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UNIQUENESS THEOREMS FOR PERIODIC SOLUTIONS OF CERTAIN FOURTH AND FIFTH ORDER DIFFERENTIAL SYSTEMS

BY

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Abstract

In this paper, we establish sufficient conditions which guarantee the existence at the most one ω-periodic solution for certain two class of fourth and fifth order differential equations. Our results extend some well-known results carried out in the relevant literature.

1. Introduction and Statement of the Result

We consider fourth and fifth order nonlinear vector differential equations

$$X^{(4)} + A_1 \dddot{X} + F(\ddot{X}) + A_3 \dot{X} + G(X) = P_1(t), \tag{1.1}$$

and

$$X^{(5)} + B_1 X^{(4)} + B_2 \dddot{X} + \Phi(\ddot{X}) + B_4 \dot{X} + H(X) = P_2(t), \tag{1.2}$$

in the real Euclidean space R^n (with the usual norm denoted in what follows by $\|.\|$) where A_1, A_3, B_1, B_2, B_4 are constant $n \times n-$ matrices; F, G, Φ, $H \in C^1[R^n, R^n]$ and $P_1, P_2 \in C^0[R, R^n]$. The matrices A_1, A_3, B_1, B_2 and B_4 that appeared in (1.1) and (1.2) are symmetric and the functions P_1, P_2 are both ω-periodic in t, that is $P_i(t+\omega) = P_i(t)$, $(i = 1, 2)$, for some $\omega > 0$ and all $t > 0, t \in R$. Let $J_f(\ddot{X})$, $J_g(X)$, $J_\phi(\ddot{X})$, $J_h(X)$ denote the

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Jacobian matrices corresponding to the functions $F(\ddot{X})$, $G(X)$, $\Phi(\ddot{X})$, $H(X)$ respectively, that is $J_f(\ddot{X}) = (\frac{\partial f_i}{\partial \ddot{x}_j})$, $J_g(X) = (\frac{\partial g_i}{\partial x_j})$, $J_\phi(\ddot{X}) = (\frac{\partial \phi_i}{\partial \ddot{x}_j})$, $J_h(X) = (\frac{\partial h_i}{\partial x_j})$ where $(x_1, x_2, \ldots, x_n)$, $(\ddot{x_1}, \ddot{x_2}, \ldots, \ddot{x_n})$, $(f_1, f_2, \ldots, f_n)$, $(g_1, g_2, \ldots, g_n)$, $(\phi_1, \phi_2, \ldots, \phi_n)$ and $(h_1, h_2, \ldots, h_n)$ are the components of X, $\ddot{X}$, F, G, Φ and H, respectively. It will be further assumed as basic throughout the paper that $J_f(\ddot{X})$, $J_g(X)$, $J_\phi(\ddot{X})$, $J_h(X)$ are symmetric (for arbitrary $X \in R^n$), so that their eigenvalues, which we denote respectively by $\lambda_i(J_g(X))$, $\lambda_i(J_h(X))$, $(i = 1, 2, \ldots, n)$, are all real.

In 1983, Ezeilo [5] discussed the existence of periodic solutions of the non-linear vector differential equations

$$X^{(4)} + A_1 \dddot{X} + A_2 \ddot{X} + A_3 \dot{X} + G(X) = P_1(t)$$

and

$$X^{(5)} + B_1 X^{(4)} + B_2 \dddot{X} + B_3 \ddot{X} + B_4 \dot{X} + H(X) = P_2(t).$$

According to the our observations in the relevant literature, we did not find another research with respect to the continuation of results established by Ezeilo [5]. It should be noticed that our results extend that obtained in [5]. However, till now, in a sequence of the works periodic properties for various third-, fourth-, fifth-, sixth-, seventh and eighth order certain nonlinear differential equations have been the subject of many investigations. (See, for example, Ezeilo ([4], [5]), Tejumola [9], Tunç ([13], [14], [15], [16]), and the references cited therein.)

We establish the following results.

Theorem 1. *In addition to the fundamental assumptions imposed F and G in* (1.1)*, suppose that following condition are satisfied:*

Let $\delta_0 = \max_{i,j} \left| \frac{\partial f_i}{\partial \ddot{x}_j} \right|$ *where* $J_f(\ddot{X}) = (\frac{\partial f_i}{\partial \ddot{x}_j})$*, and suppose that there exists a constant* $\alpha_1 > \frac{1}{4} n^2 \delta_0^2$ *such that*

$$\lambda_i(J_g(X)) \geq \alpha_1 \text{ for } i = 1, 2, \ldots, n \text{ and for arbitrary } X \in R^n. \qquad (1.3)$$

Then there exists at most one ω*-periodic solution of* (1.1).

Theorem 2. *Assume that B_1 is definite (positive or negative) and let*

$$\beta_1 = \inf_i \lambda_i(B_1) \quad or \quad -\sup_i \lambda_i(B_1),$$

according as B_1 is positive or negative definite, where $\lambda_i(B_1)$ $(i = 1, \ldots, n)$ are the eigenvalues of B_1. Let

$$\gamma_0 = \max_{i,j} \left| \frac{\partial \phi_i}{\partial \ddot{x}_j} \right|, \ where \ J_\phi(\ddot{X}) = (\frac{\partial \phi_i}{\partial \ddot{x}_j}).$$

Suppose that there exists a constant $\beta_2 > \frac{1}{4} n^2 \gamma_0^2 \beta_1^{-1}$ such that

$$k_1 \lambda_i(J_h(X)) \geq \beta_2 \tag{1.4}$$

where

$$k_1 = \begin{cases} +1, & if\ B_1\ is\ positive\ definite \\ -1, & if\ B_1\ is\ negative\ definite. \end{cases}$$

Then there exists at most one ω-periodic solution of (1.2).

We need the following algebraic result

Lemma. *Let A be a real symmetric $n \times n$ matrix and*

$$a' \geq \lambda_i(A) \geq a > 0 \ (i = 1, 2, \ldots, n), \ where\ a',\ a\ are\ constants.$$

Then

$$a' \langle X, X \rangle \ \geq \ \langle AX, X \rangle \geq a \langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \ \geq \ \langle AX, AX \rangle \geq a^2 \langle X, X \rangle.$$

Proof. See [17]. □

2. Proof of the Theorem 1

Let $X_1(t)$, $X_2(t)$ be any two solutions of (1.1) and set

$$Y(t) = X_2(t) - X_1(t).$$

Then $Y = Y(t)$ satisfies the differential equation

$$Y^{(4)} + A_1 \dddot{Y} + S(t) \ddot{Y} + A_3 \dot{Y} + R(t)Y = 0 \tag{2.1}$$

where the matrices $R(t)$ and $S(t)$ here are defined by

$$R(t) = \int_0^1 J_g(X_1(t) + \sigma(X_2(t) - X_1(t)))d\sigma, \tag{2.2}$$

$$S(t) = \int_0^1 J_f(\ddot{X}_1(t) + \sigma(\ddot{X}_2(t) - \ddot{X}_1(t)))d\sigma, \tag{2.3}$$

respectively. If $\langle . \rangle$, here and in what follows, denotes the usual scalar product in R^n, that is $\langle U, V\rangle = \sum_{i=1}^n u_i v_i$ where $(u_1, u_2, \ldots, u_n)$, $(v_1, v_2, \ldots, v_n)$ are the respective components of $U, V \in R^n$, it is clear, from the fact of $J_g(X), J_f(\ddot{X})$ being symmetric for all X, $\ddot{X}$, that $R(t)$, $S(t)$ are symmetric and then from conditions of theorem that

$$\langle R(t)U, U\rangle \geq \alpha_1 \|U\|^2 \tag{2.4}$$

and

$$\langle S(t)V, W\rangle \geq -\delta_0 n\|V\|\|W\| \tag{2.5}$$

for all t and for arbitrary $U, V, W \in R^n$, respectively.

We shall now prove that, subject to (2.4) and (2.5), the equation (2.1) has no nontrivial ω-periodic solutions, which will thereby verify the theorem.

Let then $Y = Y(t)$ be an ω-periodic solution of (2.1) and consider the scalar function $\theta = \theta(t)$ defined by

$$\theta = \left\langle \dot{Y}, \ddot{Y} \right\rangle - \left\langle Y, \dddot{Y} \right\rangle - \left\langle Y, A_1 \ddot{Y} \right\rangle - \frac{1}{2}\langle Y, A_3 Y\rangle + \frac{1}{2}\left\langle \dot{Y}, A_1 \dot{Y} \right\rangle .$$

We have, by an elementary differentiation, that

$$\dot{\theta} = \| \ddot{Y} \|^2 + \left\langle S(t)Y, \ddot{Y} \right\rangle + \langle R(t)Y, Y\rangle$$

thus

$$\begin{aligned} \dot{\theta} &\geq \| \ddot{Y} \|^2 + \alpha_1 \|Y\|^2 - \delta_0 n \|Y\| \| \ddot{Y} \| \\ &= \left(\| \ddot{Y} \| - \frac{1}{2}\delta_0 n \|Y\|\right)^2 + \left(\alpha_1 - \frac{1}{4}\delta_0^2 n^2\right)\|Y\|^2 \geq 0, \end{aligned} \tag{2.6}$$

since

$$\alpha_1 > \frac{1}{4} n^2 \delta_0^2.$$

Thus $\theta(t)$ is nondecreasing in t, and, being bounded (in view of the continuity and the assumed ω-periodicity of $Y(t)$), it therefore tends to a unique limit as $t \to \infty$. In particular, since

$$\theta(t) = \theta(t + N\omega) \tag{2.7}$$

for arbitrary t and for any integer N, it follows then on letting $N \to \infty$ in (2.7) that $\theta(t) =$ constant, and therefore that

$$\dot{\theta}\ (t) = 0 \tag{2.8}$$

for all t. It is clear from (2.6) and (2.8) that

$$Y(t) \equiv 0 \text{ for all } t$$

and the theorem now follows. □

3. Proof of the Theorem 2

The procedure here is similar to that used above in section 2. If $X_1(t)$, $X_2(t)$ are any two solutions of (1.2), then $Y = Y(t) = X_2(t) - X_1(t)$ satisfies the equation

$$Y^{(5)} + B_1 Y^{(4)} + B_2 \dddot{Y} + M(t) \ddot{Y} + B_4 \dot{Y} + N(t) Y = 0 \tag{3.1}$$

where $N(t)$ and $M(t)$ are the symmetric matrices defined by

$$N(t) = \int_0^1 J_h\Big(X_1(t) + \sigma(X_2(t) - X_1(t))\Big) d\sigma \tag{3.2}$$

and

$$M(t) = \int_0^1 J_\phi\Big(\ddot{X}_1\ (t) + \sigma(\ddot{X}_2\ (t) - \ddot{X}_1\ (t))\Big) d\sigma, \tag{3.3}$$

respectively.

If (1.4) holds, then

$$k_1 \langle N(t)U, U\rangle \geq \beta_2 \|U\|^2 \text{ for all } t \text{ and for arbitrary } U \in R^n; \tag{3.4}$$

and the objective once again will be to show that, subject to (3.4), there are no nontrivial $\omega-$periodic solutions whatever of (3.1).

Let then $Y = Y(t)$ be any $\omega-$periodic solution of (3.1) and consider the scalar function $\psi = \psi(t)$ defined by

$$\begin{aligned} \psi &= \left\langle \dot{Y}, \dddot{Y} \right\rangle + \left\langle \dot{Y}, B_1 \ddot{Y} \right\rangle - \left\langle Y, \dddot{Y} + B_1 \ddot{Y} + B_2 \dot{Y} \right\rangle \\ &\quad + \frac{1}{2}\left\langle B_2 \dot{Y}, \dot{Y} \right\rangle - \frac{1}{2}\left\langle \ddot{Y}, \ddot{Y} \right\rangle - \frac{1}{2}\langle B_4 Y, Y\rangle . \end{aligned}$$

It is a straightforward matter to verify that

$$\dot{\psi} = \left\langle B_1 \ddot{Y}, \ddot{Y} \right\rangle + \langle N(t)Y, Y\rangle + \left\langle M(t) \ddot{Y}, Y \right\rangle ,$$

so that, by (3.3) and the definition of γ_0,

$$\begin{aligned} \dot{\psi} &\geq \beta_1 \| \ddot{Y} \|^2 + \beta_2 \|Y\|^2 - \gamma_0 n \| \ddot{Y} \| \|Y\| \\ &= \beta_1 \left(\| \ddot{Y} \| - \frac{1}{2} n \gamma_0 \beta_1^{-1} \|Y\| \right)^2 + \left(\beta_2 - \frac{1}{4} n^2 \gamma_0^2 \beta_1^{-1} \right) \|Y\|^2 \end{aligned} \tag{3.5}$$

if B_1 is positive definite, and

$$\begin{aligned} \dot{\psi} &\leq -\beta_1 \| \ddot{Y} \|^2 - \beta_2 \|Y\|^2 + \gamma_0 n \| \ddot{Y} \| \|Y\| \\ &= -\beta_1 \left(\| \ddot{Y} \| - \frac{1}{2} n \gamma_0 \beta_1^{-1} \|Y\| \right)^2 - \left(\beta_2 - \frac{1}{4} n^2 \gamma_0^2 \beta_1^{-1} \right) \|Y\|^2 \end{aligned} \tag{3.6}$$

if B_1 is negative definite. Thus, since $\beta_2 > \frac{1}{4} n^2 \gamma_0^2 \beta_1^{-1}$, $\psi(t)$ is monotone (increasing or decreasing according as B_1 is positive or negative definite) in t, and, being bounded, thus tends to a limit as $t \to \infty$. As before this implies that $\psi(t) =$ constant for all t, and in turn, therefore, that

$$\dot{\psi}(t) = 0 \text{ for all } t. \tag{3.7}$$

It is evident from (3.5)-(3.7) that $Y(t) = 0$ for all t, and the theorem follows. □

Remark. In the special case when the matrix $J_f(\ddot{X}) = (\frac{\partial f_i}{\partial \ddot{x}_j})$ is diagonal, the estimate (2.6) can be readily refined to

$$\dot{\theta} \geq \left(\| \ddot{Y} \| - \frac{1}{2}\delta_0 \|Y\|\right)^2 + \left(\alpha_1 - \frac{1}{4}\delta_0^2\right) \|Y\|^2$$

so that Theorem 1 holds here subject to the weaker condition $\alpha_1 > \frac{1}{4}\delta_0^2$ on G.

Similarly if the matrix $J_\phi(\ddot{X}) = (\frac{\partial \phi_i}{\partial \ddot{x}_j})$ is diagonal, the estimates (3.5) and (3.6) can be relaxed respectively to

$$\dot{\psi} \geq \beta_1 \left(\| \ddot{Y} \| - \frac{1}{2}\gamma_0 \beta_1^{-1} \|Y\|\right)^2 + \left(\beta_2 - \frac{1}{4}\gamma_0^2 \beta_1^{-1}\right) \|Y\|^2,$$

$$\dot{\psi} \leq -\beta_1 \left(\| \ddot{Y} \| - \frac{1}{2}\gamma_0 \beta_1^{-1} \|Y\|\right)^2 - \left(\beta_2 - \frac{1}{4}\gamma_0^2 \beta_1^{-1}\right) \|Y\|^2$$

so that Theorem 2 in this case holds subject to the weaker condition $\beta_2 > \frac{1}{4}\gamma_0^2 \beta_1^{-1}$.

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