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SPHERICAL COMPLETENESS OF THE NON-ARCHIMEDEAN RING OF COLOMBEAU GENERALIZED NUMBERS

BY

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Abstract

We show spherical completeness of the ring of Colombeau generalized real numbers endowed with the sharp norm. As an application, we establish a Hahn-Banach extension theorem for ultra-pseudo-normed modules of generalized functions in the sense of Colombeau.

1. Introduction

Let (M, d) be an ultrametric space. For given $x \in M$, $r \in \mathbb{R}^+$, we call $B_{\leq r}(x) := \{y \in M \mid d(x, y) \leq r\}$ the dressed ball with center x and radius r. Throughout $\mathbb{N} := \{1, 2, \ldots\}$ denote the positive integers. Let $(x_i)_i \in M^{\mathbb{N}}$ and $(r_i)_i$ be a sequence of positive reals. We call $(B_i)_i$, $B_i := B_{\leq r_i}(x_i)$ $(i \geq 1)$ a nested sequence of dressed balls, if $r_1 \geq r_2 \geq r_3 \cdots$ and $B_1 \supseteq B_2 \supseteq \cdots$. Following standard ultrametric literature (cf. [11]), nested sequences of dressed balls might have an empty intersection. The converse property is defined as follows:

Definition 1.1. (M, d) is called spherically complete, if every nested sequence of dressed balls has a non-empty intersection.

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It is evident that any spherically complete ultrametric space is complete with respect to the topology induced by its metric (using the well known fact that topological completeness of (M, d) is equivalent to the property of Definition 1.1 with radii $r_i \searrow 0$) . However, there are popular non-trivial examples in the literature, for which the converse is not true. As an example we mention the completion $\mathbb{C}_p$ of the algebraic closure of the field of rational p-adic numbers. Due to Krasner, this field has nice algebraic properties (as it is algebraically closed, and even isomorphic to the complex numbers cf. [11], pp. 134–145), but it also has been shown, that $\mathbb{C}_p$ is not spherically complete. This is mainly due to the fact that the complex p-adic numbers are a separable, complete ultrametric space with dense valuation (cf. [11] pp. 143–144). However, for an ultrametric field K, spherical completeness is necessary in order to ensure K has the Hahn-Banach extension property (to which we refer as HBEP), that is, any ultra-normed K-vector space E admits continuous linear functionals previously defined on a strict subspace V of E to be extended to the whole space under conservation of their norm (this is due to W. Ingleton, [6]). Since spherical completeness fails, it is natural to ask if the p-adic numbers could at least be spherically completed, i.e., if there existed a spherically complete ultrametric field Ω into which $\mathbb{C}_p$ can be embedded. This question has a positive answer (cf. [11]). The necessity of spherical completeness for the HBEP of $K = \mathbb{C}_p$ is evident: even the identity map

$$\varphi : \quad \mathbb{C}_p \to \mathbb{C}_p, \quad \varphi(x) := x$$

cannot be extended to a functional $\psi : \Omega \to \mathbb{C}_p$ under conservation of its norm $\|\varphi\| = 1$ (here we consider Ω as a $\mathbb{C}_p$- vector space).[1]

The present paper is motivated by the question if a HBEP for the ring $\widetilde{\mathbb{R}}$ (resp. $\widetilde{\mathbb{C}}$) of generalized numbers holds. Even though a first version of Hahn-Banach's Theorem is given in ([4], Proposition 3.23), a general version of the latter has not been established yet in the literature.

[1] To check this, let $B_i := B_{\leq r_i}(x_i)$ be a nested sequence of dressed balls in $\mathbb{C}_p$ with empty intersection. Then $\hat{B}_i := B_{\leq r_i}(x_i) \subseteq \Omega$ have nonempty intersection, say $\Omega \ni \alpha \in \bigcap_{i=1}^{\infty} \hat{B}_i$. Assume further, the identity φ on $\mathbb{C}_p$ can be extended to some linear map $\psi : \Omega \to \mathbb{C}_p$ under conservation of its norm. Then

$$|\psi(\alpha) - x_i|_\Omega = |\psi(\alpha) - \varphi(x_i)|_\Omega \leq \|\psi\| |\alpha - x_i|_{\mathbb{C}_p} = |\alpha - x_i|_{\mathbb{C}_p},$$

therefore $\psi(\alpha) \in \bigcap_{i=1}^{\infty} B_i$ which is a contradiction and we are done.

The analogy with the p-adic case lies at hand, since the ring of generalized numbers can naturally be endowed with an ultrametric pseudo-norm. However, the presence of zero-divisor in $\widetilde{\mathbb{R}}$ as well as the failing multiplicativity of the pseudo-norm turns the question into a non-trivial one and Ingleton's ultrametric version of the Hahn-Banach Theorem cannot be carried over to our setting unrestrictedly.

On our first step tackling this question we discuss spherical completeness of the ring of generalized numbers endowed with the given ultrametric (induced by the respective ultra-pseudo-norm, cf. the preliminary section).

$\widetilde{\mathbb{R}}$ was first introduced as the set of values of generalized functions at standard points; however, a subring consisting of compactly supported generalized numbers turned out to be the set of points for which evaluation determines uniqueness, whereas standard points do not suffice do determine generalized functions uniquely (cf. [7, 8] as well as section 1.2.4 in [5]). A hint that $\widetilde{\mathbb{R}}$ (or $\widetilde{\mathbb{C}}$), is spherically complete, is that contrary to the above outlined situation on $\mathbb{C}_p$, the generalized numbers endowed with the topology induced by the sharp ultra-pseudo-norm are not separable. This follows from the fact that the restriction of the sharp valuation (cf. Section 2) to the real (or complex) numbers is discrete.

Having motivated our work by now, we may formulate the aim of this paper, which is to prove the following:

Theorem 1.2. *The ring of generalized numbers is spherically complete.*

We therefore have an independent proof of the fact (cf. [4], Proposition 1. 31 and Proposition 3.4):

Corollary 1.3. *The ring of generalized numbers is topologically complete.*

In the last section of this paper we present a modified version of Hahn-Banach's Theorem which bases on spherically completeness of $\widetilde{\mathbb{R}}$ (resp. $\widetilde{\mathbb{C}}$). Finally, a remark on the applicability of the ultrametric version of Banach's fixed point theorem can be found in the appendix.

2. Preliminaries

In what follows we repeat the definitions of the ring of (real or complex) generalized numbers along with its non-archimedean valuation function v. The material is taken from different sources; as references we recommend the recent works due to C. Garetto ([3, 4]) and A. Delcroix et al ([1]) as well as one of the original sources of this topic due to D. Scarpalezos (cf. [2]).

Let $I := (0,1] \subseteq \mathbb{R}$, and let $\mathbb{K}$ denote $\mathbb{R}$ resp. $\mathbb{C}$. The ring of generalized numbers over $\mathbb{K}$ is constructed in the following way: given the ring of moderate (nets of) numbers

$$\mathcal{E}_M := \{(x_\varepsilon)_\varepsilon \in \mathbb{K}^I \mid \exists\, N : |x_\varepsilon| = O(\varepsilon^{-N})\ (\varepsilon \to 0)\}$$

and, similarly, the ideal of negligible nets in $\mathcal{E}_M$ which are of the form

$$\mathcal{N} := \{(x_\varepsilon)_\varepsilon \in \mathbb{K}^I \mid \forall\, m : |x_\varepsilon| = O(\varepsilon^{m})\ (\varepsilon \to 0)\},$$

we define the generalized numbers as the factor ring $\widetilde{\mathbb{K}} := \mathcal{E}_M/\mathcal{N}$. We define a valuation function v on $\mathcal{E}_M$ with values in $(-\infty, \infty]$ in the following way:

$$\mathrm{v}((u_\varepsilon)_\varepsilon) := \sup\{b \in \mathbb{R} \mid |u_\varepsilon| = O(\varepsilon^b)\ \ (\varepsilon \to 0)\}.$$

This valuation can be carried over to the ring of generalized numbers in a well defined way, since for two representatives of a generalized number, the valuation above coincides (cf. [4], Section 1). We then endow $\widetilde{\mathbb{K}}$ with an ultra-pseudo-norm ('pseudo' refers to non-multiplicativity) $|\ \ |_{\mathrm{e}}$ in the following way: $|0|_{\mathrm{e}} := 0$, and whenever $x \neq 0$, $|x|_{\mathrm{e}} := \mathrm{e}^{-\mathrm{v}(x)}$. With the ultrametric d_{e} induced by the above ultra-pseudo-norm, $\widetilde{\mathbb{K}}$ turns out to be a non-discrete ultrametric space, with the following topological properties:

(i) $(\widetilde{\mathbb{K}}, d_{\mathrm{e}})$ is topologically complete (cf. [4]),
(ii) $(\widetilde{\mathbb{K}}, d_{\mathrm{e}})$ is not separable, since the restriction of d_{e} onto $\mathbb{K}$ is discrete.

The latter property holds, since on metric spaces second countability and separability are equivalent and the well known fact that the property of second countability is inherited by subspaces (whereas separability is not in general).

In order to avoid confusion we henceforth denote closed balls in $\mathbb{K}$ by $B_{\leq r}(x) := \{y \in \mathbb{K} \mid |y - x| \leq r\}$ in distinction with dressed balls in $\widetilde{\mathbb{K}}$ which we denote by $\widetilde{B}_{\leq r}(x) := \{y \in \widetilde{\mathbb{K}} \mid |y-x|_{\mathrm{e}} \leq r\}$. Similarly stripped balls and the sphere in the ring of generalized numbers are denoted by $\widetilde{B}_{<r}(x) := \{y \in \widetilde{\mathbb{K}} \mid |y-x|_{\mathrm{e}} < r\}$ resp. $\widetilde{S}_r(x) := \{y \in \widetilde{\mathbb{K}} \mid |y-x|_{\mathrm{e}} = r\}$.

3. Euclidean Models of Sharp Neighborhoods

Throughout, a net of real numbers $(C_\varepsilon)_\varepsilon$ is said to *increase monotonically with* $\varepsilon \to 0$, if the following holds:

$$\forall \eta, \eta' \in I : \ (\eta \leq \eta' \Rightarrow C_\eta \geq C_{\eta'}).$$

To begin with we formulate the following condition:

Condition (E).

A net $(C_\varepsilon)_\varepsilon$ is said to satisfy condition (E), if it is

(i) positive for each ε and
(ii) monotonically increasing with $\varepsilon \to 0$, and finally, if
(iii) the sharp norm is $|(C_\varepsilon)_\varepsilon|_e = 1$.

Next, we introduce the notion of euclidean models for sharp neighborhoods of generalized points:

Definition 3.1. Let $x \in \widetilde{\mathbb{K}}$, $\rho \in \mathbb{R}, r := \exp(-\rho)$. Let further $(C_\varepsilon)_\varepsilon \in \mathbb{R}^I$ be a net of real numbers satisfying condition (E) and let $(x_\varepsilon)_\varepsilon$ be a representative of x. Then we call the net of closed balls $(B_\varepsilon)_\varepsilon \subseteq \mathbb{K}^I$ given by

$$B_\varepsilon := B_{\leq C_\varepsilon \varepsilon^\rho}(x_\varepsilon)$$

for each $\varepsilon \in I$ an euclidean model for $\widetilde{B}_{\leq r}(x)$.

Note that every dressed ball admits an euclidean model: let $(x_\varepsilon)_\varepsilon$ be a representative of x and define $(C_\varepsilon)_\varepsilon$ by $C_\varepsilon := 1$ for each $\varepsilon \in I$; then $B_{\leq C_\varepsilon \varepsilon^\rho}(x_\varepsilon)$ determines an euclidean model for $\widetilde{B}_{\leq r}(x)$ when $\rho = -\log r$. We need to mention that whenever we write $(B_\varepsilon^{(1)})_\varepsilon \subseteq (B_\varepsilon^{(2)})_\varepsilon$, we mean the inclusion relation $\subseteq$ holds component wise (that is for each $\varepsilon \in I$), and we say $(B_\varepsilon^{(2)})_\varepsilon$ contains $(B_\varepsilon^{(1)})_\varepsilon$.

The following lemma is basic; however, in order to get familiar with the concept of euclidean neighborhoods, we include a detailed proof:

Lemma 3.2. *For $x \in \widetilde{\mathbb{K}}$ and $r > 0$ let $(B_\varepsilon)_\varepsilon$ be an euclidean model for $\widetilde{B}_{\leq r}(x)$. Then,*

(i) *for any $y \in \widetilde{B}_{<r}(x)$ there exists a representative $(y_\varepsilon)_\varepsilon$ such that $y_\varepsilon \in B_\varepsilon$ for all $\varepsilon \in I$.*

(ii) *There exist $y \in \widetilde{S}_r(x)$ fulfilling the following property: for every representative $(y_\varepsilon)_\varepsilon$ of y there exists $\varepsilon_0 \in I$ such that $y_{\varepsilon_0} \notin B_{\varepsilon_0}$. However, for all $y \in \widetilde{S}_r(x)$ and for all representatives $(y_\varepsilon)_\varepsilon$ of y there exists an euclidean model $\hat{B}_\varepsilon := \hat{B}_{\leq \hat{C}_\varepsilon \varepsilon^\rho}(x_\varepsilon)$ for $\widetilde{B}_{\leq r}(x)$ containing $(B_\varepsilon)_\varepsilon$ such that $y_\varepsilon \in \hat{B}_\varepsilon$ and $d(\partial \hat{B}_\varepsilon, y_\varepsilon) \geq \frac{C_\varepsilon}{2}\varepsilon^\rho$ for all $\varepsilon \in I$.*

Proof. (i): By definition of the sharp norm, $|y - x|_e < r$ is equivalent to the situation that for each representative $(y_\varepsilon)_\varepsilon$ of y and for each representative $(x_\varepsilon)_\varepsilon$ of x, we have

$$\sup\{b \in \mathbb{R} \mid |y_\varepsilon - x_\varepsilon| = O(\varepsilon^b)(\varepsilon \to 0)\} > \rho,$$

and this implies that there exists some $\rho' > \rho$ such that for any representative $(y_\varepsilon)_\varepsilon$ of y and any representative $(x_\varepsilon)_\varepsilon$ of x we have

$$|y_\varepsilon - x_\varepsilon| = o(\varepsilon^{\rho'}), \quad \varepsilon \to 0.$$

This further implies that for any choice of representatives of x resp. of y, there exists some $\eta \in I$ with

$$|y_\varepsilon - x_\varepsilon| \leq \varepsilon^{\rho'} \tag{3.1}$$

for each $\varepsilon < \eta$. Since $C_\varepsilon > 0$ for each $\varepsilon \in I$ and C_ε is monotonically increasing with $\varepsilon \to 0$, we have $\varepsilon^{\rho'} \leq C_\varepsilon \varepsilon^\rho$ for sufficiently small ε. Therefore, a suitable choice of η and of y_ε for $\varepsilon \geq \eta$ yields the first claim (for instance, one can set $y_\varepsilon := x_\varepsilon$ whenever $\varepsilon \geq \eta$).

We go on by proving (ii): For the first part, set

$$y_\varepsilon := 2C_\varepsilon \varepsilon^\rho + x_\varepsilon$$

Let y denote the class of $(y_\varepsilon)_\varepsilon$. It is evident that $y \in \widetilde{S}_r(x)$. However, $(y_\varepsilon) \notin B_\varepsilon$ for each $\varepsilon \in I$. Indeed,

$$\forall\, \varepsilon \in I : |y_\varepsilon - x_\varepsilon| = 2C_\varepsilon \varepsilon^\rho > C_\varepsilon \varepsilon^\rho,$$

since $C_\varepsilon > 0$ for each ε. We further show that the same holds for any representative $(\bar{y}_\varepsilon)_\varepsilon$ of y for sufficiently small index ε. Indeed, the difference of two representatives being negligible implies that for any $N > 0$ we have

$$y_\varepsilon - \hat{y}_\varepsilon = o(\varepsilon^N) \ \ (\varepsilon \to 0).$$

Therefore, for $N > \rho$ and sufficiently small ε, we have:

$$|\hat{y}_\varepsilon - y_\varepsilon| \geq \Big| |\hat{y}_\varepsilon - y_\varepsilon| - |y_\varepsilon - x_\varepsilon| \Big| \geq 2C_\varepsilon \varepsilon^\rho - \varepsilon^N \geq \frac{3}{2} C_\varepsilon \varepsilon^\rho > C_\varepsilon \varepsilon^\rho .$$

Therefore we have shown the first part of (ii). Let us take an arbitrary $y \in \widetilde{S}_r(x)$. We demonstrate how to blow up $(B_\varepsilon)_\varepsilon$ to catch some fixed representative $(y_\varepsilon)_\varepsilon$ of y. Since $|y - x| = e^{-\rho} = r$, there is a net $C'_\varepsilon \geq 0$ ($|(C'_\varepsilon)_\varepsilon|_e = 1$) such that

$$\forall \varepsilon \in I: \ |y_\varepsilon - x_\varepsilon| = C'_\varepsilon \varepsilon^\rho$$

Set $C''_\varepsilon = \max_{\eta \geq \varepsilon}\{1, C'_\eta\}$. This ensures that (C''_ε) is a monotonically increasing with $\varepsilon \to 0$, above 1 for each $\varepsilon \in I$, and $|(C''_\varepsilon)|_e = 1$ is preserved. The same holds for the net $C'''_\varepsilon := C''_\varepsilon + C_\varepsilon$. Define $B'_\varepsilon := B_{\leq C'''_\varepsilon \varepsilon^\rho}(x_\varepsilon)$. Then $(B'_\varepsilon)_\varepsilon$ is a new model for $\widetilde{B}_{\leq r}(x)$ containing the old model and $(y_\varepsilon)_\varepsilon$ as well, since the sum C'''_ε satisfies the required properties (of condition (E)), and

$$|y_\varepsilon - x_\varepsilon| \leq C''_\varepsilon \varepsilon^\rho \leq C'''_\varepsilon \varepsilon^\rho .$$

Setting $\hat{C}_\varepsilon := 2C'''_\varepsilon$ we obtain a model $\hat{B}_\varepsilon := B_{\leq \hat{C}_\varepsilon \varepsilon^\rho}(x_\varepsilon)$ for $\widetilde{B}_{\leq r}(x)$ with the further property that $|y_\varepsilon - x_\varepsilon| \leq \frac{C'''_\varepsilon}{2}\varepsilon^\rho$ for each $\varepsilon \in I$ which finishes the proof of (ii). □

Remark 3.3. The preceding lemma can be reformulated in the following way: For all $y \in \widetilde{B}_{\leq r}(x)$ there exists an euclidean model $B_\varepsilon := B_{\leq C_\varepsilon \varepsilon^\rho}(x_\varepsilon)$ and a representative $(y_\varepsilon)_\varepsilon$ of y such that $y_\varepsilon \in B_\varepsilon$ and $d(\partial B_\varepsilon, y_\varepsilon) \geq \frac{C_\varepsilon}{2}\varepsilon^\rho$ for all $\varepsilon \in I$.

Before going on by establishing the crucial statement which will allow us to translate decreasing sequences of closed balls in the given ultrametric space $\widetilde{\mathbb{K}}$ to decreasing sequences of their euclidean models, we introduce a useful term:

Definition 3.4. Suppose, we have a nested sequence $(\widetilde{B}_i)_{i=1}^{\infty}$ of closed balls with centers x_i and radii r_i in $\widetilde{\mathbb{K}}$. Let $(B_\varepsilon^{(i)})_\varepsilon$ be an euclidean model for $\widetilde{B}_i$ $(i \in \mathbb{N})$. We say that this associated sequence of euclidean models is proper, if $((B_\varepsilon^{(i)})_\varepsilon)_{i=1}^{\infty}$ is nested as well, that is, if we have:

$$(B_\varepsilon^{(1)})_\varepsilon \supseteq (B_\varepsilon^{(2)})_\varepsilon \supseteq (B_\varepsilon^{(3)})_\varepsilon \supseteq \cdots .$$

4. Proof of the Main Theorem

In order to prove the main statement, we proceed by establishing two important preliminary statements. First, a remark on the notation adopted in the sequel: if $(x_i)_i$, a sequence of points in the ring of generalized numbers, is considered, then $(x_\varepsilon^{(i)})_\varepsilon$ denote representatives of the x_i's. Furthermore, for subsequent choices of nets of real numbers $(C_\varepsilon^{(i)})_\varepsilon$, and positive radii r_i, we denote by ρ_i the negative logarithms of the r_i's $(i = 1, 2, \ldots ,)$ while the euclidean models for the balls $\widetilde{B}_{\le r_i}(x_i)$ with radii $r_\varepsilon^{(i)} := C_\varepsilon^{(i)} \varepsilon^{\rho_i}$ to be constructed are denoted by

$$B_\varepsilon^{(i)} := B_{\le r_\varepsilon^{(i)}}(x_\varepsilon^{(i)}).$$

We start with the fundamental proposition:

Proposition 4.1. *Let $x_1, x_2 \in \widetilde{\mathbb{K}}$, and r_1, r_2 be positive numbers such that $\widetilde{B}_{\le r_1}(x_1) \supseteq \widetilde{B}_{\le r_2}(x_2)$. Let $(x_\varepsilon^{(1)})_\varepsilon$ be a representative of x_1. Then the following holds:*

(i) *There exists a net $(C_\varepsilon^{(1)})_\varepsilon$ satisfying condition (E) and a representative $(x_\varepsilon^{(2)})_\varepsilon$ of x_2 such that*

$$x_\varepsilon^{(2)} \in B_{\le \frac{C_\varepsilon^{(1)} \varepsilon^{\rho_1}}{2}}(x_\varepsilon^{(1)}) \tag{4.2}$$

for each $\varepsilon \in I$.

(ii) *Furthermore, for each net $(C_\varepsilon^{(2)})_\varepsilon$ satisfying condition (E) there exists $\varepsilon_0^{(1)} \in I$ such that $B_\varepsilon^{(2)} \subseteq B_\varepsilon^{(1)}$ for all $\varepsilon \in (0, \varepsilon_0^{(1)})$.*

Proof. Proof of (i): We distinguish the following two cases:

- $x_2 \in \widetilde{S}_{r_1}(x_1)$, that is $|x_2 - x_1|_{\mathrm{e}} = r_1$. For a given representative $(x_\varepsilon^{(2)})_\varepsilon$ of x_2, define $\hat{C}_\varepsilon^{(1)} := |x_\varepsilon^{(1)} - x_\varepsilon^{(2)}|$. Now, set $C_\varepsilon^{(1)} := 2\max(\{\hat{C}_\eta^{(1)} | \eta > \varepsilon\}, 1)$. Then not only $C_\varepsilon^{(1)} > 0$ for each parameter ε, but also the net $C_\varepsilon^{(1)} > 0$ is monotonically increasing with $\varepsilon \to 0$, furthermore (4.2) holds, and we are done with this case.
- $x_2 \notin \widetilde{S}_{r_1}(x_1)$, that is $|x_2 - x_1|_e < r_1$. Set, for instance, $C_\varepsilon^{(1)} = 1$. For each representative $(x_\varepsilon^{(2)})_\varepsilon$ of x_2 it follows that

$$|x_\varepsilon^{(2)} - x_\varepsilon^{(1)}| = o(\varepsilon^{\rho_1})$$

and a representative satisfying the desired properties is easily found.

Proof of (ii): To show this we consider the asymptotic growth of $(C_\varepsilon^{(1)})_\varepsilon$, $(C_\varepsilon^{(2)})_\varepsilon$, ε^{ρ_1}, ε^{ρ_2} as well as the monotonicity of $C_\varepsilon^{(1)}$. Let $y \in B_{\le C_\varepsilon^{(2)}\varepsilon^{\rho_2}}(x_\varepsilon^{(2)})$. By the triangle inequality we have that

$$|y - x_\varepsilon^{(1)}| \le |y - x_\varepsilon^{(2)}| + |x_\varepsilon^{(2)} - x_\varepsilon^{(1)}| \le C_\varepsilon^{(2)}\varepsilon^{\rho_2} + \frac{C_\varepsilon^{(1)}\varepsilon^{\rho_1}}{2}, \tag{4.3}$$

for all $\varepsilon \in I$. We know further that by the monotonicity $\forall \varepsilon \in I : C_\varepsilon^{(1)} \ge C_{\varepsilon=1}^{(1)} =: C_1$ so that

$$\frac{C_\varepsilon^{(2)}}{C_\varepsilon^{(1)}}\varepsilon^{\rho_2-\rho_1} \le C_1 C_\varepsilon^{(2)}\varepsilon^{\rho_2-\rho_1}. \tag{4.4}$$

Moreover, since the sharp norm of $C_\varepsilon^{(2)}$ equals 1, for any $\alpha > 0$ we have that

$$C_\varepsilon^{(2)} = o(\varepsilon^{-\alpha}), \quad (\varepsilon \to 0),$$

which in conjunction with the fact that $\rho_2 > \rho_1$ allows us to further estimate the right hand side of (4.4): We obtain

$$\frac{C_\varepsilon^{(2)}}{C_\varepsilon^{(1)}}\varepsilon^{\rho_2-\rho_1} = o(1),\ (\varepsilon \to 0).$$

We plug this information into (4.3). This yields for sufficiently small ε, say $\varepsilon < \varepsilon_0^{(1)}$:

$$|y - x_\varepsilon^{(1)}| \le \frac{C_\varepsilon^{(1)}\varepsilon^{\rho_1}}{2} + \frac{C_\varepsilon^{(1)}\varepsilon^{\rho_1}}{2} = C_\varepsilon^{(1)}\varepsilon^{\rho_1} \tag{4.5}$$

and completes the proof. □

Proposition 4.2. *Any nested sequence of closed balls in $\widetilde{\mathbb{K}}$ admits a proper sequence of associated euclidean models.*

Proof. We proceed step by step so that we can easily read off the inductive argument of the proof in the end.

We may assume that for each $i \geq 1$, $r_i > r_{i+1}$. Define $\rho_i := -\log(r_i)$ (so that $\rho_i < \rho_{i+1}$ for each $i \geq 1$).

Step 1.

Choose a representative $(x_\varepsilon^{(1)})_\varepsilon$ of x_1.

Step 2.

Due to Proposition (4.1)(i) we can choose a representative $(x_\varepsilon^{(2)})_\varepsilon$ of x_2 and a net $(C_\varepsilon^{(1)})_\varepsilon$ of real numbers satisfying condition (E) such that

$$x_\varepsilon^{(2)} \in B_{\leq \frac{C_\varepsilon^{(1)} \varepsilon^{\rho_1}}{2}}(x_\varepsilon^{(1)})$$

for all $\varepsilon \in I$.

Step 3.

Similarly, take a representative $(\hat{x}_\varepsilon^{(3)})_\varepsilon$ of x_3 and a net $(\hat{C}_\varepsilon^{(2)})_\varepsilon$ of real numbers satisfying condition (E) such that such that for each $\varepsilon \in I$

$$\hat{x}_\varepsilon^{(3)} \in B_{\leq \frac{\hat{C}_\varepsilon^{(2)} \varepsilon^{\rho_2}}{2}}(x_\varepsilon^{(2)}). \tag{4.6}$$

Denote by $\varepsilon_0^{(1)} \in I$ be the maximal ε such that the inclusion relation $B_\varepsilon^{(2)} \subseteq B_\varepsilon^{(1)}$ holds (cf. (ii) of Proposition 4.1). We show now, how to adjust our choice of $\hat{x}_\varepsilon^{(3)}$, $\hat{C}_\varepsilon^{(2)}$ such that condition (E) as well as the inclusion relation (4.6) is preserved, however, we do this in a way such that we moreover achieve the inclusion relation

$$B_\varepsilon^{(2)} \subseteq B_\varepsilon^{(1)} \tag{4.7}$$

for *each* ε. For $\varepsilon < \varepsilon_0^{(1)}$ we leave the choice unchanged, that is, we set

$$x_\varepsilon^{(3)} := \hat{x}_\varepsilon^{(3)}, \quad C_\varepsilon^{(2)} := \hat{C}_\varepsilon^{(2)}.$$

For $\varepsilon \geq \varepsilon_0^{(1)}$, however, we set

$$x_\varepsilon^{(3)} := x_\varepsilon^{(2)}, \quad C_\varepsilon^{(2)} := \min(\frac{C_\varepsilon^{(1)}}{2}\varepsilon^{\rho_1-\rho_2}, \ \hat{C}_\varepsilon^{(2)}). \tag{4.8}$$

Therefore, $(C_\varepsilon^{(2)})_\varepsilon$ still satisfies condition (E), since it is still positive and monotonically increasing with $\varepsilon \to 0$. Next, it is evident that

$$x_\varepsilon^{(3)} \in B_{\leq \frac{C_\varepsilon^{(2)}\varepsilon^{\rho_2}}{2}}(x_\varepsilon^{(2)}).$$

holds for each $\varepsilon \in I$. Finally, by (4.8) it follows that the inclusion relation (4.7) holds now for each $\varepsilon \in I$. For the inductive proof of the statement one formally proceeds as in Step 3. Let $k > 1$. Assume we have representatives

$$(x_\varepsilon^{(1)})_\varepsilon, \ldots, (x_\varepsilon^{(k+1)})_\varepsilon$$

and nets of positive numbers

$$(C_\varepsilon^{(j)})_\varepsilon, (1 \leq j \leq k),$$

satisfying condition (E), such that for each $\varepsilon \in I$ we have:

$$B_{\leq C_\varepsilon^{(1)}\varepsilon^{\rho_1}}(x_\varepsilon^{(1)}) \supseteq B_{\leq C_\varepsilon^{(2)}\varepsilon^{\rho_2}}(x_\varepsilon^{(2)}) \supseteq \cdots \supseteq B_{\leq C_\varepsilon^{(k-1)}\varepsilon^{\rho_{k-1}}}(x_\varepsilon^{(k-1)}),$$

and for some $\varepsilon_0^{(k-1)}$ we have for each $\varepsilon < \varepsilon_0^{(k-1)}$

$$B_{\leq C_\varepsilon^{(k-1)}\varepsilon^{\rho_{k-1}}}(x_\varepsilon^{(k-1)}) \supseteq B_{\leq C_\varepsilon^{(k)}\varepsilon^{\rho_k}}(x_\varepsilon^{(k)}).$$

Furthermore we suppose the following additional property is satisfied: For each $\varepsilon \in I$ we have:

$$x_\varepsilon^{(k+1)} \in B_{\leq \frac{C_\varepsilon^{(k)}}{2}\varepsilon^{\rho_k}}(x_\varepsilon^{(k)}),$$

where $\rho_k := -\log r_k$. In the very same manner as above, we can now find a representative $(x_\varepsilon^{(k+2)})_\varepsilon$ of x_{k+2} and a net of numbers $(C_\varepsilon^{(k+1)})_\varepsilon$ satisfying condition (E) such that the above sequential construction can be enlarged by one $(k \to k+1)$. □

The preceding proposition is a key ingredient in the proof of our main statement Theorem 1.2:

Proof. Let $(\widetilde{B}_i)_{i=1}^{\infty}$, $B_i := \widetilde{B}_{\leq r_i}(x_i)$ $(i \geq 1)$ be the given nested sequence of dressed balls; due to Proposition 4.2, there exists a proper sequence of associated euclidean models

$$(B_\varepsilon^{(i)})_\varepsilon$$

such that for representatives $(x_\varepsilon^{(i)})_\varepsilon$ of x_i $(i \geq 1)$ the above nets are given by

$$B_\varepsilon^{(i)} := B_{\leq C_\varepsilon^{(i)} \varepsilon^{\rho_i}}(x_\varepsilon^{(i)}), \quad \rho_i := -\log r_i, \quad C_\varepsilon^{(i)} \in \mathbb{R}_+$$

for each $(\varepsilon, i) \in I \times \mathbb{N}$. Since $\mathbb{K}$ is locally compact, for each $\varepsilon \in I$ we can choose some $x_\varepsilon \in \mathbb{R}$ such that

$$x_\varepsilon \in \bigcap_{i=1}^{\infty} B_\varepsilon^{(i)}$$

since for each $\varepsilon \in I$ we have $B_\varepsilon^{(1)} \supseteq B_\varepsilon^{(2)} \supseteq \cdots$. By the construction of the net $(x_\varepsilon)_\varepsilon$, we have

$$|x_\varepsilon - x_\varepsilon^{(i)}| \leq C_\varepsilon^{(i)} \varepsilon^{\rho_i}$$

for each $\varepsilon \in I$. This shows that not only the net $(x_\varepsilon)_\varepsilon$ is moderate (use the triangle inequality), but also gives rise to a generalized number $x := (x_\varepsilon)_\varepsilon + \mathcal{N}(\mathbb{K})$ with the property

$$|x - x_i|_{\mathrm{e}} \leq r_i$$

for each i. Therefore we have that

$$x \in \bigcap_{i=1}^{\infty} \widetilde{B}_i \neq \emptyset$$

which yields the claim: $\widetilde{\mathbb{K}}$ is spherically complete. $\square$

5. A Hahn-Banach Theorem

Let L be a subfield of $\widetilde{\mathbb{K}}$. Let $(E, \|\cdot\|)$ be an ultrametric normed L-linear space. We call φ an L- linear functional on E, if φ is an L- linear mapping on E with values in $\widetilde{\mathbb{K}}$. φ is continuous if and only if

$$\|\varphi\| := \sup_{0\neq x\in E} \frac{|\varphi(x)|_{\mathrm{e}}}{\|x\|} < \infty.$$

We denote the space of all continuous L-linear functionals on E by E'_L.

Remark 5.1. Note that nontrivial subfields L of $\widetilde{\mathbb{K}}$ exist. For instance, one can choose $\mathbb{K}(\alpha)$ with $\alpha = [(\varepsilon)_\varepsilon] \in \widetilde{\mathbb{K}}$ or its completion with respect to $|\ \ |_{\mathrm{e}}$, the Laurent series over $\widetilde{\mathbb{K}}$. Moreover, given an ultra-pseudo-normed $\widetilde{\mathbb{C}}$-module $(\mathcal{G}, \mathcal{P})$, the L-linear space E generated by elements of $\mathcal{G}$ is an an ultrametric normed L-linear space.

Having introduced these notions we show that the following version of the Hahn-Banach Theorem holds:

Theorem 5.2. *Let V be an L-linear subspace of E and $\varphi \in V'_L$. Then φ can be extended to some $\psi \in E'_L$ such that $\|\psi\| = \|\varphi\|$.*

Proof. We follow the lines of the proof of Ingleton's theorem (cf. [6]) in the fashion of ([11], pp. 194–195). To start with, let V be a strict L-linear subspace of E and let $a \in E\setminus V$. We first show that $\varphi \in V'_L$ can be extended to $\psi \in (V + La)'_L$ under conservation of its norm. To do this it is sufficient to prove that such ψ satisfies for each $x \in V$:

$$\begin{aligned} \|\psi(x-a)\| &\le \|\psi\| \cdot \|x-a\| \\ \|\varphi(x)-\psi(a)\| &\le \|\varphi\| \cdot \|x-a\| =: r_x. \end{aligned} \tag{5.9}$$

To this end define for each x in V the dressed ball

$$B_x := B_{\le r_x}(\varphi(x)).$$

Next we claim that the family $\{B_x \mid x \in V\}$ of dressed balls is nested. To see this, let $x, y \in V$. By the linearity of φ and the ultrametric (strong) triangle inequality we have

$$|\varphi(x)-\varphi(y)|_{\mathrm{e}} \le \|\varphi\| \cdot \|x-y\| \le \|\varphi\| \max(\|x-a\|, \|y-a\|) = \max(r_x, r_y).$$

Therefore we have $B_x \subseteq B_y$ or $B_y \subseteq B_x$. According to Theorem 1.2, $\widetilde{\mathbb{K}}$ is spherically complete, therefore we can choose

$$\alpha \in \bigcap_{x \in V} B_x$$

and further define $\psi(a) := \alpha$. According to (5.9) we therefore have for each $z \in V$ and for each $\lambda \in L$, [2]

$$|\psi(z-\lambda a)|_{\mathrm{e}} = |\lambda|_{\mathrm{e}} \cdot |\psi(z/\lambda - a)|_{\mathrm{e}} \le |\lambda|_{\mathrm{e}} r_{z/\lambda} = |\lambda|_{\mathrm{e}} \cdot \|\varphi\| \cdot \|z/\lambda - a\| = \|\varphi\| \cdot \|z - \lambda a\|$$

which shows that ψ is an extension of φ onto $V + La$ and $\|\psi\| = \|\varphi\|$.

The rest of the proof is the standard one-an application of Zorn's Lemma. □

Let $(\mathcal{G}, \|\cdot\|)$ be an ultra-pseudo-normed $\widetilde{\mathbb{K}}$ module and denote by $\mathcal{L}(\mathcal{G}, \widetilde{\mathbb{K}})$ the space of continuous linear functionals on $\mathcal{G}$ (according to the notation in [3, 4]). We end this section by posing the following conjecture:

Conjecture 5.3. *Let $\mathcal{V}$ be a submodule of $\mathcal{G}$ and let $\varphi \in \mathcal{L}(\mathcal{V}, \widetilde{\mathbb{K}})$. Then φ can be extended to some element $\psi \in \mathcal{L}(\mathcal{G}, \widetilde{\mathbb{K}})$ such that $\|\psi\| = \|\varphi\|$.*

Appendix

Finally, it is worth mentioning that apart from the standard Fixed Point Theorem due to Banach, a non-archimedean version is available in spherically complete ultrametric spaces (therefore, also on $\widetilde{\mathbb{K}}$, cf. [9], and for a recent generalization cf. [10]):

Theorem 5.4. *Let (M, d) be a spherically complete ultrametric space and $f : M \to M$ be a mapping having the property*

$$\forall x, y \in M : d(f(x), f(y)) < d(x, y).$$

Then f has a unique fixed point in M.

[2]Note that since E is a normed L-linear space, the restriction of $|\cdot|_{\mathrm{e}}$ to L is multiplicative.

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