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L^1-CONTINUOUS DEPENDENCE OF MILD SOLUTIONS TO THE FOKKER-PLANCK-BOLTZMANN EQUATION

BY

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Abstract

We present a uniform L^1-stability estimate for mild solutions to the Fokker-Planck-Boltzmann equation. For stability estimate, we derive a Gronwall type estimate using dispersion estimates for mild solutions due to the hypoelliptic structure of the Vlasov-Fokker-Planck operator.

1. Introduction

This paper is devoted to the uniform L^1-stability estimate of mild solutions to the frictionless Fokker-Planck-Boltzmnn (in short FPB) equation governing the dynamics of dilute gas particles interacting with its environment such as a thermal bath. Let $f = f(x, \xi, t)$ be the phase space density of a dilute gas whose local macroscopic quantities are given as the moments of f in velocity space. In the high temperature limit, the dynamics of a phase space density is governed by

$$\begin{aligned} \partial_t f + \xi \cdot \nabla_x f &= \sigma \Delta_\xi f + Q(f, f), \quad x, \xi \in \mathbb{R}^3, \quad t > 0, \\ f(x, \xi, 0) &= f^{in}(x, \xi). \end{aligned} \tag{1.1}$$

Here σ is a diffusion coefficient (see [2] for explicit formula), and $Q(f, f)$ is the collision operator whose explicit form will be given below. Let (ξ, ξ_*) and (ξ', ξ'_*) be pairs of pre-collisional and post-collisional velocities satisfying a

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collision transformation:

$$\xi' = \xi - [(\xi - \xi_*) \cdot \omega)]\omega \quad \text{and} \quad \xi'_* = \xi_* + [(\xi - \xi_*) \cdot \omega]\omega, \quad \omega \in \mathbf{S}^2_+, \tag{1.2}$$

where $v \cdot w$ is the standard inner product between v and w in $\mathbb{R}^3$ and $\mathbf{S}^2_+ = \{\omega \in \mathbf{S}^2 : (\xi - \xi_*) \cdot \omega \geq 0\}$. The Boltzmann collision operator $Q(f, f)$ takes the form of

$$Q(f, f)(x, \xi, t) \equiv \frac{1}{\kappa} \int_{\mathbb{R}^3 \times \mathbf{S}^2_+} q(\xi - \xi_*, \omega)(f' f'_* - f f_*) d\omega d\xi_*, \tag{1.3}$$

where κ is the Knudsen number which is the ratio between the mean free path and characteristic length of the flow, and we have used standard abbreviated notations:

$$f = f(x, \xi, t), \quad f_* = f(x, \xi_*, t), \quad f' = f(x, \xi', t), \quad f'_* = f(x, \xi'_*, t).$$

We assume that the collision kernel $q(\xi - \xi_*, \omega)$ satisfies an angular cut-off soft potential assumption:

$$q(\xi - \xi_*, \omega) = |\xi - \xi_*|^\gamma b_\gamma(\theta), \quad -2 < \gamma < 0, \quad \frac{b_\gamma(\theta)}{\cos\theta} \leq b_* < \infty, \tag{1.4}$$

where θ is the angle between $\xi - \xi_*$ and ω. In the sequel, C denotes a generic positive constant which may depend on initial data, but is independent of t. Let $G = G(x, \xi, t; y, \xi_*, s)$ be the Green's function to the Fokker-Planck equation which is the linear part of the FPB equation (1.1). Then the equation (1.1) can be rewritten as a mild form (see Section 2):

$$\begin{aligned} f(x, \xi, t) \; &= \int_{\mathbb{R}^6} G(x, \xi, t; y, \xi_*, 0) f^{in}(y, \xi_*) d\xi_* dy \\ &\quad + \int_0^t \int_{\mathbb{R}^6} G(x, \xi, t; y, \xi_*, s) Q(f, f)(y, \xi_*, s) d\xi_* dy ds. \end{aligned} \tag{1.5}$$

The definition of mild solution is given as follows.

Definition 1.1. Suppose that $f^{in} \in L^1(\mathbb{R}^3 \times \mathbb{R}^3)$ and $T > 0$. Let $f \in C([0, T); L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ be a mild solution to (1.1) corresponding to initial datum f^{in} if and only if f satisfies the mild form (1.5) pointwise a.e. $(x, \xi) \in \mathbb{R}^3 \times \mathbb{R}^3$.

The FPB equation has been used to model dissipative particle dynamics in the area of aerosols [12, 13]. However compared to other kinetic equations, the FPB equation has not been much studied in previous literatures. The global existence and vanishing viscosity limit of mild and renormalized solutions were studied in [4, 8, 9, 11], when initial datum is a perturbation of a vacuum and a global Maxwellian. In particular, the existence theory of mild solutions in [8] is restricted to the soft potential case $\gamma \in (-2, 0)$ and the corresponding theory for the hard potential case is still open. This is a rather strange situation, because when the diffusion coefficient σ is turned off, the resulting equation, "Boltzmann equation" near vacuum is well understood for the hard potential case in existence and stability aspects, hence we can expect when the good regularizing term $\sigma\Delta_\xi f$ is added to the Boltzmann equation, the resulting equation, the FPB equation should behave better than the Boltzmann equation itself. However the Illner-Shinbrot's trick [10] for the Boltzmann equation, which interchanges time-integral and velocity-integral is difficult to implement due to the complicated pointwise nature of mild solutions. We set an exponentially decaying function φ:

$$\varphi(x, \xi; \alpha, \beta, \lambda) \equiv K(x - \beta\xi, \frac{\lambda^3}{12})K(\xi, \alpha), \qquad \alpha, \beta, \lambda > 0,$$
$$\text{where} \quad K(x, t) = \frac{1}{\sqrt{(4\pi t)^3}} e^{-\frac{|x|^2}{4t}}.$$

Assumption $(\mathcal{A})$: For positive constants α, β, λ,

$$f^{in}(x, \xi) \leq \delta\varphi(x, \xi; \alpha, \beta, \lambda), \qquad 0 < \delta \ll 1.$$

The main result of this paper as follows.

Theorem 1.2. *Suppose that $\gamma \in (-2, 0)$. Let f and $\bar{f}$ be two mild solutions to* (1.1) *corresponding to initial data f^{in} and $\bar{f}^{in}$ satisfying assumption* $(\mathcal{A})$ *respectively. Then we have a uniform L^1-stability estimate:*

$$\sup_{0 \leq t < \infty} ||f(t) - \bar{f}(t)||_{L^1} \leq M||f^{in} - \bar{f}^{in}||_{L^1}.$$

Here M is a positive constant independent of time t and $||\cdot||_{L^1} = ||\cdot||_{L^1(\mathbb{R}^3 \times \mathbb{R}^3)}$.

Remark 1.3.

1. The global existence of mild solutions for soft potential case $\gamma \in (-2, 0)$ has been established in [8].

2. In the absence of Fokker-Planck term $\sigma\Delta_\xi f$ in (1.1), a uniform L^p-stability estimate has been studied in [5, 6].

The rest of the paper is organized as follows. In Section 2, we briefly review the basic properties of the FPB equation and existence framework of Hamdache in [8]. Finally, Section 3 is devoted to the uniform L^1-stability estimate of mild solutions.

2. Preliminaries

In this section, we briefly review the basics of the FPB equation (1.1) and Hamdache's existence framework on small mild solutions.

2.1. Formal balance laws and H-theorem

In this part, we study formal conservation laws and H-theorem to (1.1).

Lemma 2.1.([3]) *Let f be a solution of* (1.1). *Then we have*

(1) $$\frac{d}{dt}\int_{\mathbb{R}^3}\Big(1,\xi,\frac{|\xi|^2}{2}\Big)Q(f,f)d\xi = 0.$$

(2) $$\frac{d}{dt}\int_{\mathbb{R}^3}\log f Q(f,f)d\xi$$
$$= -\frac{1}{4}\int_{\mathbb{R}^6\times\mathbf{S}^2_+} q(\xi-\xi_*,\omega)\{f'f'_*-ff_*\}\log\frac{f'f'_*}{ff_*}d\omega d\xi_* d\xi.$$

Proof. It follows from the structure of $Q(f,f)$ that we have

$$\int_{\mathbb{R}^3}Q(f,f)\phi(\xi)d\xi = \frac{1}{4}\int_{\mathbb{R}^6\times\mathbf{S}^2_+} q(\xi-\xi_*,\omega)[f'f'_*-ff_*][\phi+\phi_*-\phi'-\phi'_*]d\omega d\xi_* d\xi.$$

We take $\phi = 1,\xi,\frac{|\xi|^2}{2},\log f$ to get the desired result. $\square$

We define local macroscopic densities (ρ,u,θ):

$$\rho \equiv \int_{\mathbb{R}^3} f d\xi \quad (\text{mass})\ ; \quad u(x,t)\equiv\frac{1}{\rho}\int_{\mathbb{R}^3}\xi f d\xi \quad (\text{bulk velocity})\ ;$$
$$\frac{1}{2}\rho(|u|^2+3\theta)\equiv\int_{\mathbb{R}^3}\frac{|\xi|^2}{2}f d\xi \quad (\text{energy})\ .$$

We now multiply $1, \xi, \frac{|\xi|^2}{2}$ to (1.1) to get a system of balanced laws:

$$\begin{aligned}
&\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\
&\partial_t (\rho u) + \nabla_x \cdot (\rho u \otimes u + P) = 0, \\
&\partial_t \Big(\rho (|u|^2 + 3\theta) \Big) + \nabla_x \cdot \Big(\frac{\rho}{2} (|u|^2 + 3\theta) u + Pu + q \Big) = 3\sigma\rho.
\end{aligned} \tag{2.1}$$

Here $P = (p_{ij})$ is the stress tensor and q is a heat flux: $1 \le i, j \le 3$,

$$p_{ij} \equiv \int_{\mathbb{R}^3} \frac{1}{2} (\xi_i - u_i)(\xi_j - u_j) f d\xi, \quad q_i = \int_{\mathbb{R}^3} \frac{1}{2} (\xi_i - u_i) |\xi - u|^2 f d\xi.$$

Note that mass and momentum are conserved, whereas the energy is increasing due to the energy input by its environment such as a heat bath. As in the Boltzmann equation [3], we define an H-functional as the phase space integral of $f \log f$:

$$H(f(t)) \equiv \int_{\mathbb{R}^6} f \log f d\xi dx.$$

Proposition 2.2. *Let f be a solution of* (1.1) *and decay fast enough in phase space. Then H-theorem holds.*

$$\begin{aligned}
\frac{d}{dt} H(f(t)) \;=\; & -\frac{1}{4} \int_{\mathbb{R}^6 \times \mathbf{S}^2_+} q(\xi - \xi_*, \omega) \{ f' f'_* - f f_* \} \log \frac{f' f'_*}{f f_*} d\omega d\xi_* d\xi \\
& - \frac{\sigma}{4} \int_{\mathbb{R}^6} |\nabla_\xi \sqrt{f}|^2 d\xi dx \le 0.
\end{aligned}$$

2.2. Hamdache's framework

In this part, we briefly review Green's function to the Vlasov-Fokker-Planck (VFP) equation and Hamdache's framework for mild solution. The explicit Green's function to the VFP equation was first constructed in [8], and was further refined and used to the study of large time behavior of solutions to the VPFP system in [1, 14, 15, 16]. The explicit presentation on the Green's function can be found in [1, 7]. Consider the VFP equation:

$$\begin{aligned}
\partial_t f + \xi \cdot \nabla_x f = \sigma \Delta_\xi f, \qquad & x, \xi \in \mathbb{R}^3, \quad t > 0, \\
f(x, \xi, 0) = f^{in}(x, \xi). &
\end{aligned} \tag{2.2}$$

We first recall the definition of Green's function to (2.2).

Definition 2.3 Let $(x, \xi, t) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+$ be given. Then Green's function $G = G(x, \xi, t; y, \xi_*, s)$ satisfies the following initial value problem for the adjoint equation in (y, v_*, s)-variables:

$$\begin{gathered}\partial_s G + \xi_* \cdot \nabla_y G + \sigma \Delta_{\xi_*} G = 0, \quad y, \xi_* \in \mathbb{R}^3, \quad 0 \le s < t, \\ G(x, \xi, t; y, \xi_*, t) = \delta(y - x)\delta(\xi_* - \xi),\end{gathered} \tag{2.3}$$

where $\delta(\cdot)$ is the three dimensional Dirac Delta function.

Then the mild solution f for (2.2) is given by the integral formula:

$$f(x, \xi, t) = \int_{\mathbb{R}^6} G(x, \xi, t; y, \xi_*, 0) f^{in}(y, \xi_*) d\xi_* dy. \tag{2.4}$$

Consider the frictionless VFP equation with a Dirac measure as initial datum:

$$\begin{aligned}\partial_t g + \xi \cdot \nabla_x g &= \sigma \Delta_\xi g, \qquad x, \xi \in \mathbb{R}^3, \quad t > 0, \\ g(x, \xi, 0) &= \delta(x)\delta(\xi).\end{aligned}$$

Then g satisfies

$$g(x, \xi, t) \equiv G(x, \xi, t; 0, 0, 0),$$

where G is Green's function to (2.3). By taking Fourier transform in x and ξ-variables, we can find the explicit form of g:

$$g = \Big(\frac{3^{\frac{1}{2}}}{2\pi\sigma t^2} \Big)^3 \exp\Big\{ - \frac{3|x|^2 + 3|x - t\xi|^2 - t^2|\xi|^2}{2\sigma t^3} \Big\}.$$

In this case, Green's function G is given by

$$\begin{aligned}&G(x, \xi, t; y, \xi_*, s) \\ &= g(x - y - \xi_*(t - s), \xi - \xi_*, t - s) \\ &= \Big(\frac{3^{\frac{1}{2}}}{2\pi\sigma(t - s)^2} \Big)^3 \exp\Big\{ - \frac{3|x - y - \frac{(t-s)(\xi+\xi_*)}{2}|^2}{\sigma(t - s)^3} - \frac{|\xi - \xi_*|^2}{4\sigma(t - s)} \Big\} \\ &= K\Big(x - y - \frac{t - s}{2}(\xi + \xi_*), \frac{\sigma(t - s)^3}{12} \Big) K(\xi - \xi_*, \sigma(t - s)).\end{aligned} \tag{2.5}$$

We use the above explicit representation (2.5) to derive the following estimates for G.

Proposition 2.4.([1, 14]) *Green's function G satisfies*

(1) $\displaystyle\int_{\mathbb{R}^6} G(x,\xi,t;z,\xi_1,\tau)G(z,\xi_1,\tau;y,\xi_*,s)dzd\xi_1 = G(x,\xi,t;y,\xi_*,s).$

(2) $\displaystyle\int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,s)d\xi dx = \int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,s)d\xi_* dy = 1.$

(3) $\displaystyle\int_{\mathbb{R}^3} G(x,\xi,t;y,\xi_*,s)d\xi \le \frac{C}{(\sigma(t-s)^3)^{\frac{3}{2}}}\exp\Big\{-\frac{3|x-y-(t-s)\xi_*|^2}{4\sigma(t-s)^3}\Big\}.$

(4) $\displaystyle\int_{\mathbb{R}^3} G(x,\xi,t;y,\xi_*,s)dx \le \frac{C}{(\sigma(t-s))^{\frac{3}{2}}}\exp\Big\{-\frac{|\xi-\xi_*|^2}{4\sigma(t-s)}\Big\}.$

Here $x,\xi,y,\xi_* \in \mathbb{R}^3$ and $t>\tau>s\ge 0$, moreover we have

(5) $$|\nabla_x G(x,\xi,t;y,\xi_*,s)|, |\nabla_y G(x,\xi,t;y,\xi_*,s)| \le \frac{C}{(\sigma(t-s)^3)^{\frac{1}{2}}}G^{[2]}(x,\xi,t;y,\xi_*,s).$$

(6) $$|\nabla_\xi G(x,\xi,t;y,\xi_*,s)|, |\nabla_{\xi_*} G(x,\xi,t;y,\xi_*,s)| \le \frac{C}{(\sigma(t-s))^{\frac{1}{2}}}G^{[2]}(x,\xi,t;y,\xi_*,s).$$

Here we used the simplified notation $G^{[k]}$ for scaled Green's function in phase space by k, i.e.,

$$G^{[k]}(x,v,t;y,v_*,s) \equiv G\Big(\frac{x}{k},\frac{v}{k},t;\frac{y}{k},\frac{v_*}{k},s\Big), \qquad k>0.$$

Remark 2.5. In L^1-stability estimate in next section, we will use the nonnegativity and the property (2) of Green's function:

$$G(x,\xi,t;y,\xi_*,s) > 0 \quad \text{and} \quad \int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,s)d\xi dx = 1.$$

Theorem 2.6.([8]) *Let α,β,λ be positive constants. Suppose that $\gamma \in (-2,0)$ and the initial datum f^{in} is majorized by φ, i.e.,*

$$f^{in}(x,\xi) \le \delta\varphi(x,\xi;\alpha,\beta,\lambda), \qquad 0<\delta \ll 1.$$

Then there exists a unique mild solution f to (1.1) satisfying a pointwise bound

$$f(x,\xi,t) \le \frac{C\delta}{E(t)^{\frac{3}{2}}}\exp\Big(-\frac{4\alpha|x-(t+\beta)\xi|^2}{E(t)}\Big),$$

where $E(t)$ is explicitly given by

$$E(t) \equiv 16\sigma\alpha t\Big(\frac{t}{2}+\beta\Big)^2 + \frac{4}{3}(\sigma t+\alpha)(\sigma t^3+\lambda^3).$$

3. Uniform L^1-Stability Estimate

In this section, we present a uniform L^1-stability estimate for the frictionless FPB equation:

$$\begin{aligned} &\partial_t f + \xi\cdot\nabla_x f - \sigma\triangle_\xi f = Q(f,f), \quad x,\xi\in\mathbb{R}^3, \quad t>0,\\ &f(x,\xi,0) = f^{in}(x,\xi). \end{aligned} \tag{3.1}$$

Below we provide key estimates to be used in L^1-stability estimate.

Lemma 3.1. *Assume $\gamma\in(-2,0)$ and let f and $\bar f$ be two mild solutions given in Theorem* 2.6*. Then we have*

(1) $\quad \Big\| \int_{\mathbb{R}^3} |\xi-\xi_*|^\gamma f(x,\xi_*,t)d\xi_* \Big\|_{L^\infty_{x,\xi}} \le C(t+\beta)^{-(\gamma+3)}, \qquad t\ge 0.$

(2) $\quad ||Q(f,f)(t) - Q(\bar f,\bar f)(t)||_{L^1} \le \dfrac{C}{(t+\beta)^{\gamma+3}}\|f(t)-\bar f(t)\|_{L^1}.$

Here $E(t)$ is the function in Theorem 2.6*.*

Proof. We first consider the following elementary estimate: For positive constants $\mu_1\in(0,3)$ and μ_2, we have

$$\begin{aligned} \mathcal{I}(\mu_1,\mu_2,\beta) &\equiv \sup_{x,\xi\in\mathbb{R}^3}\int_{\mathbb{R}^3} |\xi-\xi_*|^{-\mu_1}\exp\Big(-\mu_2|x-(t+\beta)\xi_*|^2\Big)d\xi_* \\ &\le C(t+\beta)^{\mu_1-3}(1+\mu_2^{-\frac{3}{2}}). \end{aligned} \tag{3.2}$$

The proof of (3.2). Let $x,\xi\in\mathbb{R}^3$ be fixed. We use a change of variable $\eta=(t+\beta)\xi_*$:

$$d\xi_* = (t+\beta)^{-3}d\eta \quad \text{and} \quad |\xi-\xi_*| = \frac{|\eta-(t+\beta)\xi|}{(t+\beta)}$$

to get

$$\int_{\mathbb{R}^3} |\xi-\xi_*|^{-\mu_1}\exp\Big(-\mu_2|x-(t+\beta)\xi_*|^2\Big)d\xi_*$$

$$
\begin{aligned}
&= (t+\beta)^{\mu_1-3}\int_{\mathbb{R}^3}|\eta-(t+\beta)\xi|^{-\mu_1}\exp\Big(-\mu_2|\eta-x|^2\Big)d\eta\\
&\le (t+\beta)^{\mu_1-3}\Big[\int_{|\eta-(t+\beta)\xi|\le 1}|\eta-(t+\beta)\xi|^{-\mu_1}\exp\Big(-\mu_2|\eta-x|^2\Big)d\eta\\
&\qquad +\int_{|\eta-(t+\beta)\xi|>1}|\eta-(t+\beta)\xi|^{-\mu_1}\exp\Big(-\mu_2|\eta-x|^2\Big)d\eta\Big]\\
&= (t+\beta)^{\mu_1-3}\Big(\mathcal{J}_1(x,t)+\mathcal{J}_2(x,t)\Big).
\end{aligned}
$$

We now estimate $\mathcal{J}_i(x,t)$ separately.

$$
\mathcal{J}_1(x,t) \le \int_{|\eta-(t+\beta)\xi|\le 1}|\eta-(t+\beta)\xi|^{-\mu_1}d\eta \le C,
$$

and

$$
\mathcal{J}_2(x,t) \le \int_{|\eta-(t+\beta)\xi|>1}\exp\Big(-\mu_2|\eta-x|^2\Big)d\eta \le \Big(\frac{\pi}{\mu_2}\Big)^{\frac{3}{2}}.
$$

We combine the above two estimates to get the desired result.
(1) We use the pointwise bound of f in Theorem 2.6 and (3.2) to find

$$
\begin{aligned}
&\int_{\mathbb{R}^3}|\xi-\xi_*|^\gamma f(x,\xi_*,t)d\xi_*\\
&\quad\le C\int_{\mathbb{R}^3}|\xi-\xi_*|^\gamma\frac{1}{E(t)^{\frac{3}{2}}}\exp\Big(-\frac{4\alpha|x-(t+\beta)\xi_*|^2}{E(t)}\Big)d\xi_*\\
&\quad\le \frac{C}{E(t)^{\frac{3}{2}}}\int_{\mathbb{R}^3}|\xi-\xi_*|^\gamma\exp\Big(-\frac{4\alpha}{E(t)}|x-(t+\beta)\xi_*|^2\Big)d\xi_*\\
&\quad= \frac{C}{E(t)^{\frac{3}{2}}}\mathcal{I}\Big(-\gamma,\frac{4\alpha}{E(t)},\beta\Big)\le C(t+\beta)^{-(\gamma+3)}.
\end{aligned}
$$

Here we used

$$
\frac{(1+E(t)^{\frac{3}{2}})}{E(t)^{\frac{3}{2}}}\le C.
$$

(2) Note that

$$
|(f'f'_*-ff_*)-(\bar{f}'\bar{f}'_*-\bar{f}\bar{f}_*)|\le|f-\bar{f}|'f'_*+\bar{f}'|f-\bar{f}|'_*+|f-\bar{f}|f_*+\bar{f}|f-\bar{f}|_*
$$

By straightforward calculation, we have

$$
\begin{aligned}
&\|Q(f,f)(t)-Q(\bar{f},\bar{f})(t)\|_{L^1}\\
&\quad\le \int_{\mathbb{R}^6}\Big(\int_{\mathbb{R}^3\times\mathbf{S}^2_+}q(\xi-\xi_*,\omega)\Big(|f-\bar{f}|'f'_*+\bar{f}'|f-\bar{f}|'_*\Big)d\omega d\xi_* d\xi dx
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^6}\Big(\int_{\mathbb{R}^3\times\mathbf{S}^2_+} q(\xi-\xi_*,\omega)\Big(|f-\bar f|f_*+\bar f|f-\bar f|_*\Big)d\omega d\xi_* d\xi dx\\
\equiv\ & \mathcal{K}_1(t)+\mathcal{K}_2(t).
\end{aligned}
$$

Note that the change of variable $(\xi',\xi'_*)\to(\xi,\xi_*)$ yields

$$
\mathcal{K}_1(t)=\mathcal{K}_2(t).
$$

Hence it suffices to consider the term $\mathcal{K}_2(t)$. The first term in $\mathcal{K}_2(t)$ can be estimated as follows.

$$
\begin{aligned}
&\int_{\mathbb{R}^6}\Big(\int_{\mathbb{R}^3\times\mathbf{S}^2_+} q(\xi-\xi_*,\omega)|f-\bar f|f_* d\omega d\xi_*\Big)d\xi dx\\
&\qquad\le C\Big\|\int_{\mathbb{R}^3}|\xi-\xi_*|^\gamma f_* d\xi_*\Big\|_{L^\infty}||f(t)-\bar f(t)||_{L^1}\\
&\qquad\le \frac{C}{(t+\beta)^{\gamma+3}}||f(t)-\bar f(t)||_{L^1}.
\end{aligned}
$$

By symmetry, we can treat the second term in $\mathcal{K}_2(t)$ in the same way. Finally we obtain the desired estimate.

$$
\mathcal{K}_1(t)=\mathcal{K}_2(t)\le\frac{C}{(t+\beta)^{\gamma+3}}||f(t)-\bar f(t)||_{L^1}. \qquad \square
$$

We next study the uniform L^1-stability estimate for mild solutions in Hamdache's framework in Section 2. Let f and $\bar f$ be two mild solutions corresponding to initial data given by f^{in} and $\bar f^{in}$ respectively.

Then f and $\bar f$ satisfy

$$
\begin{aligned}
f(x,\xi,t) &= \int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,0)f^{in}(y,\xi_*)d\xi_* dy\\
&\quad+\int_0^t\int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,s)Q(f,f)(y,\xi_*,s)d\xi_* dyds, \qquad (3.3)\\
\bar f(x,\xi,t) &= \int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,0)\bar f^{in}(y,\xi_*)d\xi_* dy\\
&\quad+\int_0^t\int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,s)Q(\bar f,\bar f)(y,\xi_*,s)d\xi_* dyds. \qquad (3.4)
\end{aligned}
$$

We now subtract (3.4) from (3.3) and take L^1-norm to find

$$
\|f(t)-\bar f(t)\|_{L_1}\le\Big\|\int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,0)|f^{in}(y,\xi_*)-\bar f^{in}(y,\xi_*)|dyd\xi_*\Big\|_{L^1}
$$

$$
\begin{aligned}
&+\Big\| \int_0^t \int_{\mathbb{R}^6} G(x,\xi,t;y,\xi_*,s)|Q(f,f)-Q(\bar f,\bar f)|(y,\xi_*,s)dyd\xi_*ds\Big\|_{L^1}\\
\equiv\ & \mathcal{L}_1(t)+\mathcal{L}_2(t). \qquad (3.5)
\end{aligned}
$$

Lemma 3.2. *The terms $\mathcal{L}_i(t), i=1,2$ satisfy the following estimates.*

(1) $\mathcal{L}_1(t) \le \|f^{in}-\bar f^{in}\|_{L^1}$.

(2) $\mathcal{L}_2(t) \le \displaystyle\int_0^t ||Q(f,f)(s)-Q(\bar f,\bar f)(s)||_{L^1}ds.$

Proof. We use Remark 2.5 and Hausdorff-Young's inequality to get

$$
\begin{aligned}
\mathcal{L}_1(t) &\le \|G(\cdot,\cdot,t;y,\xi_*,0)\|_{L^1}\|f^{in}-\bar f^{in}\|_{L^1} \le \|f^{in}-\bar f^{in}\|_{L^1},\\
\mathcal{L}_2(t) &\le \int_0^t ||G(\cdot,\cdot,t;y,\xi_*,s)||_{L^1}||Q(f,f)(s)-Q(\bar f,\bar f)(s)||_{L^1}ds\\
&\le \int_0^t ||Q(f,f)(s)-Q(\bar f,\bar f)(s)||_{L^1}ds. \qquad \square
\end{aligned}
$$

The proof of Theorem 1.1. Let f and $\bar f$ be mild solutions to (1.1) corresponding to initial data f_0 and $\bar f_0$ satisfying $\mathcal{A}$ respectively in Hamdache's framework. In (3.5), we use Lemma 3.2 and Lemma 3.1 (2) to see

$$
\|f(t)-\bar f(t)\|_{L^1} \le \|f^{in}-\bar f^{in}\|_{L^1} + C\int_0^t \frac{\|f(s)-\bar f(s)\|_{L^1}ds}{(s+\beta)^{\gamma+3}}.
$$

Then it follows from Gronwall's lemma that

$$
\|f(t)-\bar f(t)\|_{L^1} \le \|f^{in}-\bar f^{in}\|_{L^1}\exp\Big(\int_0^t \frac{C}{(s+\beta)^{\gamma+3}}ds\Big). \qquad (3.6)
$$

Since $-2<\gamma<0$ and $\beta>0$, Gronwall's lemma yields

$$
\|f(t)-\bar f(t)\|_{L^1} \le M\|f^{in}-\bar f^{in}\|_{L^1}. \qquad (3.7)
$$

Here M does not depend on time t. This completes the proof. $\square$

Remark 3.3. In [7], the first author and Se Eun Noh show that the Vlasov-Poisson-Fokker-Planck system is uniformly L^1-stable, when initial data is sufficiently small and decay fast enough in phase space. Hence when the self-consistent electric field is added in (1.1), the resulting system "Vlasov-Poisson-Fokker-Planck-Boltzmann system" is also L^1-stable.

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