of as the stable manifold of $M_{(\rho_\infty,u_\infty,\theta_\infty)}$, by analogy with the case where (1.1) is an evolution system with time variable x.

The case where F_b is a Maxwellian state with zero bulk velocity, i.e.

$$F_b = M_{(\rho_b,0,\theta_b)} \tag{1.7}$$

is of particular interest in the context of a phase transition, with the vapor above (i.e. for $x>0$) its condensed phase (located at $x<0$.) This problem has been investigated by Y. Sone, then with K. Aoki and their group in Kyoto, in a series of important papers. Their work gives a detailed description of

$$\{\rho_b > 0, \quad \theta_b > 0 \mid M_{(\rho_b,0,\theta_b)} \in \mathbf{S}[M_{(\rho_\infty,u_\infty,\theta_\infty)}]\}$$

based on numerical and asymptotic arguments. Specifically, their analysis shows that the set of ρ_b/ρ_∞, θ_b/θ_∞ and $\mathrm{Ma}_\infty := u_{1,\infty}/\sqrt{\frac{5}{3}\theta_\infty}$ such that $M_{(\rho_b,0,\theta_b)} \in \mathbf{S}[M_{(\rho_\infty,u_\infty,\theta_\infty)}]$ has a codimension in $\mathbf{R}^3$ that depends upon the position of the transverse Mach number at infinity Ma_∞ with respect to the singular values -1, 0 and 1. For a detailed presentation of this important series of results, see sections 3.5.2, 4.5.2, 4.10.2 and chapter 7 in [18], chapters

6 and 7 in [19] and the references therein to the original articles. For a more concise survey of these works, see [2].

Finding a complete proof of these results is truly one of the most fascinating — and challenging — mathematical problems on the Boltzmann equation, as the problem is strongly nonlinear except in the particular case where $\mathrm{Ma}_\infty \to 0$.

More recently, S. Ukai, T. Yang and S.-H. Yu [21] have studied the local structure of $\mathbf{S}[M_{(\rho_\infty,u_\infty,\theta_\infty)}]$ near $M_{(\rho_\infty,u_\infty,\theta_\infty)}$. This set is found to be a C^1 manifold of finite codimension near the Maxwellian at infinity. This finite codimension is the dimension of maximal positive subspaces of $\mathrm{Im}DM_{(\rho_\infty,u_\infty,\theta_\infty)}$ — i.e. on the linear span of collision invariants 1, v_1, v_2, v_3, $|v|^2$— for the Hessian of the entropy flux

$$F \mapsto \int_{\mathbf{R}^3} v_1 F \ln F dv \tag{1.8}$$

at the point $M_{(\rho_\infty,u_\infty,\theta_\infty)}$.

The analogous statement for the linearization of the Boltzmann equation (1.1) about the Maxwellian state at infinity $M_{(\rho_\infty,u_\infty,\theta_\infty)}$ has been conjectured by C. Cercignani [8], and subsequently proved by the author in collaboration with F. Coron and C. Sulem [10].

The method used in [21] uses the ideas in [10] — especially the energy method pioneered by C. Bardos, R. Caflisch, and B. Nicolaenko in [1] — in a way that is both elegant and of striking efficiency.

Unfortunately, the result in [21] does not solve the phase transition problem studied by Y. Sone, K. Aoki and their group. Indeed, the main theorem in [21] bears on $\mathbf{S}[M_{(\rho_\infty,u_\infty,\theta_\infty)}]$ in the neighborhood of $M_{(\rho_\infty,u_\infty,\theta_\infty)}$, whereas the phase transition problem corresponds with the stable manifold $\mathbf{S}[M_{(\rho_\infty,u_\infty,\theta_\infty)}]$ near $M_{(\rho_b,0,\theta_b)}$— i.e. far from $M_{(\rho_\infty,u_\infty,\theta_\infty)}$ in general. In fact, the dimension of maximal positive subspaces of the Hessian of (1.8) at $M_{(\rho_\infty,u_\infty,\theta_\infty)}$ in the linear span of collision invariants does not contain enough information to arrive at a qualitative picture of the phase transition problem. For instance, the existence of a supersonic condensing solution implies that the pressure ratio $\rho_\infty\theta_\infty/\rho_b\theta_b$ must exceed some threshold depending on Ma_∞. Physically, this threshold corresponds with a subsonic condensing solution confined to the boundary by a plane shock wave, a pattern that cannot be completely explained by studying (1.8) as $x \to \infty$. The presence of this threshold, which has been computed numerically by Y. Sone and his collaborators, is confirmed rigorously by a priori inequalities on the phase transition problem in [20] and [3].

However, the theorem of S. Ukai, T. Yang and S.-H. Yu [21] could be of interest for the phase transition problem when the Maxwellian states at infinity $M_{(\rho_\infty,u_\infty,\theta_\infty)}$ and on the boundary $M_{(\rho_b,0,\theta_b)}$ are close — which implies in particular that $\mathrm{Ma}_\infty \to 0$. Even in this case, there remains a difficulty: the analysis in the paper by S. Ukai, T. Yang and S.-H. Yu excludes the cases where the transverse Mach number at infinity $\mathrm{Ma}_\infty \in \{-1,0,1\}$ — see Remark 1.2 on p. 376 in [21].

In the present paper, we extend the analysis of S. Ukai, T. Yang and S.-H. Yu to the case $\mathrm{Ma}_\infty = 0$. Specifically, we prove the following

Theorem 1.1. *For each $\rho_\infty, \theta_\infty > 0$ and $u_{j,\infty}$ with $j = 2,3$, denote $u_\infty = (0, u_{2,\infty}, u_{3,\infty})$. Then, there exists $0 < \epsilon \ll 1$ so that*

$$\mathbf{S}[M_{(\rho_\infty,u_\infty,\theta_\infty)}] \cap \left\{ F_b \middle| \sup_v (1+|v|)^3 \left| \frac{F_b - M_{(\rho_\infty,u_\infty,\theta_\infty)}}{M^{1/2}_{(\rho_\infty,u_\infty,\theta_\infty)}} \right| \le \epsilon \right\}$$

is a set of codimension 4.

Without loss of generality, we shall restrict our attention to the case

$$\rho_\infty = \theta_\infty = 1, \quad \text{and} \quad u_{2,\infty} = u_{3,\infty} = 0. \tag{1.9}$$

Indeed, one can always reduce the general case to that one, by setting the Boltzmann equation (1.1) and its boundary condition (1.5) in the new velocity variables

$$\xi_1 = \frac{1}{\sqrt{\theta_\infty}} v_1, \quad \xi_2 = \frac{1}{\sqrt{\theta_\infty}} (v_2 - u_{2,\infty}), \quad \xi_3 = \frac{1}{\sqrt{\theta_\infty}} (v_3 - u_{3,\infty})$$

and by scaling the density and the space variable as

$$F \to \rho_\infty F, \quad x \to \rho_\infty x.$$

Our analysis follows very closely that in [21] — especially the idea of using a penalized variant of the linearization of (1.1) about the Maxwellian state at infinity, which we believe is by far the best way of handling half-space problems in kinetic theory. There are minor differences with [21] here and there — in particular, the bootstrap argument starting from the energy estimate and leading to weighted L^∞ bounds is based on our earlier analysis using the integral formulation of the linearized half-space problem in [12], together with the full force of Caflisch's estimates for the linearized collision operator in [5], which is a slight improvement of the Grad's classical estimates in [13]

However, the main difficulty in the proof of Theorem 1.1 — and the core of the present contribution — is the fact that the Hessian of the entropy flux (1.8) becomes degenerate on the linear span of collision invariants. Because of this degeneracy, the structure of the Ukai-Yang-Yu penalization has to be modified. This is done with the help of a pair of projections, $\mathbf{P}$ and $\mathbf{p}$ introduced in section 2.3, based on the extra invariants of the linearized version of (1.1) presented in section 2.2. These extra-invariants are analogues for the Boltzmann equation of the K-integral used in Radiative Transfer: see chapter 1, §10 in [9]; Chandrasekhar attributes the K-integral to A.S. Eddington. Perhaps the first occurence of these extra-invariants in the case of the half-space problem for the linearized Boltzmann equation is in Cercignani's simplification of the Bardos-Caflisch-Nicolaenko energy method presented in [8]. The key estimate in the present paper is Lemma 2.6 below, establishing the coercivity of a suitable variant of the Ukai-Yang-Yu penalization of the linearized Boltzmann collision integral.

Going back to the phase transition problem studied by Y. Sone and his school, we recall that

$$\mathbf{S}[M_{(\rho_\infty,0,\theta_\infty)}]\cap\{M_{(\rho_b,0,\theta_b)}\mid\rho_b,\theta_b>0\}=\{M_{(\rho_\infty,u_\infty,\theta_\infty)}\}$$

whenever $u_\infty=(0,u_{2,\infty},u_{3,\infty})$ — see Theorem 5.1 in [2]. This result is established by an estimate bearing on the entropy production and entropy flux for equation (1.1)[1].

The work of Y. Sone has had a considerable influence on the modern understanding of rarefied gas dynamics. The theory of boundary layers for kinetic models is just one example of a subject where Y. Sone's far reaching contributions are at the origin of many deep and fascinating problems in mathematical analysis. I am most grateful to Y. Sone's generosity in sharing his ideas — and patience in teaching me, over the last twenty years, a significant fraction of the little I know about the Boltzmann equation. With great pleasure, I am offering him this small contribution to a question that we have discussed so many times both in Kyoto and in Paris.

[1]This estimate is the nonlinear analogue of a clever argument for the uniqueness of the solution of the linearization of (1.1) about a Maxwellian state, due to C. Bardos, R. Caflisch and B. Nicolaenko [1].

2. Formulation of the Penalized Problem

2.1. Linearization

The solution of the boundary layer equation is sought in the form

$$F(x,v) = M_{(1,0,1)}(v)(1+f(x,v)). \tag{2.1}$$

From now on, the centered, reduced Maxwellian (Gaussian) state $M_{(1,0,1)}$ is abbreviated as M. The linearized collision integral is defined by

$$\begin{aligned}(\mathcal{L}\phi)(v) &= -2M^{-1}\mathcal{B}(M, M\phi)(v)\\ &= \iint (\phi+\phi_1-\phi'-\phi_1')|(v-v_1)\cdot\omega| M dv_1 d\omega;\end{aligned} \tag{2.2}$$

likewise, we introduce the quadratic operator $\mathcal{Q}$ — which is the Boltzmann collision operator intertwined with the multiplication by M:

$$\begin{aligned}(\mathcal{Q}(\phi,\phi))(v) &= M^{-1}\mathcal{B}(M\phi, M\phi)(v)\\ &= \iint (\phi'\phi_1'-\phi\phi_1)|(v-v_1)\cdot\omega| M dv_1 d\omega.\end{aligned} \tag{2.3}$$

With these notations, the original problem is recast in the form

$$\begin{aligned} v_1\partial_x f + \mathcal{L}f = \mathcal{Q}(f,f), &\qquad x>0, \quad v\in\mathbf{R}^3,\\ f(x,v)\to 0, &\qquad \text{as } x\to+\infty.\end{aligned} \tag{2.4}$$

The Hilbert space $L^2(\mathbf{R}^3, Mdv)$ will be used systematically in the sequel; in particular, we shall use the notation

$$\langle\phi\rangle = \int_{\mathbf{R}^3}\phi(v)M(v)dv.$$

The following properties of $\mathcal{L}$ and $\mathcal{Q}$ are summarized below:

Proposition 2.1.

- *The linearized collision integral $\mathcal{L}$ defines an unbounded, self-adjoint Fredholm operator on $L^2(Mdv)$, with domain*

$$\mathrm{D}(\mathcal{L}) = \{\phi\in L^2(Mdv) \,|\, |v|\phi\in L^2(Mdv)\}.$$

  Its null-space is

$$\operatorname{Ker}\mathcal{L} = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}.$$

- *The quadratic operator $\mathcal{Q}$ is continuous on* $\mathrm{D}(\mathcal{L})$ *(endowed with the norm* $\phi \mapsto \langle (1+|v|)^2 |\phi|^2 \rangle^{1/2}$*) with values in* $L^2(Mdv)$*. It satisfies the relations*

$$\langle \chi \mathcal{Q}(\phi,\phi) \rangle = 0, \quad \chi \in \operatorname{Ker}\mathcal{L}, \quad \phi \in \mathrm{D}(\mathcal{L}).$$

The problem (2.4) is solved by a fixed-point argument. Thus, the main task in the present paper is the analysis of the linear problem

$$\begin{aligned} v_1\partial_x f + \mathcal{L} f = S, &\qquad x>0, \quad v \in \mathbf{R}^3, \\ f(x,v) \to 0, &\qquad \text{as } x \to +\infty, \end{aligned} \tag{2.5}$$

where $S \equiv S(x,v)$ is a given source term that satisfies the orthogonality relations

$$\left\langle \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} S(x,\cdot) \right\rangle = 0, \quad x > 0. \tag{2.6}$$

2.2. Invariants of the linear problem

Multiplying each side of the equality in (2.5) successively by 1, v_1, v_2, v_3, $|v|^2$, and averaging in v leads to the invariance of the fluxes of mass, momentum and energy:

$$\frac{d}{dx}\left\langle v_1 \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} f(x,\cdot) \right\rangle = 0, \quad x > 0.$$

Since $f(x,\cdot) \to 0$ in $L^2(Mdv)$ as $x \to +\infty$, the invariant quantities above must satisfy

$$\left\langle v_1 \begin{pmatrix} 1 \\ v_1 \\ v_2 \\ v_3 \\ |v|^2 \end{pmatrix} f(x,\cdot) \right\rangle = 0, \quad x > 0. \tag{2.7}$$

However, the invariants in (2.7) are not enough to analyze the problem (2.5).

Before explaining why, we recall the following observation:

Lemma 2.2. (Coron-Golse-Sulem [10]) *The five functions*

$$\begin{aligned} \chi_\pm(v) &= |v|^2 \pm \sqrt{15} v_1, \\ \chi_0(v) &= |v|^2 - 5, \\ \chi_j(v) &= v_j, \quad j = 2, 3 \end{aligned} \tag{2.8}$$

span $\operatorname{Ker} \mathcal{L}$. *They are orthogonal for both the inner product of* $L^2(M dv)$ *and the bilinear form* $(\phi, \psi) \mapsto \langle v_1 \phi \psi \rangle$ *defined on* $\mathrm{D}(\mathcal{L})$. *Finally, one has*

$$\langle \chi_\pm^2 \rangle = 30, \quad \langle \chi_0^2 \rangle = 10, \quad \langle \chi_j^2 \rangle = 1, \quad j = 2, 3,$$

while

$$\langle v_1 \chi_\pm^2 \rangle = \pm 30 \sqrt{\tfrac{5}{3}}, \quad \langle v_1 \chi_0^2 \rangle = \langle v_1 \chi_j^2 \rangle = 0, \quad j = 2, 3.$$

In other words, the quadratic form $\phi \mapsto \langle v_1 \phi^2 \rangle$ restricted to $\operatorname{Ker} \mathcal{L}$ has signature $(1, 1)$ and a 3-dimensional radical spanned by $\{\chi_0, \chi_2, \chi_3\}$.

The constant $c = \sqrt{\frac{5}{3}}$ above is the (dimensionless) speed of sound that corresponds to the centered reduced Maxwellian state M in space dimension 3.

We also recall that each entry of the tensor field $v^{\otimes 2} - \frac{1}{3}|v|^2$ and of the vector field $v(|v|^2 - 5)$ belongs to $(\operatorname{Ker} \mathcal{L})^\perp$; since $\mathcal{L}$ is a Fredholm operator, there is a unique tensor field $A \equiv A(v)$ and a unique vector field $B \equiv B(v)$ such that A_{ij} and $B_j \in \mathrm{D}(\mathcal{L})$ for $i, j = 1, 2, 3$, and

$$\begin{aligned} &\mathcal{L}A = v^{\otimes 2} - \tfrac{1}{3}|v|^2, && A_{ij} \in (\operatorname{Ker} \mathcal{L})^\perp, \quad i, j = 1, 2, 3, \\ &\mathcal{L}B = v(|v|^2 - 5), && B_j \in (\operatorname{Ker} \mathcal{L})^\perp, \quad \quad j = 1, 2, 3. \end{aligned} \tag{2.9}$$

We recall (see [11]) that there exist two radial functions $a \equiv a(|v|)$ and $b \equiv b(|v|)$ such that

$$A(v) = a(|v|)(v^{\otimes 2} - \tfrac{1}{3}|v|^2), \quad B(v) = b(|v|)(|v|^2 - 5)v. \tag{2.10}$$

The components B_1, A_{12} and A_{13} are of particular interest for the analysis of (2.5), since they provide additional invariants of the problem (2.5). These invariants are analogues of Chandrasekhar's K-invariant for the half-space problem in radiative transfer (see [9], chapter I, §10) in the context of the kinetic theory of gases; their role in simplifying the analysis of Bardos-Caflisch-Nicolaenko [1] was advocated in an elegant argument by C. Cercignani [8] on linear half-space problems in the kinetic theory of gases.

Indeed, multiplying each side of the equality in (2.5) by B_1, A_{12} and A_{13}, and averaging in v, one arrives at

$$\begin{aligned}
\frac{d}{dx}\left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} f \right\rangle &= \left\langle \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} S \right\rangle - \left\langle \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \mathcal{L} f \right\rangle \\
&= \left\langle \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} S \right\rangle - \left\langle \begin{pmatrix} \mathcal{L}B_1 \\ \mathcal{L}A_{12} \\ \mathcal{L}A_{13} \end{pmatrix} f \right\rangle \\
&= \left\langle \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} S \right\rangle - \left\langle v_1 \begin{pmatrix} \chi_0 \\ \chi_2 \\ \chi_3 \end{pmatrix} f \right\rangle \\
&= \left\langle \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} S \right\rangle.
\end{aligned}$$

Since $f(x,\cdot) \to 0$ in $L^2(Mdv)$ as $x \to +\infty$, the relation above leads to

$$\left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} f(x,\cdot) \right\rangle = -\int_x^{+\infty} \left\langle \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} S(z,\cdot) \right\rangle dz, \quad x \geq 0. \tag{2.11}$$

2.3. Two projections

Our aim in this section is to reduce the problem (2.5) to the particular case where

$$\left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} f(x,\cdot) \right\rangle = \left\langle \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} S(x,\cdot) \right\rangle = 0, \quad x > 0. \tag{2.12}$$

The most natural way of achieving this is through the introduction of the following pair of projections:

$$\mathbf{P}\phi = \frac{\langle B_1 \phi \rangle}{\langle B_1 \mathcal{L} B_1 \rangle} \mathcal{L} B_1 + \frac{\langle A_{12} \phi \rangle}{\langle A_{12} \mathcal{L} A_{12} \rangle} \mathcal{L} A_{12} + \frac{\langle A_{13} \phi \rangle}{\langle A_{13} \mathcal{L} A_{12} \rangle} \mathcal{L} A_{13} \tag{2.13}$$

and

$$\mathbf{p}\phi = \frac{\langle v_1 B_1 \phi\rangle}{\langle B_1 \mathcal{L} B_1\rangle}\chi_0 + \frac{\langle v_1 A_{12}\phi\rangle}{\langle A_{12}\mathcal{L}A_{12}\rangle}\chi_2 + \frac{\langle v_1 A_{13}\phi\rangle}{\langle A_{13}\mathcal{L}A_{12}\rangle}\chi_3. \tag{2.14}$$

The families $\{\chi_0, \chi_2, \chi_3\}$ and $\{\mathcal{L}B_1, \mathcal{L}A_{12}, \mathcal{L}A_{13}\}$ are both $L^2(Mdv)$-orthogonal, so that

$$\mathbf{P}^2 = \mathbf{P}, \quad \mathbf{p}^2 = \mathbf{p},$$

meaning that $\mathbf{P}$ and $\mathbf{p}$ are indeed projections — notice that neither $\mathbf{P}$ nor $\mathbf{p}$ are self-adjoint (i.e. orthogonal) projections. The elementary properties of the projections $\mathbf{P}$ and $\mathbf{p}$ are summarized in the next lemma.

Lemma 2.3. *One has*

$$\begin{aligned}\operatorname{Im}\mathbf{P} &= \operatorname{span}\{v_1\chi_0, v_1\chi_2, v_1\chi_3\} \subset (\operatorname{Ker}\mathcal{L})^\perp,\\ \operatorname{Im}\mathbf{p} &= \operatorname{span}\{\chi_0, \chi_2, \chi_3\} \subset \operatorname{Ker}\mathcal{L}.\end{aligned}$$

The projections $\mathbf{P}$ *and* $\mathbf{p}$ *satisfy the relation*

$$\mathbf{P}(v_1\phi) = v_1\mathbf{p}(\phi); \qquad \mathbf{P}(\mathcal{L}\phi) = 0 \quad \text{if } v_1 f \perp \operatorname{Ker}\mathcal{L}.$$

Finally $\langle \mathbf{p}(\phi)\phi\rangle \geq 0$, *and the map* $\phi \mapsto \langle \mathbf{p}(\phi)\phi\rangle^{1/2}$ *defines a norm on* $\operatorname{span}\{\chi_0, \chi_2, \chi_3\}$.

Proof. Because of (2.10),

$$B_1 \perp \mathcal{L}A_{12}, \quad B_1 \perp \mathcal{L}A_{13}, \quad A_{12} \perp \mathcal{L}A_{13}.$$

Hence, if $\phi \in \operatorname{span}\{\chi_0, \chi_2, \chi_3\}$, one has

$$\begin{aligned}\langle \mathbf{p}(\phi)\phi\rangle &= \frac{\langle v_1 B_1\phi\rangle}{\langle B_1\mathcal{L}B_1\rangle}\langle\phi\chi_0\rangle + \frac{\langle v_1 A_{12}\phi\rangle}{\langle A_{12}\mathcal{L}A_{12}\rangle}\langle\phi\chi_2\rangle + \frac{\langle v_1 A_{13}\phi\rangle}{\langle A_{13}\mathcal{L}A_{13}\rangle}\langle\phi\chi_3\rangle\\ &= \frac{\langle\phi\chi_0\rangle^2}{\langle\chi_0^2\rangle} + \frac{\langle\phi\chi_2\rangle^2}{\langle\chi_2^2\rangle} + \frac{\langle\phi\chi_3\rangle^2}{\langle\chi_3^2\rangle} = \tfrac{1}{10}\langle\phi\chi_0\rangle^2 + \langle\phi\chi_2\rangle^2 + \langle\phi\chi_3\rangle^2 \quad \square\end{aligned}$$

Split f as $f = \mathbf{p}f + (I-\mathbf{p})f$; likewise, split S as $S = \mathbf{P}S + (I-\mathbf{P})S$. By applying successively $\mathbf{P}$ and $\mathbf{p}$ to each side of the equality in (2.5), on account of the various properties of $\mathbf{P}$ and $\mathbf{p}$ recalled in Lemma 2.3, one arrives at the relations

$$\begin{aligned} v_1\partial_x(I-\mathbf{p})f + \mathcal{L}(I-\mathbf{p})f &= (I-\mathbf{P})S, \qquad && x > 0, \quad v \in \mathbf{R}^3,\\ (I-\mathbf{p})f(x,\cdot) &\to 0, && \text{as } x \to +\infty;\end{aligned} \tag{2.15}$$

and

$$\begin{aligned} v_1\partial_x(\mathbf{p}f) &= \mathbf{P}S, && x>0, \quad v\in\mathbf{R}^3, \\ \mathbf{p}f(x,\cdot) &\to 0, && \text{as } x\to+\infty. \end{aligned} \tag{2.16}$$

Observe that

$$(I-\mathbf{P})S \text{ and } v_1(I-\mathbf{p})f \perp \operatorname{Ker}\mathcal{L}\oplus\operatorname{span}\{B_1,A_{12},A_{13}\} \tag{2.17}$$

which is equivalent to the orthogonality relations (2.12) leading to the introduction of the projections $\mathbf{P}$ and $\mathbf{p}$. On the other hand, the problem (2.16) is equivalent to the relations before (2.11), which means that the solution to this problem is given by (2.11).

2.4. The penalized problem

We first recall a fundamental property of the linearized collision integral. In the sequel, we designate by Π the orthogonal projection of $L^2(Mdv)$ on $\operatorname{Ker}\mathcal{L}$. Hilbert [14] proved that $\mathcal{L}$ can be split as the multiplication by a positive function $\nu\equiv\nu(|v|)$ minus a compact operator K on $L^2(Mdv)$:

$$(\mathcal{L}\phi)(v)=\nu(|v|)\phi(v)-(K\phi)(v), \quad \phi\in\mathrm{D}(\mathcal{L}). \tag{2.18}$$

The function ν can be computed explicitly in terms of the Erf function; for some constants $0<\nu_-<\nu_+$,

$$\nu_-(1+|v|)\le\nu(|v|)\le\nu_+(1+|v|), \quad v\in\mathbf{R}^3. \tag{2.19}$$

Lemma 2.4. (Bardos-Caflisch-Nicolaenko [1]) *There exists a constant* $0<\lambda<1$ *such that*

$$\langle\phi\mathcal{L}\phi\rangle\ge\lambda\langle\nu|(I-\Pi)\phi|^2\rangle, \qquad \text{for all } \phi\in\mathrm{D}(\mathcal{L}). \tag{2.20}$$

Were it not for the weight ν on the right-hand side of the above inequality, this would be equivalent to the fact that $\mathcal{L}$, being an unbounded, self-adjoint, nonnegative Fredholm operator on $L^2(Mdv)$, has a spectral gap.

In order to simplify the analysis of (2.15), we shall replace $\mathcal{L}$ with an operator that is coercive on the *whole* domain $\mathrm{D}(\mathcal{L})$, as follows. Proceeding as in [1], and especially [21], we consider the penalized operator

$$\mathcal{L}^p\phi=\mathcal{L}\phi+\alpha\Pi_+(v_1\phi)+\beta\mathbf{p}(\phi)-\gamma v_1\phi, \tag{2.21}$$

where Π_+ is the orthogonal projection on $\mathbf{R}\chi_+$ and α, β and γ are positive constants to be adjusted later.

The reason for replacing the operator $\mathcal{L}$ with $\mathcal{L}^p$ is explained by the next two lemmas.

Lemma 2.5. *Assume that f is such that $e^{\gamma x} f \in L^\infty(\mathbf{R}_+; L^2(M dv))$ for some $\gamma > 0$ and satisfies* (2.5). *Set $g = e^{\gamma x}(I - \mathbf{p})f$, and let $\mathcal{L}^p$ be defined as in* (2.21) *with the same parameter γ as in the definition of g, and with arbitrary positive constants α and β. Then g satisfies*

$$\begin{aligned} v_1 \partial_x g(x,v) + \mathcal{L}^p g(x,v) &= e^{\gamma x}(I - \mathbf{P})S(x,v), \quad x > 0, \\ g(x,v) &\in L^\infty(\mathbf{R}_+; L^2(M dv)). \end{aligned} \tag{2.22}$$

Proof. Observe first that f satisfies (2.7). Hence

$$\Pi_+(v_1 g) = e^{\gamma x}\Pi_+(v_1 f) = 0.$$

On the other hand, $\mathbf{p}(g) = e^{\gamma x}\mathbf{p}(I - \mathbf{p})f = 0$ since $\mathbf{p}^2 = \mathbf{p}$ — see Lemma 2.3. Therefore

$$\begin{aligned} v_1 \partial_x g + \mathcal{L}^p g &= v_1 \partial_x g - \gamma v_1 g + \mathcal{L} g + \alpha \Pi_+(v_1 g) + \beta \mathbf{p}(g) \\ &= e^{\gamma x}\Big(v_1 \partial_x (e^{-\gamma x} g) + \mathcal{L}(e^{-\gamma x} g)\Big) = e^{\gamma x}(I - \mathbf{P})S, \end{aligned}$$

because of (2.15). □

The second reason for replacing $\mathcal{L}$ with $\mathcal{L}^p$ is the following observation.

Lemma 2.6. *For appropriately chosen positive constants α, β and γ, the operator $\mathcal{L}^p$ is coercive on* $\mathrm{D}(\mathcal{L})$, *i.e. there exists $\lambda' = \lambda'(\alpha, \beta, \gamma, \nu_\pm) > 0$ such that*

$$\langle \phi \mathcal{L}^p \phi \rangle \geq \lambda' \langle \nu |\phi|^2 \rangle, \quad \phi \in \mathrm{D}(\mathcal{L}). \tag{2.23}$$

The conditions that the parameters α, β and γ must satisfy for (2.23) *to hold are collected in* (2.29), (2.30), *and* (2.31).

Proof. Set $w = (I - \Pi)\phi$ and $q = \Pi\phi$. We further decompose q into $q = q_+ + q_- + q_0$, following the $L^2(M dv)$-orthogonal decomposition

$$\operatorname{Ker} \mathcal{L} = \mathbf{R}\chi_+ \oplus \mathbf{R}\chi_- \oplus \operatorname{span}\{\chi_0, \chi_2, \chi_3\}.$$

Then

$$\begin{aligned}\langle\phi\mathcal{L}^p\phi\rangle &\geq \lambda\langle\nu w^2\rangle + \alpha\langle v_1(w+q)q_+\rangle + \beta\langle\mathbf{p}(\phi)q_0\rangle \\ &\quad -\gamma\langle v_1 w^2\rangle - \gamma\langle v_1 q_+^2\rangle - \gamma\langle v_1 q_-^2\rangle - 2\gamma|\langle v_1 w(q_+ + q_- + q_0)\rangle| \\ &\geq (\lambda-\gamma/\nu_-)\langle\nu w^2\rangle + (\alpha-\gamma)\langle v_1 q_+^2\rangle + \gamma|\langle v_1 q_-^2\rangle| + \beta\langle\mathbf{p}(q_0)q_0\rangle \\ &\quad -\alpha|\langle v_1 w q_+\rangle| - 2\gamma|\langle v_1 w q_+\rangle| - 2\gamma|\langle v_1 w q_-\rangle| - 2\gamma|\langle v_1 w q_0\rangle| \\ &\quad -\beta|\langle\mathbf{p}(w)q_0\rangle| - \beta|\langle\mathbf{p}(q_+)q_0\rangle| - \beta|\langle\mathbf{p}(q_-)q_0\rangle|. \end{aligned} \tag{2.24}$$

Observe that

$$\begin{aligned}&(\lambda-\tfrac{\gamma}{\nu_-})\langle\nu w^2\rangle + (\alpha-\gamma)\langle v_1 q_+^2\rangle + \gamma|\langle v_1 q_-^2\rangle| + \beta\langle\mathbf{p}(q_0)q_0\rangle \\ &\quad\geq (\lambda-\tfrac{\gamma}{\nu_-})\langle\nu w^2\rangle + (\alpha-\gamma)\frac{\langle v_1\chi_+^2\rangle}{\langle\nu\chi_+^2\rangle}\langle\nu q_+^2\rangle + \gamma\frac{|\langle v_1\chi_-^2\rangle|}{\langle\nu\chi_-^2\rangle}\langle\nu q_-^2\rangle + C_1\beta\langle\nu q_0^2\rangle \\ &\quad\geq (\lambda-\tfrac{\gamma}{\nu_-})\langle\nu w^2\rangle + \tfrac{\alpha-\gamma}{\nu_+}\langle\nu q_+^2\rangle + \tfrac{\gamma}{\nu_+}\langle\nu q_-^2\rangle + C_1\beta\langle\nu q_0^2\rangle \end{aligned} \tag{2.25}$$

where $C_1 \in (0,1)$ is such that

$$C_1\langle\nu\phi^2\rangle \leq \langle\mathbf{p}(q_0)q_0\rangle \leq \frac{1}{C_1}\langle\nu\phi^2\rangle \text{ for each } \phi\in\operatorname{span}\{\chi_0,\chi_2,\chi_3\}.$$

(We recall from Lemma 2.3 that $\phi \mapsto \langle\mathbf{p}(\phi)\phi\rangle$ defines a norm on the 3-dimensional space $\operatorname{span}\{\chi_0,\chi_2,\chi_3\}$, and that all norms on a finite dimensional space are equivalent).

On the other hand, applying repeatedly the elementary inequality

$$xy \leq \epsilon x^2 + \frac{1}{\epsilon}y^2, \qquad \text{for each } x,y\in\mathbf{R} \text{ and } \epsilon>0,$$

we see that

$$\begin{aligned}&\alpha|\langle v_1 w q_+\rangle| + 2\gamma|\langle v_1 w q_+\rangle| + 2\gamma|\langle v_1 w q_-\rangle| + 2\gamma|\langle v_1 w q_0\rangle| \\ &\quad +\beta|\langle\mathbf{p}(w)q_0\rangle| + \beta|\langle\mathbf{p}(q_+)q_0\rangle| + \beta|\langle\mathbf{p}(q_-)q_0\rangle| \\ &\leq (\tfrac{\alpha}{\epsilon_1\nu_-} + \tfrac{2\gamma}{\epsilon_2\nu_-} + \tfrac{2\gamma}{\epsilon_3\nu_-} + \tfrac{2\gamma}{\epsilon_4\nu_-} + \tfrac{\beta\|\mathbf{p}\|}{\epsilon_5\nu_-})\langle\nu w^2\rangle + (\tfrac{\alpha\epsilon_1}{\nu_-} + \tfrac{2\gamma\epsilon_2}{\nu_-} + \tfrac{\beta\|\mathbf{p}\|}{\epsilon_5\nu_-})\langle\nu q_+^2\rangle \\ &\quad +(\tfrac{2\gamma\epsilon_3}{\nu_-} + \tfrac{\beta\|\mathbf{p}\|}{\epsilon_5\nu_-})\langle\nu q_-^2\rangle + (\tfrac{2\gamma\epsilon_4}{\nu_-} + \tfrac{3\beta\epsilon_5}{\nu_-})\langle\nu q_0^2\rangle \end{aligned} \tag{2.26}$$

Choose then the constants α, β, γ, ϵ_1, ϵ_2, ϵ_3, ϵ_4, and ϵ_5 such that

$$\begin{aligned}(\tfrac{\alpha}{\epsilon_1\nu_-}+\tfrac{2\gamma}{\epsilon_2\nu_-}+\tfrac{2\gamma}{\epsilon_3\nu_-}+\tfrac{2\gamma}{\epsilon_4\nu_-}+\tfrac{\beta\|\mathbf{p}\|}{\epsilon_5\nu_-}) &< \tfrac{1}{2}(\lambda-\tfrac{\gamma}{\nu_-})\\ (\tfrac{\alpha\epsilon_1}{\nu_-}+\tfrac{2\gamma\epsilon_2}{\nu_-}+\tfrac{\beta\|\mathbf{p}\|}{\epsilon_5\nu_-}) &< \tfrac{1}{2}\tfrac{\alpha-\gamma}{\nu_+}\\ (\tfrac{2\gamma\epsilon_3}{\nu_-}+\tfrac{\beta\|\mathbf{p}\|}{\epsilon_5\nu_-}) &< \tfrac{1}{2}\tfrac{\gamma}{\nu_+}\\ (\tfrac{2\gamma\epsilon_4}{\nu_-}+\tfrac{3\beta\epsilon_5}{\nu_-}) &< \tfrac{1}{2}C_1\beta.\end{aligned} \tag{2.27}$$

To realize these constraints, assume first that $\gamma < \min(\frac{\alpha}{2}, \frac{\nu_-\lambda}{2})$, and pick $\epsilon_5 > 0$ so small that $12\epsilon_5 < C_1\nu_-$. Then set $\beta = \theta\gamma$ with $\theta > 0$ small enough so that $\frac{\theta\|\mathbf{p}\|}{\epsilon_5\nu_-} < \frac{1}{4\nu_+} < \frac{1}{4\nu_-}$. With the above assumptions, $\frac{\beta\|\mathbf{p}\|}{\epsilon_5\nu_-} < \frac{\gamma}{4\nu_+} < \frac{\alpha-\gamma}{4\nu_+}$ as well as $\frac{\beta\|\mathbf{p}\|}{\epsilon_5\nu_-} < \frac{\gamma}{4\nu_-} < \frac{1}{4}(\lambda-\frac{\gamma}{\nu_-})$. Hence, (2.27) holds provided that

$$\begin{aligned}(\tfrac{\alpha}{\epsilon_1\nu_-}+\tfrac{2\gamma}{\epsilon_2\nu_-}+\tfrac{2\gamma}{\epsilon_3\nu_-}+\tfrac{2\gamma}{\epsilon_4\nu_-}) &< \tfrac{1}{4}(\lambda-\tfrac{\gamma}{\nu_-})\\ \tfrac{\alpha\epsilon_1}{\nu_-}+\tfrac{2\gamma\epsilon_2}{\nu_-} &< \tfrac{\gamma}{8\nu_+}\\ \tfrac{2\gamma\epsilon_3}{\nu_-} &< \tfrac{1}{4}\tfrac{\gamma}{\nu_+}\\ \tfrac{2\gamma\epsilon_4}{\nu_-} &< \tfrac{1}{4}C_1\beta.\end{aligned} \tag{2.28}$$

Pick ϵ_4 small enough so that $\epsilon_4 < \frac{C_1\nu_-\theta}{8}$ and $\epsilon_2 = \epsilon_3 < \frac{\nu_+}{32\nu_-}$: the last two inequalities in (2.28) are then automatically satisfied. Set $\gamma = \theta'\alpha$ with $0 < \theta' < \frac{1}{2}$ and pick ϵ_1 small enough so that $\epsilon_1 < \frac{\theta'\nu_-}{32\nu_+}$, which implies the second inequality in (2.28). Finally, pick $\alpha < \nu_-\lambda(\frac{4}{\epsilon_1}+\frac{2}{\epsilon_2}+\frac{1}{\epsilon_4}+\frac{1}{8})^{-1}$. Observe that $\frac{\nu_-\lambda}{\alpha} > 1$ since $\frac{4}{\epsilon_5} > 256$, and hence the condition $\theta' < \frac{1}{2}$ implies that $\gamma \le \min(\frac{\alpha}{2}, \frac{\nu_-\lambda}{2})$. With the choice of α as above, on account of the condition $\theta' \in (0, \frac{1}{2})$, the first inequality in (2.28) holds. With the same choice of parameters, (2.27) also holds. To summarize, the positive constants ϵ_j, $j = 1, \ldots, 5$ are given by

$$\epsilon_5 < \frac{C_1\nu_-}{12}, \quad \epsilon_4 < \frac{C_1\nu_-\theta}{12}, \quad \epsilon_2 = \epsilon_3 < \frac{\nu_+}{32\nu_-}, \quad \epsilon_1 < \frac{\theta'\nu_-}{32\nu_+} \tag{2.29}$$

with

$$0 < \theta < \frac{\epsilon_5\nu_-}{4\|\mathbf{p}\|\nu_+}, \quad 0 < \theta' < \frac{1}{2}. \tag{2.30}$$

Finally

$$\beta = \theta\gamma, \quad \gamma = \theta'\alpha, \quad 0 < \alpha < \nu_-\lambda(\tfrac{4}{\epsilon_1}+\tfrac{2}{\epsilon_2}+\tfrac{1}{\epsilon_4}+\tfrac{1}{8})^{-1} \tag{2.31}$$

For such a choice of the parameters α, β and γ, the inequalities (2.27) hold.

The coercivity of $\mathcal{L}^p$ follows from the estimates (2.24), (2.25), (2.26), and from the choice of parameters α, β and γ satisfying the three conditions (2.29), (2.30), and (2.31). In particular, (2.31) implies that $\gamma < \frac{1}{2}\nu_-$. $\square$

We conclude this section with the detailed formulation of the penalized, linear boundary layer equation. Let α, β and γ be three positive constants satisfying (2.29), (2.30), (2.31). Let $S \equiv S(x,v)$ be such that $e^{\gamma^* x} S \in L^\infty(\mathbf{R}_+; L^2(Mdv))$ for some $\gamma^* > \gamma$, and let $g_b \in L^2(|v_1|Mdv)$. The penalized, linear boundary layer problem is:

To find $g \in L^2(\mathbf{R}_+; L^2(Mdv))$ *satisfying*

$$\begin{aligned} v_1\partial_x g(x,v) + \mathcal{L}^p g(x,v) &= e^{\gamma x}(I-\mathbf{P})S, \qquad x>0, \quad v \in \mathbf{R}^3, \\ g(0,v) &= g_b(v), \qquad v_1 > 0. \end{aligned} \tag{2.32}$$

3. Existence and Uniqueness for the Penalized Linear Problem

3.1. The L^2 theory

On the Hilbert space $\mathfrak{H} = L^2(\mathbf{R}_+; L^2(Mdv))$ we define an unbounded operator $\mathcal{T}$ by

$$\begin{aligned} &\mathrm{D}(\mathcal{T}) = \{\phi \in \mathfrak{H} | \nu\phi \in \mathfrak{H}, \quad v_1\partial_x\phi \in \mathfrak{H}, \quad \phi(0,v) = 0 \text{ for each } v_1 > 0\}, \\ &\mathcal{T}\phi = v_1\partial_x\phi + \mathcal{L}^p\phi. \end{aligned} \tag{3.1}$$

Clearly, $\mathrm{D}(\mathcal{T})$ is dense in $\mathfrak{H}$, $\mathcal{T}$ is closed and its adjoint is the unbounded operator defined on $\mathfrak{H}$ by

$$\begin{aligned} &\mathrm{D}(\mathcal{T}^*) = \{\psi \in \mathfrak{H} | \nu\psi \in \mathfrak{H},\ v_1\partial_x\psi \in \mathfrak{H},\ \psi(0,v) = 0 \text{ for each } v_1 < 0\}, \\ &\mathcal{T}^*\psi = -v_1\partial_x\psi + \mathcal{L}\psi + \alpha v_1\Pi_+\psi + \beta\mathbf{p}^*\psi - \gamma v_1\psi, \end{aligned} \tag{3.2}$$

where $\mathbf{p}^*$ is given by

$$\mathbf{p}^*\psi = \frac{\langle\chi_0\psi\rangle}{\langle B_1\mathcal{L}B_1\rangle}v_1B_1 + \frac{\langle\chi_2\phi\rangle}{\langle A_{12}\mathcal{L}A_{12}\rangle}v_1A_{12} + \frac{\langle\chi_3\phi\rangle}{\langle A_{13}\mathcal{L}A_{12}\rangle}v_1A_{13}. \tag{3.3}$$

Lemma 3.1. *Let* α, β *and* γ *be positive constants satisfying* (2.29), (2.30) *and* (2.31). *Then, there exists* $\lambda'' \equiv \lambda''(\alpha,\beta,\gamma,\nu_\pm) > 0$ *such that*

- *for each* $\phi \in \mathrm{D}(\mathcal{T}^*)$, *one has* $\lambda''\|\nu\phi\|_{\mathfrak{H}} \le \|\mathcal{T}\phi\|_{\mathfrak{H}}$,
- *for each* $\psi \in \mathrm{D}(\mathcal{T}^*)$, *one has* $\lambda''\|\nu\psi\|_{\mathfrak{H}} \le \|\mathcal{T}^*\psi\|_{\mathfrak{H}}$.

In particular

$$\operatorname{Ker}\mathcal{T}=\{0\},\qquad \operatorname{Im}\mathcal{T}=\mathfrak{H}.$$

Proof. Let $\phi\in \mathrm{D}(\mathcal{T})$. In particular $\nu\phi\in L^2(Mdvdx)$. Hence there exists $L_n\to+\infty$ such that $\nu\phi(L_n,\cdot)\to 0$ in $L^2(Mdv)$ as $n\to+\infty$. Thus

$$\begin{aligned}\|\mathcal{T}\phi\|_{\mathfrak{H}}\|\phi\|_{\mathfrak{H}}&\ge\int_0^{L_n}\langle\phi\mathcal{T}\phi\rangle dx\\&=\langle v_1\phi(L_n,\cdot)^2\rangle-\int_{v_1<0}v_1\phi(0,v)^2Mdv+\int_0^{L_n}\langle\phi\mathcal{L}^p\phi\rangle dx\\&\ge\lambda'\int_0^{L_n}\langle\nu\phi^2\rangle dx.\end{aligned}$$

Letting $L_n\to+\infty$, one arrives at the inequality

$$\lambda'\|\sqrt{\nu}\phi\|^2_{\mathfrak{H}}\le\|\mathcal{T}\phi\|_{\mathfrak{H}}\|\phi\|_{\mathfrak{H}},$$

which implies that

$$\lambda'\sqrt{\nu_-}\|\sqrt{\nu}\phi\|_{\mathfrak{H}}\le\|\mathcal{T}\phi\|_{\mathfrak{H}}.$$

Hence

$$\begin{aligned}\|\mathcal{T}\phi\|_{\mathfrak{H}}\;&\ge\;\|v_1\partial_x\phi+\nu\phi\|_{\mathfrak{H}}-\|K\phi\|_{\mathfrak{H}}\\&\quad-\alpha\|\Pi_+(v_1\phi)\|_{\mathfrak{H}}-\beta\|\mathbf{p}(\phi)\|_{\mathfrak{H}}-\gamma\|v_1\phi\|_{\mathfrak{H}}\\&\ge\;\|v_1\partial_x\phi+(\nu_--\gamma)(1+|v|)\phi\|_{\mathfrak{H}}-C\|\phi\|_{\mathfrak{H}}\\&\ge\;\|v_1\partial_x\phi+(\nu_--\gamma)(1+|v|)\phi\|_{\mathfrak{H}}-\tfrac{C}{\lambda'\nu_-}\|\mathcal{T}\phi\|_{\mathfrak{H}}.\end{aligned}$$

It remains to prove that, for some constant $C'>0$, one has

$$\|v_1\partial_x\phi+(\nu_--\gamma)(1+|v|)\phi\|_{\mathfrak{H}}\ge C'\|\nu\phi\|_{\mathfrak{H}}\tag{3.4}$$

whenever $\phi\in\mathrm{D}(\mathcal{T})$. Given $S\in\mathfrak{H}$, one solves for ϕ the equation

$$v_1\partial_x\phi+(\nu_--\gamma)(1+|v|)\phi=S,\quad \phi\in\mathrm{D}(\mathcal{T})$$

and finds

$$\begin{aligned}\phi(x,v)&=\int_0^x e^{-(\nu_--\gamma)(1+|v|)(x-y)/v_1}\frac{1}{v_1}S(y,v)dy,&&v_1>0\\ \phi(x,v)&=\int_x^\infty e^{-(\nu_--\gamma)(1+|v|)(y-x)/|v_1|}\frac{1}{|v_1|}S(y,v)dy,&&v_1<0.\end{aligned}$$

In other words,

$$\phi(\cdot,v) = G(\cdot,v) \star \Big(S(\cdot,v)\mathbf{1}_{y>0}\Big)$$

with

$$G(z,v) = \frac{1}{|v_1|}e^{-(\nu_- - \gamma)(1+|v|)|z|/|v_1|}(\mathbf{1}_{z>0}\mathbf{1}_{v_1>0} + \mathbf{1}_{z<0}\mathbf{1}_{v_1<0}).$$

Since

$$\int_{-\infty}^{\infty} G(z,v)dz = \frac{\mathbf{1}_{v_1>0} + \mathbf{1}_{v_1<0}}{(\nu_- - \gamma)(1+|v|)}$$

we conclude by the Hausdorff-Young inequality that

$$\|(\nu_- - \gamma)(1+|v|)g(\cdot,v)\|_{L^2_x} \leq \|S(\cdot,v)\|_{L^2_x}.$$

Integrating each side of this inequality for the measure Mdv leads to the inequality (3.4) with $C' = \frac{\nu_- - \gamma}{\nu_+}$ which entails in turn the inequality

$$\lambda''\|\nu\phi\|_{\mathfrak{H}} \leq \|\mathcal{T}\phi\|_{\mathfrak{H}}$$

with $\lambda'' = \frac{\nu_- - \gamma}{\nu_+}(1 + \frac{C}{\lambda'\nu_-})^{-1}$.

The analogous statement on $\mathcal{T}^*$ is proved in the same way.

The first inequality clearly implies that $\operatorname{Ker}\mathcal{T} = \{0\}$. The second inequality implies that $\operatorname{Im}\mathcal{T} = \mathfrak{H}$ (see [4], Theorem II.19). $\square$

As a consequence, we have the following

Proposition 3.2. *Let $\alpha > 0$, $\beta > 0$, and $\gamma > 0$ be constants satisfying* (2.29), (2.30), *and* (2.31). *Let $S \equiv S(x,v)$ be such that $e^{\gamma x}S \in L^2(\mathbf{R}_+; L^2(Mdv))$ and let $g_b \in L^2(\nu^2\mathbf{1}_{v_1>0}Mdv)$. Then, there exists a unique solution $g \in L^2(\mathbf{R}_+; L^2(\nu^2 Mdv))$ to the penalized problem* (2.32). *This solution satisfies the estimate*

$$\begin{aligned}\lambda''\|\nu g\|_{\mathfrak{H}} \leq \quad & \|e^{\gamma x}S\|_{\mathfrak{H}} + \tfrac{1}{\sqrt{2\gamma}}\|\mathcal{L}^p(g_b\mathbf{1}_{v_1>0})\|_{L^2(Mdv)} \\ & + \tfrac{1}{\nu_-}\Big(\sqrt{\tfrac{\gamma}{2}} + \tfrac{\lambda''}{\sqrt{2\gamma}}\Big)\|\nu g_b\mathbf{1}_{v_1>0}\|_{L^2(Mdv)}.\end{aligned}$$

Proof. Let $h(x,v) = g(x,v) - g_b(v)\mathbf{1}_{v_1>0}e^{-\gamma x}$; one easily checks that

$h \in \mathrm{D}(\mathcal{T})$ if and only if $g \in \mathrm{D}(\mathcal{T})$, and that

$$\begin{aligned}\mathcal{T}h &= e^{\gamma x}(I-\mathbf{P})S + \gamma e^{-\gamma x}\mathbf{1}_{v_1>0}v_1 g_b(v) - e^{-\gamma x}\mathcal{L}^p(g_b\mathbf{1}_{v_1>0}),\\ &\in L^2(\mathbf{R}_+;L^2(Mdv))\end{aligned}$$

if and only if g is a solution to (2.32). By Lemma 3.1, this problem has a unique solution in $L^2(\mathbf{R};L^2(Mdv))$. Hence there exists a unique solution $g \in L^2(\mathbf{R}_+;L^2(\nu^2 Mdv))$ to the penalized problem (2.32). □

3.2. The L^∞ theory

The material in this section is essentially adapted from [12]. For $v_1 \neq 0$, let $a(v) = \frac{\nu(|v|)}{v_1} - \gamma$; denote by v^R the "reflected" velocity vector $v^R = (-v_1, v_2, v_3)$. The decomposition (2.18) for $\mathcal{L}$ gives the analogous decomposition $(\mathcal{L}^p\phi)(v) = v_1 a(v)\phi(v) - (\mathcal{K}^p\phi)(v)$; in other words

$$\mathcal{K}^p\phi = \mathcal{K}\phi - \alpha\Pi_+(v_1\phi) - \beta\mathbf{P}(\phi), \quad \phi \in \mathrm{D}(\mathcal{L}). \tag{3.5}$$

Finally, the notation $\tilde{S}$ designates $\tilde{S}(x,v) = e^{\gamma x}(I-\mathbf{P})S(x,v)$.

With these notations, let g be the solution of the penalized problem (2.32) in $L^2(\mathbf{R}_+;L^2(Mdv))$. Then, for a.e. $x>0$, one has

$$\begin{aligned} g(x,v) &= e^{-a(v)x}g_b(v) + \int_0^x e^{-a(v)(x-z)}\frac{1}{v_1}(\mathcal{K}^p g + \tilde{S})(z,v)dz, && v_1>0,\\ g(x,v) &= \int_x^{+\infty} e^{-a(v^R)(z-x)}\frac{1}{|v_1|}(\mathcal{K}^p g + \tilde{S})(z,v)dz, && v_1<0.\end{aligned} \tag{3.6}$$

(In the case of $v_1>0$, the formula above is obvious; the formula for the case $v_1<0$ is proved in [10] — see flas (3.47) and (3.48) on p. 90 there).

The main estimates needed in this section are summarized below. We define two operators by the formulas

$$\begin{aligned} (\mathcal{A}_+h)(x,v) &= \int_0^x e^{-a(v)(x-z)}\frac{1}{v_1}h(z,v)dz, && v_1>0,\\ (\mathcal{A}_-h)(x,v) &= \int_x^{+\infty} e^{-a(v^R)(z-x)}\frac{1}{|v_1|}h(z,v)dz, && v_1<0.\end{aligned} \tag{3.7}$$

Lemma 3.3. *The following estimates hold:*

- *if* $h(\cdot,v)\in L^\infty(\mathbf{R}_+)$ *for a.e.* $v\in\mathbf{R}^3$*, then*

$$\begin{aligned}\|\mathcal{A}_+h(\cdot,v)\|_{L^\infty}&\le\frac{1}{\nu(|v|)-\gamma|v_1|}\|h(\cdot,v)\|_{L^\infty},\\ \|\mathcal{A}_-h(\cdot,v)\|_{L^\infty}&\le\frac{1}{\nu(|v|)-\gamma|v_1|}\|h(\cdot,v)\|_{L^\infty};\end{aligned}\tag{3.8}$$

- *if* $h(\cdot,v)\in L^2(\mathbf{R}_+)$ *for a.e.* $v\in\mathbf{R}^3$*, then*

$$\begin{aligned}\|\mathbf{1}_{v_1>1}\mathcal{A}_+h(\cdot,v)\|_{L^\infty}&\le\frac{1}{\sqrt{2(\nu(|v|)-\gamma|v_1|)}}\|h(\cdot,v)\|_{L^2},\\ \|\mathbf{1}_{v_1<1}\mathcal{A}_-h(\cdot,v)\|_{L^\infty}&\le\frac{1}{\sqrt{2(\nu(|v|)-\gamma|v_1|)}}\|h(\cdot,v)\|_{L^2},\end{aligned}\tag{3.9}$$

- *if* $h(\cdot,v)\in L^2\cap L^\infty(\mathbf{R}_+)$ *for a.e.* $v\in\mathbf{R}^3$ *and each* $\epsilon>0$*, one has*

$$\begin{aligned}\|\mathbf{1}_{0<v_1<1}\mathcal{A}_+h(\cdot,v)\|_{L^\infty}\le&\frac{\|h(\cdot,v)\|_{L^2}}{2\sqrt{\epsilon e}(\nu(|v|)-\gamma|v_1|)}\\ &+\frac{\epsilon^{1/4}\|h(\cdot,v)\|_{L^\infty}}{(\nu(|v|)-\gamma|v_1|)^{3/4}|v_1|^{1/4}},\\ \|\mathbf{1}_{-1<v_1<0}\mathcal{A}_-h(\cdot,v)\|_{L^\infty}\le&\frac{\|h(\cdot,v)\|_{L^2}}{2\sqrt{\epsilon e}(\nu(|v|)-\gamma|v_1|)}\\ &+\frac{\epsilon^{1/4}\|h(\cdot,v)\|_{L^\infty}}{(\nu(|v|)-\gamma|v_1|)^{3/4}|v_1|^{1/4}}.\end{aligned}\tag{3.10}$$

The proof of these estimates is given in [12] (see the proof of Proposition 3.4 there, on pp. 90–93).

Here is how these estimates are used on the problem (2.32). First, the integral equations (3.6) are recast as

$$\mathbf{1}_{v_1>0}g=e^{-ax}g_b+\mathcal{A}_+(\mathcal{K}^pg+\tilde S),\quad \mathbf{1}_{v_1<0}g=\mathcal{A}_-(\mathcal{K}^pg+\tilde S).$$

Then

$$\begin{aligned}\|g\|_{L^2(Mdv;L^\infty_x)}\le\;&\|g_b\|_{L^2(Mdv)}+\tfrac{1}{\nu_--\gamma}\|(1+|v|)^{-1}\tilde S\|_{L^2(Mdv;L^\infty_x)}\\ &+\tfrac{1}{\sqrt{2(\nu_--\gamma)}}\|\mathcal{K}^pg\|_{L^2(Mdvdx)}+\tfrac{1}{2\sqrt{\epsilon e}(\nu_--\gamma)}\|\mathcal{K}^pg\|_{L^2(Mdvdx)}\\ &+\tfrac{1}{(\nu_--\gamma)^{3/4}}\epsilon^{1/4}\left\|\mathbf{1}_{|v_1|\le1}\frac{\|\mathcal{K}^pg(\cdot,v)\|_{L^\infty_x}}{|v_1|^{1/4}(1+|v|)^{3/4}}\right\|_{L^2(Mdv)}.\end{aligned}\tag{3.11}$$

Next we recall a fundamental decay property verified by $\mathcal{K}$.

Lemma 3.4. *If $\phi \in L^2(Mdv)$, then*[2] $\sqrt{M}\mathcal{K}\phi \in L_v^{\infty,1/2}$. *More precisely, there exists a positive constant denoted $\|\mathcal{K}\|_{L^2,L^{\infty,1/2}}$ such that, for each $\phi \in L^2(Mdv)$*

$$\|(1+|v|)^{1/2}\sqrt{M}\mathcal{K}\phi\|_{L_v^\infty} \le \|\mathcal{K}\|_{L^2,L^{\infty,1/2}}\|\phi\|_{L^2(Mdv)}.$$

See Proposition 6.1 (especially formula (6.1)) in [5] for a proof of this classical result. Obviously, $\mathcal{K}^p$ shares the same property.

Corollary 3.5. *For each $\alpha > 0$ and $\beta > 0$, there exists $\|\mathcal{K}^p\|_{L^2,L^{\infty,1/2}} > 0$ such that the operator $\mathcal{K}^p$ defined in* (3.5) *satisfies*

$$\|(1+|v|)^2\sqrt{M}\mathcal{K}^p\phi\|_{L_v^\infty} \le \|\mathcal{K}^p\|_{L^2,L^{\infty,1/2}}\|\phi\|_{L^2(Mdv)}.$$

Proof. One has

$$|\sqrt{M}\Pi_+(v_1\phi)| = |\sqrt{M}\chi_+|\frac{|\langle v_1\phi\chi_+\rangle|}{|\langle\chi_+^2\rangle|} \le \tfrac{\langle|v_1\chi_+|^2\rangle^{1/2}}{|\langle\chi_+^2\rangle|}\|\phi\|_{L^2(Mdv)}|\sqrt{M}\chi_+|$$

and $\sqrt{M}(1+|v|)^s\chi_+(v) \in L_v^\infty$ for all $s > 0$. Since there is a similar estimate for $\mathbf{p}(\phi)$, the announced inequality follows from Lemma 3.4. □

The last term in the right hand side of (3.11) is estimated as follows:

$$\begin{aligned}\left\|\mathbf{1}_{|v_1|\le 1}\frac{\sqrt{M}\|\mathcal{K}^p g(\cdot,v)\|_{L_x^\infty}}{|v_1|^{1/4}(1+|v|)^{3/4}}\right\|_{L_v^2} &\le \left\|\mathbf{1}_{|v_1|\le 1}\frac{\|\mathcal{K}^p\|_{L^2,L^{\infty,1/2}}\|g\|_{L_x^\infty(L^2(Mdv))}}{|v_1|^{1/4}(1+|v|)^{3/41/2}}\right\|_{L_v^2}\\ &\le \|\mathcal{K}^p\|_{L^2,L^{\infty,1/2}}\||u|^{-1/4}\|_{L^2(-1,1)}\\ &\quad\times\|(1+|w|)^{-5/4}\|_{L^2(\mathbf{R}^2)}\|g\|_{L^2(Mdv;L_x^\infty)}.\end{aligned}$$

Set $C_0 = \|\mathcal{K}^p\|_{L^2,L^{\infty,1/2}}\||u|^{-1/4}\|_{L^2(-1,1)}\|(1+|w|)^{-5/4}\|_{L^2(\mathbf{R}^2)}$; the estimate (3.11) becomes

$$\begin{aligned}\|g\|_{L^2(Mdv;L_x^\infty)} \le\ & \|g_b\|_{L^2(Mdv)} + \tfrac{1}{\nu_- - \gamma}\|(1+|v|)^{-1}\tilde{S}\|_{L^2(Mdv;L_x^\infty)}\\ &+\Big(\tfrac{1}{\sqrt{2(\nu_- - \gamma)}} + \tfrac{1}{2\sqrt{\epsilon e}(\nu_- - \gamma)}\Big)\|\mathcal{K}^p\|_{L^2,L^2}\|g\|_{L^2(Mdvdx)}\\ &+\tfrac{C_0}{(\nu_- - \gamma)^{3/4}}\epsilon^{1/4}\|g\|_{L^2(Mdv;L_x^\infty)}\end{aligned} \tag{3.12}$$

[2] The notation $L_v^{p,s}$ designates the class of functions $\{f \in L_v^p | |v|^s f \in L_v^p\}$.

By choosing $\epsilon > 0$ small enough so that $\frac{C_0}{(\nu_- - \gamma)^{3/4}}\epsilon^{1/4} = \frac{1}{2}$ in the inequality above, we have proved the following

Proposition 3.6. *Assume that $\alpha > 0$, $\beta > 0$ and $\gamma > 0$ satisfy* (2.29), (2.30), *and* (2.31). *Let $g_b \in L^2(\nu^2 \mathbf{1}_{v_1>0} M dv)$ and $S \equiv S(x, v)$ be such that $e^{\gamma^* x} S \in L^2(Mdv; L^\infty_x)$ for some $\gamma^* > \gamma$. Then the solution g of the penalized problem* (2.32) *belongs to $L^2(Mdv; L^\infty_x)$ and one has*

$$\begin{aligned}\|g\|_{L^2(Mdv;L^\infty_x)} \;\le\; & 2\|g_b\|_{L^2(Mdv)} + \tfrac{2}{\nu_- - \gamma}\|(1+|v|)^{-1}\tilde{S}\|_{L^2(Mdv;L^\infty_x)} \\ & + \Big(\tfrac{\sqrt{2}}{\sqrt{\nu_- - \gamma}} + \tfrac{4C_0^2}{\sqrt{e}(\nu_- - \gamma)^{5/2}}\Big)\|\mathcal{K}^p\|_{L^2,L^2}\|g\|_{L^2(Mdvdx)}.\end{aligned}$$

Applying Corollary 3.5 once again, together with the bounds (3.8), one finds that

$$\begin{aligned}\|\sqrt{M}g\|_{L^\infty_{x,v}} \;\le\; & \|\sqrt{M}g_b\mathbf{1}_{v_1>0}\|_{L^\infty_v} + \tfrac{1}{\nu_- - \gamma}\|(1+|v|)^{-1}\sqrt{M}\tilde{S}\|_{L^\infty_{x,v}} \\ & + \tfrac{1}{\nu_- - \gamma}\|\mathcal{K}^p\|_{L^2,L^{\infty,1/2}}\|g\|_{L^2(Mdv,L^\infty_x)}\end{aligned}$$

Next, we improve the decay in v of the solution, by iterating on the following classical estimate:

Lemma 3.7. *For $s \ge 0$, if $\sqrt{M}\phi \in L^{\infty,s}(Mdv)$, then $\sqrt{M}\mathcal{K}\phi \in L^{\infty,s+1}_v$. Moreover, there exists a positive constant denoted $\|\mathcal{K}\|_{L^{\infty,s},L^{\infty,s+1}}$ such that*

$$\|(1+|v|)^{s+1}\sqrt{M}\mathcal{K}\phi\|_{L^\infty_v} \le \|\mathcal{K}\|_{L^{\infty,s},L^{\infty,s+1}}\|(1+|v|)^s\sqrt{M}\phi\|_{L^\infty_v}$$

whenever $\sqrt{M}\phi \in L^{\infty,s}(Mdv)$

See formula (6.2) in Proposition 6.1 of [5]. Obviously, the same is true of $\mathcal{K}^p$:

Corollary 3.8. *For $\alpha > 0$ and $\beta > 0$, there exists $\|\mathcal{K}^p\|_{L^{\infty,s},L^{\infty,s+1}} > 0$ such that the operator $\mathcal{K}^p$ defined in* (3.5) *satisfies*

$$\|(1+|v|)^{s+1}\sqrt{M}\mathcal{K}^p\phi\|_{L^\infty_v} \le \|\mathcal{K}^p\|_{L^{\infty,s},L^{\infty,s+1}}\|(1+|v|)^s\sqrt{M}\phi\|_{L^\infty_v}$$

whenever $\sqrt{M}\phi \in L^{\infty,s}(Mdv)$.

The proof is the same as that of Corollary 3.5.

Applying Corollary 3.8 together with the bounds (3.8) leads to

$$\|(1+|v|)^{s+1}\sqrt{M}g\|_{L^\infty_{x,v}} \le \|(1+|v|)^{s+1}\sqrt{M}g_b\mathbf{1}_{v_1>0}\|_{L^\infty_v}$$

$$+\tfrac{1}{\nu_- - \gamma}\|(1+|v|)^s\sqrt{M}\tilde{S}\|_{L^\infty_{x,v}}$$
$$+\tfrac{1}{\nu_- - \gamma}\|\mathcal{K}^p\|_{L^{\infty,s},L^{\infty,s+1}}\|(1+|v|)^s\sqrt{M}g\|_{L^\infty_{x,v}}. \tag{3.13}$$

By induction, applying successively Proposition 3.2, Proposition 3.6 and the estimate (3.13) we arrive at the

Proposition 3.9. *Assume that* $\alpha > 0$*,* $\beta > 0$ *and* $\gamma > 0$ *satisfy* (2.29), (2.30), *and* (2.31). *Assume that*

$$\sqrt{M}g_b\mathbf{1}_{v_1>0} \in L^{\infty,3}_v \text{ and that } e^{\delta x}(1+|v|)^2\sqrt{M}\tilde{S} \in L^\infty_{x,v}$$

for some $\delta > 0$*. Then, the solution* g *of the penalized problem* (2.32) *satisfies an estimate of the form*

$$\|(1+|v|)^3\sqrt{M}g\|_{L^\infty_{x,v}} \;\leq\; C\|\sqrt{M}g_b\mathbf{1}_{v_1>0}\|_{L^{\infty,3}_v} + C\|e^{\delta x}(1+|v|)^2\sqrt{M}\tilde{S}\|_{L^\infty_{x,v}}$$

for some constant

$$C \equiv C(\nu_+,\nu_-,\alpha,\beta,\gamma,\delta,\|\mathcal{K}^p\|_{L^2,L^{\infty,1/2}},\max_{s=0,1,2}\|\mathcal{K}^p\|_{L^{\infty,s},L^{\infty,s+1}}).$$

4. The Nonlinear Problem

4.1. The penalized, nonlinear problem

Given $f_b \equiv f_b(v)$ such that $\sqrt{M}f_b\mathbf{1}_{v_1>0} \in L^{\infty,3}_v$, we consider the following problem: to find (g,h) such that

$$\begin{aligned} v_1\partial_x g + \mathcal{L}^p g &= e^{-\gamma x}(I-\mathbf{P})\mathcal{Q}(g+h,g+h)\,, \quad x>0\,,\ v\in\mathbf{R}^3, \\ h(x,v) &= -e^{\gamma x}\int_x^{+\infty} e^{-2\gamma z}\mathcal{P}\mathcal{Q}(g+h,g+h)(z,v)dz,\ x>0,\ v\in\mathbf{R}^3, \\ g(0,v) &= f_b(v) - h(0,v)\,, \quad v_1>0,\ v\in\mathbf{R}^3 \end{aligned} \tag{4.1}$$

where the operator $\mathcal{P}$ is defined by the formula

$$\mathcal{P}\phi = v_1^{-1}\mathbf{P}\phi = \frac{\langle B_1\phi\rangle}{\langle B_1\mathcal{L}B_1\rangle}\chi_0 + \frac{\langle A_{12}\phi\rangle}{\langle A_{12}\mathcal{L}A_{12}\rangle}\chi_2 + \frac{\langle A_{13}\phi\rangle}{\langle A_{13}\mathcal{L}A_{12}\rangle}\chi_3\,.$$

This is the nonlinear penalized problem.

By Proposition 3.9, a solution (g,h) of the nonlinear penalized problem satisfies the bound

$$\begin{aligned}
&\|(1+|v|)^3\sqrt{M}g\|_{L^\infty_{x,v}}\\
&\quad\le\ C\|\sqrt{M}f_b\mathbf{1}_{v_1>0}\|_{L^{\infty,3}_v}+C\|(1+|v|)^2\sqrt{M}\mathcal{Q}(g+h,g+h)\|_{L^\infty_{x,v}}\\
&\qquad+C\left\|(1+|v|)^3\sqrt{M}\int_0^{+\infty}e^{-2\gamma z}\mathcal{P}\mathcal{Q}(g+h,g+h)(z,v)dz\right\|_{L^\infty_v}
\end{aligned}$$

by choosing $\delta=\gamma$ in the estimate of Proposition 3.9. Besides, observe that one has the trivial continuity bound

$$\|(1+|v|)^3\sqrt{M}\mathcal{P}\phi\|_{L^\infty_v}\le C'\|\sqrt{M}\phi\|_{L^\infty_v}$$

for some positive constant C', so that the estimate above becomes

$$\begin{aligned}
&\|(1+|v|)^3\sqrt{M}g\|_{L^\infty_{x,v}}\\
&\quad\le\ C\|\sqrt{M}f_b\mathbf{1}_{v_1>0}\|_{L^{\infty,3}_v}\\
&\qquad+(C+\tfrac{1}{2\gamma}CC')\|(1+|v|)^2\sqrt{M}\mathcal{Q}(g+h,g+h)\|_{L^\infty_{x,v}}\,.
\end{aligned}\tag{4.2}$$

By the same token, we find that

$$\begin{aligned}
&\|(1+|v|)^3\sqrt{M}h\|_{L^\infty_{x,v}}\\
&\quad=\ \left\|(1+|v|)^3\sqrt{M}\int_x^{+\infty}e^{-\gamma(2z-x)}\mathcal{P}\mathcal{Q}(g+h,g+h)(z,v)dz\right\|_{L^\infty_{x,v}}\\
&\quad\le\ \tfrac{1}{2\gamma}C'\|(1+|v|)^2\sqrt{M}\mathcal{Q}(g+h,g+h)\|_{L^\infty_{x,v}}\,.
\end{aligned}\tag{4.3}$$

Let us recall the following classical bound on the Boltzmann collision integral:

Lemma 4.1. *For each $s>0$, there exists $C_s>0$ such that*

$$\|(1+|v|)^{s-1}\sqrt{M}\mathcal{Q}(\phi,\phi)\|_{L^\infty_v}\le C_s\|(1+|v|)^s\sqrt{M}\phi\|^2_{L^\infty_v}\,.$$

See Proposition 5.1 in [6] for a proof of this result.

Proposition 4.2. *Assume that $\alpha>0$, $\beta>0$ and $\gamma>0$ satisfy* (2.29), (2.30), *and* (2.31). *There exists $\epsilon>0$ such that, for each $f_b\equiv f_b(v)$ satisfying*

$$\|\sqrt{M}f_b\mathbf{1}_{v_1>0}\|_{L^{\infty,3}_v}\le\epsilon\tag{4.4}$$

the nonlinear penalized problem (4.1) *has a unique solution such that*

$$\|(1+|v|)^3\sqrt{M}g\|_{L^\infty_{x,v}}+\|(1+|v|)^3\sqrt{M}h\|_{L^\infty_{x,v}}<\infty\,. \tag{4.5}$$

Proof. Consider the map $(g,h)\mapsto(g_1,h_1)$, where (g_1,h_1) is the solution of

$$\begin{aligned}&v_1\partial_x g_1+\mathcal{L}^p g_1=e^{-\gamma x}(I-\mathbf{P})\mathcal{Q}(g+h,g+h)\,,\quad x>0\,,\ v\in\mathbf{R}^3\,,\\&\qquad h_1(x,v)=-e^{\gamma x}\int_x^{+\infty}e^{-2\gamma z}\mathcal{P}\mathcal{Q}(g{+}h,g{+}h)(z,v)dz,\ \ x{>}0,\ v{\in}\mathbf{R}^3,\\&\qquad g_1(0,v)=f_b(v)-h_1(0,v)\,,\quad v_1>0\,,\ v\in\mathbf{R}^3\,.\end{aligned} \tag{4.6}$$

Similarly to the bounds (4.2) and (4.3) above, one has, in view of Lemma 4.1

$$\begin{aligned}&\|(1+|v|)^3\sqrt{M}g_1\|_{L^\infty_{x,v}}+\|(1+|v|)^3\sqrt{M}h_1\|_{L^\infty_{x,v}}\\&\le C\|\sqrt{M}f_b\mathbf{1}_{v_1>0}\|_{L^{\infty,3}_v}+(C{+}\tfrac{1}{2\gamma}C'{+}\tfrac{1}{2\gamma}CC')\|(1{+}|v|)^2\sqrt{M}\mathcal{Q}(g{+}h,g{+}h)\|_{L^\infty_{x,v}}\\&\le C\|\sqrt{M}f_b\mathbf{1}_{v_1>0}\|_{L^{\infty,3}_v}\\&\quad+(C+\tfrac{1}{2\gamma}C'+\tfrac{1}{2\gamma}CC')C_3\|(1+|v|)^3\sqrt{M}(g+h)\|^2_{L^\infty_{x,v}}\,.\end{aligned}$$

Hence, setting

$$\epsilon=\Big(4C(C+\tfrac{1}{2\gamma}C'+\tfrac{1}{2\gamma}CC')\Big)^{-1}$$

one sees that the condition

$$\|(1+|v|)^3\sqrt{M}g\|_{L^\infty_{x,v}}+\|(1+|v|)^3\sqrt{M}h\|_{L^\infty_{x,v}}\le C\epsilon$$

implies that

$$\|(1+|v|)^3\sqrt{M}g_1\|_{L^\infty_{x,v}}+\|(1+|v|)^3\sqrt{M}h_1\|_{L^\infty_{x,v}}\le C\epsilon\,.$$

The existence and uniqueness of the solution of the nonlinear penalized problem (4.1) follows from the fixed point theorem in the complete space

$$\{(g,h)\,|\,\|(1+|v|)^3\sqrt{M}g\|_{L^\infty_{x,v}}+\|(1+|v|)^3\sqrt{M}h\|_{L^\infty_{x,v}}\le C\epsilon\}\,. \qquad \square$$

4.2. The nonlinear Knudsen layer

Assume that $\alpha>0$, $\beta>0$ and $\gamma>0$ satisfy (2.29), (2.30), and (2.31), and let $\epsilon > 0$ be as in Proposition 4.2. For $f_b \equiv f_b(v)$ satisfying (4.4), the nonlinear penalized problem (4.1) has a unique solution (g, h) satisfying (4.5). Set

$$f(x, v) = e^{-\gamma x} g(x, v) + e^{-\gamma x} h(x, v)\,.$$

This function f satisfies

$$\begin{aligned} v_1 \partial_x f + \mathcal{L} f &= \mathcal{Q}(f, f) - \alpha e^{-\gamma x} \Pi_+(v_1 g) - \beta e^{-\gamma x} \mathbf{p}(g) \\ f(0, v) &= f_b(v) \text{ for } v_1 > 0 \\ \sqrt{M} |f(x, v)| &\leq c(1 + |v|)^{-3} e^{-\gamma x}\,, \qquad x > 0\,,\ v \in \mathbf{R}^3\,. \end{aligned}$$

for some constant $c > 0$.

Therefore, f is a solution of the nonlinear Knudsen layer problem (2.4) if

$$\Pi_+(v_1 g) = 0 \text{ and } \mathbf{p}(g) = 0\,. \tag{4.7}$$

Conversely, if f is a solution of (2.4) such that

$$\sqrt{M} |f(x, v)| \leq c(1 + |v|)^{-3} e^{-\gamma x}\,, \quad x > 0\,,\ v \in \mathbf{R}^3\,,$$

we set

$$g = e^{\gamma x} (I - \mathbf{p}) f \text{ and } h = e^{\gamma x} \mathbf{p} f\,.$$

By Lemma 2.5 — see also the proof of (2.16) — (g, h) is then a solution of the nonlinear penalized problem (4.1) that satisfies (4.7).

Hence the proof of Theorem 1.1 the conditions (4.7) define a subset of boundary data f_b of codimension 4 in the ball $B(0, \epsilon)$ of $L^{\infty,3}_v$.

For $f_b \equiv f_b(v)$ satisfying (4.4), let (g, h) be the solution of the nonlinear penalized problem (4.5), and let

$$\tilde{g}(x, v) = e^{-\gamma x} g(x, v) \text{ and } \tilde{h}(x, v) := e^{-\gamma x} h(x, v)\,.$$

Then

$$v_1 \partial_x \tilde{g} + \mathcal{L} \tilde{g} = (I - \mathbf{P}) \mathcal{Q}(\tilde{g} + \tilde{h}, \tilde{g} + \tilde{h}) - \alpha \Pi_+(v_1 \tilde{g}) - \beta \mathbf{p}(\tilde{g})\,. \tag{4.8}$$

Multiplying both side of this equality by $\chi_+(v)$ as in (2.8) and integrating for the measure Mdv leads to the identity

$$\frac{d}{dx}\langle v_1\chi_+\tilde{g}\rangle = -\alpha\langle v_1\chi_+\tilde{g}\rangle\,, \quad x>0\,,$$

so that

$$\langle v_1\chi_+\tilde{g}\rangle(x) = \langle v_1\chi_+\tilde{g}\rangle(0)e^{-\alpha x}\,, \quad x>0\,.$$

Hence the first condition in (4.7) is equivalent to

$$\langle v_1\chi_+\tilde{g}\rangle(0) = 0\,. \tag{4.9}$$

Multiplying both side of (4.8) by $\chi_j(v)$ for $j=0,2,3$ (see (2.8) for the definition of these functions) and integrating for the measure Mdv leads to the identity

$$\frac{d}{dx}\left\langle v_1\begin{pmatrix}\chi_0\\ \chi_2\\ \chi_3\end{pmatrix}\tilde{g}\right\rangle = -\beta\begin{pmatrix}\lambda_0\langle v_1B_1\tilde{g}\rangle\\ \lambda_2\langle v_1A_{12}\tilde{g}\rangle\\ \lambda_3\langle v_1A_{13}\tilde{g}\rangle\end{pmatrix}\,, \quad x>0\,, \tag{4.10}$$

where

$$\lambda_0 = \frac{\langle\chi_0^2\rangle}{\langle B_1\mathcal{L}B_1\rangle}>0\,, \quad \lambda_j = \frac{\langle\chi_j^2\rangle}{\langle A_{1j}\mathcal{L}A_{1j}\rangle}>0\,,\ j=2,3\,.$$

Moreover, multiplying both side of (4.8) by B_1, A_{12} and A_{13} and integrating for the measure Mdv leads to

$$\frac{d}{dx}\left\langle v_1\begin{pmatrix}B_1\\ A_{12}\\ A_{13}\end{pmatrix}\tilde{g}\right\rangle + \left\langle\begin{pmatrix}B_1\\ A_{12}\\ A_{13}\end{pmatrix}\mathcal{L}\tilde{g}\right\rangle = \left\langle\begin{pmatrix}B_1\\ A_{12}\\ A_{13}\end{pmatrix}(I-\mathbf{P})\mathcal{Q}(\tilde{g}+\tilde{h},\tilde{g}+\tilde{h})\right\rangle.$$

Since

$$\left\langle\begin{pmatrix}B_1\\ A_{12}\\ A_{13}\end{pmatrix}\mathcal{L}\tilde{g}\right\rangle = \left\langle v_1\begin{pmatrix}\chi_0\\ \chi_2\\ \chi_3\end{pmatrix}\tilde{g}\right\rangle$$

and

$$\left\langle\begin{pmatrix}B_1\\ A_{12}\\ A_{13}\end{pmatrix}\mathbf{P}\phi\right\rangle = \left\langle\begin{pmatrix}B_1\\ A_{12}\\ A_{13}\end{pmatrix}\phi\right\rangle$$

which implies that

$$\left\langle \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} (I-\mathbf{P})\mathcal{Q}(\tilde g+\tilde h,\tilde g+\tilde h) \right\rangle = 0$$

we conclude that

$$\frac{d}{dx}\left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \tilde g \right\rangle + \left\langle v_1 \begin{pmatrix} \chi_0 \\ \chi_2 \\ \chi_3 \end{pmatrix} \tilde g \right\rangle = 0\,. \tag{4.11}$$

Putting together (4.10) and (4.11), we finally obtain

$$\frac{d^2}{dx^2}\left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \tilde g \right\rangle = -\beta \begin{pmatrix} \lambda_0 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \tilde g \right\rangle .$$

Since $\beta > 0$, and $\lambda_j > 0$ for $j = 0, 2, 3$, and $\sqrt{M}\tilde g$ is bounded on $\mathbf{R}_x \times \mathbf{R}^3_v$, we conclude that

$$\left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \tilde g \right\rangle(x) = \begin{pmatrix} e^{-\sqrt{\beta\lambda_0}x} & 0 & 0 \\ 0 & e^{-\sqrt{\beta\lambda_2}x} & 0 \\ 0 & 0 & e^{-\sqrt{\beta\lambda_3}x} \end{pmatrix} \left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \tilde g \right\rangle(0)$$

Hence the second condition in (4.7), which is

$$\left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \tilde g \right\rangle(x) = 0 \quad \text{for each } x \geq 0\,,$$

is in turn equivalent to

$$\left\langle v_1 \begin{pmatrix} B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \tilde g \right\rangle(0) = 0\,. \tag{4.12}$$

Setting $f = \tilde g + \tilde h$, we see that the conditions (4.9) and (4.12) are equivalent to

$$\left\langle \begin{pmatrix} \chi_+ \\ B_1 \\ A_{12} \\ A_{13} \end{pmatrix} \left(v_1 f(0,\cdot) + \int_0^\infty \mathcal{Q}(f,f)(z,\cdot)dz \right) \right\rangle = 0\,.$$

As the functions $\chi_+, B_1, A_{12}, A_{13}$ are linearly independent, this condition defines indeed a codimension 4 subset of boundary data f_b in the ball $B(0,\epsilon)$ of $L^{\infty,3}_v$. This completes the proof of Theorem 1.1.

5. Conclusion

Thus we have extended the Ukai-Yang-Yu theorem in [21] to the case where the Mach number at infinity $\mathrm{Ma}_\infty = 0$. The proof of this extension uses

(a) the variant of the Bardos-Caflisch-Nicolaenko energy method based on a suitable penalization of the linearized collision integral proposed by [21], and

(b) the analogue in the case of the Boltzmann equation of the K-integral for the half-space problem in radiative transfer — i.e. the projections $\mathbf{p}$ and $\mathbf{P}$ in Section 2.3.

The same method can be also applied in the cases $\mathrm{Ma}_\infty = \pm 1$ which is not covered by the method in [21].

Again without loss of generality, we can restrict our attention to the cases where

$$\rho_\infty = \theta_\infty = 1\,, \quad u_{1,\infty} = \pm\sqrt{\tfrac{5}{3}}\,, \quad u_{2,\infty} = u_{3,\infty} = 0\,. \tag{5.1}$$

and set the problem (1.1)-(1.5) in the variables

$$\xi_1 = v_1 \pm \sqrt{\tfrac{5}{3}}\,, \quad \xi_2 = v_2\,, \quad \xi_3 = v_3\,.$$

Setting $c = \sqrt{5/3}$, we observe that

$$\langle(\xi_1 + c)\chi_-^2\rangle = 0\,, \quad \langle(\xi_1 - c)\chi_+^2\rangle = 0\,.$$

Hence

$$(\xi_1 + c)\chi_- \perp \operatorname{Ker}\mathcal{L}\,, \text{ and } (\xi_1 - c)\chi_+ \perp \operatorname{Ker}\mathcal{L}\,,$$

and the Fredholm alternative for the linearized collision operator implies the existence and uniqueness of

$$X_+ \in \operatorname{Ker}\mathcal{L} \text{ such that } \mathcal{L}X_+ = (\xi_1 + c)\chi_-\,,$$
$$X_- \in \operatorname{Ker}\mathcal{L} \text{ such that } \mathcal{L}X_- = (\xi_1 - c)\chi_+\,.$$

This suggests introducing the projections

$$\mathbf{P}_\pm\phi = \frac{\langle X_\pm\phi\rangle}{\langle X_\pm\mathcal{L}X_\pm\rangle}\mathcal{L}X_\pm\,, \quad \mathbf{p}_\pm\phi = \frac{\langle(\xi_1\pm c)X_\pm\phi\rangle}{\langle X_\pm\mathcal{L}X_\pm\rangle}\frac{\mathcal{L}X_\pm}{\xi_1\pm c}$$

together with the following penalized versions of the linearized collision integral:

$$\begin{aligned}\mathcal{L}^{p,+}\phi &= \mathcal{L}\phi + \alpha\Pi_+^+((\xi_1+c)\phi) + \beta\mathbf{p}_+(\phi) - \gamma(\xi_1+c)\phi\\ \mathcal{L}^{p,-}\phi &= \mathcal{L}\phi + \beta\mathbf{p}_-(\phi) - \gamma(\xi_1-c)\phi\end{aligned}$$

where Π_+^+ is the $L^2(Mdv)$-orthogonal projectionon the linear span of $\{\chi_+,\chi_0,$ $\chi_2,\chi_3\}$.

Following the method presented above, we consider the penalized nonlinear problems

$$\begin{aligned}(\xi_1\pm c)\partial_x g + \mathcal{L}^{p,\pm}g &= e^{-\gamma x}(I-\mathbf{P}_\pm)\mathcal{Q}(g+h,g+h)\,,\\ h(x,\xi) = -e^{\gamma x}\int_x^{+\infty} & e^{-2\gamma z}\frac{1}{\xi_1\pm c}\mathbf{P}_\pm\mathcal{Q}(g+h,g+h)(z,v)dz\,,\\ g(0,\xi) &= f_b(\xi) - h(0,\xi)\,,\quad \xi_1\pm c>0\,,\end{aligned}$$

and one sees that $\mathbf{S}[M_{(1,(+c,0,0),1)}]$ is the set of f_bs satisfying

$$\left\langle\begin{pmatrix}\chi_+\\ \chi_0\\ \chi_2\\ \chi_3\\ X_+\end{pmatrix}\left((\xi_1+c)f(0,\cdot) + \int_0^\infty \mathcal{Q}(f,f)(z,\cdot)dz\right)\right\rangle = 0\,,$$

while $\mathbf{S}[M_{(1,(-c,0,0),1)}]$ is the set of f_bs satisfying

$$\left\langle X_-\left((\xi_1+c)f(0,\cdot) + \int_0^\infty \mathcal{Q}(f,f)(z,\cdot)dz\right)\right\rangle = 0\,.$$

In other words, $\mathbf{S}[M_{(1,(+c,0,0),1)}]$ and $\mathbf{S}[M_{(1,(-c,0,0),1)}]$ are respectively of codimension 5 near $M_{(1,(+c,0,0),1)}$ and 1 near $M_{(1,(-c,0,0),1)}$.

Finally, observe that Theorem 1.1 does not guarantee that the solution F of (1.1) with boundary condition (1.5) defined by

$$F_b = M_{(1,0,1)}(1+f_b) \text{ with } f_b\in\mathbf{S}[M_{(1,0,1)}]$$

is nonnegative. This is a major shortcoming of all near-equilibrium existence and uniqueness results for the steady Boltzmann equation proved by a fixed-point argument. Such was the case, for instance, of the weakly nonlinear shock profiles for the Boltzmann equation constructed by B. Nicolaenko [16, 17] in the case of a hard-sphere gas and by Caflisch-Nicolaenko [7] for softer cut-off potentials. Fortunately, the positivity issue was recently settled (by a stability argument) by T.-P. Liu and S.-H. Yu in [15] in the case of the shock profile problem. One could hope that a similar method would eventually establish the positivity also in the case of boundary layers. As a matter of fact, the work of Y. Sone and his collaborators on the condensation-evaporation half-space problem is indeed based on a time-marching method, which guarantees that the solution so obtained is nonnegative. In the case of a supersonic condensation flow ($\mathrm{Ma}_\infty = -1$) S. Ukai, T. Yang and S.-H. Yu [22] have indeed proved the stability — and therefore the positivity — of the solution of the boundary layer problem (1.1)-(1.5) with asymptotic behavior at infinity given by (1.6).

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