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THE SUPERHARMONIC INSTABILITY OF FINITE-AMPLITUDE INTERFACIAL WAVES

BY

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Abstract

The linear stability of finite-amplitude interfacial waves in a two-layer fluid is investigated for superharmonic disturbances on the basis of the Euler set of equations. In the previous study (*J. Fluid Mech.* 2006, vol.547, p.175), the author proved analytically for surface waves that the superharmonic instability first occurs when the wave energy density becomes stationary as a function of wave speed for fixed mean surface height. This analysis is here extended to the two-layer-fluid system. It is found that the above law is true of any interfacial waves by replacing the word 'surface' by 'interface'.

1. Introduction

We consider the linear stability of periodic waves to disturbances that are periodic in one wavelength of the basic wave, so called superharmonic stability. The previous studies on the superharmonic stability of periodic waves are mainly for surface gravity waves on deep fluid. Longuet-Higgins (1978) was the first to treat this problem and found that the superharmonic instability first occurs at the crest-to-trough height of 0.1366 times the wavelength. Tanaka (1983) later found numerically that, at this critical point, the wave energy density becomes stationary as a function of wave speed. This fact is proved analytically by Saffman (1985).

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For surface waves on fluid of finite depth, Kataoka (2006a) proved analytically that an exchange of stability to superharmonic disturbances occurs when the wave energy density becomes stationary as a function of wave speed for fixed mean surface height. This fixed parameter must be given, since the periodic surface waves on fluid of finite depth are characterized by two parameters, unlike those on deep fluid that are characterized by a single parameter.

For interfacial waves in a two-layer fluid, on the other hand, there are no previous studies that focused on the superharmonic instability. Pullin & Grimshaw (1985) undertook numerical stability analysis of interfacial waves under the Boussinesq approximation. They paid attention not to the superharmonic instability but to the subharmonic instabilities like resonance instabilities (Phillips 1960; Benjamin & Feir 1967) and wave-induced Kelvin-Helmholtz instabilities. In the present study, therefore, we investigate the superharmonic instability of interfacial waves analytically (without using the Boussinesq approximation). The analytical method is based on Kataoka (2006a). That is, we solve the linear eigenvalue problem for disturbances using the asymptotic analysis for small eigenvalues, or small growth rates of linear disturbances. It is then found that an exchange of stability to superharmonic disturbances occurs when the wave energy density is stationary as a function of wave speed for fixed mean interface height.

It should be mentioned here that the asymptotic analysis used in several surface and interfacial-wave problems by the author (Kataoka & Tsutahara 2004; Kataoka 2006a,b) is systematically explained in Sone (2002, 2007) in association with its application to molecular-gas-dynamics problems.

2. Basic Equations

Consider a two-layer-fluid system where a lighter fluid lies above a heavier fluid under the uniform acceleration g due to gravity as shown in figure 1(a). The density of the upper fluid is ρ_U and that of the lower $\rho_L (> \rho_U)$. The fluids are incompressible, inviscid, and occupy a channel between two horizontal rigid boundaries of distance d apart. In what follows, index U refers to the upper fluid and index L to the lower, and all variables are non-dimensionalized using g, ρ_L, and a reference length l. For the moment we leave l unspecified (it is specified to be a wavelength of a periodic wave in the statement after (13)). The flow in each layer is irrotational and the effect of interfacial tension is neglected. Introducing the two-dimensional Cartesian coordinates x and y with y pointed vertically upward and their origin located at the bottom (see figure 1(b)), we obtain the following set of non-dimensional governing equations for the fluid motion:

$$\nabla^2 \phi_U = 0 \qquad \text{for } \eta < y < D, \tag{1}$$

$$\nabla^2 \phi_L = 0 \qquad \text{for } 0 < y < \eta, \tag{2}$$

$$\frac{\partial \phi_U}{\partial y} = 0 \qquad \text{at } y = D, \tag{3}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi_U}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi_U}{\partial y} \qquad \text{at } y = \eta, \tag{4}$$

$$\frac{\partial \eta}{\partial t} + \frac{\partial \phi_L}{\partial x}\frac{\partial \eta}{\partial x} = \frac{\partial \phi_L}{\partial y} \qquad \text{at } y = \eta, \tag{5}$$

$$-\rho\left\{\frac{\partial \phi_U}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi_U}{\partial x}\right)^2 + \left(\frac{\partial \phi_U}{\partial y}\right)^2\right]\right\} + \frac{\partial \phi_L}{\partial t} + \frac{1}{2}\left[\left(\frac{\partial \phi_L}{\partial x}\right)^2 + \left(\frac{\partial \phi_L}{\partial y}\right)^2\right] + (1-\rho)\eta = f(t) \quad \text{at } y = \eta, \tag{6}$$

$$\frac{\partial \phi_L}{\partial y} = 0 \qquad \text{at } y = 0, \tag{7}$$

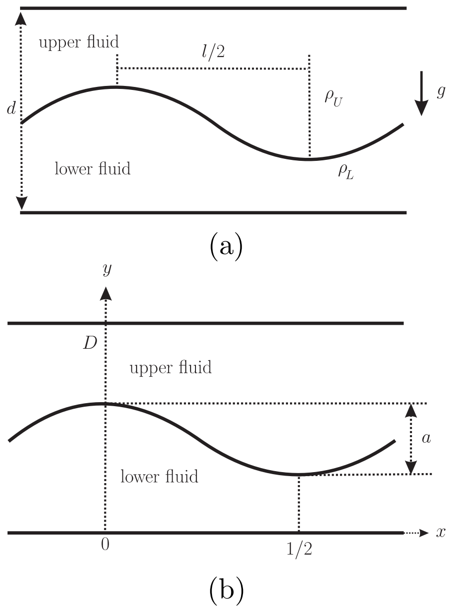

Figure 1. Schematic of the two-layer fluid system: (a) dimensional system, (b) non-dimensional system.

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{8}$$

is the Laplacian, t is the time, $\phi_U(x, y, t)$ and $\phi_L(x, y, t)$ are the velocity potentials of the upper and lower fluids, respectively, $\eta(x, t)$ is the height of the interface from the bottom, and $f(t)$ is a given function of t. ρ and D are, respectively, the density ratio of the two fluids and the non-dimensional total depth of a channel defined by

$$\rho = \frac{\rho_U}{\rho_L}, \quad D = \frac{d}{l}. \tag{9}$$

The cases of $0 \leq \rho < 1$ (statically stable) are considered throughout this paper.

Let us consider a steady solution of (1)-(7) in the following form:

$$\phi_U = -cx + \Phi_U(x, y; b, c, \rho, D), \tag{10}$$

$$\phi_L = -cx + \Phi_L(x, y; b, c, \rho, D), \tag{11}$$

$$\eta = \eta_I(x; b, c, \rho, D), \tag{12}$$

with

$$f(t) = b + \frac{1-\rho}{2}c^2, \tag{13}$$

where b, c, ρ, and D are positive parameters of the steady solution, while Φ_U, Φ_L, and η_I are functions of x (and y) which are periodic in x with unit period. The solution (10)-(12) represents a periodic wave that propagates steadily against a uniform stream of constant velocity $-c$ in the x direction in a two-layer fluid of density ratio ρ, non-dimensional total depth D, and Bernoulli constant (relative to the uniform stream $\phi_U = \phi_L = -cx$) b. We consider the solution of this class with all crests being of the same height and all troughs of a different same height, and call it 'basic wave solution'. Since the non-dimensional wavelength of this wave is set at unity, the reference length l of the system mentioned at the first paragraph of this section is the wavelength of this basic wave (see also figure 1). Substituting (10)-(12) with (13) into (1)-(7), we obtain a set of governing equations for Φ_U, Φ_L, and η_I as

$$\frac{\partial^2 \Phi_U}{\partial x^2} + \frac{\partial^2 \Phi_U}{\partial y^2} = 0 \quad \text{for } \eta_I < y < D, \tag{14}$$

$$\frac{\partial^2 \Phi_L}{\partial x^2} + \frac{\partial^2 \Phi_L}{\partial y^2} = 0 \quad \text{for } 0 < y < \eta_I, \tag{15}$$

$$\frac{\partial \Phi_U}{\partial y} = 0 \quad \text{at } y = D, \tag{16}$$

$$\left(-c + \frac{\partial \Phi_U}{\partial x}\right)\frac{\mathrm{d}\eta_I}{\mathrm{d}x} = \frac{\partial \Phi_U}{\partial y} \quad \text{at } y = \eta_I, \tag{17}$$

$$\left(-c + \frac{\partial \Phi_L}{\partial x}\right)\frac{\mathrm{d}\eta_I}{\mathrm{d}x} = \frac{\partial \Phi_L}{\partial y} \quad \text{at } y = \eta_I, \tag{18}$$

$$-\rho\left\{-c\frac{\partial \Phi_U}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \Phi_U}{\partial x}\right)^2 + \left(\frac{\partial \Phi_U}{\partial y}\right)^2\right]\right\} - c\frac{\partial \Phi_L}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial \Phi_L}{\partial x}\right)^2 + \left(\frac{\partial \Phi_L}{\partial y}\right)^2\right] + (1-\rho)\eta_I \;=\; b \quad \text{at } y = \eta_I, \tag{19}$$

$$\frac{\partial \Phi_L}{\partial y} = 0 \quad \text{at } y = 0. \tag{20}$$

Since Φ_U, Φ_L, and η_I are periodic in x with unit period, we can impose the condition $\int_{-1/2}^{1/2} \Phi_U \mathrm{d}x = 0$ at one particular level $y =$ constant above the wave crests, and hence (from (14) and (16)) at all levels within the upper fluid. In the same way, $\int_{-1/2}^{1/2} \Phi_L \mathrm{d}x = 0$ can be imposed at all levels within the lower fluid. According to previous numerical studies (Holyer 1979; Turner & Vanden-Broeck 1986), the basic wave is symmetric about its crest and trough, so that restricting our attention to such symmetric waves, we can choose the origin of the x coordinate such that Φ_U and Φ_L are odd and η_I is even in x.

In order to investigate the linear stability of the above basic wave with respect to disturbances that are periodic in one wavelength of the basic wave, the solution of (1)-(7) is expressed as summation of the basic wave solution (10)-(12) and a disturbance to it:

$$\phi_U = -cx + \Phi_U + \hat{\phi}_U(x,y)\exp(\lambda\, t), \tag{21}$$

$$\phi_L = -cx + \Phi_L + \hat{\phi}_L(x,y)\exp(\lambda\, t), \tag{22}$$

$$\eta = \eta_I + \hat{\eta}(x)\exp(\lambda\, t), \tag{23}$$

where λ is a complex constant to be determined, and $\hat{\phi}_U$, $\hat{\phi}_L$, and $\hat{\eta}$ are periodic in x with unit period. Substituting (21)–(23) into (1)-(7) and linearizing with respect to $(\hat{\phi}_U, \hat{\phi}_L, \hat{\eta})$, we obtain the following set of linear equations for $(\hat{\phi}_U, \hat{\phi}_L, \hat{\eta})$:

$$\nabla^2 \hat{\phi}_U = 0 \quad \text{for } \eta_I < y < D, \tag{24}$$

$$\nabla^2 \hat{\phi}_L = 0 \quad \text{for } 0 < y < \eta_I, \tag{25}$$

$$\frac{\partial \hat{\phi}_U}{\partial y} = 0 \quad \text{at } y = D, \tag{26}$$

$$\mathrm{L}_U[\hat{\phi}_U, \hat{\eta}] = -\lambda \hat{\eta} \quad \text{at } y = \eta_I, \tag{27}$$

$$\mathrm{L}_L[\hat{\phi}_L, \hat{\eta}] = -\lambda \hat{\eta} \quad \text{at } y = \eta_I, \tag{28}$$

$$\mathrm{L}_I[\hat{\phi}_U, \hat{\phi}_L, \hat{\eta}] = \lambda(\rho \hat{\phi}_U - \hat{\phi}_L) \quad \text{at } y = \eta_I, \tag{29}$$

$$\frac{\partial \hat{\phi}_L}{\partial y} = 0 \quad \text{at } y = 0, \tag{30}$$

where L_U, L_L, and L_I are the linear operators defined by

$$\begin{aligned} \mathrm{L}_U[\hat{\phi}_U, \hat{\eta}] \; = \; & \left(-\frac{\partial}{\partial y} + \frac{\mathrm{d}\eta_I}{\mathrm{d}x} \frac{\partial}{\partial x} \right) \hat{\phi}_U \\ & + \left[\left(\frac{\partial^2 \Phi_U}{\partial x^2} + \frac{\partial^2 \Phi_U}{\partial x \partial y} \frac{\mathrm{d}\eta_I}{\mathrm{d}x} \right) + \left(-v + \frac{\partial \Phi_U}{\partial x} \right) \frac{\mathrm{d}}{\mathrm{d}x} \right] \hat{\eta}, \end{aligned} \tag{31}$$

$$\begin{aligned} \mathrm{L}_L[\hat{\phi}_L, \hat{\eta}] \; = \; & \left(-\frac{\partial}{\partial y} + \frac{\mathrm{d}\eta_I}{\mathrm{d}x} \frac{\partial}{\partial x} \right) \hat{\phi}_L \\ & + \left[\left(\frac{\partial^2 \Phi_L}{\partial x^2} + \frac{\partial^2 \Phi_L}{\partial x \partial y} \frac{\mathrm{d}\eta_I}{\mathrm{d}x} \right) + \left(-v + \frac{\partial \Phi_L}{\partial x} \right) \frac{\mathrm{d}}{\mathrm{d}x} \right] \hat{\eta}, \end{aligned} \tag{32}$$

$$\begin{aligned} \mathrm{L}_I[\hat{\phi}_U, \hat{\phi}_L, \hat{\eta}] \; = \; & -\rho \left[\left(-v + \frac{\partial \Phi_U}{\partial x} \right) \frac{\partial}{\partial x} + \frac{\partial \Phi_U}{\partial y} \frac{\partial}{\partial y} \right] \hat{\phi}_U \\ & + \left[\left(-v + \frac{\partial \Phi_L}{\partial x} \right) \frac{\partial}{\partial x} + \frac{\partial \Phi_L}{\partial y} \frac{\partial}{\partial y} \right] \hat{\phi}_L \\ & + \left\{ -\rho \left[\left(-v + \frac{\partial \Phi_U}{\partial x} \right) \frac{\partial^2 \Phi_U}{\partial x \partial y} + \frac{\partial \Phi_U}{\partial y} \frac{\partial^2 \Phi_U}{\partial y^2} \right] \right. \\ & \left. + \left(-v + \frac{\partial \Phi_L}{\partial x} \right) \frac{\partial^2 \Phi_L}{\partial x \partial y} + \frac{\partial \Phi_L}{\partial y} \frac{\partial^2 \Phi_L}{\partial y^2} + 1 - \rho \right\} \hat{\eta}. \end{aligned} \tag{33}$$

Equations (24)-(30) together with periodic conditions at $x = \pm 1/2$ constitute an eigenvalue problem for $(\hat{\phi}_U, \hat{\phi}_L, \hat{\eta})$ whose eigenvalue is λ. When this problem possesses a solution whose eigenvalue λ has a positive real part, the corresponding basic wave is superharmonically unstable.

Before proceeding to the next section, we define two properties of the basic wave, that is, the mean interface height from the bottom and the wave energy density. The mean interface height $\bar{\eta}_I$ from the bottom is

$$\bar{\eta}_I\,(b,c,\rho,D) \equiv \int_{-1/2}^{1/2} \eta_I \mathrm{d}x. \tag{34}$$

For the wave energy density, we must decide how the potential energy is defined, and it is defined in terms of the interfacial displacement measured from the mean interface height $y = \bar{\eta}_I$. The corresponding wave energy density E is

$$\begin{aligned} E\,(b,c,\rho,D) \equiv\; & \frac{\rho}{2}\int_{-1/2}^{1/2}\mathrm{d}x\int_{\eta_I}^{D}\left[\left(\frac{\partial\Phi_U}{\partial x}\right)^2+\left(\frac{\partial\Phi_U}{\partial y}\right)^2\right]\mathrm{d}y \\ & +\frac{1}{2}\int_{-1/2}^{1/2}\mathrm{d}x\int_{0}^{\eta_I}\left[\left(\frac{\partial\Phi_L}{\partial x}\right)^2+\left(\frac{\partial\Phi_L}{\partial y}\right)^2\right]\mathrm{d}y \\ & +\frac{1-\rho}{2}\int_{-1/2}^{1/2}\left(\eta_I-\bar{\eta}_I\right)^2\mathrm{d}x. \end{aligned} \tag{35}$$

3. Asymptotic Analysis

We make an asymptotic analysis of (24)-(30) for small $|\lambda|$ to study how an exchange of stability occurs. To this end, we seek a solution $(\hat{\phi}_U, \hat{\phi}_L, \hat{\eta})$ of (24)-(30) whose appreciable variation occurs in x and y of the order of unity (i.e. $\partial\hat{h}/\partial x = O(\hat{h})$ and $\partial\hat{h}/\partial y = O(\hat{h})$, where $\hat{h}$ represents $(\hat{\phi}_U, \hat{\phi}_L, \hat{\eta})$), in the following power series of λ:

$$\hat{\phi}_U = \hat{\phi}_{U0} + \lambda\hat{\phi}_{U1} + \lambda^2\hat{\phi}_{U2} + \cdots, \tag{36}$$

$$\hat{\phi}_L = \hat{\phi}_{L0} + \lambda\hat{\phi}_{L1} + \lambda^2\hat{\phi}_{L2} + \cdots, \tag{37}$$

$$\hat{\eta} = \hat{\eta}_0 + \lambda\hat{\eta}_1 + \lambda^2\hat{\eta}_2 + \cdots, \tag{38}$$

where the component functions $\hat{\phi}_{Un}$, $\hat{\phi}_{Ln}$, and $\hat{\eta}_n$ are of the order of unity. Substituting the series (36)–(38) into (24)-(30) and arranging the same-order terms in $|\lambda|$, we obtain a series of sets of equations for $(\hat{\phi}_{Un}, \hat{\phi}_{Ln}, \hat{\eta}_n)$ $(n = 0, 1, 2, \ldots)$:

$$\nabla^2\hat{\phi}_{Un} = 0 \qquad \text{for } \eta_I < y < D, \tag{39}$$

$$\nabla^2\hat{\phi}_{Ln} = 0 \qquad \text{for } 0 < y < \eta_I, \tag{40}$$

$$\frac{\partial\hat{\phi}_{Un}}{\partial y} = 0 \qquad \text{at } y = D, \tag{41}$$

$$\mathrm{L}_U[\hat{\phi}_{Un}, \hat{\eta}_n] = -\hat{\eta}_{n-1} \quad \text{at } y = \eta_I, \tag{42}$$

$$\mathrm{L}_L[\hat{\phi}_{Ln}, \hat{\eta}_n] = -\hat{\eta}_{n-1} \quad \text{at } y = \eta_I, \tag{43}$$

$$\mathrm{L}_I[\hat{\phi}_{Un}, \hat{\phi}_{Ln}, \hat{\eta}_n] = \rho\hat{\phi}_{Un-1} - \hat{\phi}_{Ln-1} \quad \text{at } y = \eta_I, \tag{44}$$

$$\frac{\partial \hat{\phi}_{Ln}}{\partial y} = 0 \quad \text{at } y = 0, \tag{45}$$

where ∇^2, L_U, L_L, and L_I are defined by (8) and (31)–(33). Note that $\hat{\phi}_{Un}$, $\hat{\phi}_{Ln}$, and $\hat{\eta}_n$ ($n = 0, 1, 2, \ldots$) are periodic in x with unit period, and $\hat{\phi}_{U-1} = \hat{\phi}_{L-1} = \hat{\eta}_{-1} = 0$.

At $n = 0$, the above set of equations (39)–(45) is homogeneous, and has a solution

$$\hat{\phi}_{U0} = \alpha_U + \beta\frac{\partial \Phi_U}{\partial x}, \quad \hat{\phi}_{L0} = \alpha_L + \beta\frac{\partial \Phi_L}{\partial x}, \quad \hat{\eta}_0 = \beta\frac{\mathrm{d}\eta_I}{\mathrm{d}x}, \tag{46}$$

where α_U, α_L, and β are undetermined constants. The fundamental solution $(\partial\Phi_U/\partial x, \partial\Phi_L/\partial x, \mathrm{d}\eta_I/\mathrm{d}x)$ in (46) arises from the invariance of the system (14)–(20) under the horizontal shift, or the shift in x.

At $n = 1, 2, \ldots$, the set of equations (39)–(45) is inhomogeneous. Since the homogeneous part satisfies

$$\int_{-1/2}^{1/2} \mathrm{d}x \int_{\eta_I}^{D} \nabla^2 \hat{\phi}_{Un} \mathrm{d}y + \int_{-1/2}^{1/2} \Big[\mathrm{L}_U[\hat{\phi}_{Un}, \hat{\eta}_n]\Big]_{y=\eta_I} \mathrm{d}x \; = \; 0, \tag{47}$$

$$\int_{-1/2}^{1/2} \mathrm{d}x \int_{0}^{\eta_I} \nabla^2 \hat{\phi}_{Ln} \mathrm{d}y + \int_{-1/2}^{1/2} \Big[\mathrm{L}_L[\hat{\phi}_{Ln}, \hat{\eta}_n]\Big]_{y=\eta_I} \mathrm{d}x \; = \; 0, \tag{48}$$

$$\begin{aligned}\int_{-1/2}^{1/2} &\left(\int_{\eta_I}^{D} \rho\frac{\partial \Phi_U}{\partial x}\nabla^2\hat{\phi}_{Un}\mathrm{d}y + \int_0^{\eta_I}\frac{\partial \Phi_L}{\partial x}\nabla^2\hat{\phi}_{Ln}\mathrm{d}y\right)\mathrm{d}x \\ &+ \int_{-1/2}^{1/2}\Bigg[-\rho\frac{\partial \Phi_U}{\partial x}\mathrm{L}_U[\hat{\phi}_{Un}, \hat{\eta}_n] + \frac{\partial \Phi_L}{\partial x}\mathrm{L}_L[\hat{\phi}_{Ln}, \hat{\eta}_n] \\ &\qquad -\frac{\mathrm{d}\eta_I}{\mathrm{d}x}\mathrm{L}_I[\hat{\phi}_{Un}, \hat{\phi}_{Ln}, \hat{\eta}_n]\Bigg]_{y=\eta_I}\mathrm{d}x \; = \; 0,\end{aligned} \tag{49}$$

the corresponding inhomogeneous terms $-\hat{\eta}_{n-1}$ and $\rho\hat{\phi}_{Un-1} - \hat{\phi}_{Ln-1}$ ($n = 1, 2, \ldots$) on the right-hand sides of (42)–(44) must satisfy the solvability conditions:

$$\sum_{m=1}^{n} \lambda^m \int_{-1/2}^{1/2} \hat{\eta}_{m-1} \mathrm{d}x = o\left(|\lambda|^n\right), \tag{50}$$

$$\sum_{m=1}^{n} \lambda^m \int_{-1/2}^{1/2} \left[\left(\rho \frac{\partial \Phi_U}{\partial x} - \frac{\partial \Phi_L}{\partial x} \right) \hat{\eta}_{m-1} - \frac{\mathrm{d}\eta_I}{\mathrm{d}x} \left(\rho \hat{\phi}_{Um-1} - \hat{\phi}_{Lm-1} \right) \right]_{y=\eta_I} \mathrm{d}x = o\left(|\lambda|^n\right), \quad (51)$$

where the first two relations (47)–(48) reduce to a single condition (50). The quantities in the square brackets with subscript $y = \eta_I$ (or $[\ ]_{y=\eta_I}$) are evaluated at $y = \eta_I$, and $o\left(|\lambda|^n\right)$ represents terms of order smaller than $|\lambda|^n$.

Substituting (46) into (50)–(51) at $n = 1$, we find that the solvability conditions at $n = 1$ are identically satisfied, and a solution of (39)–(45) at $n = 1$ is

$$\begin{aligned} \hat{\phi}_{U1} &= (\rho\alpha_U - \alpha_L) \frac{\partial \Phi_U}{\partial b} - \beta \frac{\partial \Phi_U}{\partial c}, \\ \hat{\phi}_{L1} &= (\rho\alpha_U - \alpha_L) \frac{\partial \Phi_L}{\partial b} - \beta \frac{\partial \Phi_L}{\partial c}, \\ \hat{\eta}_1 &= (\rho\alpha_U - \alpha_L) \frac{\partial \eta_I}{\partial b} - \beta \frac{\partial \eta_I}{\partial c}, \end{aligned} \quad (52)$$

where $\partial/\partial b$ denotes the derivative with respect to b for fixed x, y, c, ρ, D, and $\partial/\partial c$ denotes that with respect to c for fixed x, y, b, ρ, and D. The homogeneous solution (46) multiplied by an arbitrary constant is omitted in (52), since it can be included in the leading-order solution (46). One can check that the solution (52) satisfies (39)–(45) at $n = 1$ by differentiating (14)–(20) with respect to b or c.

Substituting (52) into (50)–(51) at $n = 2$, we find that the solvability conditions at $n = 2$ reduce to

$$\lambda^2 \left[(\alpha_L - \rho\alpha_U) \frac{\partial \bar{\eta}_I}{\partial b} + \beta \frac{\partial \bar{\eta}_I}{\partial c} \right] = o\left(|\lambda|^2\right), \quad (53)$$

$$\begin{aligned} \frac{\lambda^2}{c} \Bigg\{ (\alpha_L - \rho\alpha_U) \left[\frac{\partial E}{\partial b} + [(1-\rho)\bar{\eta}_I - b] \frac{\partial \bar{\eta}_I}{\partial b} \right] & \\ + \beta \left[\frac{\partial E}{\partial c} + [(1-\rho)\bar{\eta}_I - b] \frac{\partial \bar{\eta}_I}{\partial c} \right] \Bigg\} &= o\left(|\lambda|^2\right), \end{aligned} \quad (54)$$

where $\bar{\eta}_I$ and E are defined by (34) and (35), respectively. The key to deriving (53)–(54) is to note that differentiation of E with respect to b or c gives

$$\frac{\partial E}{\partial b} = c \int_{-1/2}^{1/2} \left[\left(-\rho \frac{\partial \Phi_U}{\partial x} + \frac{\partial \Phi_L}{\partial x} \right) \frac{\partial \eta_I}{\partial b} + \frac{\mathrm{d}\eta_I}{\mathrm{d}x} \left(\rho \frac{\partial \Phi_U}{\partial b} - \frac{\partial \Phi_L}{\partial b} \right) \right]_{y=\eta_I} \mathrm{d}x$$

$$-\left[(1-\rho)\bar{\eta}_I - b\right]\frac{\partial \bar{\eta}_I}{\partial b}, \qquad (55)$$

$$\frac{\partial E}{\partial c} = c\int_{-1/2}^{1/2}\left[\left(-\rho\frac{\partial \Phi_U}{\partial x} + \frac{\partial \Phi_L}{\partial x}\right)\frac{\partial \eta_I}{\partial c} + \frac{\mathrm{d}\eta_I}{\mathrm{d}x}\left(\rho\frac{\partial \Phi_U}{\partial c} - \frac{\partial \Phi_L}{\partial c}\right)\right]_{y=\eta_I}\mathrm{d}x$$
$$-\left[(1-\rho)\bar{\eta}_I - b\right]\frac{\partial \bar{\eta}_I}{\partial c}, \qquad (56)$$

with the aid of (14)–(20). From (53)–(54), the condition for the existence of solution $(\hat{\phi}_{U2}, \hat{\phi}_{L2}, \hat{\eta}_2)$ with nonzero $(\alpha_L - \rho\alpha_U, \beta)$ is

$$\lambda^2 W_1 = o\left(|\lambda|^2\right), \qquad (57)$$

where

$$W_1 = \frac{\partial E}{\partial c} - \left(\frac{\partial \bar{\eta}_I}{\partial b}\right)^{-1}\frac{\partial \bar{\eta}_I}{\partial c}\frac{\partial E}{\partial b}. \qquad (58)$$

When the condition (57) is satisfied, the solution $(\hat{\phi}_{U2}, \hat{\phi}_{L2}, \hat{\eta}_2)$ of (39)–(45) at $n = 2$ exists, although its explicit form cannot be obtained. Here we only find that $\hat{\phi}_{U2}$ and $\hat{\phi}_{L2}$ are even and $\hat{\eta}_2$ is odd in x by examining the order of the differential operators with respect to x of (39)–(45) and noting the parity in x of the basic wave solution (Φ_U, Φ_L, η_I) (Φ_U and Φ_L are odd and η_I is even; see the statement after (20)) and that of the inhomogeneous terms $-\hat{\eta}_1$ and $\rho\hat{\phi}_{U1} - \hat{\phi}_{L1}$ ($-\hat{\eta}_1$ is even and $\rho\hat{\phi}_{U1} - \hat{\phi}_{L1}$ is odd).

Let us proceed to the next order $n = 3$. From the parity in x of (Φ_U, Φ_L, η_I) and $(\hat{\phi}_{U2}, \hat{\phi}_{L2}, \hat{\eta}_2)$ mentioned above, the integrands of $O\left(|\lambda|^3\right)$ in the solvability conditions (50)–(51) are odd in x, so that the corresponding integrals vanish. The solvability conditions at $n = 3$, therefore, remain the same form as those at $n = 2$ and the solution $(\hat{\phi}_{U3}, \hat{\phi}_{L3}, \hat{\eta}_3)$ of (39)–(45) at $n = 3$ exists. By examining the order of differential operators with respect to x of (39)–(45) and noting the parity in x of the basic wave solution (Φ_U, Φ_L, η_I) (Φ_U and Φ_L are odd and η_I is even) and that of the inhomogeneous terms $-\hat{\eta}_2$ and $\rho\hat{\phi}_{U2} - \hat{\phi}_{L2}$ ($-\hat{\eta}_2$ is odd and $\rho\hat{\phi}_{U2} - \hat{\phi}_{L2}$ is even), we find that $\hat{\phi}_{U3}$ and $\hat{\phi}_{L3}$ are odd and $\hat{\eta}_3$ is even in x.

Let us proceed to the next order $n = 4$. The solvability conditions at $n = 4$ are

$$\lambda^2\left[(\alpha_L - \rho\alpha_U)\frac{\partial \bar{\eta}_I}{\partial b} + \beta\frac{\partial \bar{\eta}_I}{\partial c}\right] + \lambda^4\left[(\alpha_L - \rho\alpha_U)S_1 + \beta S_2\right] = o\left(|\lambda|^4\right), \qquad (59)$$

$$\frac{\lambda^2}{c}\left\{(\alpha_L-\rho\alpha_U)\left[\frac{\partial E}{\partial b}+[(1-\rho)\bar{\eta}_I-b]\,\frac{\partial\bar{\eta}_I}{\partial b}\right]+\beta\left[\frac{\partial E}{\partial c}+[(1-\rho)\bar{\eta}_I-b]\,\frac{\partial\bar{\eta}_I}{\partial c}\right]\right\}$$
$$+\lambda^4\left[(\alpha_L-\rho\alpha_U)\,S_3+\beta S_4\right]=o\left(|\lambda|^4\right), \qquad (60)$$

where S_1, S_2, S_3, and S_4 are obtained by equating terms with $\alpha_L-\rho\alpha_U$ and those with β of the following equations:

$$(\alpha_L-\rho\alpha_U)S_1+\beta S_2=\int_{-1/2}^{1/2}-\hat{\eta}_3\mathrm{d}x, \qquad (61)$$

$$(\alpha_L-\rho\alpha_U)S_3+\beta S_4=\int_{-1/2}^{1/2}\left[\left(\rho\frac{\partial\Phi_U}{\partial x}-\frac{\partial\Phi_L}{\partial x}\right)\hat{\eta}_3-\frac{\mathrm{d}\eta_I}{\mathrm{d}x}\left(\rho\hat{\phi}_{U3}-\hat{\phi}_{L3}\right)\right]_{y=\eta_I}\mathrm{d}x. \qquad (62)$$

The condition for the existence of solution $(\hat{\phi}_{U4},\hat{\phi}_{L4},\hat{\eta}_4)$ with nonzero $(\alpha_L-\rho\phi_U,\beta)$ is

$$\lambda^2W_1+\lambda^4W_2=o\left(|\lambda|^4\right), \qquad (63)$$

where W_1 is given by (58), and

$$\begin{aligned}W_2 \;=\; &\left[\frac{\partial E}{\partial c}+[(1-\rho)\bar{\eta}_I-b]\,\frac{\partial\bar{\eta}_I}{\partial c}\right]\left(\frac{\partial\bar{\eta}_I}{\partial b}\right)^{-1}S_1\\ &-\left[\frac{\partial E}{\partial b}\left(\frac{\partial\bar{\eta}_I}{\partial b}\right)^{-1}+[(1-\rho)\bar{\eta}_I-b]\right]S_2-c\left(\frac{\partial\bar{\eta}_I}{\partial b}\right)^{-1}\frac{\partial\bar{\eta}_I}{\partial c}S_3+cS_4.\end{aligned} \qquad (64)$$

Since $\hat{\phi}_{U3}$, $\hat{\phi}_{L3}$, Φ_U, and Φ_L are odd, and $\hat{\eta}_3$ and η_I are even in x, the integrands on the right-hand sides of (61)–(62) are even in x, so that the corresponding integrals rarely vanish. Therefore, W_2 defined by (64) is nonzero in general, and (63) gives

$$\lambda=\begin{cases}0,\ \pm\sqrt{\left|\dfrac{W_1}{W_2}\right|} & \text{when } \dfrac{W_1}{W_2}<0,\\[2ex] 0,\ \pm\mathrm{i}\sqrt{\dfrac{W_1}{W_2}} & \text{when } \dfrac{W_1}{W_2}>0.\end{cases} \qquad (65)$$

The solution (65), which is valid for $|\lambda| << 1$, indicates that an exchange of stability occurs at $W_1=0$ where W_1 is defined by (58). If we denote the derivative with respect to c for fixed $\bar{\eta}_I$, ρ, and D, by $\partial/\partial c|_{\bar{\eta}_I}$, or

$$\left.\frac{\partial}{\partial c}\right|_{\bar{\eta}_I}\equiv\frac{\partial}{\partial c}-\left(\frac{\partial\bar{\eta}_I}{\partial b}\right)^{-1}\frac{\partial\bar{\eta}_I}{\partial c}\frac{\partial}{\partial b},$$

W_1 is rewritten as

$$W_1 = \left.\frac{\partial E}{\partial c}\right|_{\bar{\eta}_I}. \tag{66}$$

Thus, an exchange of stability occurs at the stationary value of the wave energy density E for fixed $\bar{\eta}_I$, ρ, and D, where E is defined by (35), $\bar{\eta}_I$ is the mean interface height defined by (34), and ρ and D are, respectively, the density ratio and the non-dimensional total depth of the two fluids defined by (9). This is the main result of this paper. Wave speeds c and crest-to-trough heights $a \equiv |\eta_I(0) - \eta_I(1/2)|$ (see also figure 1(b)) at which an exchange of stability occurs for $\rho = 0.05$ and $D = 1$ are shown in table 1 as specific examples. We have obtained these results from numerical computation of the basic wave on the basis of Turner & Vanden-Broeck (1988).

Table 1. Wave speeds c and crest-to-trough heights $a \equiv |\eta_I(0) - \eta_I(1/2)|$ of periodic interfacial waves at which an exchange of stability occurs for various values of $b/(1-\rho)$, when $\rho = 0.05$ and $D = 1$. The corresponding mean interface heights $\bar{\eta}_I$ defined by (34) are also shown.

$b/(1-\rho)$	c	a	$\bar{\eta}_I$
0.01	0.134	0.0121	0.009963
0.02	0.183	0.0233	0.01987
0.03	0.218	0.0338	0.02975
0.05	0.267	0.0530	0.04947
0.1	0.338	0.0936	0.09887
0.2	0.402	0.147	0.1989
0.3	0.425	0.170	0.2995
0.5	0.435	0.180	0.49996

If we introduce the impulse density I of the basic wave defined by

$$I(b, c, \rho, D) \equiv \int_{-1/2}^{1/2} \left(\rho \int_{\eta_I}^{D} \frac{\partial \Phi_U}{\partial x} \mathrm{d}y + \int_0^{\eta_I} \frac{\partial \Phi_L}{\partial x} \mathrm{d}y \right) \mathrm{d}x, \tag{67}$$

we have the following relations:

$$c\frac{\partial I}{\partial b} = \frac{\partial E}{\partial b} + [(1-\rho)\bar{\eta}_I - b] \frac{\partial \bar{\eta}_I}{\partial b}, \tag{68}$$

$$c\frac{\partial I}{\partial c} = \frac{\partial E}{\partial c} + [(1-\rho)\bar{\eta}_I - b] \frac{\partial \bar{\eta}_I}{\partial c}, \tag{69}$$

with the aid of (14)–(20). Thus, W_1 defined by (58) or (66) is expressed as

$$W_1 = c \left.\frac{\partial I}{\partial c}\right|_{\bar{\eta}_I}, \tag{70}$$

which indicates that an exchange of stability occurs at the stationary value of I for fixed $\bar{\eta}_I$, ρ, and D, or $\partial I/\partial c|_{\bar{\eta}_I} = 0$.

4. Concluding Remarks

The linear stability of periodic interfacial waves to superharmonic disturbances is examined analytically on the basis of the Euler set of equations. It is found that an exchange of stability occurs when the wave energy density is stationary as a function of wave speed for fixed mean interface height. The physical mechanism for this law is, however, still an open problem both for surface and interfacial waves. See a series of papers by Longuet-Higgins (Longuet-Higgins & Cleaver 1994; Longuet-Higgins, Cleaver & Fox 1994; Longuet-Higgins & Dommermuth 1997) and statement in the last paragraph of Kataoka (2006a) for suggestions and discussion on this point.

References

1. Benjamin, T. B. and Feir, J. E., The disintegration of wave trains on deep water. Part 1. Theory, *J. Fluid Mech.*, **27**(1967), 417-430.

2. Holyer, J. Y., Large amplitude progressive interfacial waves, *J. Fluid Mech.*, **93**(1979), 433-448.

3. Kataoka, T., On the superharmonic instability of surface gravity waves on fluid of finite depth, *J. Fluid Mech.*, **547**(2006a), 175-184.

4. Kataoka, T., The stability of finite-amplitude interfacial solitary waves, *Fluid Dyn. Res.*, **38**(2006b), 831-867.

5. Kataoka, T. and Tsutahara, M., Transverse instability of surface solitary waves, *J. Fluid Mech.*, **512**(2004), 211-221.

6. Longuet-Higgins, M. S., The instabilities of gravity waves of finite amplitude in deep water. I. Superharmonics, *Proc. R. Soc. Lond. Ser. A*, **360**(1978), 471-488.

7. Longuet-Higgins, M. S. and Cleaver, R. P., Crest instabilities of gravity waves. Part 1. The almost-highest wave, *J. Fluid Mech.*, **258**(1994), 115-129.

8. Longuet-Higgins, M. S., Cleaver, R. P. and Fox, M. J. H., Crest instabilities of gravity waves. Part 2. Matching and asymptotic analysis, *J. Fluid Mech.*, **259**(1994), 333-344.

9. Longuet-Higgins, M. S. and Dommermuth, D. G., Crest instabilities of gravity waves. Part 3. Nonlinear development and breaking, *J. Fluid Mech.*, **336**(1997), 33-50.

10. Phillips, O. M., On the dynamics of unsteady gravity waves of finite amplitude. Part 1. The elementary interactions, *J. Fluid Mech.*, **9**(1960), 193-217.

11. Pullin, D. I. and Grimshaw, R. H. J., Stability of finite-amplitude interfacial waves. Part 2. Numerical results, *J. Fluid Mech.* **160**(1985), 317-336.

12. Saffman, P. G., The superharmonic instability of finite-amplitude water waves, *J. Fluid Mech.*, **159**(1985), 169-174.

13. Sone, Y., *Kinetic Theory of Fluid Dynamics*, Birkhäuser, 2002.

14. Sone, Y., *Molecular Gas Dynamics*, Birkhäuser, 2007.

15. Tanaka, M., The stability of steep gravity waves. *J. Phys. Soc. Jpn.* **52**(1983), 3047-3055.

16. Turner, R. E. L. and Vanden-Broeck, J.-M., The limiting configuration of interfacial gravity waves, *Phys. Fluids*, **29**(1986), 372-375.

17. Turner, R. E. L. and Vanden-Broeck, J.-M. Broadning of interfacial solitary waves, *Phys. Fluids*, **31**(1988), 2486-2490.

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