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MACH-NUMBER UNIFORM ASYMPTOTIC-PRESERVING GAUGE SCHEMES FOR COMPRESSIBLE FLOWS

BY

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Abstract

We present novel algorithms for compressible flows that are efficient for all Mach numbers. The approach is based on several ingredients: semi-implicit schemes, the gauge decomposition of the velocity field and a second order formulation of the density equation (in the isentropic case) and of the energy equation (in the full Navier-Stokes case). Additionally, we show that our approach corresponds to a micro-macro decomposition of the model, where the macro field corresponds to the incompressible component satisfying a perturbed low Mach number limit equation and the micro field is the potential component of the velocity. Finally, we also use the conservative variables in order to obtain a proper conservative formulation of the equations when the Mach number is order unity. We successively consider the isentropic case, the full Navier-Stokes case, and the isentropic Navier-Stokes-Poisson case. In this work, we only concentrate on the question of the time discretization and show that the proposed method leads to Asymptotic Preserving schemes for compressible flows in the low Mach number limit.

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1. Introduction

Simulation of Low-Mach number flows has been the subject of a considerable amount of literature. Low Mach number flows occur in numerous situations in geophysics (such as atmospheric modeling), industrial processes (e.g., in CVD or Chemical Vapor Deposition), nuclear reactor engineering (in the water-vapor circuitry) or in every day applications (such as the air flow around a vehicle), and combustion [45]. However, the search for numerical algorithms which are efficient uniformly in the Mach number is still highly desirable since no completely satisfactory solution has been achieved yet.

In this paper, we present novel algorithms for compressible flows that are valid for all Mach numbers. The approach is based on several ingredients: semi-implicit schemes, the gauge decomposition of the velocity field and a second order formulation of the density equation (in the isentropic case) and of the energy equation (in the full Navier-Stokes case). Additionally, we show that our approach corresponds to a micro-macro decomposition of the model, where the macro field corresponds to the incompressible component satisfying a perturbed Low-Mach number limit equations and the micro field is the potential component of the velocity. Finally, we also use the conservative variables in order to obtain a proper conservative formulation of the equations for shock capturing when the Mach number is of order unity. We successively consider the isentropic case, the full Navier-Stokes case, and the isentropic Navier-Stokes-Poisson case.

In the literature, many papers are concerned with the low Mach number limit. Only recently, new approaches have considered algorithms that are expected to be efficient for all range of Mach numbers. Additionally, most of the early literature on the subject deals with steady-state computations and are based on preconditioning techniques that are not consistent with the model in the time-dependent case. The prototype of this approach can be found in the seminal paper of Chorin [9], where an artificial compressibility approach was proposed. It was first recognized by Turkel [55] that this approach could be viewed as a preconditioning technique. These pioneering works have been followed by an abundant literature, see e.g., [2, 8, 13, 40, 56, 34, 51]. Further studies provided the accurate asymptotic expansions of the numerical fluxes in terms of the Mach number [26, 25].

Then, it was recognized that consistency of the time difference scheme required the use of implicit schemes and that preconditioning techniques could often be viewed as a predictor-corrector version of these schemes. Full backward Euler time integration schemes have been investigated in [46] and [58]. However, it was soon realized that taking all fluxes implicit is probably too numerically demanding for a uniform stability with respect to the Mach number and that only those fluxes which correspond to the propagation of acoustic waves need to be taken implicitly. For instance, this was achieved in [1, 27, 38] through splitting methods.

In [21], only the pressure flux in the momentum equation is taken implicitly. In a number of references, [47, 48, 57, 35, 42, 59, 49], both the mass flux and the pressure contribution to the momentum fluxes are taken implicitly. In doing so, the convection flux and pressure flux are treated separately, in the spirit of the earlier Advection Upstream Splitting Method (AUSM) [43, 41]. In most of the cases, the pressure (or more often, the perturbation to the hydrostatic pressure) solves an elliptic or Helmholtz equation, which corresponds to acoustic wave propagation at high speeds. In the low Mach number limit, the momentum is updated using the perturbation pressure to avoid an ill-conditioning of the pressure term. In these works, the pressure equation is solved instead of the full energy one but the use of non-conservative variables may lead to incorrect shock speeds for finite Mach numbers.

In the present work, we use a similar semi-implicit methodology and a wave-equation formulation of either the density or the energy equation, which, after time discretization leads to an elliptic problem for the pressure. However, the rationale is fairly different. Indeed, we do work in the total pressure (or total energy) variable rather than in the perturbation pressure variable. The reason for that is that we bypass the ill-conditioned pressure term by using a gauge (or Hodge) decomposition of the velocity field (actually, of the momentum variable in order to preserve the conservativity of the scheme). The solenoidal (divergence-free) component of the velocity is updated using the momentum conservation equation, in which the total pressure combines with the potential associated with the irrotational part of the velocity field. In this way, the ill-conditioning of the pressure term disappears as it becomes only the Lagrange multiplier of the divergence-free constraint of the solenoidal part of the velocity.

The use of the Hodge decomposition as a projection method in the context of incompressible fluids has been proposed in [3, 4, 39] and later used in e.g., [49, 10]. It is based on an earlier theoretical work of Roberts [53] and Oseledets [50] who introduced a Velocity-Impulse-Density Formulation for Navier-Stokes equation. Roberts's primary motivation was to write the incompressible Euler equations in a Hamiltonian form. Buttke and Chorin [6] used the impulse density variable as a numerical tool in the computation of incompressible flows. E and Liu [18] showed that the original velocity-impulse density formulation of Oseledets is marginally ill-posed for the inviscid flow, and this has the consequence that some ordinarily stable numerical methods in other formulations become unstable in the velocity-impulse density formulation. To remove this marginal ill-posedness, they [18] then introduced a class of numerical method based on a simplified velocity-impulse density formulation. This class of numerical method was later renamed as the gauge method [36, 19, 20]. Unconditional stability of the gauge method was shown by Wang and Liu [60]. Maddocks and Pego [44] also introduced an unconstrained Velocity-Impulse-Density/Hamiltonian formulation for incompressible fluid which has better stability properties. A systematic comparison of different gauge choices in this content was studied by Russo and Smereka [54].

In the present work, we only concentrate on the question of the time discretization. Our goal is to show that the combination of a semi-implicit scheme with a second order formulation of the density or energy equation and, more importantly, with a gauge (Hodge-like) decomposition of the momentum field, can lead to an Asymptotic Preserving scheme for compressible flows in the Low-Mach number limit. An Asymptotic Preserving scheme for a model (M_ε) depending on a parameter ε and converging to the limit model (M) as $\varepsilon \to 0$ is a scheme that *preserves the discrete analogy of the asymptotic passage from model* (M_ε) *to model* (M). Specifically, the method is a consistent and stable discretization of (M_ε) when the time step Δt (and possibly the mesh size Δx) resolve the scales associated with ε and which is consistent and stable discretization of the limit model (M) when Δt (and possibly Δx) is fixed and $\varepsilon \to 0$. The latter situation is called 'underresolved' in the sense that Δt (and Δx) are unable to resolve the scales associated with ε. Additionally, if the scheme is stable uniformly with respect to ε, the scheme is said Asymptotically Stable. Of course, the two concepts are linked, but it may happen that a scheme enjoys one of the properties without the other one.

If an Asymptotically Stable and Asymptotic Preserving scheme is used, a uniform accuracy for all range of ε is expected. A rigorous proof of this claim, in the context of linear transport equation in the diffusive regime, was given in [22]. For transitional values of ε (i.e., when ε is small without being very small), the uniform stability guarantees that the discrepancy remains bounded with time. An Asymptotic Preserving scheme enjoys many advantages compared with a model coupling strategy (i.e., solving model (M_ε) where ε is finite and model (M) where ε is very small). Indeed, a domain decomposition strategy requires the design of appropriate transmission conditions between the models, together with a geometric approximation of the interfacial region (which, in many situation, is moving with time) and a specific adaptive meshing strategy. With an Asymptotically Stable and Asymptotic Preserving scheme, the scheme itself switches from one model to the other one when it is needed, without any intervention of the user.

Asymptotic Preserving schemes have been proposed in a variety of contexts, such as hydrodynamic or diffusion limits of kinetic model [7, 32, 33, 29, 52, 5, 23], relaxation limits of hyperbolic models [30, 31, 24], quasi-neutral limits of Euler-Poisson or Vlasov-Poisson systems [11, 12, 17, 14], Dirac-Maxwell systems in the non-relativistic regime [28], fluid limit of Complex-Ginzburg-Landau equations [15], ...

The paper is organized as follows. In Section 2, we propose an Asymptotic Preserving time discretization of the isentropic Navier-Stokes equations, based on a gauge formulation of the model. In passing, we show that the gauge formulation provides the proper macro-micro decomposition of the model. Indeed, the set of macro variables evolve according to a system which corresponds to the limit model, perturbed by small terms which depend on the micro variables. Here the macro variables are the constant density and the solenoidal component of the velocity, while the micro variables correspond to the perturbation density and the potential component of the velocity. In Section 3, a similar methodology is applied to the full Navier-Stokes equations. Finally, the case of the isentropic Navier-Stokes-Poisson problem is investigated in Section 4. A conclusion is drawn in Section 5.

Again, we stress the fact that this paper is devoted to the time-discretization only and the investigation of its Asymptotic-Preserving property. We defer the investigation of the space discretization and numerical simulations to future works.

2. Isentropic Navier-Stokes Equation

2.1. The model and the Low-Mach number limit

Consider the isentropic Navier-Stokes equations:

$$\partial_t \rho^\varepsilon + \nabla \cdot q^\varepsilon = 0\,, \tag{2.1}$$

$$\partial_t q^\varepsilon + \nabla\Big(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}\Big) + \frac{1}{\varepsilon^2}\nabla p(\rho^\varepsilon) = \nabla(\mu\sigma(u^\varepsilon))\,, \tag{2.2}$$

where $\rho^\varepsilon(x,t)$ is the volumic mass density, $q^\varepsilon = \rho^\varepsilon u^\varepsilon(x,t)$ the volumic momentum density depending on the position $x \in \mathbb{R}^d$ (d being the dimension) and the time $t > 0$, $p(\rho)$ is the isentropic pressure-density relationship, μ is the viscosity and $\sigma(u)$ is the rate of strain tensor:

$$\sigma(u) = \nabla u + (\nabla u)^T - \frac{2}{d}(\nabla \cdot u)\, I.$$

For scalar, vector and tensor fields φ, a and S, we denote by $\nabla\varphi$, $\nabla \cdot a$ and ∇S the gradient of φ, divergence of a and gradient of S respectively. The exponent T denotes the transpose of a tensor, I, the unit tensor and for two vectors a and b, $a \otimes b$ denotes the tensor product of a and b.

Here, the equations have already been put in the scaled form, with ε being the Mach number. In order to understand the significance of ε, we come back to the system in dimensional physical quantities

$$\partial_{\bar t}\bar\rho + \nabla_{\bar x} \cdot \bar q = 0\,, \tag{2.3}$$

$$\partial_{\bar t}\bar q + \nabla_{\bar x}\Big(\frac{\bar q \otimes \bar q}{\bar\rho}\Big) + \nabla_{\bar x}\bar p(\bar\rho) = \nabla_{\bar x}(\bar\mu\bar\sigma(\bar u))\,, \tag{2.4}$$

where the barred quantities denote quantities expressed in physical units. Now, let x_0, t_0, ρ_0, q_0, p_0, μ_0, u_0 be a set of scaling units for the position, time, mass density, momentum density, pressure, viscosity and velocity respectively. We link these units by the natural relations $u_0 = x_0/t_0$, $q_0 = \rho_0 u_0$ and note that $\bar\sigma(\bar u) = \sigma(u)/t_0$. Then, the dimensionless variables are defined by the relations $\bar x = x_0 x$, $\bar t = t_0 t$, $\bar\rho = \rho_0\rho$, $\ldots$. Using these changes of variables and unknowns, we find:

$$\partial_t \rho + \nabla \cdot q = 0\,, \tag{2.5}$$

$$\frac{\rho_0 u_0^2}{p_0} \left(\partial_t q + \nabla (\frac{q \otimes q}{\rho}) \right) + \nabla p(\rho) = \frac{\mu_0}{p_0 t_0} \nabla (\mu \sigma(u))\,. \tag{2.6}$$

The first dimensionless parameter $\rho_0 u_0^2/p_0$ appears as the ratio of the the drift energy of the fluid to its internal energy (up to a numerical factor). Writing that p_0 is up to a numerical factor equal to the product of the density ρ_0 and the square of the speed of sound c_s^2, we can also identify this dimensionless parameter as the square of the Mach number $\mathcal{M} = u_0/c_s$. Therefore, we denote this dimensionless parameter by ε^2. For the second dimensionless parameter, we write

$$\frac{\mu_0}{p_0 t_0} = \frac{\mu_0}{\rho_0 u_0 x_0} \frac{\rho_0 u_0^2}{p_0} = \frac{\mu_0}{\rho_0 u_0 x_0} \varepsilon^2..$$

The dimensionless quantity $\frac{\mu_0}{\rho_0 u_0 x_0}$ measures the ratio of the strenth of the viscosity term to that of the transport term. We suppose that this ratio is of order unity i.e., that the time and space gradients of momentum are balanced by viscosity terms of similar order of magnitude. Therefore, we take a viscosity scale such that

$$\frac{\mu_0}{\rho_0 u_0 x_0} = 1.$$

Now, inserting these values of the parameters into (2.6), we recover the dimensionless system (2.1), (2.2).

The low-Mach number limit is the regime where $\varepsilon \to 0$ [37]. Letting $\varepsilon \to 0$, in the momentum conservation equation (2.2), we get:

$$\nabla p(\rho) = 0\,. \tag{2.7}$$

Therefore, the solution of (2.19) is given by:

$$\rho = \rho_0, \quad p(\rho) = p(\rho_0) := p_0 \tag{2.8}$$

where ρ_0 is a constant (with respect to both space and time) given by the boundary conditions. That ρ_0 is uniform in space is a necessary condition for the existence of a low Mach number limit. That it is a constant in time is an assumption that we make for the sake of simplicity, but which can be easily relaxed.

Then, letting $\varepsilon \to 0$ in the momentum equation (2.2) again leads to

$$\partial_t q + \nabla(\frac{q \otimes q}{\rho_0}) + \nabla P = \nabla(\mu\sigma(u))\,, \tag{2.9}$$

where $P = \lim_{\varepsilon\to 0} \varepsilon^{-2}(p(\rho^\varepsilon) - p_0)$ is the hydrostatic pressure. We assume μ is constant. Then,

$$\nabla(\mu\sigma(u)) = \mu(\Delta u + \frac{d-2}{d}\nabla(\nabla \cdot u)) \tag{2.10}$$

In the low-mach number model, P is the Lagrange multiplier of the divergence free constraint

$$\nabla \cdot u = 0. \tag{2.11}$$

which follows from the limit of the mass conservation eq. (2.1) together with (2.8). Eq. (2.9) is the incompressible Navier-Stokes equation. Using the incompressibility constraint (2.11), it can be recast in the more familiar form:

$$\rho_0(\partial_t u + (u \cdot \nabla)u) + \nabla P = \mu \Delta u\,.$$

Using this form and the fact that ρ_0 is uniform, it is easy to see that P satisfies the following elliptic equation:

$$-\Delta P = \nabla^2 : (\rho_0 u \otimes u)\,. \tag{2.12}$$

To construct an Asymptotic-Preserving scheme for the density, it suffices to use (2.2) in conjunction with a backwards Euler scheme. The difficulty with the low mach number limit is rather the determination of the velocity. To obtain an AP method, we must derive a scheme for the original model in which, to some extent, the limit eq. (2.9) is built in the scheme. For that purpose, we use the gauge methodology, which is developed in the next section.

2.2. Gauge decomposition of the momentum field

We introduce the decomposition of q^ε into a solenoidal field a^ε and an irrotational one $\nabla\varphi^\varepsilon$:

$$q^\varepsilon = a^\varepsilon - \nabla\varphi^\varepsilon\,, \quad \nabla \cdot a^\varepsilon = 0, \tag{2.13}$$

Introducing (2.13) into (2.2), we get

$$\partial_t a^\varepsilon + \nabla(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) + \nabla P^\varepsilon = \nabla(\mu\sigma(u^\varepsilon))\,, \tag{2.14}$$

where P^ε is a 'quasi-hydrostatic pressure' defined by

$$P^\varepsilon = \frac{1}{\varepsilon^2}(p(\rho^\varepsilon) - p_0) - \partial_t \varphi^\varepsilon\,. \tag{2.15}$$

The gauge potential can be determined from the mass conservation equation (2.1): using that $\nabla \cdot q^\varepsilon = -\Delta\varphi^\varepsilon$ where Δ denotes the Laplacian operator, we get:

$$\Delta\varphi^\varepsilon = \partial_t \rho^\varepsilon\,. \tag{2.16}$$

On the other hand, the quasi-hydrostatic pressure P^ε is obtained by taking the divergence of (2.14), which gives

$$-\Delta P^\varepsilon = \nabla^2 : (\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) - \nabla^2 : (\mu\sigma(u^\varepsilon))\,. \tag{2.17}$$

where, for a tensor S we denote by $\nabla^2 : S = \sum_{k,l} \partial_{x_k x_l} T_{kl}$. We assume that $\partial_n \varphi^\varepsilon = 0$ at the boundary. This implies that $\partial_n P^\varepsilon = \mathbf{n} \cdot \nabla(\mu\sigma(u^\varepsilon))$ at the boundary.

Before inserting the gauge decomposition into the Navier-Stokes equation, we also transform the mass conservation equation (2.1) into a wave equation; Indeed, we first take the time derivative of the mass conservation equation and the divergence of the momentum conservation equation, and obtain:

$$\partial_t^2 \rho^\varepsilon - \nabla^2 : (\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) - \frac{1}{\varepsilon^2}\Delta p(\rho^\varepsilon) = -\nabla^2 : (\mu\sigma(u^\varepsilon))\,, \tag{2.18}$$

This wave equation formulation will be useful for the design of the numerical scheme, because, in the limit $\varepsilon \to 0$, it reduces to an elliptic equation

$$\Delta p(\rho) = 0\,, \tag{2.19}$$

which, with the condition $\rho = \rho_0$ at the boundary, leads to (2.8). Also, taking the Laplacian term in (2.18) implicit will be easy because ρ^ε will be computed by solving an elliptic problem.

Then, we transform the compressible isentropic Navier-Stokes equations into the following equivalent system:

$$\partial_t^2 \rho^\varepsilon - \nabla^2 : (\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) - \frac{1}{\varepsilon^2}\Delta p(\rho^\varepsilon) = -\nabla^2 : (\mu\sigma(u^\varepsilon))\,, \tag{2.20}$$

$$\Delta\varphi^\varepsilon = \partial_t \rho^\varepsilon\,, \tag{2.21}$$

$$\Delta P^\varepsilon = -\nabla^2 : (\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) + \nabla^2 : (\mu\sigma(u^\varepsilon))\,, \tag{2.22}$$

$$\partial_t a^\varepsilon + \nabla(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) + \nabla P^\varepsilon = \nabla(\mu\sigma(u^\varepsilon))\,, \tag{2.23}$$

$$q^\varepsilon = a^\varepsilon - \nabla\varphi^\varepsilon\,. \tag{2.24}$$

Eqs (2.20) to (2.24) have been put in chronological order in an one time-step computation cycle: we first update ρ^ε using (2.20). Then we compute φ^ε and P^ε by solving the elliptic equations (2.21) and (2.22). From P^ε, we can update a^ε with (2.23). Then, we reconstruct a new momentum q^ε thanks to (2.24).

We first check that formulation (2.20)-(2.24) is equivalent to the initial one (2.1), (2.2), provided that $\nabla \cdot a^\varepsilon|_{t=0} = 0$. First, taking the divergence of (2.23) and using (2.22), we obtain that $\partial_t(\nabla \cdot a^\varepsilon) = 0$, which, with the assumption that the divergence is zero initially, implies that $\nabla \cdot a^\varepsilon = 0$ for all times. Then, taking the divergence of (2.24) and using (2.21) leads to the mass conservation equation (2.1). Next, combining (2.20) and (2.22) with the time derivative of (2.21) leads to

$$\Delta\left(\partial_t\varphi^\varepsilon - \frac{1}{\varepsilon^2}(p(\rho^\varepsilon) - p_0) + P^\varepsilon\right) = 0.$$

Note that the addition of p_0 is possible since it is a constant. By the choice of the boundary conditions on φ^ε and P^ε, the quantity inside the Laplacian is zero on the boundary. We deduce that it is identically zero, and we recover (2.15). Then, (2.15) inserted into (2.23) with (2.24) leads to the momentum conservation equation (2.2).

Now, we check that the low-Mach number limit problem is imbedded into formulation (2.20)-(2.24). More precisely, this formulation corresponds to a micro-macro decomposition of the problem, where 'microscopic' refers to the compressible Navier-Stokes equations while 'macroscopic' refers to the incompressible one. We develop the micro-macro formalism in the next section.

2.3. The gauge method viewed as a 'macro-micro' decomposition

We define ρ_1^ε and φ_1^ε by

$$\rho^\varepsilon = \rho_0 + \varepsilon^2 \rho_1^\varepsilon \,, \quad \varphi^\varepsilon = \varepsilon^2 \varphi_1^\varepsilon \,. \tag{2.25}$$

Then

$$q^\varepsilon = a^\varepsilon - \varepsilon^2 \nabla \varphi_1^\varepsilon \,. \tag{2.26}$$

The pair (ρ_0, a^ε) corresponds to the macro-field (or, in the dynamical systems terminology, the 'slow variables'), whereas $(\rho_1^\varepsilon, -\nabla\varphi_1^\varepsilon)$ corresponds to the micro-field or fast variables. The state of the system i.e., the pair (density, momentum) is constructed as the sum of the macro and macro field. The micro field is of order ε^2 if the macro field is of order 1.

We introduce the scalar p_1^ε, the tensor Π_1^ε and the vectors u_0^ε and u_1^ε according to

$$p(\rho^\varepsilon) = p_0 + \varepsilon^2 p_1^\varepsilon, \quad \frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon} = \frac{a^\varepsilon \otimes a^\varepsilon}{\rho_0} + \varepsilon^2 \Pi_1^\varepsilon, \tag{2.27}$$

$$u^\varepsilon = u_0^\varepsilon + \varepsilon^2 u_1^\varepsilon, \quad u_0^\varepsilon = \frac{a^\varepsilon}{\rho_0}\,. \tag{2.28}$$

All quantities with index '1' are $O(1)$ and, due to the prefactor ε^2, the corresponding terms in the expansions are $O(\varepsilon^2)$. In (2.27), (2.28), the leading order terms only depend on the macro variables, while the $O(\varepsilon^2)$ terms depend on both the macro and the micro variables:

$$p_1^\varepsilon = p_1^\varepsilon(\rho_0; \rho_1^\varepsilon), \quad \Pi_1^\varepsilon = \Pi_1^\varepsilon(\rho_0, a^\varepsilon; \rho_1^\varepsilon, \varphi_1^\varepsilon), \quad u_1^\varepsilon = u_1^\varepsilon(\rho_0, a^\varepsilon; \rho_1^\varepsilon, \varphi_1^\varepsilon).$$

We insist on the fact that eqs. (2.27), (2.28) are *exact* and not approximations. They are the relations defining the index '1' quantities .

Now, eq. (2.23) can be written as

$$\partial_t a^\varepsilon + \nabla(\frac{a^\varepsilon \otimes a^\varepsilon}{\rho_0}) + \nabla P^\varepsilon - \nabla(\mu\sigma(u_0^\varepsilon)) = \varepsilon^2[-\nabla\Pi_1^\varepsilon + \nabla(\mu\sigma(u_1^\varepsilon))]\,, \tag{2.29}$$

$$\nabla \cdot a^\varepsilon = 0\,. \tag{2.30}$$

If the order $O(\varepsilon^2)$ terms at the right-hand side of (2.26) and (2.29) are omitted, we find the low Mach number equations (2.9), (2.11). Now, the right-hand side of (2.29) depends on the micro variables $(\rho_1^\varepsilon, \varphi_1^\varepsilon)$ via eqs. (2.20), (2.21). More precisely, $(\rho_1^\varepsilon, \varphi_1^\varepsilon)$ are solutions of

$$\begin{aligned} -\Delta p_1^\varepsilon - \nabla^2 : \left(\frac{a^\varepsilon \otimes a^\varepsilon}{\rho_0}\right) + \nabla^2 : (\mu\sigma(u_0^\varepsilon)) \\ = \varepsilon^2[-\partial_t^2 \rho_1^\varepsilon + \Pi_1^\varepsilon - \nabla^2 : (\mu\sigma(u_1^\varepsilon))]\,, \end{aligned} \tag{2.31}$$

$$-\Delta \varphi_1^\varepsilon = -\partial_t \rho_1^\varepsilon\,. \tag{2.32}$$

We note from (2.32) that φ_1^ε is actually an $O(1)$ quantity, which was not completely obvious from the scaling (2.25).

This remark a posteriori justifies all previous considerations about the magnitude of the various terms. Now, when $\varepsilon \to 0$ in (2.26), (2.31), we deduce that

$$\Delta(p_1^\varepsilon - P^\varepsilon) \to 0.$$

Since both p_1^ε and P^ε vanish at the boundary, we deduce that

$$p_1^\varepsilon - P^\varepsilon \to 0$$

and that $p_1^\varepsilon \to P$, the hydrostatic pressure. This is a known fact about the low-Mach number limit, that the first-order perturbation of the pressure to the constant pressure converges to the hydrostatic pressure as the Mach number goes to zero. As a by-product of (2.27), we obtain that $p(\rho^\varepsilon) \to p_0$, which implies that $\rho^\varepsilon \to \rho_0$.

To summarize, the pair (ρ_0, a^ε) is the macro-field and eqs. (2.22), (2.23) provide the evolution of this macro-field. When ε is finite, this evolution depends (at the order $O(\varepsilon^2)$) on the micro variables $(\rho_1^\varepsilon, \varphi_1^\varepsilon)$ which are determined by eqs. (2.20), (2.21). Therefore, system (2.20)-(2.24) provides a micro-macro decomposition, which will be the starting point of our numerical methodology. Again, we point out that these equations are *exact* and not approximations of the original problem. In particular, they are equivalent to the original problem whatever the value of ε, be it small or not.

Now, we propose an Asymptotic-Preserving time-discretization of system (2.20)-(2.24). Indeed, in designing AP-schemes, the crucial issue is that of the time-discretization. Once a proper time-discretization is defined, the question of the space-discretization is a technical one. In particular, for hyperbolic systems, it can be any standard shock capturing method [30]. The issue of spatial discretization is outside the scope of the present work.

2.4. Asymptotic-Preserving time-discretization of the gauge formulation

The key point is an implicit time-discretization of eq. (2.20) which, in the limit $\varepsilon \to 0$, leads to an approximation of the Lapace equation (2.19). Let Δt be the time-step. For any time dependent quantity $f(t)$, an approximation at time $t^m = m\Delta t$ is defined by f^m. Then, we propose the following time-discretization of system (2.20)-(2.24):

$$\frac{1}{\Delta t^2}(\rho^{\varepsilon,m+1} - 2\rho^{\varepsilon,m} + \rho^{\varepsilon,m-1}) - \nabla^2 : (\frac{q^{\varepsilon,m} \otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}}) - \frac{1}{\varepsilon^2}\Delta p(\rho^{\varepsilon,m+1}) = -\nabla^2 : (\mu\sigma(u^{\varepsilon,m}))\,, \tag{2.33}$$

$$\Delta\varphi^{\varepsilon,m+1} = \frac{1}{\Delta t}(\rho^{\varepsilon,m+1} - \rho^{\varepsilon,m})\,, \tag{2.34}$$

$$\Delta P^{\varepsilon,m+1} = -\nabla^2 : (\frac{q^{\varepsilon,m} \otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}}) + \nabla^2 : (\mu\sigma(u^{\varepsilon,m}))\,, \tag{2.35}$$

$$\frac{1}{\Delta t}(a^{\varepsilon,m+1} - a^{\varepsilon,m}) + \nabla(\frac{q^{\varepsilon,m} \otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}}) + \nabla P^{\varepsilon,m+1} = \nabla(\mu\sigma(u^{\varepsilon,m}))\,, \tag{2.36}$$

$$q^{\varepsilon,m+1} = a^{\varepsilon,m+1} - \nabla\varphi^{\varepsilon,m+1}\,. \tag{2.37}$$

We make a few comments about this discretization procedure. First, we notice that (2.33) can be obtained from the following semi-implicit discretization of the original formulation (2.1), (2.2):

$$\frac{1}{\Delta t}(\rho^{\varepsilon,m+1} - \rho^{\varepsilon,m}) + \nabla \cdot q^{\varepsilon,m+1} = 0\,, \tag{2.38}$$

$$\frac{1}{\Delta t}(q^{\varepsilon,m+1} - q^{\varepsilon,m}) + \nabla(\frac{q^{\varepsilon,m} \otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}}) + \frac{1}{\varepsilon^2}\nabla p(\rho^{\varepsilon,m+1}) = \nabla(\mu\sigma(u^{\varepsilon,m}))\,. \tag{2.39}$$

Indeed, taking the time difference of eq. (2.38) at steps $m+1$ and m and combining it with the divergence of (2.39) leads to (2.33). Then, (2.35) for $P^{\varepsilon,m+1}$ follows from the application of the constraint $\nabla\cdot a^{\varepsilon,m+1} = 0$ to (2.36). Similarly, (2.34) follows from the insertion of the decomposition (2.37) into (2.38). Therefore, we find that the scheme (2.33)-(2.37) is a mere application of the gauge decomposition to the conservative semi-implicit scheme (2.38), (2.39).

We now show that, in the limit $\varepsilon \to 0$, this scheme gives a consistent approximation of the Low-Mach number equations (2.8), (2.9), (2.11). We drop the exponent ε to indicate that we have taken the limit. We proceed by induction and suppose that at time step m, we already have proven that $\rho^m = \rho_0$ and $\varphi^m = 0$. The latter implies that $q^m = a^m$ and consequently that $\nabla \cdot a^m = \nabla \cdot q^m = 0$. First, the limit $\varepsilon \to 0$ in (2.33) gives

$$\Delta p(\rho^{m+1}) = 0, \tag{2.40}$$

which, with the condition $\rho = \rho_0$ at the boundary gives $\rho^{m+1} = \rho_0$. Then, the limit $\varepsilon \to 0$ in (2.34) leads to

$$-\Delta \varphi^{m+1} = 0\,. \tag{2.41}$$

Again, with the condition $\partial_n \varphi^{m+1} = 0$ at the boundary, we get $\varphi^{m+1} = 0$. Then, we deduce from (2.37) that $q^{m+1} = a^{m+1}$. Eqs. (2.35) and (2.36) are formally unchanged in the limit $\varepsilon \to 0$ but eq. (2.35) inserted into (2.36) ensures that $\nabla \cdot a^{m+1} = \nabla \cdot a^m = 0$. Eq. (2.36) then appears as a time discretization of the Low-Mach number eq. (2.9).

Compared to a straightforward explicit discretization of (2.1), (2.2), the computation of one time step using (2.33), (2.37) involves considerably more computational work. Indeed, it requires the inversion of three elliptic equations: eq. (2.33) for finding $\rho^{\varepsilon,m+1}$, eq. (2.34) for $\varphi^{\varepsilon,m+1}$ and eq. (2.35) for $P^{\varepsilon,m+1}$. Additionally, eq. (2.33) is nonlinear and requires inner iterations. This is the price to pay for a scheme whose time step does not collapse to zero as $\varepsilon \to 0$. One way to make it more efficient is to use the upscaling technique which was designed in [16] for the coupling of Boltzmann and Euler models. This technique allows to switch from a standard scheme when the Mach number is of order unity or large to this AP scheme when the Mach number has values significantly below unity. The development of such a strategy will be the subject of future work.

An important remark is about the conservativity of the scheme when ε is finite. Indeed, when ε is finite, discontinuous solutions in the form of shock waves can appear. The use of non-conservative variables, i.e., variables other than the mass and momentum densities, may lead to incorrect shock speeds. Here, the scheme uses the conservative variables and is not subject to this problem. However, a question remains about whether the various transformations used to pass from the classical formulation (2.1), (2.2) to the gauge formulation (2.20)-(2.24) will not alter this property. In fact, the best way of having the final scheme (with time and space both discrete) satisfy the conservation property is to derive it from a usual shock capturing methodology (such as a Godunov or a Roe scheme). To this aim, one must start from the semi-implicit discretization (2.38), (2.39) of the original fomulation (2.1), (2.2) and reproduce the same computations as those used in the derivation of the gauge formulation of the continuous model. Indeed, we have seen that in the time-semi discrete case, the scheme (2.33)-(2.37) is a mere application of the gauge decomposition to the conservative semi-implicit scheme (2.38), (2.39). Using the same methodology for a fully discrete scheme will produce a gauge decomposed version of a fully conservative scheme. The resulting scheme will therefore produce the correct shock speeds. We shall not pursue this direction however and refer to future works for the details. Instead, we would like to show how the methodology can be extended to more complex models. In the Section 3, we will discuss the case of the full Navier-Stokes equations.

Finally, we note that a variant to the second order (in time) formulation (2.33) can be found, within the framework of a first order (in time) formulation. Indeed, we can merely eliminate $q^{\varepsilon,m+1}$ from (2.38) using (2.39) and get

$$\begin{aligned}&\frac{1}{\Delta t}(\rho^{\varepsilon,m+1}-\rho^{\varepsilon,m})+\nabla\cdot q^{\varepsilon,m}\\&\quad+\Delta t\left\{\nabla^2:(\frac{q^{\varepsilon,m}\otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}})+\frac{1}{\varepsilon^2}\Delta p(\rho^{\varepsilon,m+1})-\nabla^2:(\mu\sigma(u^{\varepsilon,m}))\right\}=0. \quad (2.42)\end{aligned}$$

This also leads to an elliptic equation for $\rho^{\varepsilon,m+1}$.

Also,another observation is that the nonlinearity in the elliptic operator can be linearized by using the approximation:

$$\Delta p(\rho^{\varepsilon,m+1})\approx\nabla\cdot(p'(\rho^{\varepsilon,m})\nabla\rho^{\varepsilon,m+1}),$$

without altering the AP character of the scheme.

3. Full Compressible Navier-Stokes Equations

3.1. The model and the low Mach number limit

In this section, we consider the full compressible Navier-Stokes equations

$$\partial_t\rho^\varepsilon+\nabla\cdot q^\varepsilon=0\,, \quad (3.1)$$

$$\partial_t q^\varepsilon + \nabla(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) + \frac{1}{\varepsilon^2}\nabla p^\varepsilon = \nabla(\mu^\varepsilon \sigma(u^\varepsilon)), \tag{3.2}$$

$$\partial_t W^\varepsilon + \nabla \cdot ((W^\varepsilon + p^\varepsilon)u^\varepsilon) = \varepsilon^2 \nabla \cdot (\mu^\varepsilon \sigma(u^\varepsilon)u^\varepsilon) + \varepsilon^2 \nabla \cdot (\kappa^\varepsilon \nabla T^\varepsilon), \tag{3.3}$$

$$W^\varepsilon = \frac{1}{2}\varepsilon^2 \rho^\varepsilon |u^\varepsilon|^2 + e^\varepsilon, \quad e^\varepsilon = \frac{1}{\gamma - 1}p^\varepsilon, \quad p^\varepsilon = \rho^\varepsilon T^\varepsilon. \tag{3.4}$$

The functions ρ^ε, q^ε, p^ε and u^ε, having the same meaning as in the previous section, are respectively the mass density, momentum density, pressure and velocity. In addition, we also have the total energy density W^ε, the internal energy density e^ε and the temperature T^ε. The viscosity μ^ε and the heat conductivity κ^ε are generally functions of ρ^ε and T^ε, and are indexed by ε. γ is the ratio of specific heats and is a given constant, equal to 5/3 for a perfect gas. Again, ε is a measure of the Mach number, the ratio of the typical velocity of the fluid to the typical velocity of sound.

We justify this scaling by going back to the physical variables. In addition to eqs (2.3), (2.4) (where $\bar p$ is now a function of $\bar\rho$ and $\bar T$), we have

$$\partial_{\bar t} \bar W + \nabla_{\bar x} \cdot ((\bar W + \bar p)\bar u) = \nabla_{\bar x} \cdot (\bar\mu \bar\sigma(\bar u)\bar u) + \nabla_{\bar x} \cdot (\bar\kappa \nabla_{\bar x} \bar T), \tag{3.5}$$

$$\bar W = \frac{1}{2}\bar\rho|\bar u|^2 + \bar e, \quad \bar e = \frac{1}{\gamma - 1}\bar p, \quad \bar p = \bar\rho \frac{k_B \bar T}{m}, \tag{3.6}$$

where k_B is the Boltzmann constant and m is the particle mass. We introduce an additional set of scaling units W_0, e_0 and T_0 for the total and internal energies and the temperature respectively and link them by the natural relations $W_0 = e_0 = p_0 = \rho_0 k_B T_0/m$. Then, after passage to the dimensionless variables, (3.5), (3.6) lead to (3.4) and to

$$\partial_t W + \nabla \cdot ((W + p)u) = \varepsilon^2 \nabla \cdot (\mu\sigma(u)u) + \frac{\kappa_0 T_0}{p_0 x_0 u_0} \nabla \cdot (\kappa \nabla T). \tag{3.7}$$

The dimensionless parameter $\frac{\kappa_0 T_0}{p_0 x_0 u_0}$ measures the ratio of the heat diffusion term compared to the energy transport term. We suppose that this ratio is of order ε^2, i.e., that in the limit $\varepsilon \to 0$, we obtain pure transport of the energy. We note that the term corresponding to the work of the viscosity force (the first term at the right-hand side) is of order ε^2 with the chosen scaling of the momentum equation.

To find the low Mach number limit, we expand $p^\varepsilon = p + \varepsilon^2 P + o(\varepsilon^2)$. At leading order in ε, eq. (3.2) leads to

$$\nabla p = 0, \tag{3.8}$$

which implies that

$$p = \rho T = p_0\,, \tag{3.9}$$

where p_0 is a constant which is independent of space and which we also take independent of time for simplicity. Therefore, ρ and T are now linked together. At the next order in ε, eq. (3.2) leads to

$$\partial_t q + \nabla(\frac{q \otimes q}{\rho}) + \nabla P = \nabla(\mu\sigma(u))\,, \tag{3.10}$$

and P is the hydrostatic pressure.

Next, we need to find the constraint which allows to compute P. For that purpose, we use a classical procedure to rewrite the energy equation (3.3) into an equation for the pressure p^ε. Of course, this manipulation is only valid for smooth solutions but in the limit $\varepsilon \to 0$, we do not expect shocks to appear. This equation is written as follows:

$$\begin{aligned}\frac{1}{\gamma-1}(\partial_t p^\varepsilon + u^\varepsilon \cdot \nabla p^\varepsilon) + \frac{\gamma}{\gamma-1} p^\varepsilon(\nabla \cdot u^\varepsilon)\\ = \varepsilon^2 \frac{\mu^\varepsilon}{2} |\sigma(u^\varepsilon)|^2 + \varepsilon^2 \nabla \cdot (\kappa^\varepsilon \nabla T^\varepsilon)\,,\end{aligned} \tag{3.11}$$

where for a tensor A, we denote by $|A|^2 = A : A$, the contracted product of A with itself. Letting $\varepsilon \to 0$ and using that p_0 is a constant in time and space, we find

$$\nabla \cdot u = 0\,, \tag{3.12}$$

Finally, because of (3.12), the mass conservation eq. (3.1) can be written as a transport equation at the limit:

$$\partial_t \rho + u \cdot \nabla \rho = 0\,. \tag{3.13}$$

The incompressible model consists of eqs. (3.9), (3.10), (3.12) and (3.13). The momentum equation (3.10) is equivalent to

$$\rho(\partial_t u + (u \cdot \nabla)u) + \nabla P = \nabla(\mu\sigma(u))\,. \tag{3.14}$$

But, because ρ is no more constant in space, the equation for the hydrostatic pressure P is more complicated than in the isentropic case and is given by:

$$-\nabla\cdot\left(\frac{1}{\rho}\nabla P\right)=\nabla\cdot\left(\frac{1}{\rho}(u\cdot\nabla)u\right)-\nabla\cdot\left(\frac{1}{\rho}\nabla(\mu\sigma(u))\right). \quad (3.15)$$

Finally, the energy has the following expansion

$$W^\varepsilon=\frac{p_0}{\gamma-1}+\varepsilon^2(\frac{1}{2}\rho|u|^2+P)+o(\varepsilon^2). \quad (3.16)$$

3.2. Gauge decomposition of the momentum field

The gauge decomposition of q^ε is modified as compared with the isentropic case, to take into account the fact that now the leading order density is no more a constant. We introduce a pseudo-solenoidal field a^ε and an irrotational one $\nabla\varphi^\varepsilon$ according to

$$q^\varepsilon=a^\varepsilon-\nabla\varphi^\varepsilon\,,\quad \nabla\cdot\left(\frac{a^\varepsilon}{\rho^\varepsilon}\right)=0, \quad (3.17)$$

Introducing (3.17) into (3.2), we get

$$\partial_t a^\varepsilon+\nabla(\frac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon})+\nabla P^\varepsilon=\nabla(\mu^\varepsilon\sigma(u^\varepsilon))\,, \quad (3.18)$$

where P^ε is a 'quasi-hydrostatic pressure' defined by

$$P^\varepsilon=\frac{1}{\varepsilon^2}(p^\varepsilon-p_0)-\partial_t\varphi^\varepsilon\,. \quad (3.19)$$

The mass conservation equation (3.1) is no longer useful to compute φ^ε and will actually determine ρ^ε. We shall see below how to determine φ^ε. In doing so, we need to transform $\nabla\cdot(\frac{1}{\rho^\varepsilon}\partial_t a^\varepsilon)$. First, using the second equation of (3.17) and the mass conservation equation (3.1) in the following way:

$$\nabla\cdot\left(\frac{1}{\rho^\varepsilon}\partial_t a^\varepsilon\right)\;=\;-\nabla\cdot\left(a^\varepsilon\partial_t\left(\frac{1}{\rho^\varepsilon}\right)\right). \quad (3.20)$$

We do not develop the time derivative of $1/\rho^\varepsilon$ using the mass conservation equation (3.1) because $1/\rho^\varepsilon$ is not a conservative variable and the resulting equation would not be valid for discontinuous solutions. The quasi-hydrostatic pressure P^ε is obtained by multiplying $1/\rho^\varepsilon$ to the momentum equation (2.14), and then taking the divergence by using 3.20 to get

$$
\begin{aligned}
-\nabla\cdot\left(\frac{1}{\rho^\varepsilon}\nabla P^\varepsilon\right) &= -\nabla\cdot\left(a^\varepsilon\partial_t\left(\frac{1}{\rho^\varepsilon}\right)\right)+\nabla\cdot\left(\frac{1}{\rho^\varepsilon}\nabla(\frac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon})\right)\\
&\quad -\nabla\cdot\left(\frac{1}{\rho^\varepsilon}\nabla(\mu^\varepsilon\sigma(u^\varepsilon))\right). \qquad (3.21)
\end{aligned}
$$

We still assume that $\partial_n\varphi^\varepsilon=0$ at the boundary and with (3.19), this implies that $P^\varepsilon=\mu\mathbf{n}\cdot\Delta u^\varepsilon+\frac{\mu}{3}\partial_n(\nabla\cdot u^\varepsilon)$ at the boundary.

Like in the case of the isentropic model, we need to find a second order formulation which reduces the low Mach number limit problem to an elliptic problem. But, by contrast to the isentropic case, this second order formulation involves the energy and not the mass density.

Taking the time derivative of the energy equation (3.3), we obtain

$$
\begin{aligned}
&\partial^2_{tt}W^\varepsilon+\nabla\cdot(h^\varepsilon\,\partial_t q^\varepsilon)+\nabla\cdot(\partial_t h^\varepsilon q^\varepsilon)\\
&\quad = \varepsilon^2\nabla\cdot(\partial_t(\mu^\varepsilon\sigma(u^\varepsilon)u^\varepsilon))+\varepsilon^2\nabla\cdot(\partial_t(\kappa^\varepsilon\nabla T^\varepsilon)), \qquad (3.22)
\end{aligned}
$$

where we have defined the enthalpy

$$h^\varepsilon=\frac{W^\varepsilon+p^\varepsilon}{\rho^\varepsilon}. \qquad (3.23)$$

On the other hand, using the momentum equation (3.2) to eliminate the time derivative of q^ε in (3.22) and using (3.4) to express the pressure in terms of the total energy, we get:

$$
\begin{aligned}
&\partial^2_{tt}W^\varepsilon-\frac{\gamma-1}{\varepsilon^2}\nabla\cdot(h^\varepsilon\,\nabla W^\varepsilon)+\frac{\gamma-1}{2}\nabla\cdot\big(h^\varepsilon\,\nabla(\rho^\varepsilon|u^\varepsilon|^2)\big)\\
&\quad -\nabla\cdot\left(h^\varepsilon\,\nabla(\frac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon})\right)+\nabla\cdot(\partial_t h^\varepsilon\,q^\varepsilon)+\nabla\cdot(h^\varepsilon\,\nabla(\mu^\varepsilon\sigma(u^\varepsilon)))\\
&\quad =\varepsilon^2\left[\nabla\cdot(\partial_t(\mu^\varepsilon\sigma(u^\varepsilon)u^\varepsilon))+\nabla\cdot(\partial_t(\kappa^\varepsilon\nabla T^\varepsilon))\right]. \qquad (3.24)
\end{aligned}
$$

Incidentally, we check that the limit of (3.24) when $\varepsilon\to 0$ leads to a constant pressure, as it should in the Low-Mach number limit. Indeed, in the limit $\varepsilon\to 0$, we formally find that $W=p/(\gamma-1)$ from (3.4), and $h=\frac{\gamma p}{(\gamma-1)\rho}$ satisfy:

$$-(\gamma-1)\nabla\cdot(h\nabla W)=-\gamma\nabla\cdot\left(\frac{p}{\rho}\nabla p\right)=0. \qquad (3.25)$$

Assuming that $p = p_0$ at the boundary where p_0 is uniform in space along the boundary and constant in time (for convenience), the solution of this equation is

$$p = p_0 \,. \tag{3.26}$$

Indeed, since p_0 is a constant, it is a solution (even if ρ is not a constant) and since the elliptic problem is well-posed, it is the unique one. Therefore, (3.24) gives the right Low-Mach number limit for the energy.

Now, to find φ^ε, we use the original, first-order formulation of the energy eq. (3.3), which we rewrite:

$$\partial_t W^\varepsilon + \nabla \cdot (h^\varepsilon (a^\varepsilon - \nabla \varphi^\varepsilon)) = \varepsilon^2 \nabla \cdot (\mu^\varepsilon \sigma(u^\varepsilon) u^\varepsilon) + \varepsilon^2 \nabla \cdot (\kappa^\varepsilon \nabla T^\varepsilon). \tag{3.27}$$

Knowing W^ε, this equation can be recast into an equation for φ^ε:

$$-\nabla \cdot (h^\varepsilon \nabla \varphi^\varepsilon) = -\partial_t W^\varepsilon - \nabla \cdot (h^\varepsilon a^\varepsilon) + \varepsilon^2 \nabla \cdot (\mu^\varepsilon \sigma(u^\varepsilon) u^\varepsilon) + \varepsilon^2 \nabla \cdot (\kappa^\varepsilon \nabla T^\varepsilon). \tag{3.28}$$

To summarize, the full-Navier-Stokes problem is formally equivalent to the following gauge formulation:

$$\begin{aligned}
&\partial_t^2 W^\varepsilon - \frac{\gamma-1}{\varepsilon^2} \nabla \cdot (h^\varepsilon \, \nabla W^\varepsilon) + \frac{\gamma-1}{2} \nabla \cdot \big(h^\varepsilon \, \nabla (\rho^\varepsilon |u^\varepsilon|^2) \big) \\
&\qquad - \nabla \cdot \left(h^\varepsilon \, \nabla (\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) \right) + \nabla \cdot (\partial_t h^\varepsilon \, q^\varepsilon) + \nabla \cdot (h^\varepsilon \, \nabla (\mu^\varepsilon \sigma(u^\varepsilon))) \\
&\qquad\qquad = \varepsilon^2 \left[\nabla \cdot (\partial_t (\mu^\varepsilon \sigma(u^\varepsilon) u^\varepsilon)) + \nabla \cdot (\partial_t (\kappa^\varepsilon \nabla T^\varepsilon)) \right], &(3.29)\\
&\partial_t \rho^\varepsilon + \nabla \cdot q^\varepsilon = 0 \,, &(3.30)\\
&-\nabla \cdot \left(\frac{1}{\rho^\varepsilon} \nabla P^\varepsilon \right) = -\nabla \cdot \left(a^\varepsilon \partial_t \left(\frac{1}{\rho^\varepsilon} \right) \right) + \nabla \cdot \left(\frac{1}{\rho^\varepsilon} \nabla (\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) \right) \\
&\qquad\qquad - \nabla \cdot \left(\frac{1}{\rho^\varepsilon} \nabla (\mu^\varepsilon \sigma(u^\varepsilon)) \right) \,, &(3.31)\\
&\partial_t a^\varepsilon + \nabla (\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon}) + \nabla P^\varepsilon = \nabla (\mu^\varepsilon \sigma(u^\varepsilon)) \,, &(3.32)\\
&-\nabla \cdot (h^\varepsilon \nabla \varphi^\varepsilon) = -\partial_t W^\varepsilon - \nabla \cdot (h^\varepsilon a^\varepsilon) + \varepsilon^2 \nabla \cdot (\mu^\varepsilon \sigma(u^\varepsilon) u^\varepsilon) \\
&\qquad\qquad + \varepsilon^2 \nabla \cdot (\kappa^\varepsilon \nabla T^\varepsilon), &(3.33)\\
&q^\varepsilon = a^\varepsilon - \nabla \varphi^\varepsilon \,. &(3.34)
\end{aligned}$$

Again, we have listed these equations in the natural order of a time-step loop, as we will see later on.

3.3. The gauge method viewed as a 'macro-micro' decomposition

Again, we introduce the following definitions: we define the macro-scale density ρ_0^ε as the solution of

$$\partial_t \rho_0^\varepsilon + \nabla \cdot a^\varepsilon = 0\,. \tag{3.35}$$

The quantitiy $W_0 = p_0/(\gamma-1)$ is the macro-scale energy, and is a *constant.* The macro-scale velocity u_0^ε and temperature T_0^ε are defined by $u_0^\varepsilon = a^\varepsilon/\rho_0^\varepsilon$, $T_0^\varepsilon = p_0/\rho_0^\varepsilon$.

We now define a set of micro-scale quantities. First, let

$$\varphi^\varepsilon = \varepsilon^2 \varphi_1^\varepsilon\,. \tag{3.36}$$

We shall see below why φ^ε is actually an $O(\varepsilon^2)$ quantity, which justifies defining φ_1^ε this way. Then, of course, the relation

$$q^\varepsilon = a^\varepsilon - \varepsilon^2 \nabla \varphi_1^\varepsilon \tag{3.37}$$

defines a macro-micro decomposition of q^ε.

Similarly, we define the micro-components of the pressure p_1^ε, density ρ_1^ε, energy W_1^ε, temperature T_1^ε, velocity u_1^ε, according to

$$p^\varepsilon = p_0^\varepsilon + \varepsilon^2 \nabla p_1^\varepsilon, \quad \rho^\varepsilon = \rho_0^\varepsilon + \varepsilon^2 \nabla \rho_1^\varepsilon, \quad \text{etc.}\,. \tag{3.38}$$

Here

$$W_1^\varepsilon = \frac{1}{2}\rho^\varepsilon |u^\varepsilon|^2 + \frac{1}{\gamma-1} p_1^\varepsilon\,. \tag{3.39}$$

We also introduce the decompositions of the enthalpy h^ε and of the specific volume $\tau^\varepsilon = 1/\rho^\varepsilon$:

$$h^\varepsilon = h_0^\varepsilon + \varepsilon^2 h_1^\varepsilon, \quad h_0^\varepsilon = \frac{W_0 + p_0}{\rho_0^\varepsilon} = \frac{\gamma}{\gamma-1}\frac{p_0}{\rho_0^\varepsilon}\,, \tag{3.40}$$

$$\tau^\varepsilon = \tau_0^\varepsilon + \varepsilon^2 \tau_1^\varepsilon, \quad \tau_0^\varepsilon = \frac{1}{\rho_0^\varepsilon}\,. \tag{3.41}$$

Finally, we introduce auxiliary expansion terms, such as the tensors Π_1^ε and

$(\mu\sigma(u))_1^\varepsilon$ defined by:

$$
\begin{aligned}
&\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon} = \frac{a^\varepsilon \otimes a^\varepsilon}{\rho_0^\varepsilon} + \varepsilon^2 \Pi_1^\varepsilon, \\
&\mu^\varepsilon \sigma(u^\varepsilon) = \mu_0^\varepsilon \sigma(u_0^\varepsilon) + \varepsilon^2 (\mu\sigma(u))_1^\varepsilon, \\
&\mu_0^\varepsilon = \mu(\rho_0^\varepsilon, T_0^\varepsilon)\,.
\end{aligned}
\tag{3.42}
$$

The independent (conservative) variable being the mass, momentum and energy densities written as a single vector $U^\varepsilon = (\rho^\varepsilon, q^\varepsilon, W^\varepsilon)$, the macroscopic field is given by $U_0^\varepsilon = (\rho_0^\varepsilon, a^\varepsilon, W_0)$ and the microscopic one is $U_1^\varepsilon = (\rho_1^\varepsilon, -\nabla\varphi_1^\varepsilon, W_1^\varepsilon)$. Obviously, $U^\varepsilon = U_0^\varepsilon + \varepsilon^2 U_1^\varepsilon$ and this relation is *exact* and not an approximation.

Now, we show that formulation (3.29)-(3.30) can be put in a form such that the dependence of the macroscopic variables on the microscopic ones only appear in $O(\varepsilon^2)$. Indeed, one can easily write the momentum conservation equations together with the definition (3.35) as

$$
\partial_t a^\varepsilon + \nabla(\frac{a^\varepsilon \otimes a^\varepsilon}{\rho_0^\varepsilon}) + \nabla P^\varepsilon - \nabla(\mu_0^\varepsilon \sigma(u_0^\varepsilon)) = \varepsilon^2 \left[-\nabla \Pi_1^\varepsilon - \nabla(\mu\sigma(u))_1^\varepsilon\right] \,, \tag{3.43}
$$

$$
\nabla \cdot \left(\frac{a^\varepsilon}{\rho_0^\varepsilon}\right) = -\varepsilon^2 \nabla \cdot (a^\varepsilon \tau_1^\varepsilon)\,, \tag{3.44}
$$

$$
\partial_t \rho_0^\varepsilon + \nabla \cdot a^\varepsilon = 0\,. \tag{3.45}
$$

Eq. (3.44) is nothing but the constraint (3.17) in which the decomposition of (3.41) has been used. We recall that the constraint (3.17) is easily deduced from (3.31) as soon as the constraint is satisfied initially. Now, we see that the macroscopic variables evolve according to equations in which the microscopic variables only enter the $O(\varepsilon^2)$ terms.

Now, we turn to the microscopic variables equations and begin with φ_1^ε. Noting that

$$
\begin{aligned}
\nabla \cdot (h^\varepsilon\, a^\varepsilon) &= \frac{\gamma p_0}{\gamma - 1} \nabla \cdot \left(\frac{a^\varepsilon}{\rho_0^\varepsilon}\right) + \varepsilon^2 \nabla \cdot (h_1^\varepsilon\, a^\varepsilon) \\
&= -\varepsilon^2 \frac{\gamma p_0}{\gamma - 1} \nabla \cdot (\tau_1^\varepsilon a^\varepsilon) + \varepsilon^2 \nabla \cdot (h_1^\varepsilon\, a^\varepsilon)\,,
\end{aligned}
\tag{3.46}
$$

by using (3.44). Inserting the definitions of the macro and micro variables into eq. (3.33) yields

$$\begin{aligned} -\nabla\cdot(h^\varepsilon\nabla\varphi_1^\varepsilon) &= -\partial_t W_1^\varepsilon - \frac{\gamma p_0}{\gamma-1}\nabla\cdot(\tau_1^\varepsilon a^\varepsilon) - \nabla\cdot(h_1^\varepsilon\, a^\varepsilon) \\ &\quad +\nabla\cdot(\mu^\varepsilon\sigma(u^\varepsilon)u^\varepsilon) + \nabla\cdot(\kappa^\varepsilon\nabla T^\varepsilon)\,. \end{aligned} \tag{3.47}$$

The equation for ρ_1^ε is deduced from (3.30), (3.45) and the decompositions (3.37), (3.38) and is given by

$$\partial_t\rho_1^\varepsilon - \Delta\varphi_1^\varepsilon = 0\,. \tag{3.48}$$

Then, the equation for W_1^ε follows from (3.29), which can be written equivalently as

$$\begin{aligned} &\partial_t^2 W^\varepsilon - \frac{1}{\varepsilon^2}\nabla\cdot(h^\varepsilon\nabla p^\varepsilon) - \nabla\cdot\left(h^\varepsilon\nabla(\frac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon})\right) + \nabla\cdot(\partial_t h^\varepsilon q^\varepsilon) \\ &\qquad +\nabla\cdot(h^\varepsilon\,\nabla(\mu^\varepsilon\sigma(u^\varepsilon))) \\ &\quad = \varepsilon^2\left[\nabla\cdot(\partial_t(\mu^\varepsilon\sigma(u^\varepsilon)u^\varepsilon)) + \nabla\cdot(\partial_t(\kappa^\varepsilon\nabla T^\varepsilon))\right], \end{aligned} \tag{3.49}$$

Inserting the decomposition (3.38) as well as (3.40), (3.43), we find that eq. (3.49) is equivalent to:

$$\begin{aligned} &-\nabla\cdot(h^\varepsilon\,\nabla p_1^\varepsilon) - \nabla\cdot\left(h^\varepsilon\,\nabla(\frac{q^\varepsilon\otimes q^\varepsilon}{\rho^\varepsilon})\right) + \nabla\cdot(\partial_t h^\varepsilon\, q^\varepsilon) + \nabla\cdot(h^\varepsilon\,\nabla(\mu^\varepsilon\sigma(u^\varepsilon))) \\ &\quad = \varepsilon^2\left[-\partial_t^2 W_1^\varepsilon + \nabla\cdot(\partial_t(\mu^\varepsilon\sigma(u^\varepsilon)u^\varepsilon)) + \nabla\cdot(\partial_t(\kappa^\varepsilon\nabla T^\varepsilon))\right], \end{aligned} \tag{3.50}$$

and gives an equation for p_1^ε. Then, W_1^ε is deduced through (3.39).

To summarize, the macroscopic equations (equations for the macroscopic variables $U_0^\varepsilon = (\rho_0^\varepsilon, a^\varepsilon, W_0))$ are (3.43), (3.44) and (3.45) (remember, W_0 is a constant and is given by the boundary conditions), while the microscopic equations (equations for the microscopic variables $U_1^\varepsilon = (\rho_1^\varepsilon, -\nabla\varphi_1^\varepsilon, W_1^\varepsilon))$ are (3.47), (3.48) and (3.50). From these considerations, the low Mach number limit is obvious. First we see that the equations for the microscopic variables involve terms of order 1 or order ε^2 but no term of order ε^{-2}. Consequently, the microscopic variables stay bounded as $\varepsilon\to 0$. Consequently, in the limit $\varepsilon\to 0$, it is legitimate to merely drop the order $O(\varepsilon^2)$ terms in the macroscopic equations (3.43), (3.44) and (3.45), which leads to the Low-Mach number limit system.

It is interesting to investigate the limit of p_1^ε when $\varepsilon \to 0$. To this aim, we further transform eq. (3.50) by using (3.40) and (3.37) and we find

$$-\nabla \cdot \left(\frac{1}{\rho^\varepsilon} \nabla p_1^\varepsilon \right) - \nabla \cdot \left(\frac{1}{\rho^\varepsilon} \nabla \left(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho^\varepsilon} \right) \right)$$
$$+\nabla \cdot \left(\partial_t \left(\frac{1}{\rho^\varepsilon} \right) a^\varepsilon \right) + \nabla \cdot \left(\frac{1}{\rho^\varepsilon} \nabla (\mu^\varepsilon \sigma(u^\varepsilon)) \right) = O(\varepsilon^2), \qquad (3.51)$$

Comparing with (3.31), we find that:

$$-\nabla \cdot \left(\frac{1}{\rho^\varepsilon} \nabla (p_1^\varepsilon - P^\varepsilon) \right) = O(\varepsilon^2), \qquad (3.52)$$

which shows that

$$P^\varepsilon = p_1^\varepsilon + O(\varepsilon^2). \qquad (3.53)$$

Therefore, the quasi-hydrostatic pressure is, with an error of order $O(\varepsilon^2)$, equal to the first order corrector of the fluid pressure. But where no simple elliptic equation for the pressure corrector is found, a nice elliptic equation for the quasi-hydrostatic pressure exists. An additional remark is that, since $P^\varepsilon \to P$ as $\varepsilon \to 0$, we similarly have $p_1^\varepsilon \to P$. This is again a known fact that the pressure corrector converges to the hydrostatic pressure in the low Mach number limit.

For practical applications and in particular, for numerical discretizations, it is preferable to use the set of equations (3.29)-(3.30) which is more compact. In the next section, we propose an Asymptotic-Preserving time-discretization of system (3.29)-(3.30). Again, we will only focus on the time-discretization.

3.4. Asymptotic-Preserving time-discretization of the gauge formulation

Following the same ideas as in Section 2.4, we propose the following scheme:

$$\frac{1}{\Delta t^2}(W^{\varepsilon,m+1} - 2W^{\varepsilon,m} + W^{\varepsilon,m-1}) - \frac{\gamma-1}{\varepsilon^2} \nabla \cdot \left(h^{\varepsilon,m} \nabla W^{\varepsilon,m+1} \right)$$
$$+\frac{\gamma-1}{2} \nabla \cdot \left(h^{\varepsilon,m} \nabla (\rho^{\varepsilon,m} |u^{\varepsilon,m}|^2) \right) - \nabla \cdot \left(h^{\varepsilon,m} \nabla \left(\frac{q^{\varepsilon,m} \otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}} \right) \right)$$

$$
\begin{aligned}
&+\nabla\cdot\left(\frac{1}{\Delta t}(h^{\varepsilon,m}-h^{\varepsilon,m-1})\,q^{\varepsilon,m}\right)+\nabla\cdot(h^{\varepsilon,m}\,\nabla(\mu^{\varepsilon,m}\sigma(u^{\varepsilon,m})))\\
=\ &\varepsilon^2\Big[\nabla\cdot(\frac{1}{\Delta t}(\mu^{\varepsilon,m}\sigma(u^{\varepsilon,m})u^{\varepsilon,m}-\mu^{\varepsilon,m-1}\sigma(u^{\varepsilon,m-1})u^{\varepsilon,m-1}))\\
&+\nabla\cdot(\frac{1}{\Delta t}(\kappa^{\varepsilon,m}\nabla T^{\varepsilon,m}-\kappa^{\varepsilon,m-1}\nabla T^{\varepsilon,m-1}))\Big]\,,
\end{aligned}
\tag{3.54}
$$

$$
\frac{1}{\Delta t}(\rho^{\varepsilon,m+1}-\rho^{\varepsilon,m})+\nabla\cdot q^{\varepsilon,m}=0\,,
\tag{3.55}
$$

$$
\begin{aligned}
-\nabla\cdot\left(\frac{1}{\rho^{\varepsilon,m+1}}\nabla P^{\varepsilon,m+1}\right)=&-\nabla\cdot\left(a^{\varepsilon,m}\frac{1}{\Delta t}\left(\frac{1}{\rho^{\varepsilon,m+1}}-\frac{1}{\rho^{\varepsilon,m}}\right)\right)\\
&+\nabla\cdot\left(\frac{1}{\rho^{\varepsilon,m+1}}\nabla(\frac{q^{\varepsilon,m}\otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}})\right)-\nabla\cdot\left(\frac{1}{\rho^{\varepsilon,m+1}}\nabla(\mu^{\varepsilon,m}\sigma(u^{\varepsilon,m}))\right),
\end{aligned}
\tag{3.56}
$$

$$
\frac{1}{\Delta t}(a^{\varepsilon,m+1}-a^{\varepsilon,m})+\nabla(\frac{q^{\varepsilon,m}\otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}})+\nabla P^{\varepsilon,m+1}=\nabla(\mu^{\varepsilon,m}\sigma(u^{\varepsilon,m})),
\tag{3.57}
$$

$$
\begin{aligned}
-\nabla\cdot\left(h^{\varepsilon,m}\,\nabla\varphi^{\varepsilon,m+1}\right)=&-\frac{1}{\Delta t}(W^{\varepsilon,m+1}-W^{\varepsilon,m})-\nabla\cdot\left(h^{\varepsilon,m}\,a^{\varepsilon,m+1}\right)\\
&+\varepsilon^2\nabla\cdot(\mu^{\varepsilon,m}\sigma(u^{\varepsilon,m})u^{\varepsilon,m})+\varepsilon^2\nabla\cdot(\kappa^{\varepsilon,m}\nabla T^{\varepsilon,m}),
\end{aligned}
\tag{3.58}
$$

$$
q^{\varepsilon,m+1}=a^{\varepsilon,m+1}-\nabla\varphi^{\varepsilon,m+1}\,.
\tag{3.59}
$$

We have note $h^{\varepsilon,m}=(W^{\varepsilon,m}+p^{\varepsilon,m})/\rho^{\varepsilon,m}$.

Now, we make some comments about this scheme. First, (3.54) can be deduced from the following scheme for the first order formulation of the momentum and energy equations (the mass equation scheme (3.55) is already of the time discretization of the first order equation (3.1)):

$$
\begin{aligned}
&\frac{1}{\Delta t}(q^{\varepsilon,m+1}-q^{\varepsilon,m})+\nabla(\frac{q^{\varepsilon,m}\otimes q^{\varepsilon,m}}{\rho^{\varepsilon,m}})+\frac{\gamma-1}{\varepsilon^2}\nabla W^{\varepsilon,m+1}\\
&\qquad-\frac{\gamma-1}{2}\nabla(\rho^{\varepsilon,m}|u^{\varepsilon,m}|^2)=\nabla(\mu^{\varepsilon,m}\sigma(u^{\varepsilon,m}))\,,
\end{aligned}
\tag{3.60}
$$

$$
\begin{aligned}
&\frac{1}{\Delta t}(W^{\varepsilon,m+1}-W^{\varepsilon,m})+\nabla\cdot(h^{\varepsilon,m}q^{\varepsilon,m+1})\\
&\quad=\ \varepsilon^2\nabla\cdot(\mu^{\varepsilon,m}\sigma(u^{\varepsilon,m})u^{\varepsilon,m})+\varepsilon^2\nabla\cdot(\kappa^{\varepsilon,m}\nabla T^{\varepsilon,m})\,.
\end{aligned}
\tag{3.61}
$$

Indeed, taking the difference of (3.61) at time $m+1$ and at time m and combining with the divergence of (3.60) leads to (3.54). We see that this scheme is based on taking the energy flux implicit by taking the momentum implicit and the enthalpy explicit, on the one hand, and implicit the part of the momentum flux which corresponds to the gradient of the energy on the other hand. As in the isentropic case, this scheme is based on taking an appropriate selection of flux terms implicitly. By contrast to the isentropic case, the mass flux term is taken explicitly in (3.1).

Next, we easily see that (3.56) follows from applying the constraint $\nabla\cdot a^{\varepsilon,m+1}/\rho^{\varepsilon,m+1}=0$ to (3.57). Finally, (3.58) is obtained by inserting the decomposition (3.59) into the first order formulation (3.61). Therefore, the whole scheme is based on a gauge decomposition of the semi-implicit scheme (3.55), (3.60), (3.61).

Now, we show that, in the limit $\varepsilon\to 0$, this scheme gives a consistent approximation of the low Mach number equations (3.9), (3.10), (3.12), (3.13). We drop the exponent ε to indicate that we have taken the limit. We proceed by induction and suppose that at time step m, we already have proven that $W^m = p_0/(\gamma-1)$ and $\varphi^m=0$. The latter implies that $q^m=a^m$ and consequently that $\nabla\cdot(a^m/\rho^m)=\nabla\cdot(q^m/\rho^m)=0$.

First, let $\varepsilon\to 0$ in (3.54) and find

$$\gamma p_0 \nabla\cdot\left(\frac{1}{\rho}\nabla W^{m+1}\right)=0\,. \tag{3.62}$$

Since $W^{m+1}=p_0/(\gamma-1)$ at the boundary and that p_0 is uniform along the boundary and constant in time, we deduce that $W^{m+1}=p_0/(\gamma-1)$ everywhere. Indeed, beeing a constant, this function satisfies both eq. (3.62) inside the domain and the boundary condition. Similarly, the limit $\varepsilon\to 0$ in (3.58) leads to:

$$-\frac{\gamma p_0}{\gamma-1}\nabla\cdot\left(\frac{1}{\rho}\nabla\varphi^{m+1}\right)=0\,. \tag{3.63}$$

Since $\partial_n\varphi^{m+1}=0$ along the boundary, we have $\varphi^{m+1}=0$ everywhere. Then, $q^{m+1}=a^{m+1}$ and eq. (3.56) is just equivalent to saying that $\nabla\cdot(a^{m+1}/\rho^{m+1})=0$, as soon as $\nabla\cdot(a^m/\rho^m)=0$ which is the case by induction hypothesis. Therefore, the scheme (3.54)-(3.59) reduces to the only equations (3.55), (3.56) and (3.57) with $q^{\varepsilon,m}\equiv a^{\varepsilon,m}$ for all m, and is obviouslyl consistent with the Low-Mach number model when $\varepsilon\to 0$.

About the numerical cost of this scheme, the same remarks as in the isentropic case can be made. The computational cost involves the inversion of three elliptic operators: one in (3.54) for finding $W^{\varepsilon,m+1}$, one in (3.56) for finding $P^{\varepsilon,m+1}$ and one in (3.58) for finding $\varphi^{\varepsilon,m+1}$. By contrast to the isentropic case, the diffusion coefficients of these elliptic operators change in the course of time. However, two of the three operators have the same diffusion coefficient $h^{\varepsilon,m}$ and the third diffusion coefficient is $\rho^{\varepsilon,m}$, which, when $\varepsilon \ll 1$, is nearly proportional to $h^{\varepsilon,m}$. Beside the inversion of these three elliptic operators, this scheme leads to explicit computations, since the various unknowns can be computed recursively, following the order of exposition of the equations in (3.54)-(3.59). This is a big advantage over other implicit approaches leading to more complex nonlinear iterations.

Finally, we remark that, like in the isentropic case, the conservativity of the scheme is enforced by the use of the conservative variables $(\rho^\varepsilon, q^\varepsilon, W^\varepsilon)$ and the use of the conservative scheme (3.60), (3.61). To be more specific about this point, one needs to use a shock capturing based discretization. The investigation of the space discretization will be the subject to future work.

Also, like in the isentropic case, the second order (in time) formulation (3.54) can be replaced by a first order formulation by eliminating $q^{\varepsilon,m+1}$ from (3.61) using (3.60). We leave this computation to the reader.

In the next section, we investigate another example of this methodology, the isentropic Navier-Stokes-Poisson system.

4. Isentropic Navier-Stokes-Poisson System

4.1. The model and the small Mach number / Debye length limits

The isentropic Navier-Stokes-Poisson system is written:

$$\partial_t \rho^{\varepsilon,\lambda} + \nabla \cdot q^{\varepsilon,\lambda} = 0\,, \tag{4.1}$$

$$\partial_t q^{\varepsilon,\lambda} + \nabla\Big(\frac{q^{\varepsilon,\lambda} \otimes q^{\varepsilon,\lambda}}{\rho^{\varepsilon,\lambda}}\Big) + \frac{1}{\varepsilon^2}\nabla p(\rho^{\varepsilon,\lambda}) = -\frac{1}{\varepsilon^2}\rho^{\varepsilon,\lambda}\nabla\phi^{\varepsilon,\lambda} + \nabla(\mu\sigma(u^{\varepsilon,\lambda}))\,, \tag{4.2}$$

$$-\lambda^2 \Delta \phi^{\varepsilon,\lambda} = \rho^{\varepsilon,\lambda} - \rho_B\,, \tag{4.3}$$

where $\phi^{\varepsilon,\lambda}(x,t)$ is the potential energy, $\rho_B(x,t) \geq 0$ is a given non-negative neutralizing background density and λ^2 is a dimensionless parameter representing the square of the ratio of the Debye length to the characteristic length. For instance, the considered species are electrons, the dimensionless electric potential is $-\phi^{\varepsilon,\lambda}$ and the neutralizing species are positive ions.

The scaling of this system repeats many of the considerations of Section 2.1 and we refer to that section for the notations. The equations in physical variables are written

$$\partial_{\bar t}\bar\rho + \nabla_{\bar x}\cdot\bar q = 0\,, \tag{4.4}$$

$$\partial_{\bar t}\bar q + \nabla_{\bar x}(\frac{\bar q\otimes\bar q}{\bar\rho}) + \nabla_{\bar x}\bar p(\bar\rho) = -\frac{\bar\rho}{m}\nabla_{\bar x}\bar\phi + \nabla_{\bar x}(\bar\mu\bar\sigma(\bar u))\,, \tag{4.5}$$

$$-\Delta_{\bar x}\bar\phi = \frac{q}{\epsilon_0}(\frac{q\bar\rho}{m} - \frac{q_b\bar\rho_B}{m_B})\,. \tag{4.6}$$

where q is the charge of the considered particle species and m is its mass, while q_B and m_B are the charge and mass of the neutralizing background species.

In Section 2.1, we supposed that the scales are related by the relations:

$$u_0 = \frac{x_0}{t_0}, \quad q_0 = \rho_0 u_0, \quad \rho_0 u_0^2/p_0 = \varepsilon^2, \quad \frac{\mu_0}{\rho_0 u_0 x_0} = 1. \tag{4.7}$$

Now, we introduce a potential scale ϕ_0. Using Section 2.1, the scaling of (4.4), (4.5), (4.6) leads to

$$\partial_t\rho + \nabla\cdot q = 0\,, \tag{4.8}$$

$$\partial_t q + \nabla(\frac{q\otimes q}{\rho}) + \frac{1}{\varepsilon^2}\nabla p(\rho) = -\left(\frac{\phi_0}{m u_0^2}\right)\rho\nabla\phi + \nabla(\mu\sigma(u))\,, \tag{4.9}$$

$$-\left(\frac{\epsilon_0\phi_0 m}{\rho_0 q^2 x_0^2}\right)\Delta\phi = \rho - \rho_B\,, \tag{4.10}$$

where $\rho_B = \frac{q m_B}{q_B m}\bar\rho_B$. In doing so, we assume that q/q_B and $m\bar\rho_B/(m_B\rho_0)$ are order unity. That q/q_B is of order unity is not restrictive in general, because the charge levels of the ions are generally just a few unities above the electron charge. Similarly, the ratio $m\bar\rho_B/(m_B\rho_0)$ is close to one in all cases close to quasineutrality, which encompasses a large number of situations in plasma physics.

Now, we discuss the values of the two other dimensionless parameters. We assume that the electric potential energy scale is equal to the thermal energy scale: $\phi_0 = m p_0/\rho_0$. This implies that

$$\frac{\phi_0}{m u_0^2} = \frac{\phi_0\rho_0}{m p_0}\,\frac{p_0}{\rho_0 u_0^2} = \frac{1}{\varepsilon^2}. \tag{4.11}$$

The second parameter can be written

$$\frac{\epsilon_0\phi_0 m}{\rho_0 q^2 x_0^2} = \frac{\epsilon_0\phi_0}{n_0 q^2}\,\frac{1}{x_0^2} = \frac{\lambda_D^2}{x_0^2}\,, \tag{4.12}$$

where we have introduced the density scale $n_0 = \rho_0/m$ and recognized the definition of the Debye length $\lambda_D = \frac{\epsilon_0\phi_0}{n_0 q^2}$. Setting $\lambda^2 = \frac{\lambda_D^2}{x_0^2}$, we find system (4.1)-(4.3).

In all what follows, we want to derive an Asymptotic Preserving scheme with respect to both limits $\varepsilon \to 0$ and $\lambda \to 0$. The limit $\varepsilon \to 0$ alone was investigated in a series of papers [11], [12], [17]. These papers deal with the Euler case but they can be straightforwardly extended to the Navier-Stokes case.

We now investigate the successive limits $\varepsilon \to 0$ (low Mach number limit) and $\lambda \to 0$ (Quasineutral limit) in both orders.

First case: $\lambda \to 0$ then $\varepsilon \to 0$: When $\lambda \to 0$ first, we get the following system:

$$\partial_t \rho_B + \nabla \cdot q^\varepsilon = 0\,, \tag{4.13}$$

$$\partial_t q^\varepsilon + \nabla\Big(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho_B}\Big) + \frac{1}{\varepsilon^2}\nabla p(\rho_B) = -\frac{1}{\varepsilon^2}\rho_B \nabla \phi^\varepsilon + \nabla(\mu\sigma(u^\varepsilon))\,, \tag{4.14}$$

$$\rho^\varepsilon = \rho_B\,, \tag{4.15}$$

In this limit, we find that the particle density ρ^ε is everywhere equal to the background density ρ_B. Then, the mass equation (4.13) becomes a divergence constraint for the momentum q^ε while ϕ^ε appears as the Lagrange multiplier of this constraint.

We note that in the simple case where ρ_B is a constant, independent of position and time (say $\rho_B = 1$ to make it easier) the model simplifies into

$$\nabla \cdot q^\varepsilon = 0\,, \tag{4.16}$$

$$\partial_t q^\varepsilon + \nabla(q^\varepsilon \otimes q^\varepsilon) = -\frac{1}{\varepsilon^2}\nabla\phi^\varepsilon + \nabla(\mu\sigma(u^\varepsilon))\,, \tag{4.17}$$

and we recognize the incompressible Navier-Stokes equation with hydrostatic pressure $P = \frac{1}{\varepsilon^2}\phi^\varepsilon$. It is interesting to note that the rescaling $\tilde{\phi}^\varepsilon = \frac{1}{\varepsilon^2}\phi^\varepsilon$ makes the model independent of ε and thus it coincides with its limit $\varepsilon \to 0$.

A similar feature holds in the case of a non-constant ρ_B. To see this, we introduce the enthalpy function $h(\rho)$ such that $h'(\rho) = p'(\rho)/\rho$ and we define $\psi^\varepsilon = \frac{1}{\varepsilon^2}(\phi^\varepsilon + h(\rho_B))$ Then, the model can be written:

$$\partial_t \rho_B + \nabla \cdot q^\varepsilon = 0\,, \tag{4.18}$$

$$\partial_t q^\varepsilon + \nabla(\frac{q^\varepsilon \otimes q^\varepsilon}{\rho_B}) = -\rho_B \nabla \psi^\varepsilon + \nabla(\mu\sigma(u^\varepsilon))\,. \tag{4.19}$$

This is actually not the incompressible Navier-Stokes equation (with non-constant density ρ_B) because of the pressure term replaced by $\rho_B \nabla \psi^\varepsilon$. In some sense, the projection of the momentum equation onto the divergence free fields is not the same as in the true Navier-Stokes equation but nonetheless bears a strong similarity. Like in the constant ρ_B case, the model does not depend on ε and coincides with its limit $\varepsilon \to 0$.

The pseudo-pressure term ψ^ε can be computed by taking the divergence of (4.19) and using (4.18) to eliminate $\nabla \cdot q^\varepsilon$, which leads to

$$\nabla \cdot (\rho_B \nabla \psi^\varepsilon) = -\partial_t^2 \rho_B + \nabla^2 : (\frac{q^\varepsilon \otimes q^\varepsilon}{\rho_B}) - \nabla(\mu\sigma(u^\varepsilon))\,, \tag{4.20}$$

Second case: $\varepsilon \to 0$ then $\lambda \to 0$: To derive the $\varepsilon \to 0$ limit, we rewrite the momentum equation (4.2) using the enthalpy function and get

$$\partial_t q^{\varepsilon,\lambda} + \nabla(\frac{q^{\varepsilon,\lambda} \otimes q^{\varepsilon,\lambda}}{\rho^{\varepsilon,\lambda}}) + \frac{1}{\varepsilon^2}\rho^{\varepsilon,\lambda}\nabla(h(\rho^{\varepsilon,\lambda}) + \phi^{\varepsilon,\lambda}) = \nabla(\mu\sigma(u^{\varepsilon,\lambda}))\,, \tag{4.21}$$

Therefore, when $\varepsilon \to 0$, we get at leading order that

$$h(\rho^\lambda) + \phi^\lambda = 0\,, \tag{4.22}$$

We assume that $\rho \to h(\rho)$ is an increasing function from $\mathbb{R}_+$ into $\mathbb{R}_+$ and denote by h^{-1} its inverse function. Then, $\rho^\lambda = h^{-1}(-\phi^\lambda)$ and the Poisson equation becomes:

$$-\lambda^2 \Delta \phi^\lambda - h^{-1}(-\phi^\lambda) = -\rho_B\,, \tag{4.23}$$

This is a nonlinear elliptic equation. The nonlinearity $-h^{-1}(-\phi^\lambda)$ being an increasing function of ϕ^λ, this problem is well-posed, provided appropriate boundary conditions are given, which we shall leave unspecified here. In the limit $\varepsilon \to 0$, the mass equation (4.1) remains unchanged, but becomes a constraint for q^λ since ρ^λ is specified by (4.22). To find an equation for q^λ, we look at the next order in ε of the momentum equation (4.2) and we find:

$$\partial_t q^\lambda + \nabla(\frac{q^\lambda \otimes q^\lambda}{\rho^\lambda}) + \rho^\lambda \nabla \psi^\lambda = \nabla(\mu\sigma(u^\lambda))\,, \tag{4.24}$$

where $\psi^\lambda = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}(h(\rho^{\varepsilon,\lambda}) + \phi^{\varepsilon,\lambda})$.

To summarize, the low Mach number limit $\varepsilon \to 0$ of the Navier-Stokes-Poisson system (4.1)-(4.3) is the following system:

$$\partial_t \rho^\lambda + \nabla \cdot q^\lambda = 0\,, \tag{4.25}$$

$$\partial_t q^\lambda + \nabla(\frac{q^\lambda \otimes q^\lambda}{\rho^\lambda}) + \rho^\lambda \nabla \psi^\lambda = \nabla(\mu\sigma(u^\lambda))\,, \tag{4.26}$$

$$-\lambda^2 \Delta \phi^\lambda - h^{-1}(-\phi^\lambda) = -\rho_B\,, \tag{4.27}$$

$$\rho^\lambda = h^{-1}(-\phi^\lambda)\,. \tag{4.28}$$

Again, we find a kind of incompressible Navier-Stokes system but with an unusual projection $\rho^\lambda \nabla \psi^\lambda$ to the divergence constraint. The divergence constraint involves a non-zero right-hand side which is found via the resolution of a nonlinear elliptic problem.

In the limit $\lambda \to 0$ of this system, we find the same modified incompressible Navier-Stokes problem (4.18)

$$\partial_t \rho_B + \nabla \cdot q = 0\,, \tag{4.29}$$

$$\partial_t q + \nabla(\frac{q \otimes q}{\rho_B}) = -\rho_B \nabla \psi + \nabla(\mu\sigma(u))\,, \tag{4.30}$$

$$\rho = \rho_B, \quad \phi = -h(\rho_B)\,. \tag{4.31}$$

4.2. Gauge methodology

Here, the momentum field already satisfies the right gauge. Indeed, in either limits $\lambda \to 0$ or $\varepsilon \to 0$ or both, the momentum field satisfies the constraint given by the mass conservation equation (4.1). Therefore, there is no need to decompose the momentum field in a gauge satisfying field and a small remainder, which in the previous cases was a gradient field. However, we will borrow from the gauge methodology that we shall interpret the mass equation as a gauge constraint for the momentum equation. More precisely, we shall write them

$$\partial_t \rho^{\varepsilon,\lambda} + \nabla \cdot q^{\varepsilon,\lambda} = 0\,, \tag{4.32}$$

$$\partial_t q^{\varepsilon,\lambda} + \nabla(\frac{q^{\varepsilon,\lambda} \otimes q^{\varepsilon,\lambda}}{\rho^{\varepsilon,\lambda}}) + \rho^{\varepsilon,\lambda} \nabla \psi^{\varepsilon,\lambda} = \nabla(\mu\sigma(u^{\varepsilon,\lambda})) \tag{4.33}$$

with the gauge pressure defined as

$$\psi^{\varepsilon,\lambda} = \frac{1}{\varepsilon^2}(h(\rho^{\varepsilon,\lambda}) + \phi^{\varepsilon,\lambda}). \tag{4.34}$$

Then, if the mass conservation equation is used as a gauge, we need another equation to find $\rho^{\varepsilon,\lambda}$. For this purpose, we derive a wave-equation formulation of the system by taking the time derivative of (4.1) and the divergence of (4.2) and subtracting them, and using the following identity, which follows from Poisson's equation (4.3):

$$\nabla \cdot (\rho^{\varepsilon,\lambda}\nabla\phi^{\varepsilon,\lambda}) = \nabla\rho^{\varepsilon,\lambda} \cdot \nabla\phi^{\varepsilon,\lambda} - \frac{1}{\lambda^2}\rho^{\varepsilon,\lambda}(\rho^{\varepsilon,\lambda} - \rho_B), \tag{4.35}$$

we find

$$\begin{aligned} &\lambda^2\big[\varepsilon^2\big(\partial_t^2\rho^{\varepsilon,\lambda} - \nabla^2 : (\frac{q^{\varepsilon,\lambda}\otimes q^{\varepsilon,\lambda}}{\rho^{\varepsilon,\lambda}}) + \nabla(\mu\sigma(u^{\varepsilon,\lambda}))\big) \\ &\quad -\nabla\cdot(\rho^{\varepsilon,\lambda}h'(\rho^{\varepsilon,\lambda})\nabla\rho^{\varepsilon,\lambda}) - \nabla\rho^{\varepsilon,\lambda}\cdot\nabla\phi^{\varepsilon,\lambda}\big] + \rho^{\varepsilon,\lambda}(\rho^{\varepsilon,\lambda} - \rho_B) = 0 \end{aligned} \tag{4.36}$$

Then, of course, knowing $\psi^{\varepsilon,\lambda}$ and $\rho^{\varepsilon,\lambda}$, we recover $\phi^{\varepsilon,\lambda}$ thanks to (4.34), i.e.,

$$\phi^{\varepsilon,\lambda} = -h(\rho^{\varepsilon,\lambda}) + \varepsilon^2\psi^{\varepsilon,\lambda}, \tag{4.37}$$

To summarize, our gauge method is based on the following formulation:

$$\begin{aligned} &\lambda^2\big[\varepsilon^2\big(\partial_t^2\rho^{\varepsilon,\lambda} - \nabla^2 : (\frac{q^{\varepsilon,\lambda}\otimes q^{\varepsilon,\lambda}}{\rho^{\varepsilon,\lambda}}) + \nabla(\mu\sigma(u^{\varepsilon,\lambda}))\big) - \nabla\cdot(\rho^{\varepsilon,\lambda}h'(\rho^{\varepsilon,\lambda})\nabla\rho^{\varepsilon,\lambda}) \\ &\quad -\nabla\rho^{\varepsilon,\lambda}\cdot\nabla\phi^{\varepsilon,\lambda}\big] + \rho^{\varepsilon,\lambda}(\rho^{\varepsilon,\lambda} - \rho_B) = 0, \end{aligned} \tag{4.38}$$

$$\partial_t\rho^{\varepsilon,\lambda} + \nabla\cdot q^{\varepsilon,\lambda} = 0, \tag{4.39}$$

$$\partial_t q^{\varepsilon,\lambda} + \nabla(\frac{q^{\varepsilon,\lambda}\otimes q^{\varepsilon,\lambda}}{\rho^{\varepsilon,\lambda}}) + \rho^{\varepsilon,\lambda}\nabla\psi^{\varepsilon,\lambda} = \nabla(\mu\sigma(u^{\varepsilon,\lambda})). \tag{4.40}$$

$$\phi^{\varepsilon,\lambda} = -h(\rho^{\varepsilon,\lambda}) + \varepsilon^2\psi^{\varepsilon,\lambda}, \tag{4.41}$$

We will not develop the viewpoint of the macro-micro decomposition. Indeed, there are two parameters, and we should develop such an approach for each parameter separately, which would be cumbersome. But this is not very difficult and this point is left to the reader. In the next section, we propose an Asymptotic Preserving discretization with respect to both limits

$\varepsilon \to 0$ and $\lambda \to 0$.

4.3. Asymptotic-Preserving time-discretization of the gauge formulation

We propose the following time-discretization scheme of the formulation (4.38)-(4.41):

$$\lambda^2 \big[\varepsilon^2 \big(\frac{1}{\Delta t^2}(\rho^{\varepsilon,\lambda,m+1} - 2\rho^{\varepsilon,\lambda,m} + \rho^{\varepsilon,\lambda,m-1}) - \nabla^2 : (\frac{q^{\varepsilon,\lambda,m} \otimes q^{\varepsilon,\lambda,m}}{\rho^{\varepsilon,\lambda,m}}) + \nabla(\mu\sigma(u^{\varepsilon,\lambda,m}))\big) - \nabla \cdot (\rho^{\varepsilon,\lambda,m} h'(\rho^{\varepsilon,\lambda,m}) \nabla \rho^{\varepsilon,\lambda,m+1}) - \nabla \rho^{\varepsilon,\lambda,m} \cdot \nabla \phi^{\varepsilon,\lambda,m}\big] + \rho^{\varepsilon,\lambda,m}(\rho^{\varepsilon,\lambda,m+1} - \rho_B^{m+1}) = 0\,, \tag{4.42}$$

$$\frac{1}{\Delta t}(\rho^{\varepsilon,\lambda,m+1} - \rho^{\varepsilon,\lambda,m}) + \nabla \cdot q^{\varepsilon,\lambda,m+1} = 0\,, \tag{4.43}$$

$$\frac{1}{\Delta t}(q^{\varepsilon,\lambda,m+1} - q^{\varepsilon,\lambda,m}) + \nabla(\frac{q^{\varepsilon,\lambda,m} \otimes q^{\varepsilon,\lambda,m}}{\rho^{\varepsilon,\lambda,m}}) + \rho^{\varepsilon,\lambda,m} \nabla \psi^{\varepsilon,\lambda,m+1} = \nabla(\mu\sigma(u^{\varepsilon,\lambda,m}))\,. \tag{4.44}$$

$$\phi^{\varepsilon,\lambda,m+1} = -h(\rho^{\varepsilon,\lambda,m+1}) + \varepsilon^2 \psi^{\varepsilon,\lambda,m+1}\,, \tag{4.45}$$

In fact, we show that this scheme is derived from the following scheme for the standard formulation:

$$\frac{1}{\Delta t}(\rho^{\varepsilon,\lambda,m+1} - \rho^{\varepsilon,\lambda,m}) + \nabla \cdot q^{\varepsilon,\lambda,m+1} = 0\,, \tag{4.46}$$

$$\frac{1}{\Delta t}(q^{\varepsilon,\lambda,m+1} - q^{\varepsilon,\lambda,m}) + \nabla(\frac{q^{\varepsilon,\lambda,m} \otimes q^{\varepsilon,\lambda,m}}{\rho^{\varepsilon,\lambda,m}}) + \rho^{\varepsilon,\lambda,m} \nabla \psi^{\varepsilon,\lambda,m+1} = \nabla(\mu\sigma(u^{\varepsilon,\lambda,m}))\,. \tag{4.47}$$

$$-\lambda^2 \Delta \phi^{\varepsilon,\lambda,m+1} = \rho^{\varepsilon,\lambda,m+1} - \rho_B^{m+1}\,, \tag{4.48}$$

$$\phi^{\varepsilon,\lambda,m+1} = -h(\rho^{\varepsilon,\lambda,m+1}) + \varepsilon^2 \psi^{\varepsilon,\lambda,m+1}\,, \tag{4.49}$$

Indeed, by taking the time difference of the mass equation (4.46) at times $m+1$ and m and combining with the divergence of (4.47), using (4.49) and the identity (4.35), we find

$$\lambda^2 \big[\varepsilon^2 \big(\frac{1}{\Delta t^2}(\rho^{\varepsilon,\lambda,m+1} - 2\rho^{\varepsilon,\lambda,m} + \rho^{\varepsilon,\lambda,m-1}) - \nabla^2 : (\frac{q^{\varepsilon,\lambda,m} \otimes q^{\varepsilon,\lambda,m}}{\rho^{\varepsilon,\lambda,m}}) + \nabla(\mu\sigma(u^{\varepsilon,\lambda,m}))\ \big) - \nabla \cdot (\rho^{\varepsilon,\lambda,m} h'(\rho^{\varepsilon,\lambda,m+1}) \nabla \rho^{\varepsilon,\lambda,m+1}) - \nabla \rho^{\varepsilon,\lambda,m} \cdot \nabla \phi^{\varepsilon,\lambda,m+1}\big] + \rho^{\varepsilon,\lambda,m}(\rho^{\varepsilon,\lambda,m+1} - \rho_B^{m+1}) = 0\,, \tag{4.50}$$

The difference between (4.50) and (4.42) is a time shift by one time step in two terms, the second one on the second line (inside the function h) and the third one of the second line (inside the $\nabla\phi$).

Going backwards, we easily deduce that the scheme (4.42)-(4.45) is consistent with the Poisson equation up to terms of order $O(\lambda^2\Delta t)$. More precisely, from (4.42) and the combination of (4.43), (4.44), we get

$$\begin{aligned}\lambda^2\big[\varepsilon^2\nabla\cdot(\rho^{\varepsilon,\lambda,m}\nabla\psi^{\varepsilon,\lambda,m+1})-\nabla\cdot(\rho^{\varepsilon,\lambda,m}h'(\rho^{\varepsilon,\lambda,m})\nabla\rho^{\varepsilon,\lambda,m+1})\\ -\nabla\rho^{\varepsilon,\lambda,m}\cdot\nabla\phi^{\varepsilon,\lambda,m}\big]+\rho^{\varepsilon,\lambda,m}(\rho^{\varepsilon,\lambda,m+1}-\rho_B^{m+1})=0\,, \qquad (4.51)\end{aligned}$$

which we can equivalently write:

$$\begin{aligned}\lambda^2\big[\varepsilon^2\nabla\cdot(\rho^{\varepsilon,\lambda,m}\nabla\psi^{\varepsilon,\lambda,m+1})-\nabla\cdot(\rho^{\varepsilon,\lambda,m}\nabla h(\rho^{\varepsilon,\lambda,m+1}))-\nabla\rho^{\varepsilon,\lambda,m}\cdot\nabla\phi^{\varepsilon,\lambda,m+1}\big]\\ +\rho^{\varepsilon,\lambda,m}(\rho^{\varepsilon,\lambda,m+1}-\rho_B^{m+1})+\lambda^2\big[-\nabla\cdot(\rho^{\varepsilon,\lambda,m}(h'(\rho^{\varepsilon,\lambda,m})\\ -h'(\rho^{\varepsilon,\lambda,m+1}))\nabla\rho^{\varepsilon,\lambda,m+1})-\nabla\rho^{\varepsilon,\lambda,m}\cdot(\nabla\phi^{\varepsilon,\lambda,m}-\nabla\phi^{\varepsilon,\lambda,m+1})\big]=0\,. \quad (4.52)\end{aligned}$$

Then, using (4.45), we find

$$\begin{aligned}\lambda^2\big[-\nabla\cdot(\rho^{\varepsilon,\lambda,m}\nabla\phi^{\varepsilon,\lambda,m+1})-\nabla\rho^{\varepsilon,\lambda,m}\cdot\nabla\phi^{\varepsilon,\lambda,m+1}\big]+\rho^{\varepsilon,\lambda,m}(\rho^{\varepsilon,\lambda,m+1}-\rho_B^{m+1})\\ +\lambda^2\big[-\nabla\cdot(\rho^{\varepsilon,\lambda,m}(h'(\rho^{\varepsilon,\lambda,m})-h'(\rho^{\varepsilon,\lambda,m+1}))\nabla\rho^{\varepsilon,\lambda,m+1})\\ -\nabla\rho^{\varepsilon,\lambda,m}\cdot(\nabla\phi^{\varepsilon,\lambda,m}-\nabla\phi^{\varepsilon,\lambda,m+1})\big]=0\,, \qquad (4.53)\end{aligned}$$

or

$$\begin{aligned}\rho^{\varepsilon,\lambda,m}(\lambda^2\Delta\phi^{\varepsilon,\lambda,m+1}+\rho^{\varepsilon,\lambda,m+1}-\rho_B^{m+1})\\ +\lambda^2\big[-\nabla\cdot(\rho^{\varepsilon,\lambda,m}(h'(\rho^{\varepsilon,\lambda,m})-h'(\rho^{\varepsilon,\lambda,m+1}))\nabla\rho^{\varepsilon,\lambda,m+1})\\ -\nabla\rho^{\varepsilon,\lambda,m}\cdot(\nabla\phi^{\varepsilon,\lambda,m}-\nabla\phi^{\varepsilon,\lambda,m+1})\big]=0\,. \qquad (4.54)\end{aligned}$$

Now, the last two lines are of order $O(\lambda^2\Delta t)$. Therefore, we find that $\phi^{\varepsilon,\lambda,m+1}$ satisfies a Laplace equation of the form

$$\rho^{\varepsilon,\lambda,m}(\lambda^2\Delta\phi^{\varepsilon,\lambda,m+1}+\rho^{\varepsilon,\lambda,m+1}-\rho_B^{m+1})=O(\lambda^2\Delta t)\,. \qquad (4.55)$$

We see that the potential is all the more close to a solution of the Poisson equation than λ is small, i.e., that we are close to the quasineutral regime. We also see that the potential equation is independent of ε and that it remains true in the limit $\varepsilon\to 0$.

Now, we investigate the limits $\lambda\to 0$ and $\varepsilon\to 0$ of this scheme.

First case: $\lambda \to 0$ then $\varepsilon \to 0$: When $\lambda \to 0$ first, (4.56) leads to

$$\rho^{\varepsilon,m}(\rho^{\varepsilon,m+1} - \rho_B^{m+1}) = 0\,, \tag{4.56}$$

while the other equations remain unchanged. Clearly, this means that

$$\rho^{\varepsilon,m+1} = \rho_B^{m+1}\,, \tag{4.57}$$

unless $\rho^{\varepsilon,m} = 0$, a situation in which all equations degenerate anyhow, and which we shall disregard. Then, we clearly get a scheme consistent with (4.18), (4.19). We also keep relation (4.45) which we gives us the value of the electric potential. However, since the electric potential is no longer coupled with the other variables, and that the equations (4.43), (4.44) for the momentum and the gauge potential ψ are independent of ε, the scheme is also obviously AP in the limit $\varepsilon \to 0$.

Second case: $\varepsilon \to 0$ then $\lambda \to 0$: When $\varepsilon \to 0$, we get

$$\begin{aligned} &\lambda^2\big[-\nabla\cdot(\rho^{\lambda,m}h'(\rho^{\lambda,m})\nabla\rho^{\lambda,m+1}) - \nabla\rho^{\lambda,m}\cdot\nabla\phi^{\lambda,m}\big] \\ &\quad +\rho^{\lambda,m}(\rho^{\lambda,m+1} - \rho_B^{m+1}) = 0\,, &(4.58)\\ &\phi^{\lambda,m+1} = -h(\rho^{\lambda,m+1})\,, &(4.59)\end{aligned}$$

while equations (4.43), (4.44) stay unchanged. To see that this scheme is consistent with system (4.25)-(4.28), it remains to show that (4.58) is consistent with Poisson's equation (4.27). But the approximate Poisson equation (4.54) which was derived previously from the scheme with finite ε does not depend on ε and is therefore also true for its limit $\varepsilon \to 0$. This shows that Asymptotic Preserving character of the scheme in the limit $\varepsilon \to 0$.

Then, the limit $\lambda \to 0$ is obvious and leads to (4.57) (provided $\rho^{\varepsilon,m} \neq 0$), which shows that the resulting scheme is also consistent with the limit $\lambda \to 0$ performed after the limit $\varepsilon \to 0$.

The computational complexity of this scheme involves the inversion of two elliptic operators. The first one is needed for the computation of $\rho^{\varepsilon,\lambda,m+1}$ by means of (4.42). The elliptic operator to be inverted is

$$A\rho^{\varepsilon,\lambda,m+1} := -\nabla\cdot(p'(\rho^{\varepsilon,\lambda,m})\nabla\rho^{\varepsilon,\lambda,m+1}) + \left(\frac{\lambda^2\varepsilon^2}{\Delta t^2} + \rho^{\varepsilon,\lambda,m}\right)\rho^{\varepsilon,\lambda,m+1}\,, \tag{4.60}$$

and is well-posed, provided boundary conditions for $\rho^{\varepsilon,\lambda,m+1}$ are provided. The second one concerns the computation of $\psi^{\varepsilon,\lambda,m+1}$. The equation for

$\psi^{\varepsilon,\lambda,m+1}$ is obtained by taking the divergence of (4.44) and eliminating $\nabla \cdot q^{\varepsilon,\lambda,m+1}$ by using the mass equation (4.43). It leads to:

$$\begin{aligned} -\nabla\cdot(\rho^{\varepsilon,\lambda,m}\nabla\psi^{\varepsilon,\lambda,m+1}) &= -\frac{1}{\Delta t^2}(\rho^{\varepsilon,\lambda,m+1}-2\rho^{\varepsilon,\lambda,m}+\rho^{\varepsilon,\lambda,m-1}) \\ &\quad +\nabla^2:(\frac{q^{\varepsilon,\lambda,m}\otimes q^{\varepsilon,\lambda,m}}{\rho^{\varepsilon,\lambda,m}})-\nabla(\mu\sigma(u^{\varepsilon,\lambda,m}))\,, \end{aligned} \tag{4.61}$$

Again, this elliptic equation is well-posed. The boundary conditions in the low mach number limit should be such that $\psi^{\varepsilon,\lambda,m+1}$ stays of order $O(1)$ otherwise the Low-Mach number limit is not valid.

The fact that the scheme does not satisfy exactly the Poisson equation can be viewed as a drawback in the cases where accuracy in the computation of the electrostatic interaction is important. In the next section, we propose a variant of this scheme with exact enforcement of the Poisson equation.

4.4. A variant with exact enforcement of the Poisson equation

This variant is based on a reformulation (in a gauge like framework) of the scheme (4.46), (4.49) in which we slightly modify the gauge equation (4.49) into

$$\nabla\phi^{\varepsilon,\lambda,m+1} = -h'(\rho^{\varepsilon,\lambda,m})\nabla\rho^{\varepsilon,\lambda,m+1}+\varepsilon^2\nabla\psi^{\varepsilon,\lambda,m+1}\,. \tag{4.62}$$

Indeed, only the gradients of these quantities are needed and this gauge equation is an order Δt approximation of (4.49). In this scheme, we are not going to compute the density first, like in the first one, but the electrostatic potential. Since the direct use (4.48) does not lead to an AP scheme when $\lambda \to 0$, we need to reformulate the Poisson equation. We perform it in the spirit of what has already been proposed in [11], [12], [17].

For that purpose, we take the time difference of the mass equations (4.46) at time $m+1$ and m, take the divergence of the momentum equation (4.47) and subtract the resulting equations but, instead of using (4.35) to transform the term $\nabla\cdot(\rho^{\varepsilon,\lambda,m}\nabla\phi^{\varepsilon,\lambda,m+1})$ like we did in the derivation of (4.50), we just directly use Poisson's equation (4.48) to eliminate $\rho^{\varepsilon,\lambda,m+1}$ in favor of $\phi^{\varepsilon,\lambda,m+1}$. This leads to the following scheme:

$$\lambda^2\nabla\cdot(p'(\rho^{\varepsilon,\lambda,m})\nabla\Delta\phi^{\varepsilon,\lambda,m+1})-\frac{\lambda^2\varepsilon^2}{\Delta t^2}\Delta\phi^{\varepsilon,\lambda,m+1}-\nabla\cdot(\rho^{\varepsilon,\lambda,m}\nabla\phi^{\varepsilon,\lambda,m+1})$$

$$+\varepsilon^2 \Big(\frac{1}{\Delta t^2}(\rho_B^{m+1} - 2\rho^{\varepsilon,\lambda,m} + \rho^{\varepsilon,\lambda,m-1}) - \nabla^2 : (\frac{q^{\varepsilon,\lambda,m} \otimes q^{\varepsilon,\lambda,m}}{\rho^{\varepsilon,\lambda,m}})$$
$$+\nabla(\mu\sigma(u^{\varepsilon,\lambda,m}))\Big) - \nabla \cdot (p'(\rho^{\varepsilon,\lambda,m})\nabla \rho_B^{m+1}) = 0\,, \qquad (4.63)$$

This equation is a fourth order elliptic equation which allows us to find $\phi^{\varepsilon,\lambda,m+1}$ as a function of known data. It is completely equivalent to the scheme (4.46), (4.48) with modified gauge equation (4.62). Once $\phi^{\varepsilon,\lambda,m+1}$ is known, $\rho^{\varepsilon,\lambda,m+1}$ can be computed by using the Poisson equation (4.48) directly. However, this operation might be unstable because of the laplacian of $\phi^{\varepsilon,\lambda,m+1}$ in the source term. Also, it is not possible to extend the method to the multispecies case. We prefer to use the wave-like reformulation (4.50), which, because of the gauge change, takes the form

$$\lambda^2\Big[\varepsilon^2 \Big(\frac{1}{\Delta t^2}(\rho^{\varepsilon,\lambda,m+1} - 2\rho^{\varepsilon,\lambda,m} + \rho^{\varepsilon,\lambda,m-1}) - \nabla^2 : (\frac{q^{\varepsilon,\lambda,m} \otimes q^{\varepsilon,\lambda,m}}{\rho^{\varepsilon,\lambda,m}})$$
$$+\nabla(\mu\sigma(u^{\varepsilon,\lambda,m}))\Big) - \nabla \cdot (p'(\rho^{\varepsilon,\lambda,m})\nabla \rho^{\varepsilon,\lambda,m+1}) - \nabla\rho^{\varepsilon,\lambda,m} \cdot \nabla\phi^{\varepsilon,\lambda,m+1}\Big]$$
$$+\rho^{\varepsilon,\lambda,m}(\rho^{\varepsilon,\lambda,m+1} - \rho_B^{m+1}) = 0\,, \qquad (4.64)$$

The only change with (4.42) is that last term of the second line involves $\phi^{\varepsilon,\lambda,m+1}$, which is known from the previous step, and not $\phi^{\varepsilon,\lambda,m}$. Again, this equation for $\rho^{\varepsilon,\lambda,m+1}$ is completely equivalent to the scheme (4.46), (4.48) with modified gauge equation (4.62). Once $\rho^{\varepsilon,\lambda,m+1}$ is known, we can solve for $q^{\varepsilon,\lambda,m+1}$ and $\psi^{\varepsilon,\lambda,m+1}$ like previously.

This scheme enforces the Poisson exactly. It is AP when $\varepsilon \to 0$ and/or $\lambda \to 0$ in either order, as can be easily seen (this point is left to the reader). However, this scheme is more complicated because it involves the resolution of three elliptic problems instead of two: problem (4.63) for $\phi^{\varepsilon,\lambda,m+1}$, problem (4.64) for $\rho^{\varepsilon,\lambda,m+1}$ and problem (4.61) for $\psi^{\varepsilon,\lambda,m+1}$. It also involves the resolution of a fourth order elliptic problem (problem (4.63) for $\phi^{\varepsilon,\lambda,m+1}$). Finally, it bears a slight inconsistency in the gauge equation, because it is impossible to satisfy the gauge condition (4.62) unless $\rho^{\varepsilon,\lambda,m}$ is a function of $\rho^{\varepsilon,\lambda,m+1}$, which is obviously not true. However, we note that we never use this gauge condition explicitly. Also, it is an approximation to the true gauge condition (4.49) (of order $O(\Delta t)$). We note that the use of the true gauge condition (4.49) is possible but transforms both problems (4.63) and (4.64) into nonlinear elliptic problems (the first one being fourth order).

5. Conclusion

In this paper, we have proposed new semi-implicit time discretizations for the compressible Navier-Stokes equations. These schemes are Asymptotic-Preserving in the low Mach number limit, i.e., they are consistent with the compressible Navier-Stokes equations when the Mach number is finite and are consistent with the incompressible equations (or Low-Mach number limit of the compressible Navier-Stokes equations) when the Mach number is small. To achieve Asymptotic-Preservation, we use a gauge decomposition of the momentum field which can be interpreted as a macro-micro decomposition of the problem. Additionally, a second order formulation in time is used for the density or the energy, giving rise to an easy numerical resolution of the implicitness, through the inversion of elliptic operators. A similar approach has been applied to the isentropic Navier-Stokes-Poisson system. In future work, we will investigate the effect of the space discretization and search for solvers which have good properties respective to the chosen time-stepping strategies. For this purpose, intensive numerical simulations will be carried out.

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References

1. S. Abarbanel, P. Duth and D. Gottlieb, Splitting methods for low Mach number Euler and Navier-Stokes equations, *Computers and Fluids*, **17**(1989), 1-12.

2. W.Z. Bao and S. Jin, Weakly compressible high-order I-stable central diffeence schemes for incompressible viscous flows, *Comput. Methods Appl. Mech. Engrg.*, **190**(2001), 5009-5026.

3. J. B. Bell, P. Colella and H. M. Glaz, A second-order projection method for the incompressible navier-stokes equations, *J. Comput. Phys.*, **85**(1989), 257-283.

4. J. B. Bell and D. L. Marcus, A second-order projection method for variable-density flows, *J. Comput. Phys.*, **101**(1992), 334-348.

5. C. Buet, S. Cordier, B. Lucquin-Desreux and S. Mancini, Diffusion limit of the Lorentz model: asymptotic preserving schemes, M2AN **36**(2002), 631-656.

6. T. Buttke and A. Chorin, Turbulence calculation using magnetization variables, *Appl. Numer. Math.*, **12**(1993), 47-54.

7. R. Caflisch, S. Jin and G. Russo, Uniformly Accurate Schemes for Hyperbolic Systems with Relaxations, *SIAM J. Num. Anal.*, **34**(1997), 246-281.

8. Y. -H. Choi and C. L. Merkle, The Application of Preconditioning in Viscous Flows, *J. Comput. Phys.*, **105**(1993), 207-223.

9. A. J. Chorin, A numerical method for solving incompressible viscous flow problems, *J. Comput. Phys.*, **2**(1967), 12-26.

10. P. Colella and K. Pao, A projection method for low speed flows, *J. Comput. Phys.*, **149**(1999), 245-269.

11. P. Crispel, P. Degond, M-H. Vignal, An asymptotically stable discretization for the Euler-Poisson system in the quasineutral limit, *C. R. Acad. Sci. Paris*, **341**(2005), 341-346.

12. P. Crispel, P. Degond, M-H. Vignal, An asymptotic preserving scheme for the two-fluid Euler-Poisson model in the quasineutral limit, *J. Comput. Phys.*, **223**(2007), 208-234.

13. D. L. Darmofal and P. J. Schmid, The importance of eigenvectors for local preconditioners of the Euler equations, *J. Comput. Phys.*, **127**(1996), 346-362.

14. P. Degond, F. Deluzet and L. Navoret, An asymptotically stable Particle-in-Cell (PIC) scheme for collisionless plasma simuations near quasineutrality, *C. R. Acad. Sci. Paris Ser. I*, **343**(2006), 613-618.

15. P. Degond, S. Jin and M. Tang, An Asymptotic-Preserving for the complex Ginzburg-Landau equation in the large time and space scale limit, in preparation.

16. P. Degond, J.-G. Liu, L. Mieussens, Macroscopic fluid models with localized kinetic upscaling effects, *Multiscale Model. Simul.*, **5**(2006), 940-979.

17. P. Degond, J.-G. Liu, M-H. Vignal, Analysis of an asymptotic preserving scheme for the Euler-Poisson system in the quasineutral limit, submitted.

18. W. E and J.-G. Liu, Finite difference schemes for incompressible flows in the impulse-velocity formulation, *J. Comput. Phys.*, **130**(1997), 67-76.

19. W. E and J.-G. Liu, Gauge finite element method for incompressible flows, *Int. J. Num. Meth. Fluids*, **34**(2000), 701-710.

20. W. E and J.-G. Liu, Gauge method for viscous incompressible flows, *Comm. Math. Sci.*, **1**(2003), 317-332.

21. J.- F. Gerbeau, N. Glinsky-Olivier and B. Larrouturou, Semi-implicit Roe-type fluxes for low-Mach number flows, INRIA Research report # 3132 (1997).

22. F. Golse, S. Jin and C.D. Levermore, The Convergence of Numerical Transfer Schemes in Diffusive Regimes I: The Discrete-Ordinate Method, *SIAM J. Numer. Anal.*, **36**(1999), 1333-1369.

23. L. Gosse and G. Toscani, Asymptotic preserving and well-balanced schemes for radiative transfer and the Rosseland approximation, *Numer. Math.* **98**(2004), 223-250

24. L. Gosse and G. Toscani, An asymptotic preserving well-balanced scheme for the hyperbolic heat equation, *C. R. Acad. Sci. Paris Ser. I*, **334**(2002), 1-6.

25. H. Guillard and A. Murrone, On the behaviour of upwind schemes in the low Mach number limit: II. Godunov type schemes, INRIA research report # 4189 (2001).

26. H. Guillard and C. Viozat, On the behaviour of upwind schemes in the low Mach number limit, *Comput. & Fluids*, **28** (1999), 63-86.

27. B. Gustafsson and H. Stoor, Navier-Stokes Equations for Almost Incompressible Flow, *SIAM J. Numer. Anal.*, **28**(1991), 1523-1547.

28. Z. Huang, S. Jin, P. A. Markowich, C. Sparber and C. Zheng, A time-splitting spectral scheme for the Maxwell-Dirac system, *J. Comput. Phys.*, **208**(2005), 761-789.

29. S. Jin, Efficient Asymptotic-Preserving (AP) Schemes for Some Multiscale Kinetic Equations, *SIAM J. Sci. Comput.*, **21**(1999), 441-454.

30. S. Jin, Runge-Kutta Methods for hyperbolic conservation laws with stiff relaxation terms, *J. Comput. Phys.*, **122**(1995), 51-67.

31. S. Jin and C. D. Levermore, Numerical schemes for hyperbolic conservation laws with stiff relaxation terms, *J. Comput. Phys.*, **126**(1996), 449-467.

32. S. Jin, L. Pareschi and G. Toscani, Diffusive Relaxation Schemes for Discrete-Velocity Kinetic Equations, *SIAM J. Numer. Anal.*, **35**(1998), 2405-2439.

33. S. Jin, L. Pareschi and G. Toscani, Uniformly accurate diffusive relaxation schemes for multiscale transport equations, *SIAM J. Numer. Anal.* **38**(2000), 913-936.

34. P. Jenny and B. Müler, Convergence acceleration for computing steady-state compressible flow at low Mach numbers, *Comput. & Fluids*, **28** (1999), 951-972.

35. S. Y. Kadioglu, M. Sussman, S. Osher, J. P. Wright and M. Kang, A second order primitive preconditioner for solving all speed multi-phase flows, *J. Comput. Phys.*, **209**(2005), 477-503.

36. G. Karniadakis and S. J. Sherwin, *Spectral/hp Element Methods for CFD*, Oxford Univ. Press, New York, 1999.

37. S. Klainerman and A. Majda, Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids, *Comm. Pure Appl. Math.*, **34**(1981), 481-524.

38. R. Klein, Semi-implicit extension of a godunov-type scheme based on low mach number asymptotics I: One-dimensional flow, *J. Comput. Phys.*, **121** (1995), 213-237.

39. M. Lai, H. B. Bell, P. Colella, A projection method for combustion in the zero Mach number limit, AIAA paper 1993-3369 (1993).

40. S. -H. Lee, Cancellation problem of preconditioning method at low Mach numbers, *J. Comput. Phys.*, In Press, Corrected Proof, Available online 9 April 2007,

41. M.- S. Liou, Mass flux schemes and connection to shock instability, *J. Comput. Phys.*, **160**(2000), 623-648.

42. M.- S. Liou, A sequel to AUSM, Part II: AUSM+-up for all speeds, J. Comput. Phys., **214**, (2006), 137-170.

43. M.-S. Liou and C. J. Steffen, A New Flux Splitting Scheme, *J. Comput. Phys.*, **107**(1993), 23-39.

44. J. H. Maddocks and R. L. Pego, An unconstrained Hamiltonian formulation for incompressible fluid , *Comm. Math. Phys.*, **170**(1995), 207-217.

45. A. Majda and J. Sethian, The derivation and numerical solution of the equations for zero Mach number combustion, *J. Combustion Science and Technology*, **42**(1985), 185-205.

46. W. A. Mulder and B. Van Leer, Experiments with implicit upwind methods for the Euler equations, *J. Comput. Phys.*, **59**(1985), 232-246.

47. K. Nerinckx, J. Vierendeels and E. Dick, Mach-uniformity through the coupled pressure and temperature correction algorithm, *J. Comput. Phys.*, **206**(2005), 597-623.

48. K. Nerinckx, J. Vierendeels and E. Dick, A Mach-uniform algorithm: Coupled versus segregated approach, *J. Comput. Phys.*, **224**(2007), 314-331.

49. F. Nicoud, Conservative high-order finite-difference schemes for low-mach number flows, *J. Comput. Phys.*, **158**(2000), 71-97.

50. V. I. Oseledets, On a new way of writing the Navier-Stokes equation. The Hamiltonian formalism, *Russ. Math. Surveys*, **44**(1989), 210-211.

51. H. Paillere, C. Viozat, A. Kumbaro, I. Toumi, Comparison of low Mach number models for natural convection problems, *Heat and Mass Transfer*, **36**(2000), 567-573.

52. L. Pareschi, G. Russo, Asymptotic preserving Monte Carlo methods for the Boltzmann equation, *Transport Theory Statist. Phys.*, **29**(2000), 415-430.

53. P. Roberts, A Hamiltonain theory for weakly interacting vortices, *Mathematica*, **19**(1972), 169-179.

54. G. Russo and P. Smereka, Impulse formulation of the Euler equations: general properties and numerical methods, *J. Fluid Mech.*, **391**(1999), 189-209.

55. E. Turkel, Preconditioned methods for solving the incompressible and low speed compressible equations, *J. Comput. Phys.*, **72**(1987), 277-298.

56. B. Van Leer, W. -T. Lee and P. Roe, Characteristic time-stepping or local preconditioning of the Euler equations, In *AIAA Computational Fluid Dynamics Conference*, 10th, Honolulu, HI, June 24-27, 1991, Technical Papers (A91-40701 17-34). Washington, DC, American Institute of Aeronautics and Astronautics, 1991, 260-282.

57. D. Vidovic, A. Segal and P. Wesseling, A superlinearly convergent Mach-uniform finite volume method for the Euler equations on staggered unstructured grids, *J. Comput. Phys.*, **217**(2006), 277-294.

58. C. Viozat, Implicit upwind schemes for low Mach number compressible flows, INRIA Research report # 3084 (1997).

59. C. Wall, C. D. Pierce and P. Moin, A Semi-implicit Method for Resolution of Acoustic Waves in Low Mach Number Flows, *J. Comput. Phys.*, **181**(2002), 545-563.

60. C. Wang and J.-G. Liu, Convergence of gauge method for incompressible flow, *Math. Comp.*, **69**(2000), 1385-1407.

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