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POMPEIU PROBLEM FOR COMPLEX ELLIPSOIDS ON THE HEISENBERG GROUP

BY

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Abstract

We extend results of the Pompeiu problem on the Heisenberg group $\mathbf{H}^n$ from spheres to complex ellipsoids. These results also tell us what happens for spheres and complex ellipsoids on the anisotropic Heisenberg group, $\mathbf{H}^n_{\mathbf{a}}$. The results for L^2, L^p, and L^∞ have the same character as previous results for spheres on $\mathbf{H}^n$. However, when moving to L^∞ and including rotations, we maintain the result from Euclidean space that only one complex ellipsoid is needed.

1. Introduction

In its most basic form, the Pompeiu problem asks under what conditions will the vanishing of integrals

$$\int_{\gamma S} f(\mathbf{x}) d\sigma(\mathbf{x}) = 0 \quad \text{for all} \quad \gamma \in \{\text{rigid motions}\},$$

allow us to conclude that $f \equiv 0$. In general we are asking about properties of the set S, as well as the function space for f. Sets for which the above integral conditions imply $f \equiv 0$ are said to possess the Pompeiu property. It is known, for instance, that if S is a ball, it does not possess this property.

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However, the property is recovered by using two balls of appropriately chosen radii, [20]. In extending the Pompeiu problem to the setting of $\mathbf{H}^n$, this is the result that has been studied. A similar theorem of two radii (with conditions for appropriately chosen radii) has been established when working with the space of functions L^∞, [1, 3]. But for the space $L^p, 1 \leq p < \infty$, one radius is enough for the property to hold, [6, 8]. However, in the setting of Euclidean space, much more is known about what sets do or do not possess the Pompeiu property. The paper of [10] established that sets with a corner possess the Pompeiu property. The results and methods there also lead to the conjecture [19] that the ball is the only set, among those whose boundary is homeomorphic to a sphere, which does not possess the Pompeiu property. One main aspect of the ball is the invariance under rotation, and thus we do not pick up the extra information in the integrals. The paper of [10] also proves that, although ellipses do not have any corners, they do possess the Pompeiu property. This is the result we extend to the setting of $\mathbf{H}^n$.

To handle this question on $\mathbf{H}^n$ it will be necessary to move beyond the radial case. Up to this time, all the research for this problem on $\mathbf{H}^n$ has dealt either with the ball (radial) or the solid torus (polyradial) in $\mathbf{C}^n$. This is a first effort in an attempt to look at the Pompeiu problem on $\mathbf{H}^n$ for other regions. In this paper we consider only complex ellipsoids. Nevertheless, we are able to bring forth an important aspect of the Pompeiu problem to this setting, the issue of rotations of the set. Rotations are an essential aspect of the Pompeiu problem, but not previously considered in $\mathbf{H}^n$.

We mention the close connection between our problem on complex ellipsoids in $\mathbf{H}^n$ and the Pompeiu problem for spheres in the anisotropic Heisenberg group. This connection is not surprising since a sphere, when dilated separately in its variables, gives an ellipsoid. Anisotropic Heisenberg groups arise naturally when we study geometry of strongly pseudoconvex domains. Throughout the paper, our results will also be translated into this setting, where possible.

The paper is organized as follows. In Section 2, we will recall some basic properties of anisotropic Heisenberg group and Laguerre calculus on the group from [4, 9, 15]. Inspired by a method developed in [2], we show that in L^2 the Pompeiu property holds for one complex ellipsoid (without rotation) in Section 3. By establishing a convolution relation between a bounded spherical function and the distribution representing the Radon measure on the complex ellipsoid, we are able to extend to the case of L^p, $1 \leq p < \infty$

in Section 4 by applying arguements used in [6]. Finally approaching the L^∞ case we describe the Gelfand transform appropriate for the anisotropic Heisenberg group. Applying it, we get a two radii theorem for complex ellipsoids (without rotations) in $\mathbf{H}^n$. We then address the issue of rotation and prove a one radius theorem when rotations are included in Section 6.

2. Anisotropic Heisenberg Group

We introduce anisotropic Heisenberg groups $\mathbf{H}^n_{\mathbf{a}}$ for $\mathbf{a} \in \mathbf{R}^n, a_j > 0$, followed by some of the techniques used for analysis on this space. The special case where $\mathbf{a} = (1, \ldots, 1)$ gives the isotropic Heisenberg group $\mathbf{H}^n$, used in previous work on the Pompieu problem, such as [1, 2, 3, 6, 7, 9, 11]. We begin by giving the group law for the Heisenberg coordinates: $\{[\mathbf{z}, t] \in \mathbf{C}^n \times \mathbf{R}\}$ with the (non-commutative) group law

$$[\mathbf{z}, t] \cdot [\mathbf{w}, s] = \big[\mathbf{z} + \mathbf{w}, t + s + 2\mathrm{Im} \sum_j a_j z_j \bar{w}_j\big]. \tag{1}$$

Let us define, for $\mathbf{a}$ as above, the anisotropic norm $\|\mathbf{z}\|^2_{\mathbf{a}} = \sum_{j=1}^n a_j |z_j|^2$ on $\mathbf{C}^n$. We note that the space $\mathbf{H}^n_{\mathbf{a}}$ may be identified with the boundary of the upper-half space $\Omega_{n+1} = \{(\mathbf{z}, z') \in \mathbf{C}^{n+1} : \mathrm{Im}\, z' > \|\mathbf{z}\|^2_{\mathbf{a}}\}$ by the mapping $[\mathbf{z}, t] \to (\mathbf{z}, t + i\|\mathbf{z}\|^2_{\mathbf{a}})$, and that (1) defines a group action on Ω_{n+1}. We note the space of left-invariant vector fields on $\mathbf{H}^n_{\mathbf{a}}$ are spanned by $Z_j = \frac{\partial}{\partial z_j} + i a_j \bar{z}_j \frac{\partial}{\partial t}$ and $\bar{Z}_j = \frac{\partial}{\partial \bar{z}_j} - i a_j z_j \frac{\partial}{\partial t}$, for $j = 1, \ldots, n$, together with the "missing" direction $T = \frac{\partial}{\partial t}$ generated by the commutators $[\bar{Z}_j, Z_k] = (a_j + a_k) i \delta_{j,k} T$. We define the sub-Laplacian for $\mathbf{H}^n_{\mathbf{a}}$ as follows:

$$\begin{aligned} \Box_{\mathbf{a}} &= -\sum_{j=1}^n (Z_j \bar{Z}_j + \bar{Z}_j Z_j) \\ &= -\sum_{j=1}^n \left(\frac{\partial^2}{\partial z_j \partial \bar{z}_j} + a_j^2 |z_j|^2 \frac{\partial^2}{\partial t^2} + \frac{\partial}{\partial t} a_j \left(z_j \frac{\partial}{\partial z_j} - \bar{z}_j \frac{\partial}{\partial \bar{z}_j} \right) \right). \end{aligned}$$

In the case where $\phi \in C^2_{\mathbf{0}}(\mathbf{H}^n)$, this reduces to

$$\Box_{\mathbf{a}} \phi = -\sum_{j=1}^n \left(\frac{\partial^2}{\partial z_j \partial \bar{z}_j} \phi + a_j^2 |z_j|^2 \frac{\partial^2}{\partial t^2} \phi \right),$$

since $z_j \frac{\partial}{\partial z_j}\phi = \bar{z}_j \frac{\partial}{\partial \bar{z}_j}\phi$ for such polyradial ϕ.

We now express two versions of the Pompeiu problem which are nearly equivalent. In Sections 3 to 5, these are considered in the context of using only translations. However, in the final section, we consider what happens to these issues when rotations are also included. For now we restrict our attention to translations only, for which the integral conditions are given below. In the first case, we consider a solid complex ellipsoid $E_{\mathbf{b}}$,

$$E_{\mathbf{b}} = \{(\mathbf{z}, 0) \in \mathbf{C}^n \times \{0\} : |z_1/b_1|^2 + \cdots + |z_n/b_n|^2 \leq 1\},$$

translated by the isotropic Heisenberg group $\mathbf{H}^n$. This corresponds to integral conditions of

$$\int_{E_{\mathbf{b}}} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_{\mathbf{b}}(\mathbf{z}) = 0 \quad \text{for all } \mathbf{g} \in \mathbf{H}^n \tag{2}$$

in the Pompeiu problem. Here $L_{\mathbf{g}}$ is left translation by the element $\mathbf{g}$ and $\mu_{\mathbf{b}}$ is volume measure on the solid complex ellipsoid. The second case we consider is integration over a ball B_r

$$B_r = \{(\mathbf{z}, 0) \in \mathbf{C}^n \times \{0\} : |\mathbf{z}| \leq r\},$$

translated by the anisotropic Heisenberg group $\mathbf{H}^n_{\mathbf{a}}$ for some $\mathbf{a} \in \mathbf{R}^n, a_j > 0$. The integral conditions for the Pompeiu problem are then expressed

$$\int_{B_r} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_r(\mathbf{z}) = 0 \quad \text{for all } \mathbf{g} \in \mathbf{H}^n_{\mathbf{a}}. \tag{3}$$

We note that, for $\mathbf{b}^2 = \mathbf{a}$, i.e. $a_j = b_j^2$, the $E_{\mathbf{b}}$ can be expressed as $\{(\mathbf{z}, 0) : \|\mathbf{z}\|_{1/\mathbf{a}} \leq 1\}$, and this establishes a connection between the integrals in (2) and (3). Throughout the paper, we will use the convention that $\mathbf{b}^2 = \mathbf{a}$. It is also possible to consider integration over a solid complex ellipsoid translated by the anisotropic group. In this case we write $E_{\mathbf{d}}$, with no general relation between $\mathbf{d}$ and $\mathbf{a}$. We then have integral conditions

$$\int_{E_{\mathbf{d}}} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_{\mathbf{d}}(\mathbf{z}) = 0 \quad \text{for all } \mathbf{g} \in \mathbf{H}^n_{\mathbf{a}} \tag{4}$$

for the Pompeiu problem. We will focus on integral conditions (2), but will also look at consequences of such computations for cases (3) and (4).

The techniques of our analysis come from harmonic analysis for the group $\mathbf{H}^n$, and more generally $\mathbf{H}^n_{\mathbf{a}}$. We now address some of the tools which will be used within the paper. Define the partial Fourier transform in the t variable as follows:

$$\hat{f}^\lambda(\mathbf{z}) = \int_{\mathbf{R}} e^{-2\pi i\lambda t} f(\mathbf{z}, t)dt,$$

We also define a λ-twisted convolution, which displays an important relation between convolution and the partial Fourier transform. Given $f, g \in L^2(\mathbf{H}^n_{\mathbf{a}})$, define the twisted convolution as follows:

$$\left(f \star^\lambda g\right)(\mathbf{z}) = \int_{\mathbf{C}^n} e^{-4\pi i\lambda \mathrm{Im}\,\langle \mathbf{z},\mathbf{w}\rangle} f(\mathbf{z}-\mathbf{w})g(\mathbf{w})dm(\mathbf{w}),$$

where $\langle \mathbf{z}, \mathbf{w}\rangle = \sum_{j=1}^n a_j z_j \bar{w}_j$. It is then a straightforward computation to show that

$$(f * g)^{\frown\lambda}(\mathbf{z}) = (\hat{f}^\lambda \star^\lambda \hat{g}^\lambda)(\mathbf{z}).$$

We introduce an orthonormal basis that behaves well under this twisted convolution. We have the following dilations of exponential Laguerre functions.

$$\mathcal{W}^\lambda_{\mathbf{k}}(\mathbf{z}) = c^\lambda_{\mathbf{0}} e^{-2\pi|\lambda|\|\mathbf{z}\|^2_{\mathbf{a}}} \prod_{j=1}^n a_j L^{(0)}_{k_j}(4\pi|\lambda|a_j|z_j|^2),$$

where $c^\lambda_{\mathbf{0}} = (\sqrt{4|\lambda|})^n$. For each $\lambda \in \mathbf{R}^* = \mathbf{R}\setminus\{0\}$, the set $\{\mathcal{W}^\lambda_{\mathbf{k}}(\mathbf{z}) : \mathbf{k} \geq \mathbf{0}\}$ forms an orthonormal basis for $L^2_{\mathbf{0}}(\mathbf{C}^n) = \{f \in L^2 : f(e^{i\phi}\mathbf{z}) = f(\mathbf{z}) \text{ for all } \phi \in [0,2\pi)^n, \mathbf{z} \in \mathbf{C}^n\}$. Furthermore they relate under λ-twisted convolution as follows:

$$(\mathcal{W}^\lambda_{\mathbf{j}} \star^\lambda \mathcal{W}^\lambda_{\mathbf{k}})(\mathbf{z}) = \frac{c^\lambda_{\mathbf{j},\mathbf{k}}}{(c^\lambda_{\mathbf{0}})^2}\delta_{\mathbf{j},\mathbf{k}}\mathcal{W}^\lambda_{\mathbf{j},\mathbf{k}}(\mathbf{z})$$

where $c^\lambda_{\mathbf{j},\mathbf{k}} = \prod_{i=1}^n \left(\frac{\pi}{(4\pi|\lambda|)^{|j_i-k_i|+1}}\frac{(\max(j_i,k_i))!}{(\min(j_i,k_i))!}\right)^{-1/2}$. The function $\mathcal{W}^\lambda_{\mathbf{j},\mathbf{k}}(\mathbf{z})$ is defined by

$$\mathcal{W}^\lambda_{\mathbf{j},\mathbf{k}}(\mathbf{z}) = \begin{cases} c^\lambda_{\mathbf{j},\mathbf{k}} e^{-2\pi|\lambda|\|\mathbf{z}\|^2_{\mathbf{a}}} \displaystyle\prod_{i=1}^n a_i \chi_{j_i-k_i}(z_i) L^{(|k_i-j_i|)}_{\max\{j_i,k_i\}}(4\pi|\lambda|a_i|z_i|^2) & \text{for } \lambda \geq 0 \\ \mathcal{W}^\lambda_{\mathbf{k},\mathbf{j}}(\mathbf{z}) & \text{for } \lambda < 0 \end{cases},$$

for $\chi_m(z) = z^m$ for $m > 0$ or $\bar{z}^{|m|}$ for $m < 0$. Note when $\mathbf{j} = \mathbf{k}$, the only case of non-vanishing of the convolution, $\mathcal{W}^\lambda_{\mathbf{j},\mathbf{k}} = \mathcal{W}^\lambda_{\mathbf{j}} = \mathcal{W}^\lambda_{\mathbf{k}}$, which is in the original orthonormal basis. We will make use of this orthonormal basis in conjunction with the partial Fourier transform for the case of functions in L^2.

In analysis of $L^p(\mathbf{H}^n_{\mathbf{a}})$, $1 < p < \infty$, we begin with the joint L^p spectrum of the operators $\mathcal{L}$ and iT. This spectrum is the complement of the set of $(\lambda, \mu) \in \mathbf{C}^2$ such that there exist L^p bounded operators A, B with $A(\lambda\mathbf{I} - \mathcal{L}) + B(\mu\mathbf{I} - iT) = 1$. This may be described as (λ, μ) such that $(\lambda\mathbf{I} - \mathcal{L})$ and $(\mu\mathbf{I} - iT)$ are not invertible. Using Laguerre calculus, we find the joint eigenfunctions

$$\phi^{(\omega\lambda)}_{\mathbf{k},\pm}(\mathbf{z}, t) = c(\omega\lambda)^n e^{\pm i(\omega\lambda)t} e^{-(\omega\lambda)\|\mathbf{z}\|^2_{\mathbf{a}}} \prod L^{(0)}_{k_j}(2(\omega\lambda)a_j|z_j|^2),$$

where $\lambda > 0$ and where $\omega = \frac{1}{\sum a_j(2k_j+1)}$, as in [9, 12]. It is shown that spectral projection along rays

$$\mathbf{P}_{\mathbf{k},\pm}(f) = \int_0^\infty f * \phi^\lambda_{\mathbf{k},\pm}(\mathbf{z}, t) d\lambda$$

is bounded in H^p, for $0 < p < \infty$ for every $\mathbf{k} \in (\mathbf{Z}_+)^n$. Using these spectral projections, we may form the Abel means

$$f(\mathbf{z}, t) = \lim_{r \to 1^-} \sum_{\mathbf{k} \geq \mathbf{0}} r^{|\mathbf{k}|} \int_0^\infty f(\mathbf{z}, t) * \left(\phi^\lambda_{\mathbf{k},+} + \phi^\lambda_{\mathbf{k},-}\right) d\lambda,$$

as in [9, 12], which are bounded in L^p for $1 < p < \infty$.

For the function space $L^\infty(\mathbf{H}^n)$, our tool will be the Gelfand transform on $L^1_{\mathbf{0}}(\mathbf{H}^n)$, as used in [3]. This transformation is defined by the characters on the commutative Banach algebra $L^1_{\mathbf{0}}(\mathbf{H}^n)$, which are determined by $m(f) = \int_{\mathbf{H}^n} f(\mathbf{g})\psi(\mathbf{g})dm(\mathbf{g})$, where the ψ are the bounded $\mathbf{T}^n$-spherical functions on $\mathbf{H}^n$. These are

$$\psi^\lambda_{\mathbf{k}}(\mathbf{z}, t) = e^{2\pi i\lambda t} e^{-2\pi|\lambda||\mathbf{z}|^2} \prod_{j=1}^n L^{(0)}_{k_j}(4\pi|\lambda||z_j|^2) \quad \text{for } (\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n,$$

and

$$\mathcal{J}_\rho(\mathbf{z}) = \prod_{j=1}^n J_0(\rho_j|z_j|) \quad \text{for } \rho \in (\mathbf{R}^*).$$

Thus for $f \in L^1_{\mathbf{0}}(\mathbf{H}^n)$, the Gelfand transform $\tilde{f}$ is defined by

$$\tilde{f}(\lambda, \mathbf{k}) = \int_{\mathbf{H}^n} f(\mathbf{g})\psi^\lambda_{\mathbf{k}}(\mathbf{g})dm(\mathbf{g}) \quad \text{and} \quad \tilde{f}(\mathbf{0};\rho) = \int_{\mathbf{H}^n} f(\mathbf{g})\mathcal{J}_\rho(\mathbf{g})dm(\mathbf{g}).$$

This transform $\tilde{f}$ is a function on the Heisenberg brush,

$$\begin{aligned}\mathcal{H}_{\mathbf{0}} \;=\; & \bigcup_{\mathbf{k}\in(\mathbf{Z}_+)^n} \{(\lambda, |\lambda|(4k_1+2), \ldots |\lambda|(4k_n+2)) : \lambda \in \mathbf{R}^*\} \\ & \bigcup \{(0, \rho_1^2, \ldots, \rho_n^2) : \rho \in (\mathbf{R}_+)^n\},\end{aligned}$$

where the point $(\lambda, |\lambda|(4k_1+2), \ldots, |\lambda|(4k_n+2))$ coresponds to $\psi^\lambda_{\mathbf{k}}$ and the point $(0, \rho_1^2, \ldots, \rho_n^2)$ corresponds to $\mathcal{J}_\rho$. Finally, we require the following Tauberian theorem in our application of this transform, [17].

Theorem 2.1. *Let $\mathcal{I}$ be a closed ideal of $L^1_{\mathbf{0}}(\mathbf{H}^n)$ such that*

(1) *For all $(\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$, there exists $f \in \mathcal{I}$ such that $\tilde{f}(\lambda, \mathbf{k}) \neq 0$.*
(2) *For all $\rho \in (\mathbf{R}_+)^n$, there exists $f \in \mathcal{I}$ such that $\tilde{f}(\mathbf{0};\rho) \neq 0$.*

Then $\mathcal{I} = L^1_{\mathbf{0}}(\mathbf{H}^n)$.

This completes the description of tools we will use in analysis on $\mathbf{H}^n$. In Section 5.2, we will also describe a similar Gelfand transform for the space $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$.

3. Case of L^p, $1 \leq p \leq 2$

We begin with the case of L^2 and follow the methods of [2], which are based on the partial Fourier transform and its relation to the twisted convolution, along with use of Laguerre series, as described in Section 2. We have the following theorem.

Theorem 3.1. *Let $f \in C(\mathbf{H}^n) \cap L^2(\mathbf{H}^n)$. Let $\mathbf{b} \in \mathbf{R}^n$, each $b_j > 0$, and $E_{\mathbf{b}}$ the complex ellipsoid, as defined above, with volume measure $\mu_{\mathbf{b}}$. Assume the vanishing of the integrals in* (2)*, that is,*

$$\int_{E_{\mathbf{b}}} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_{\mathbf{b}}(\mathbf{z}) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^n.$$

It follows that $f \equiv 0$.

We now observe that to prove Theorem 3.1, it is sufficient to prove it in the case where f is polyradial, i.e. $f(e^{i\theta_1}z_1,\ldots,e^{i\theta_n}z_n,t)=f(z_1,\ldots,z_n,t)$ for each $(\theta_1,\ldots,\theta_n)\in[0,2\pi)^n$. To achieve this reduction, we use the approach of [2]. Apply the radialization operrator $\mathcal{R}_{\mathbf{0}}$ defined by $(\mathcal{R}_{\mathbf{0}}f)(\mathbf{z},t)=\int_0^{2\pi}\cdots\int_0^{2\pi}f(e^{i\phi}\mathbf{z},t)\frac{d\phi_1\cdots d\phi_n}{(2\pi)^n}$. The result $\mathcal{R}_{\mathbf{0}}f$ still satisfies the integral conditions (2) and furthermore is now polyradial. Proposition 3.2 will then imply $(\mathcal{R}_{\mathbf{0}}f)(\mathbf{z},t)=0$. And this implies in particular that $(\mathcal{R}_{\mathbf{0}}f)(\mathbf{0},0)=f(\mathbf{0},0)=0$. Since $L_{\mathbf{g}}f$, for each $\mathbf{g}\in\mathbf{H}^n$, will also satisfy (2), we find that $(\mathcal{R}_{\mathbf{0}}L_{\mathbf{g}}f)(\mathbf{0},0)=(L_{\mathbf{g}}f)(\mathbf{0},0)=f(\mathbf{g})$ by applying the above to $L_{\mathbf{g}}f$. Thus by using radialization operators, we may assume f is $\mathbf{0}$-homogeneous. Thus, the proof of Theorem 3.1 will be complete once we establish Proposition 3.2, below.

To prove the following proposition, we follow the method of [2] to demonstrate that the integral conditions will imply $f\equiv 0$.

Proposition 3.2. *Suppose $f\in C(\mathbf{H}^n)\cap L^2(\mathbf{H}^n)$ and f is $\mathbf{0}$-homogeneous. If f satisfies integral conditions*

$$\int_{E_{\mathbf{b}}}L_{\mathbf{g}}f(\mathbf{z},0)d\mu_{\mathbf{b}}(\mathbf{z})=0\quad\textit{for all }\mathbf{g}\in\mathbf{H}^n,$$

then $f\equiv 0$.

Proof. We write the integral conditions (2) as the convolution equation $f*T_{\mathbf{b}}(\mathbf{g})\equiv 0$, where $T_{\mathbf{b}}$ is the distribution defined by the measure $\mu_{\mathbf{b}}(\mathbf{z})$ on the complex ellipsoid $E_{\mathbf{b}}$ as follows: $\langle\phi,T_{\mathbf{b}}\rangle=\int_{E_{\mathbf{b}}}\phi(\mathbf{z},0)d\mu_{\mathbf{b}}(\mathbf{z})$. Since $f\in L^2$ and $T_{\mathbf{b}}$ has compact support, it follows that $f*T_{\mathbf{b}}\in L^2$, and we apply the partial Fourier transform in the t variable.

$$\begin{aligned}(f*T_{\mathbf{b}})^{\frown\lambda}(\mathbf{z},\lambda)&=\int_{\mathbf{R}}e^{-2\pi i\lambda t}\int_{E_{\mathbf{b}}}f(\mathbf{z}-\mathbf{w},t-2\mathrm{Im}\sum z_j\bar{w}_j)d\mu_{\mathbf{b}}(\mathbf{w})dt\\&=\int_{E_{\mathbf{b}}}e^{-4\pi i\lambda\mathrm{Im}\,\langle\mathbf{z},\mathbf{w}\rangle}\left(\int_{\mathbf{R}}e^{-2\pi i\lambda t}f(\mathbf{z}-\mathbf{w},t)dt\right)d\mu_{\mathbf{b}}(\mathbf{w})\\&=\left(\hat{f}^\lambda\star^\lambda T_{\mathbf{b}}\right)(\mathbf{z}),\end{aligned}$$

where $\star^\lambda$ represents the λ-twisted convolution defined above. Using that $\hat{f}^\lambda$ is also $\mathbf{0}$-homogeneous, we expand in the Laguerre series $\hat{f}^\lambda(\mathbf{z})=\sum_{\mathbf{j}\geq\mathbf{0}}c_{\mathbf{j}}(\lambda)$

$\mathcal{W}^\lambda_{\mathbf{j}}(\mathbf{z})$. The convolution equation may then be rewritten as

$$(f * T_{\mathbf{b}})^{\frown\lambda}(\mathbf{z}) = \sum_{\mathbf{j}\geq\mathbf{0}} c_{\mathbf{j}}(\lambda)\left(\mathcal{W}^\lambda_{\mathbf{j}} \star^\lambda T_{\mathbf{b}}\right)(\mathbf{z}).$$

Our goal is to show $c_{\mathbf{j}}(\lambda) = 0$ for all $\mathbf{j} \geq \mathbf{0}$ and a.e. $\lambda \in \mathbf{R}^*$. After expressing $(\mathcal{W}^\lambda_{\mathbf{j}} \star^\lambda T_{\mathbf{b}})(\mathbf{z})$ as a series $\sum_{\mathbf{k}\geq\mathbf{0}} w_{\mathbf{j},\mathbf{k}}(\lambda)\mathcal{W}^\lambda_{\mathbf{k}}(\mathbf{z})$, we find

$$\left(f \star^\lambda T_{\mathbf{b}}\right)(\mathbf{z}) = \sum_{\mathbf{j}\geq\mathbf{0}} c_{\mathbf{j}}(\lambda)\Big(\sum_{\mathbf{k}\geq\mathbf{0}} w_{\mathbf{j},\mathbf{k}}(\lambda)\mathcal{W}^\lambda_{\mathbf{k}}(\mathbf{z})\Big).$$

The coefficients $w_{\mathbf{j},\mathbf{k}}(\lambda)$ are determined as

$$\begin{aligned} w_{\mathbf{j},\mathbf{k}}(\lambda) &= \int_{\mathbf{C}^n} \left(\mathcal{W}^\lambda_{\mathbf{j}} \star^\lambda T_{\mathbf{b}}\right)(\mathbf{z})\mathcal{W}^\lambda_{\mathbf{k}}(\mathbf{z})dm(\mathbf{z}) \\ &= \int_{\mathbf{C}^n} \left(\int_{E_{\mathbf{b}}} e^{-4\pi i\lambda \mathrm{Im}\,\langle z,w\rangle}\mathcal{W}^\lambda_{\mathbf{j}}(\mathbf{z}-\mathbf{w})d\mu_{\mathbf{b}}(\mathbf{w})\right)\mathcal{W}^\lambda_{\mathbf{k}}(\mathbf{z})dm(\mathbf{z}) \\ &= \int_{E_{\mathbf{b}}} \overline{\mathcal{W}^\lambda_{\mathbf{j}} \star^\lambda \mathcal{W}^\lambda_{\mathbf{k}}(\mathbf{w})}d\mu_{\mathbf{b}}(\mathbf{w}) \\ &= \delta_{\mathbf{j},\mathbf{k}}\frac{c^\lambda_{\mathbf{j},\mathbf{k}}}{(c^\lambda_{\mathbf{0}})^2}\int_{E_{\mathbf{b}}} \mathcal{W}^\lambda_{\mathbf{k},\mathbf{j}}(\mathbf{w})d\mu_{\mathbf{b}}(\mathbf{w}) \end{aligned}$$

These only contribute to the series when $\mathbf{k} = \mathbf{j}$, and the series reduces to

$$\hat{f}^\lambda(\mathbf{z}) = \sum_{\mathbf{k}\geq\mathbf{0}} c_{\mathbf{k}}(\lambda)\frac{1}{c^\lambda_{\mathbf{0}}}I(\lambda,\mathbf{a},\mathbf{k})\mathcal{W}^\lambda_{\mathbf{k}}(\mathbf{z}),$$

where the integral

$$I(\lambda,\mathbf{a},\mathbf{k}) = \int_{E_{\mathbf{b}}} e^{-2\pi|\lambda||\mathbf{w}|^2}\prod_{j=1}^{n} L^{(0)}_{k_j}(4\pi|\lambda||w_j|^2)d\mu_{\mathbf{b}}(\mathbf{w})$$

remains to be evaluated. With some simplification the integral becomes

$$\begin{aligned} I(\lambda,\mathbf{a},\mathbf{k}) &= \int_{E_{\mathbf{b}}} e^{-2\pi|\lambda||\mathbf{w}|^2}\prod_{j=1}^{n} L^{(0)}_{k_j}(4\pi|\lambda||w_j|^2)d\mu_{\mathbf{b}}(\mathbf{w}) \\ &= \int_{|\mathbf{z}|\leq 1}\prod_{j=1}^{n} e^{-2\pi|\lambda|b_j^2|z_j|^2}b_j^2 L^{(0)}_{k_j}(4\pi|\lambda|b_j^2|z_j|^2)d\mu_1(\mathbf{z}) \end{aligned}$$

$$= \frac{\pi^n}{(4\pi|\lambda|)^n} \int_{\sum x_j \le 4\pi|\lambda|} \prod_{j=1}^{n} e^{-a_j x_j/2} a_j L_{k_j}^{(0)}(a_j x_j) dx_n \dots dx_1$$

$$= \frac{1}{(4|\lambda|)^n} \int_0^{4\pi|\lambda|} e^{-a_1 x_1/2} a_1 L_{k_1}^{(0)}(a_1 x_1)$$

$$\times \int_0^{4\pi|\lambda|-x_1} e^{-a_2 x_2/2} a_2 L_{k_2}^{(0)}(a_2 x_2) \times \cdots$$

$$\times \int_0^{4\pi|\lambda|-x_1-\cdots-x_{n-1}} e^{-a_n x_n/2} a_n L_{k_n}^{(0)}(a_n x_n) dx_n \dots dx_2 dx_1. \quad \square$$

We now make the following claim

Lemma 3.3. *The above integral $I(\lambda, \mathbf{a}, \mathbf{k})$ is real analytic in the variable λ and thus has isolated zeros in this variable. The integral may also be expressed as*

$$I(\lambda, \mathbf{a}, \mathbf{k}) = \frac{1}{(4|\lambda|)^n} \Big(\sum_{j=1}^{n} e^{-2\pi|\lambda|a_j} P_{\mathbf{a},\mathbf{k},j}(4\pi|\lambda|) + C_{\mathbf{a},\mathbf{k}} \Big),$$

where $P_{\mathbf{a},\mathbf{k},j}(x)$ is the polynomial of degree $k_j + n - 1$ defined in the proof below.

Proof. We first show real-analytic by using power series. Then we make more explicit computations to show the integral yields an exponential polynomial, as claimed. First, look at the integrand as a power series in the variable $|\lambda|$, which is convergent for $\lambda \in \mathbf{R}^*$. After successive integrations, we observe the result remains a power series, convergent on the same region. It follows that $I(\lambda, \mathbf{a}, \mathbf{k})$ is real-analytic in the variable λ, as claimed. Since there are n integrations, we perform the first and establish the pattern for the others. First observe that for $n = 1$, the result that $a \int_0^1 e^{-2\pi|\lambda|ax} L_k^{(0)}(4\pi|\lambda|ax) dx$ is real-analytic, and in fact gives an exponential polynomial, was demonstrated in [11]. For arbitrary n, we can write the integral

$$\int_0^1 e^{-2\pi|\lambda|a_1 x_1} L_{k_1}^{(0)}(4\pi|\lambda|a_1 x_1) \int_0^{1-x_1} e^{-2\pi|\lambda|a_2 x_2} L_{k_2}^{(0)}(4\pi|\lambda|a_2 x_2) \cdots$$

$$\times \int_0^{1-x_1-\cdots-x_{n-1}} e^{-2\pi|\lambda|a_n x_n} L_{k_n}^{(0)}(4\pi|\lambda|a_n x_n) dx_n \dots dx_2 dx_1$$

as a power series in λ. We use that

$$\begin{aligned} e^{-2\pi|\lambda|a_m x_m} L_{k_m}^{(0)}(4\pi|\lambda|a_m x_m) &= \sum_{j=0}^{\infty} \frac{1}{j!} \sum_{\ell=0}^{k_m} c_{k_m,\ell} \frac{(4\pi a_m)^{j+\ell}}{(-2)^j} x_m^{j+\ell} |\lambda|^{j+\ell} \\ &= P + \sum_{j=0}^{\infty} \frac{b_{m,j}}{j!} x_m^{j+k_m} |\lambda|^{j+k_m} \end{aligned}$$

for $m = 1, \dots, n$, where $c_{k_m,\ell} = \frac{(-1)^\ell k_m!}{\ell!\ell!(k_m-\ell)!}$ is the ℓ^{th} coefficient of $L_{k_m}^{(0)}(x)$, and

$$b_{m,j} = \frac{(4\pi a_m)^{j+k_m}}{(-2)^j} \sum_{\ell=0}^{k_m} \frac{c_{k_m,\ell}}{(-2)^{k_m-\ell}} \frac{j!}{(j+\ell)!}.$$

For our purposes it is good enough to observe this is bounded by $|b_{m,j}| \leq c(2\pi a_m)^j 2^{k_m}$. Note that P is a polynomial of degree $k_m - 1$ and does not affect issues of convergence. To simplify matters, we leave out these first $k_m - 1$ terms of the series. Now we need to look at the integral

$$\begin{aligned} I(\lambda, \mathbf{a}, \mathbf{k}) = \int_0^1 e^{-2\pi|\lambda|a_1 x_1} L_{k_1}^{(0)}(4\pi|\lambda|a_1 x_1) \int_0^{1-x_1} e^{-2\pi|\lambda|a_2 x_2} L_{k_2}^{(0)}(4\pi|\lambda|a_2 x_2) \cdots \\ \times \int_0^{1-x_1-\cdots-x_{n-1}} e^{-2\pi|\lambda|a_n x_n} L_{k_n}^{(0)}(4\pi|\lambda|a_n x_n) dx_n \dots dx_2 dx_1 \end{aligned}$$

by rewriting as series and evaluating, beginning with the inner integral.

$$\begin{aligned} &\int_0^{1-x_1-\cdots-x_{n-1}} \sum_{j_n=0}^{\infty} \frac{b_{n,j_n}}{j_n!} x_n^{j_n+k_n} |\lambda|^{j_n+k_n} \\ &= \sum_{j_n=0}^{\infty} \frac{b_{n,j}}{j_n!(j_n+k_n+1)} (1-x_1-\cdots-x_{n-1})^{j_n+k_n+1} |\lambda|^{j_n+k_n}. \end{aligned}$$

By writing as a series and multiplying, the next integral

$$\begin{aligned} &\int_0^{S_{n-2}} e^{-2\pi|\lambda|x_{n-1}} L_{k_{n-1}}^{(0)}(4\pi|\lambda||x_{n-1}|^2) \\ &\quad \times \Big(\sum_{j_n=0}^{\infty} \frac{b_{n,j_n}}{j_n!(j_n+k_n+1)} (S_{n-2} - x_{n-1})^{j_n+k_n+1} |\lambda|^{j_n+k_n} \Big) dx_{n-1} \end{aligned}$$

then becomes

$$\int_0^{S_{n-2}} \Big(\sum_{j_{n-1}=0}^{\infty} \frac{b_{n-1,j_{n-1}}}{j_{n-1}!} x_{n-1}^{j_{n-1}+k_{n-1}} |\lambda|^{j_{n-1}+k_{n-1}} \Big)$$
$$\times \Big(\sum_{j_n=0}^{\infty} \frac{b_{n,j_n}}{j_n!(j_n+k_n+1)} (S_{n-2}-x_{n-1})^{j_n+k_n+1} |\lambda|^{j_n+k_n} \Big) dx_{n-1}$$
$$= \sum_{j_{n-1},j_n=0}^{\infty} \frac{b_{n-1,j_{n-1}} b_{n,j_n} |\lambda|^{j_{n-1}+j_n+k_{n-1}+k_n}}{j_{n-1}! j_n! (j_n+k_n+1)}$$
$$\times \Big(\int_0^{S_{n-2}} x_{n-1}^{j_{n-1}+k_{n-1}} (S_{n-2}-x_{n-1})^{j_n+k_n+1} dx_{n-1} \Big),$$

where $S_m = 1 - x_1 - \cdots - x_m$. Iterating for all n integrals, we have $I(\lambda, \mathbf{a}, \mathbf{k})$ expressed in terms of a series. Except for certain lower order terms, which do not affect convergence and are excluded, it is given by

$$\sum_{j_1,\ldots,j_n=0}^{\infty} \frac{b_{1,j_1}\cdots b_{n,j_n}}{j_1!\ldots j_n!(j_n+k_n+1)} (\beta_{\mathbf{j},\mathbf{k}}) |\lambda|^{|\mathbf{j}|+|\mathbf{k}|}$$
$$= \sum_{j_1,\ldots,j_n=0}^{\infty} b_{1,j_1}\cdots b_{n,j_n} \frac{(j_1+k_1)!}{j_1!} \cdots \frac{(j_n+k_n)!}{j_n!} \cdot \frac{1}{(|\mathbf{j}|+|\mathbf{k}|+n)!} |\lambda|^{|\mathbf{j}|+|\mathbf{k}|}$$
$$= |\lambda|^{|\mathbf{k}|} \sum_{r=0}^{\infty} \left(\sum_{|(j_1,\ldots,j_n)|=r} b_{1,j_1}\cdots b_{n,j_n} \frac{(j_1+k_1)!}{j_1!} \cdots \frac{(j_n+k_n)!}{j_n!} \frac{1}{(r+|\mathbf{k}|+n)!} \right) |\lambda|^r$$

where

$$\beta_{\mathbf{j},\mathbf{k}} = \int_0^1 x_1^{j_1+k_1} \int_0^{1-x_1} x_2^{j_2+k_2} \cdots \int_0^{S_{n-2}} x_{n-1}^{j_{n-1}+k_{n-1}} (S_{n-2}-x_{n-1})^{j_n+k_n+1} dx_{n-1}$$

is a (beta) integral, which equals to $\frac{(j_1+k_1)!\cdots(j_{n-1}+k_{n-1})!(j_n+k_n+n)!}{(|\mathbf{j}|+|\mathbf{k}|+1)!}$. After collecting terms based on powers of λ, this series becomes $\sum_{r=0}^{\infty} B_r |\lambda|^{r+|\mathbf{k}|}$, where

$$B_r = \sum_{|(j_1,\ldots,j_n)|=r} b_{1,j_1}, \ldots, b_{n,j_n} \frac{(j_1+k_1)!}{j_1!} \cdots \frac{(j_n+k_n)!}{j_n!} \frac{1}{(r+|\mathbf{k}|+n)!},$$

where each $|b_{i,m_i}| \le (2\pi a_i)^{m_i}$. We now observe the series will be real-analytic in λ based on the decay of coefficients. To apply the ratio test, we reduce

$\frac{B_{r+1}|\lambda|^{r+1}}{B_r|\lambda|^r}$. This gives us

$$\frac{|\lambda|\sum_{|\mathbf{j}|=r+1} b_{1,j_1}\cdots b_{n,j_n}\frac{(j_1+k_1)!}{j_1!}\cdots\frac{(j_n+k_n)!}{j_n}}{(r+|\mathbf{k}|+2)\sum_{|\mathbf{j}|=r} b_{1,j_1}\cdots b_{n,j_n}\frac{(j_1+k_1)!}{j_1!}\cdots\frac{(j_n+k_n)!}{j_n!}}$$

To simplify the comparison, we write the sum in the numerator in the form

$$\begin{aligned}&\sum_{|(j_1,\ldots,j_n)|=r+1} b_{1,j_1}\cdots b_{n,j_n}\frac{(j_1+k_1)!}{j_1!}\cdots\frac{(j_n+k_n)!}{j_n!}\\ &= \sum_{|(j_1,\ldots,j_n)|=r} b_{1,j_1+1}b_{2,j_2}\cdots b_{n,j_n}\frac{(j_1+k_1+1)!}{(j_1+1)!}\frac{(j_2+k_2)!}{j_2!}\cdots\frac{(j_n+k_n)!}{j_n!}\\ &\quad+\sum_{|(0,j_2,\ldots,j_n)|=r} b_{2,j_2}\cdots b_{n,j_n}\frac{(j_2+k_2)!}{j_2!}\cdots\frac{(j_n+k_n)!}{j_n!}\end{aligned}$$

Thus, our ratio,

$$\frac{|\lambda|\sum_{|\mathbf{j}|=r+1} b_{1,j_1}\cdots b_{n,j_n}\frac{(j_1+k_1)!}{j_1!}\cdots\frac{(j_n+k_n)!}{j_n}}{(r+|\mathbf{k}|+n+1)\sum_{|\mathbf{j}|=r} b_{1,j_1}\cdots b_{n,j_n}\frac{(j_1+k_1)!}{j_1!}\cdots\frac{(j_n+k_n)!}{j_n!}},$$

becomes the sum of two fractions. The first is

$$\frac{\sum_{|\mathbf{j}|=r} b_{1,j_1+1}b_{2,j_2}\cdots b_{n,j_n}\frac{(j_1+k_1+1)!}{(j_1+1)!}\frac{(j_2+k_2)!}{j_2!}\cdots\frac{(j_n+k_n)!}{j_n!}}{\sum_{|\mathbf{j}|=r} b_{1,j_1}\cdots b_{n,j_n}\frac{(j_1+k_1)!}{j_1!}\cdots\frac{(j_n+k_n)!}{j_n}}$$

and for this one we may apply a term by term comparison

$$\frac{b_{1,j_1+1}b_{2,j_2}\cdots b_{n,j_n}}{b_{1,j_1}b_{2,j_2}\cdots b_{n,j_n}}\frac{\frac{(j_1+k_1+1)!}{(j_1+1)!}\frac{(j_2+k_2)!}{j_2!}\cdots\frac{(j_n+k_n)!}{j_n!}}{\frac{(j_1+k_1)!}{j_1!}\frac{(j_2+k_2)!}{j_2!}\cdots\frac{(j_n+k_n)!}{j_n!}}=\frac{b_{1,j_1+1}}{b_{1,j_1}}\frac{j_1+k_1+1}{j_1+1}.$$

Here we also use that $|\frac{b_{1,j_1+1}}{b_{1,j_1}}|=2\pi a_1$. Thus for this first term, we have

$$\left|\frac{\sum_{|\mathbf{j}|=r+1}\alpha_{\mathbf{j}}}{\sum_{|\mathbf{j}|=r}\alpha_{\mathbf{j}}}\right|\le 2\pi a_1(k_1+1),$$

where $\alpha_{\mathbf{j}}=b_{1,j_1}\cdots b_{n,j_n}\frac{(j_1+k_1)!}{j_1!}\cdots\frac{(j_n+k_n)!}{j_n!}$.

The second term is

$$\frac{\sum_{|(0,j_2,\ldots,j_n)|=r} b_{2,j_2}\cdots b_{n,j_n}\frac{(j_2+k_2)!}{j_2}\cdots\frac{(j_n+k_n)!}{j_n!}}{\sum_{|(j_1,j_2,\ldots,j_n)|=r} b_{1,j_1}\cdots b_{n,j_n}\frac{(j_1+k_1)!}{j_1!}\cdots\frac{(j_n+k_n)!}{j_n!}}$$

Here we observe that the sum $\sum_{|(j_1,j_2,\ldots,j_n)|=r}\alpha_{\mathbf{j}}$ contains each term in the sum $\sum_{|(0,j_2,\ldots,j_n)|=r}\alpha_{\mathbf{j}}$, plus many additional terms. It easily follows that

$$\sum_{|(0,j_2,\ldots,j_n)|=r}\alpha_{\mathbf{j}} < \sum_{|(j_1,j_2,\ldots,j_n)|=r}\alpha_{\mathbf{j}}.$$

It therefore follows that

$$\frac{\sum_{|(0,j_2,\ldots,j_n)|=r}\alpha_{\mathbf{j}}}{\sum_{|(j_1,j_2,\ldots,j_n)|=r}\alpha_{\mathbf{j}}} < 1$$

Adding these two terms together, we have

$$\frac{\sum_{|\mathbf{j}|=r+1}\alpha_{\mathbf{j}}}{\sum_{|\mathbf{j}|=r}\alpha_{\mathbf{j}}} < 2\pi a_1(k_1+1)+1.$$

Thus for the ratio test, we know $\frac{B_{r+1}|\lambda|^{r+1}}{B_r|\lambda|^r} < \frac{|\lambda|}{r+|\mathbf{k}|+n+1}[2\pi a_1(k_1+1)+1]$. Clearly, for any $\lambda\in\mathbf{R}^*$, this goes to 0 as r becomes infinite. This implies the convergence result.

Using the Laplace transform, we can obtain a more explicit evaluation of the integral. First rewrite the integral based on filling out the solid ellipsoid by surfaces which are ellipsoids, with a varying radius. We can then use convolution, as follows

$$\begin{aligned}
&I(\lambda,\mathbf{a},\mathbf{k}) = \int_{E_{\mathbf{b}}} e^{-2\pi|\lambda||\mathbf{z}|^2}\prod_{j=1}^n L_{k_j}^{(0)}(4\pi|\lambda||z_j|^2)d\mu_{\mathbf{b}}(\mathbf{z})\\
&= \pi^n\int_{\sum r_j^2\le 1}\prod_{j=1}^n e^{-2\pi|\lambda|a_jr_j^2}a_jL_{k_j}^{(0)}(4\pi|\lambda|a_jr_j^2)2r_1dr_1\ldots 2r_ndr_n\\
&= \frac{\pi^n}{(4\pi|\lambda|)^n}\int_0^1\Big(\int_{\sum x_j=4\pi|\lambda|r^2}\prod_{j=1}^n e^{-a_jx_j/2}a_jL_{k_j}^{(0)}(a_jx_j)dx_{n-1}\ldots dx_1\Big)r^{2n-1}dr\\
&= \frac{1}{(4|\lambda|)^n}\int_0^1\Big(I_{r,0}(\lambda,\mathbf{a},\mathbf{k})\Big)r^{2n-1}dr
\end{aligned}$$

where

$$I_{r,0}=\int_0^{4\pi|\lambda|r^2} e^{-a_1x_1/2}a_1L_{k_1}^{(0)}(a_1x_1)\Big[\int_0^{4\pi|\lambda|r^2-x_1} e^{-a_2x_2/2}a_2L_{k_2}^{(0)}(a_2x_2)\dots$$
$$\times\int_0^{4\pi|\lambda|r^2-x_1-\dots-x_{n-2}} e^{-a_{n-1}x_{n-1}/2}a_{n-1}L_{k_{n-1}}^{(0)}(a_{n-1}x_{n-1})$$
$$\times e^{-a_n(4\pi|\lambda|r^2-x_1-\dots-x_{n-1})/2}a_nL_{k_n}^{(0)}\left(a_n(4\pi|\lambda|r^2-x_1-\dots-x_{n-1})\right)$$
$$dx_{n-1}\dots dx_2\Big]dx_1$$

Thus $I(\lambda,\mathbf{a},\mathbf{k})$ may be expressed as

$$I(\lambda,\mathbf{a},\mathbf{k})=\frac{1}{(4|\lambda|)^n}\int_0^1\left[f_{a_1,k_1}*\dots*f_{a_n,k_n}\right](4\pi|\lambda|r^2)r^{2n-1}dr,$$

where the middle integral $I_{r,0}$ is evaluated as a convolution of the functions

$$f_{a_j,k_j}(t)=e^{-a_jt/2}a_jL_{k_j}^{(0)}(a_jt).$$

To compute this integral, we apply the Laplace transform, $[\mathcal{L}(f)](s)=\int_0^\infty e^{-st}f(t)dt$. Note that the formula $\mathcal{L}[L_{k_j}^{(0)}](s)=\frac{(s-1)^{k_j}}{s^{k_j+1}}$ (see [13]) implies that $\mathcal{L}[f_{a_j,k_j}](s)=\frac{(s/a_j-1/2)^{k_j}}{(s/a_j+1/2)^{k_j+1}}$. We now apply the Laplace transform to this convolution.

$$\mathcal{L}\left[f_{a_1,k_1}*\dots*f_{a_n,k_n}\right](s)=\mathcal{L}[f_{a_1,k_1}]\times\dots\times\mathcal{L}[f_{a_n,k_n}]=\prod_{j=1}^n\frac{(\frac{s}{a_j}-\frac{1}{2})^{k_j}}{(\frac{s}{a_j}+\frac{1}{2})^{k_j+1}}.$$

Then to find $I(\lambda,\mathbf{a},\mathbf{k})$ we need only apply the inverse Laplace transform. This can be achieved using residue calculus. We let

$$\Phi_{\mathbf{k},\mathbf{a},j}(s)=\frac{(\frac{s}{a_1}-\frac{1}{2})^{k_1}\cdots(\frac{s}{a_n}-\frac{1}{2})^{k_n}}{(\frac{s}{a_1}+\frac{1}{2})^{k_1+1}\cdots(\frac{s}{a_{j-1}}+\frac{1}{2})^{k_{j-1}+1}(\frac{s}{a_{j+1}}+\frac{1}{2})^{k_{j+1}+1}\cdots(\frac{s}{a_n}+\frac{1}{2})^{k_n+1}},$$

and then we have the following.

$$I_{r,0}(\lambda,\mathbf{a},\mathbf{k})\ =\ \mathcal{L}^{-1}\Big[\prod_{j=1}^n\frac{(s/a_j-1/2)^{k_j}}{(s/a_j+1/2)^{k_j+1}}\Big]_{t=4\pi|\lambda|}$$

$$= \Big(\sum_{j=1}^{n} \mathrm{Res}_{s=-a_j/2}\Big[e^{st}\Phi_{\mathbf{k},\mathbf{a},j}(s)\Big]\Big)_{t=4\pi|\lambda|}$$
$$= \Big(\sum_{j=1}^{n} e^{-2\pi|\lambda|a_j} Q_{\mathbf{a},\mathbf{k},j}(4\pi|\lambda|)\Big),$$

where each polynomial $Q_{\mathbf{a},\mathbf{k},j}(t) = \sum_{\mu=0}^{k_j} \frac{D^{k_j-\mu}\Phi_{\mathbf{k},\mathbf{a},j}(-a_j/2)}{(k_j-\mu)!\mu!} t^{\mu}$. After one more integration, we have

$$I(\lambda,\mathbf{a},\mathbf{k}) = (4|\lambda|)^{-n} \sum_{j=1}^{n} \int_0^1 e^{-2\pi|\lambda|a_j r^2} Q_{\mathbf{a},\mathbf{k},j}(4\pi|\lambda|r^2) r^{2n-1} dr$$
$$= (4|\lambda|)^{-n} \Big[\sum_{j=1}^{n} e^{-2\pi|\lambda|a_j} P_{\mathbf{a},\mathbf{k},j}(4\pi|\lambda|)\Big] + C_{\mathbf{a},\mathbf{k}},$$

where the polynomial $P_{\mathbf{a},\mathbf{k},j}(x)$ is determined by the integral

$$\int_0^1 \Big(e^{-2\pi|\lambda|a_j r^2} Q_{\mathbf{a},\mathbf{k},j}(4\pi|\lambda|r^2)\Big) r^{2n-1} dr = e^{-2\pi|\lambda|a_j} P_{\mathbf{a},\mathbf{k},j}(4\pi|\lambda|) + c_{\mathbf{a},\mathbf{k},j}.$$

Clearly, all of these terms are exponential polynomials, which also are real-analytic in λ for $\lambda \neq 0$. This completes the proof of Lemma 3.3. □

The proof of Proposition 3.2 now follows quickly, once we recall that for each $\lambda \in \mathbf{R}^*$,

$$\hat{f}^{\lambda}(\mathbf{z}) = \sum_{\mathbf{k}\geq \mathbf{0}} \frac{c_{\mathbf{k}}(\lambda)}{c_{\mathbf{0}}^{\lambda}} I(\lambda,\mathbf{a},\mathbf{k}) \mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}).$$

Since the set $\{\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}) : \mathbf{k} \in (\mathbf{Z}_+)^n\}$ forms an orthonormal basis for $L^2_{\mathbf{0}}(\mathbf{C}^n)$ for each $\lambda \in \mathbf{R}^*$, we have that $c_{\mathbf{k}}(\lambda) \cdot I(\lambda,\mathbf{a},\mathbf{k}) = 0$ for each $\mathbf{k} \in (\mathbf{Z}_+)^n$ and for each $\lambda \in \mathbf{R}^*$. Since we know the zero set of $I(\lambda,\mathbf{a},\mathbf{k})$ is isolated, we have that, for each $\mathbf{k} \in (\mathbf{Z}_+)^n$, $c_{\mathbf{k}}(\lambda) = 0$ for a.e. $\lambda \in \mathbf{R}^*$. Recalling that $\hat{f}^{\lambda}(\mathbf{z}) = \sum_{\mathbf{k}} c_{\mathbf{k}}(\lambda)\mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z})$, this implies that $\hat{f}^{\lambda} = 0$ and thus $f = 0$. This completes the proof of Proposition 3.2.

The following corollaries follow from applying the above procedure to the cases (3) and (4) of working over anisotropic Heisenberg groups.

Corollary 3.4. *Let* $\mathbf{a} \in \mathbf{R}^n$ *such that each* $a_j > 0$, *and let* $f \in L^2(\mathbf{H}^n_{\mathbf{a}})$. *Let* B_r *be the ball* $B_r = \{(\mathbf{z},0) \in \mathbf{C}^n \times \{0\} : |\mathbf{z}| = r\}$ *with volume measure*

μ_r. *Assume* f *satisfies the integral conditions* (3), *that is*

$$\int_{B_r} L_{\mathbf{g}} f(\mathbf{z},0)d\mu_r(\mathbf{z}) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^n_{\mathbf{a}}.$$

Then it follows that $f \equiv 0$.

Proof. Use the same steps as in the proof of Theorem 3.1, but when evaluating the integral, we now have

$$I(\lambda,\mathbf{a},\mathbf{k},r) \;=\; \int_{B_r} e^{-2\pi|\lambda|\|\mathbf{w}\|^2_{\mathbf{a}}} \prod_{j=1}^{n} a_j L^{(0)}_{k_j}(4\pi|\lambda|a_j|z_j|^2)d\mu_r(\mathbf{w})$$

It is a straightforward computation to see this reduces to the same integral described prior to Lemma 3.3, with λ replaced by $r\lambda$. Applying Lemma 3.3, we obtain the real-analyticity in λ that was needed for this step. The rest of the proof goes through as above. □

Corollary 3.5. *Let* $\mathbf{a},\mathbf{d} \in \mathbf{R}^n$ *such that each* $a_j, d_j > 0$, *and let* $f \in L^2(\mathbf{H}^n_{\mathbf{a}})$. *Let* $E_{\mathbf{d}}$ *be the complex ellipsoid* $E_{\mathbf{d}}$ *with volume measure* $\mu_{\mathbf{d}}$. *Assume* f *satisfies the integral conditions* (4), *that is*

$$\int_{E_{\mathbf{d}}} L_{\mathbf{g}} f(\mathbf{z},0)d\mu_r(\mathbf{z}) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^n_{\mathbf{a}}.$$

Then it follows that $f \equiv 0$.

Proof. Again the same as above, but now the integral becomes

$$\begin{aligned} I(\lambda,\mathbf{a},\mathbf{d},\mathbf{k}) =& \int_{E_{\mathbf{d}}} e^{-2\pi|\lambda|\|\mathbf{w}\|^2_{\mathbf{a}}} \prod_{j=1}^{n} a_j L^{(0)}_{k_j}(4\pi|\lambda|a_j|z_j|^2)d\mu_{\mathbf{d}}(\mathbf{w}) \\ =&(4|\lambda|)^{-n}\int_{\sum x_j \le 4\pi|\lambda|} \prod_{j=1}^{n} e^{-a_j d_j^2 x_j/2} a_j d_j^2 L^{(0)}_{k_j}(a_j d_j^2 x_j)dx_n \dots dx_1 \end{aligned}$$

which is of the same type. After reduction, we may again apply Lemma 3.3 with the $a_j d_j^2$ in the integral here becoming an a_j in the integral of the Lemma.

Note the results in this section also extend to L^p for $1 \le p \le 2$ by usual approximation argument (see [2] and [3]). □

4. Case of L^p, $1 < p < \infty$

Strichartz has demonstrated that, in regard to the joint spectrum of the operators $\square$ and iT on the isotropic Heisenberg group, the L^2 spectrum and the L^p spectrum for $1 < p < \infty$ are the same [18]. This property carries over to anisotropic Heisenberg groups as well, and thus we may expect that our theorems for the function spaces L^p for $1 < p < \infty$ will be in essence the same as that proved above for L^2. In particular, we apply the method of [6], where a one radius theorem for $L^p(\mathbf{H}^n)$ is given. This allows us to prove that one complex ellipsoid possesses the Pomepiu property for $L^p(\mathbf{H}^n)$, or equivalently anisotropic Heisenberg groups $L^p(\mathbf{H}^n_{\mathbf{a}})$ have a one radius theorem. We will use the L^p methods described above in Section 2.

The important property from this material is the L^p summability of $f \in L^p(\mathbf{H}^n_{\mathbf{a}})$ in terms of its spectral projections, as given in [12]. Although each $(\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$ is part of the spectrum, for a projection bounded in L^p, we must project along the ray $\lambda > 0$ or $\lambda < 0$. The projections are then defined by

$$P_{\mathbf{k},+}(f) = \int_0^\infty (f * \phi^\lambda_{\mathbf{k},+})(\mathbf{z}, t) d\lambda,$$

and

$$P_{\mathbf{k},-}(f) = \int_0^\infty (f * \phi^\lambda_{\mathbf{k},-})(\mathbf{z}, t) d\lambda.$$

where $\phi^\lambda_{\mathbf{k},\pm}(\mathbf{z}, t) = c|\lambda|^n e^{\pm i\lambda t} e^{-|\lambda| \|\mathbf{z}\|^2_{\mathbf{a}}} \prod_{j=1}^n L^{(0)}_{k_j}(2|\lambda| a_j |z_j|^2)$, for $\lambda > 0$. For $f \in L^p(\mathbf{H}^n)$, we may then write

$$f(\mathbf{z}, t) \;=\; \lim_{r \to 1-} \sum_{\mathbf{k} \geq \mathbf{0}} r^{|\mathbf{k}|} \Big(P_{\mathbf{k},+}(f)(\mathbf{z}, t) + P_{\mathbf{k},-}(f)(\mathbf{z}, t) \Big).$$

This summation interacts with a property we establish for the exponential Laguerre functions $\psi^\lambda_{\mathbf{k}}$ in regard convolution with the measure $T_{\mathbf{b}}$. Using these together with the real-analyticity of the integral in Lemma 3.3 we extend Theorem 3.1 to a "one radius" result for L^p.

Let us first establish the convolution relationship.

Lemma 4.1. *Let $\mathbf{b} \in (\mathbf{Z}_+)^n$, each $b_j > 0$ and $T_{\mathbf{b}}$ the Radon measure associated to the complex ellipsoid $E_{\mathbf{b}}$, as above. For each exponential*

Laguerre $(\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$, *we have*

$$\big(\psi_{\mathbf{k}}^{\lambda} * T_{\mathbf{b}}\big)(\mathbf{z}, t) = h(\lambda, \mathbf{k}, \mathbf{a}) \cdot \psi_{\mathbf{k}}^{\lambda}(\mathbf{z}, t),$$

where $h(\lambda, \mathbf{a}, \mathbf{k}) = cI(\lambda, \mathbf{a}, \mathbf{k})$ *is proportional to the integral calculated above in Lemma* 3.3.

Proof. We compute this by direct calculation using Laguerre series and the twisted convolution of such exponential Laguerre functions. Note that another way to do this would be to use properties of spherical functions, as was done in [7]. We begin by computing

$$\begin{aligned}\left(\psi_{\mathbf{k}}^{\lambda} * T_{\mathbf{b}}\right)(\mathbf{z}, t) =& \int_{E_{\mathbf{b}}} \psi_{\mathbf{k}}^{\lambda}(\mathbf{z} - \mathbf{w}, t - 2\langle \mathbf{z}, \mathbf{w}\rangle) d\mu_{\mathbf{b}}(\mathbf{w}) \\ =& (\sqrt{4|\lambda|})^{-n} \int_{E_{\mathbf{b}}} e^{2\pi i\lambda(t - 2\langle \mathbf{z}, \mathbf{w}\rangle)} \mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z} - \mathbf{w}) d\mu_{\mathbf{b}}(\mathbf{w}) \\ =& \frac{e^{2\pi i\lambda t}}{(\sqrt{4|\lambda|})^n} \int_{E_{\mathbf{b}}} e^{-4\pi i\lambda\langle \mathbf{z}, \mathbf{w}\rangle} \mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z} - \mathbf{w}) d\mu_{\mathbf{b}}(\mathbf{w}) \\ =& \frac{e^{2\pi i\lambda t}}{(\sqrt{4|\lambda|})^n} I(\lambda, \mathbf{k}, \mathbf{b})(\mathbf{z}).\end{aligned}$$

Noting that this integral is $\mathbf{0}$-homogeneous in the variable $\mathbf{z}$, we expand this integral as a Laguerre series $I(\lambda, \mathbf{k}, \mathbf{b})(\mathbf{z}) = \sum_{\mathbf{j}\geq \mathbf{0}} \gamma_{\mathbf{j},\mathbf{k},\lambda} \mathcal{W}_{\mathbf{j}}^{\lambda}(\mathbf{z})$, and compute the coefficients $\gamma_{\mathbf{j},\mathbf{k},\lambda}$, which are given by

$$\gamma_{\mathbf{j},\mathbf{k},\lambda} = \int_{\mathbf{C}^n} I(\lambda, \mathbf{k}, \mathbf{b})(\mathbf{z}) \mathcal{W}_{\mathbf{j}}^{\lambda}(\mathbf{z}) dm(\mathbf{z}).$$

We have, for $\lambda > 0$,

$$\begin{aligned}\gamma_{\mathbf{j},\mathbf{k},\lambda} =& \int_{\mathbf{C}^n} I(\lambda, \mathbf{k}, \mathbf{b})(\mathbf{z}) \mathcal{W}_{\mathbf{j}}^{\lambda}(\mathbf{z}) dm(\mathbf{z}) \\ =& \int_{E_{\mathbf{b}}} \Big(\int_{\mathbf{C}^n} \overline{e^{-4\pi i\lambda\langle \mathbf{w}, \mathbf{z}\rangle} \mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{w} - \mathbf{z}) \mathcal{W}_{\mathbf{j}}^{\lambda}(\mathbf{z})} dm(\mathbf{z}) \Big) d\mu_{\mathbf{b}}(\mathbf{w}) \\ =& \int_{E_{\mathbf{b}}} \Big(\overline{(\mathcal{W}_{\mathbf{k}}^{\lambda} \star^{\lambda} \mathcal{W}_{\mathbf{j}}^{\lambda})(\mathbf{w})} \Big) d\mu_{\mathbf{b}}(\mathbf{w}) \\ =& c_{\mathbf{j},\mathbf{k}}^{\lambda} \delta_{\mathbf{j},\mathbf{k}} \int_{E_{\mathbf{b}}} \mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{w}) d\mu_{\mathbf{b}}(\mathbf{w}).\end{aligned}$$

Thus

$$\begin{aligned}\left(\psi_{\mathbf{k}}^{\lambda} * T_{\mathbf{b}}\right)(\mathbf{z}, t) =& \frac{1}{(\sqrt{4|\lambda|})^n} I(\lambda, \mathbf{k}, \mathbf{b})(\mathbf{z}) e^{2\pi i \lambda t} \mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}) \\ =& \frac{c_{\mathbf{0}}^{\lambda}}{(\sqrt{4|\lambda|})^n} \left(\int_{E_{\mathbf{b}}} \mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{w}) d\mu_{\mathbf{b}}(\mathbf{w}) \right) e^{2\pi i \lambda t} \mathcal{W}_{\mathbf{k}}^{\lambda}(\mathbf{z}) \\ =& h(\lambda, \mathbf{a}, \mathbf{k}) \psi_{\mathbf{k}}^{\lambda}(\mathbf{z}, t),\end{aligned}$$

where

$$h(\lambda, \mathbf{a}, \mathbf{k}) = \int_{|\mathbf{z}|<1} \prod_{j=1}^{n} e^{-2\pi|\lambda| b_j^2 |z_j|^2} b_j^2 L_{k_j}^{(0)}(4\pi|\lambda| b_j^2 |z_j|^2) d\mu_1(\mathbf{z}),$$

which is evaluated in the lines prior to and within Lemma 3.3. We mention that the case of $\lambda < 0$ is similar. Since we have shown $(\psi_{\mathbf{k}}^{\lambda} * T_{\mathbf{b}})(\mathbf{z}, t) = h(\lambda, \mathbf{a}, \mathbf{k}) \psi_{\mathbf{k}}^{\lambda}(\mathbf{z}, t)$, this completes the proof of Lemma 4.1. Recall that Lemma 3.3 implies that the zero set of $h(\lambda, \mathbf{a}, \mathbf{k})$, as a function of λ, is isolated. □

We now prove the main result of this section.

Theorem 4.2. *Let* $\mathbf{b} \in \mathbf{R}^n, b_j > 0$, *and let* $f \in L^p(\mathbf{H}^n)$, $1 < p < \infty$. *Assume that* f *satisfies the integral conditions* (1), *that is*

$$\int_{E_{\mathbf{b}}} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_{\mathbf{b}}(\mathbf{z}) = 0 \quad \textit{for every } \mathbf{g} \in \mathbf{H}^n.$$

It follows that $f \equiv 0$.

We require the relationship $(\psi_{\mathbf{k}}^{\lambda} * T_{\mathbf{b}})(\mathbf{z}, t) = h(\lambda, \mathbf{a}, \mathbf{k}) \cdot \psi_{\mathbf{k}}^{\lambda}(\mathbf{z}, t)$ established in Lemma 4.1. This is applied to each term of the Abel mean

$$f(\mathbf{z}, t) = \lim_{r \to 1^-} \sum_{\mathbf{k} \geq \mathbf{0}} r^{|\mathbf{k}|} \left(\int_0^{\infty} (f * \phi_{\mathbf{k},+}^{\lambda})(\mathbf{z}, t) d\lambda + \int_0^{\infty} (f * \phi_{\mathbf{k},-}^{\lambda})(\mathbf{z}, t) d\lambda \right)$$

in the convolution equation $f * T_{\mathbf{b}} = 0$. We may obtain

$$f(\mathbf{z}, t) = \lim_{r \to 1^-} \sum_{\mathbf{k} \geq \mathbf{0}} r^{|\mathbf{k}|} \int_{-\infty}^{\infty} h(\lambda, \mathbf{a}, \mathbf{k}) (f * \phi_{\mathbf{k}}^{\lambda})(\mathbf{z}, t) d\lambda,$$

where

$$h(\lambda,\mathbf{a},\mathbf{k})=\int_{|\mathbf{z}|^2\leq 1} e^{-2\pi|\lambda|\|\mathbf{z}\|_{\mathbf{a}}^2}\prod_{j=1}^{n} a_j L_{k_j}^{(0)}(4\pi|\lambda|a_j|z_j|^2)d\mu_1(\mathbf{z})$$

is known to be real-analytic in λ from Lemma 4.1 and Lemma 3.3 above. Since the $P_{\mathbf{k}}$ are projection operators, we may apply them to the above equation to obtain that

$$\int_{-\infty}^{\infty} h(\lambda,\mathbf{a},\mathbf{k})\left(f*\psi_{\mathbf{k}}^{\lambda}\right)(\mathbf{z},t)|\lambda|^n d\lambda=0$$

for each $\mathbf{k}\in(\mathbf{Z}_+)^n$. We now choose a sequence $\{f_j\}$, each $f_j\in\mathcal{S}(\mathbf{H}^n)$, converging to f in L^p norm. Thus we have

$$\lim_{j\to\infty}\int_{-\infty}^{\infty} h(\lambda,\mathbf{a},\mathbf{k})\left(f_j*\psi_{\mathbf{k}}^{\lambda}\right)(\mathbf{z},t)|\lambda|^n d\lambda=0$$

and therefore

$$\lim_{j\to\infty}\int_{-\infty}^{\infty} h(\lambda,\mathbf{a},\mathbf{k})(\widetilde{P_{\mathbf{k}}f_j})^{\lambda}(\mathbf{z})e^{i\lambda t}d\lambda=0$$

Since the above sequence converges to 0 in the L^p-norm, the sequence of partial Fourier transforms converges to 0 in the sense of distributions.

$$\lim_{j\to\infty} h(\lambda,\mathbf{a},\mathbf{k})(\widetilde{P_{\mathbf{k}}f_j})^{\lambda}(\mathbf{z})=h(\lambda,\mathbf{a},\mathbf{k})(\widetilde{P_{\mathbf{k}}f})^{\lambda}(\mathbf{z})=0.$$

Since $h(\lambda,\mathbf{a},\mathbf{k})$ has an isolated set of zeros, $(\widetilde{P_{\mathbf{k}}f})^{\lambda}$ is almost everywhere 0, which can only happen when $P_{\mathbf{k}}f=0$. Since these projections vanish for all $\mathbf{k}\in(\mathbf{Z}_+)^n$, it follows that $f\equiv 0$. This completes the proof.

The proof of Theorem 4.2 also yields the following two corollaries.

Corollary 4.3. *Let* $\mathbf{a}\in\mathbf{R}^n$ *with each* $a_j>0$, *and let* $f\in L^p(\mathbf{H}_{\mathbf{a}}^n)$, *where* $1<p<\infty$. *Assume that* f *satisfies the integral conditions* (3), *that is*

$$\int_{B_r} L_{\mathbf{g}}f(\mathbf{z},0)d\mu_r(\mathbf{z})=0 \quad \textit{for all } \mathbf{g}\in\mathbf{H}_{\mathbf{a}}^n.$$

It follows that $f\equiv 0$.

Corollary 4.4. *Let* $\mathbf{a},\mathbf{d}\in\mathbf{R}^n$ *with each* $a_j,d_j>0$, *and let* $f\in L^p(\mathbf{H}_{\mathbf{a}}^n)$, *where* $1 < p < \infty$. *Assume that* f *satisfies the integral conditions* (4), *that is*

$$\int_{E_{\mathbf{d}}} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_{\mathbf{d}}(\mathbf{z}) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^n_{\mathbf{a}}.$$

It follows that $f \equiv 0$.

These are both immediate since the Abel means are also valid in $\mathbf{H}^n_{\mathbf{a}}$.

5. Case of L^∞

In this section we are concerned with extending the Pompeiu theorems of the previous sections for complex ellipsoids and for spheres in $\mathbf{H}^n_{\mathbf{a}}$, to the case where the function space is L^∞. This is the level where the most interesting results arise. From [3] and [6] we recognize that the theorems of two radii for balls, well known from the Euclidean case, reappear at the L^∞ level. In extension of our results to the level of L^∞, the theorems take on a similar character. However, use of complex ellipsoids allows us to consider what happens when using rotations together with translations. This direction of investigation will be taken up in the next section. In this section we focus on integral conditions (2), (3), and (4), which only involve translations, and in each case we realize a theorem of two radii. As in [3, 9] the methods of proof require use of the Gelfand transform and application of an appropriate Tauberian theorem. In Section 5.1 we apply the Galfand transform for $L^1_{\mathbf{0}}(\mathbf{H}^n)$, already used in [3, 9], to prove the Pompeiu theorem for complex ellipsoids in $\mathbf{H}^n$. Then in Section 5.2 we discuss a Gelfand transform for $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$ and describe the $\mathbf{T}^n$-spherical functions. This culminates in application of this transform to the cases of balls and complex ellipsoids in $\mathbf{H}^n_{\mathbf{a}}$.

5.1. Complex ellipsoid in $\mathbf{H}^n$ for L^∞

Applying the method of Gelfand transforms on $L^1_{\mathbf{0}}(\mathbf{H}^n)$, as described in Section 2, it is straightforward for us to prove a Pompeiu result for complex ellipsoids of two radii in $\mathbf{H}^n$. Most of the work will come down to computation of

$$\widetilde{T}_{\mathbf{b}}(\lambda; \mathbf{k}) = \int_{E_{\mathbf{b}}} \psi^\lambda_{\mathbf{k}}(\mathbf{z}, 0) d\mu_{\mathbf{b}}(\mathbf{z}) \quad \text{for } (\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n,$$

$$\widetilde{T}_{\mathbf{b}}(\rho) = \int_{E_{\mathbf{b}}} \mathcal{J}_\rho(\mathbf{z}) d\mu_{\mathbf{b}}(\mathbf{z}) \quad \text{for } \rho \in \mathbf{R}^n_+.$$

In the following theorem, we obtain the analogue of a theorem of two radii, such as [3] Theorem 6.2, for complex ellipsoids. Define $E_{\mathbf{b},r_1}$ and $E_{\mathbf{b},r_2}$ by

$$E_{\mathbf{b},r_1} = \{\mathbf{z} \in \mathbf{C}^n : (z_1/b_1)^2 + \cdots + (z_n/b_n)^2 = r_1\},$$

and

$$E_{\mathbf{b},r_2} = \{\mathbf{z} \in \mathbf{C}^n : (z_1/b_1)^2 + \cdots + (z_n/b_n)^2 = r_2\}.$$

We then state the following theorem. We now define the function

$$\Phi_{\mathbf{a},\mathbf{k}}(t) = \sum_{j=1}^n e^{-a_j x/2} P_{\mathbf{a},\mathbf{k},j}(x) + C_{\mathbf{a},\mathbf{k}},$$

where the $P_{\mathbf{a},\mathbf{k},j}$ are polynomials and $C_{\mathbf{a},\mathbf{k}}$ is the corresponding constant, as defined in Lemma 3.3.

Theorem 5.1. *Let* $\mathbf{b} \in \mathbf{R}^n$*, each* $b_j > 0$ *and* $0 < r_1 < r_2 \in \mathbf{R}$*. Let* $f \in L^\infty(\mathbf{H}^n) \cap C(\mathbf{H}^n)$*. Assume that for* $i = 1, 2$*,* f *satisfies the integral conditions* (2)*, that is*

$$\int_{E_{\mathbf{b},r_i}} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_{\mathbf{b},r_j}(\mathbf{z}) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^n.$$

Assume further that r_1, r_2 *satisfy the conditions*

1. $(r_1/r_2)^2 \notin \mathcal{Q}(\Phi_{\mathbf{a},\mathbf{k}}(t))$, for all $\mathbf{k} \in (\mathbf{Z}_+)^n$,
2. $r_1/r_2 \notin \mathcal{Q}(J_n(t))$.

We may then conclude $f \equiv 0$*. In case* r_1, r_2 *do not satisfy conditions* 1 *and* 2*, then there exists* $f \not\equiv 0$ *satisfying the integral conditions.*

Proof. We apply the Gelfand transform on $L^1_{\mathbf{0}}(\mathbf{H}^n)$ to the $T_{\mathbf{b},r}$.

$$\begin{aligned} \widetilde{T}_{\mathbf{b},r}(\lambda; \mathbf{k}) &= \int_{E_{\mathbf{b},r_j}} \psi^\lambda_{\mathbf{k}}(\mathbf{z}, 0) d\mu_{\mathbf{b}}(\mathbf{z}) \\ &= \int_{\|\mathbf{z}\|_{1/\mathbf{b}} \le r} e^{-2\pi|\lambda||\mathbf{z}|^2} \prod_{j=1}^n L^{(0)}_{k_j}(4\pi|\lambda||z_j|^2) d\mu_{\mathbf{b}}(\mathbf{z}) \end{aligned}$$

$$= \int_{|\mathbf{z}|\le r} \prod_{j=1}^{n} e^{-2\pi|\lambda|b_j^2|z_j|^2} b_j^2 L_{k_j}^{(0)}(4\pi|\lambda|b_j^2|z_j|^2)d\mu_r(\mathbf{z})$$

$$= \frac{\pi^n}{(4\pi|\lambda|)^n} \int_{\sum x_j \le 4\pi|\lambda|r^2} \prod_{j=1}^{n} e^{-a_jx_j/2} a_j L_{k_j}^{(0)}(a_jx_j)dx_n \dots dx_1.$$

Then using Lemma 3.3, above to evaluate the integral, we have

$$\begin{aligned}\widetilde{T}_{\mathbf{b},r}(\lambda;\mathbf{k}) &= \frac{1}{(4|\lambda|)^n}\Big(\sum_{j=1}^{n} e^{-2\pi|\lambda|r^2a_j} P_{\mathbf{a},\mathbf{k},j}(4\pi|\lambda|r^2) + C_{\mathbf{a},\mathbf{k}}\Big) \\ &= \pi^n \Phi_{\mathbf{a},\mathbf{k}}(4\pi|\lambda|r^2).\end{aligned}$$

We also compute

$$\begin{aligned}\widetilde{T}_{\mathbf{b},r}(\mathbf{0};\rho) &= \int_{E_{\mathbf{b}},r} \mathcal{J}_\rho(\mathbf{z})d\mu_{\mathbf{b}}(\mathbf{z}) \\ &= \int_{\|\mathbf{z}\|_{1/\mathbf{b}}\le r} \prod_{j=1}^{n} J_0(\rho_j|z_j|)d\mu_{\mathbf{b}}(\mathbf{z}) \\ &= \int_{|\mathbf{z}|\le r} \prod_{j=1}^{n} b_j^2 J_0(\rho_j b_j|z_j|)d\mu_r(\mathbf{z}) \\ &= c\int_0^R \Big(\int_{|\mathbf{z}|=r} \prod_{j=1}^{n} a_j J_0(\rho_j\sqrt{a_j}|z_j|)d\sigma_R(\mathbf{z})\Big)R^{2n-1}dR\end{aligned}$$

To evaluate the inner integral, we may either think of this integral as a Fourier-Bessel transform and apply the known result from Euclidean space, or we may use the Laplace transform. After this evaluation, we continue

$$\begin{aligned}\widetilde{T}_{\mathbf{b},r}(\mathbf{0};\rho) &= c\int_0^r \frac{J_{n-1}(R\sqrt{a_1\rho_1^2+\cdots+a_n\rho_n^2})}{(R\sqrt{a_1\rho_1^2+\cdots+a_n\rho_n^2})^{n-1}}R^{2n-1}dR \\ &= c'\frac{J_n(r\sqrt{a_1\rho_1^2+\cdots+a_n\rho_n^2})}{(r\sqrt{a_1\rho_1^2+\cdots+a_n\rho_n^2})^n}.\end{aligned}$$

We then observe that condition 1. means that $\widetilde{T}_{\mathbf{b},r_1}(\lambda;\mathbf{k})$ and $\widetilde{T}_{\mathbf{b},r_2}(\lambda,\mathbf{k})$ are not both zero for any $(\lambda,\mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$. Likewise condition 2. is the condition that $\widetilde{T}_{\mathbf{b},r_1}(\mathbf{0},\rho)$ and $\widetilde{T}_{\mathbf{b},r_2}(\mathbf{0},\rho)$ are not both zero for any $\rho \in (\mathbf{R}_+)^n$. These are the conditions needed in the Tauberian theorem of [3, 17] to imply that the closed ideal generated by $\{\eta * T_{\mathbf{b},r_1}, \eta * T_{\mathbf{b},r_2} : \eta \in L^1_{\mathbf{0}}(\mathbf{H}^n)\}$ inside of $L^1_{\mathbf{0}}(\mathbf{H}^n)$ makes up all of $L^1_{\mathbf{0}}(\mathbf{H}^n)$. The integral conditions then tell us that $f * L^1_{\mathbf{0}}(\mathbf{H}^n) \equiv 0$, and it follows that $f \equiv 0$.

If the radii do not satisfy condition 1, then there exists $(\lambda, \mathbf{k})$ such that $\widetilde{T}_{\mathbf{b},r_1}(\lambda, \mathbf{k}) = 0$ and $\widetilde{T}_{\mathbf{b},r_2}(\lambda, \mathbf{k}) = 0$. This is equivalent to the statement that $\psi^\lambda_{\mathbf{k}}$ satisfies integral conditions (1) for radii r_1 and r_2. Likewise if the radii do not satisfy condition 2, there exists ρ such that $\widetilde{T}_{\mathbf{b},r_1}(\mathbf{0}, \rho) = 0$ and $\widetilde{T}_{\mathbf{b},r_2}(\mathbf{0}, \rho) = 0$. This is equivalent to the statement that $\mathcal{J}_\rho$ satisfies integral conditions (1) for radii r_1 and r_2. The proof of Theorem 5.1 is therefore complete. □

5.2. Spherical functions and Gelfand transform for $\mathbf{H}^n_{\mathbf{a}}$

To obtain the corresponding results for balls or complex ellipsoids in $\mathbf{H}^n_{\mathbf{a}}$ a Gelfand transform for the space $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$ will be required. We first observe that $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$ is also a commutative Banach algebra since the arguement used for $\mathbf{H}^n$ [1, 3], Lemma 3.1 carries over to this case. And therefore we know the Gelfand transform is defined on $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$. The same result [16], Theorem 3.3 Chapter 4, shows the characters are here also determined by $m(f) = \int_{\mathbf{H}^n_{\mathbf{a}}} f(\mathbf{g})\psi(\mathbf{g})dm(\mathbf{g})$, where ψ are the bounded $\mathbf{T}^n$-spherical functions on $\mathbf{H}^n_{\mathbf{a}}$. Recall the $\mathbf{T}^n$-spherical functions on $\mathbf{H}^n_{\mathbf{a}}$ are functions ψ on $\mathbf{H}^n_{\mathbf{a}}$ such that $\psi(0) = 1$, and which satisfy the functional equation

$$\int_{\mathbf{T}^n} \psi(\mathbf{x} \cdot \sigma \mathbf{y}) d\sigma = \psi(\mathbf{x})\psi(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbf{H}^n_{\mathbf{a}}. \tag{5}$$

In [1, 3, 9], the $\mathbf{T}^n$-spherical functions are determined by finding joint eigenfunctions of $\square_j = -Z_j \bar{Z}_j - \bar{Z}_j Z_j$ for $j = 1, \ldots, n$ and T. When we change from $\mathbf{H}^n$ to $\mathbf{H}^n_{\mathbf{a}}$, the restrictions of these squares of the vector fields to $L^1_{\mathbf{0}}$ changes from $\square_j = -\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - |z_j|^2 \frac{\partial^2}{\partial t^2}$ to $\square_{j,\mathbf{a}} = -\frac{\partial^2}{\partial z_j \partial \bar{z}_j} - a_j^2 |z_j|^2 \frac{\partial^2}{\partial t^2}$. We now claim that the bounded $\mathbf{T}^n$-spherical functions on $\mathbf{H}^n_{\mathbf{a}}$ are given by

$$\begin{aligned} \psi^\lambda_{\mathbf{k},\mathbf{a}} &= c e^{2\pi i \lambda t} e^{-2\pi|\lambda|\|\mathbf{z}\|^2} \prod_{j=1}^n a_j L^{(0)}_{k_j}(4\pi|\lambda| a_j |z_j|^2) \\ \mathcal{J}_{\rho,\mathbf{a}} &= c \prod_{j=1}^n a_j J_0(\rho_j \sqrt{a_j} |z_j|). \end{aligned}$$

The method used in [3, 9] can also be used here to demonstrate that these functions verify the functional equation (5). We now define the Gelfand transform for $f \in L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$.

$$\begin{aligned}\tilde{f}(\lambda;\mathbf{k}) &= \int_{\mathbf{H}^n_{\mathbf{a}}} f(\mathbf{g})\psi^{\lambda}_{\mathbf{k},\mathbf{a}}(\mathbf{g})dm(\mathbf{g}) \quad \text{for } (\lambda,\mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n,\\ \tilde{f}(\mathbf{0};\rho) &= \int_{\mathbf{H}^n_{\mathbf{a}}} f(\mathbf{g})\mathcal{J}_{\rho,\mathbf{a}}(\mathbf{g})dm(\mathbf{g}) \quad \text{for } \rho \in (\mathbf{R}_+)^n.\end{aligned}$$

Whereas, for $f \in L^1_{\mathbf{0}}(\mathbf{H}^n)$, the Gelfand transform $\tilde{f}$ is a function on the Heisenberg brush

$$\begin{aligned}&\bigcup_{\mathbf{k}\in(\mathbf{Z}_+)^n} \{(\tau, |\tau|(4k_1+2), \ldots, |\tau|(4k_n+2)) \in \mathbf{R}^{n+1} : \tau \in \mathbf{R}^*\}\\ &\cup \{(0, \rho_1^2, \ldots, \rho_n^2) \in \mathbf{R}^{n+1} : \rho \in (\mathbf{R}_+)^n\}\end{aligned}$$

for $f \in L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$, the Gelfand transform $\tilde{f}$ will be a function on the anisotropic Heisenberg brush, which we now describe.

$$\begin{aligned}&\Big(\bigcup_{\mathbf{k}\in(\mathbf{Z}_+)^n} \{(\tau, |\tau|a_1(4k_1+2), \ldots, |\tau|a_n(4k_n+2)) \in \mathbf{R}^{n+1} : \tau \in \mathbf{R}^*\}\Big)\\ &\cup \{(0, a_1\rho_1^2, \ldots, a_n\rho_n^2) \in \mathbf{R}^{n+1} : \rho \in (\mathbf{R}_+)^n\}.\end{aligned}$$

Note that the proof of Theorem 5.1 depended upon use of a Tauberian theorem for the Gelfand transform on $L^1_{\mathbf{0}}(\mathbf{H}^n)$. For this theorem we were able to refer to [14, 17]. We will require a similar Tauberian theorem when applying the Gelfand transform on $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$. Although such a theorem has not previously been given explicitly, we now describe briefly why such a theorem is also valid. The proof of the Tauberian theorem for the Gelfand transform on $L^1_{\mathbf{0}}(\mathbf{H}^n)$ follows from the construction of what are called "local identities", on the Heisenberg brush. The anisotropic Heisenberg brush is nearly identical to the Heisenberg brush, with only slight modification of the slopes of the rays. But the topology is identical. The method of construction of "local identities" in the case of $L^1_{\mathbf{0}}(\mathbf{H}^n)$ carries over directly to the anisotropic case $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$, as well. Thus we conclude we have a similar Tauberian theorem for the Gelfand transform on $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$.

Theorem 5.2. *Let $\mathcal{I}$ be a closed ideal of $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$ such that*

1. *For all $(\lambda,\mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$, there exists $f \in \mathcal{I}$ such that $\tilde{f}(\lambda,\mathbf{k}) \neq 0$.*
2. *For all $\rho \in (\mathbf{R}_+)^n$, there exists $f \in \mathcal{I}$ such that $\tilde{f}(\mathbf{0};\rho) \neq 0$.*

Then $\mathcal{I} = L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$.

We are now able to prove the following results.

Theorem 5.3. *Let* $\mathbf{a} \in \mathbf{R}^n$, *each* $a_j > 0$ *and* $0 < r_1 < r_2 \in \mathbf{R}$. *Let* $f \in L^\infty(\mathbf{H}^n_{\mathbf{a}}) \cap C(\mathbf{H}^n_{\mathbf{a}})$. *Assume that for* $i = 1, 2$, f *satisfies the integral conditions* (3), *that is*

$$\int_{B_{r_i}} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_{r_j}(\mathbf{z}) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^n_{\mathbf{a}}.$$

Assume further that r_1, r_2 *satisfy the conditions*

1. $(r_1/r_2)^2 \notin \mathcal{Q}(\Phi_{\mathbf{a},\mathbf{k}}(t))$, *for all* $\mathbf{k} \in (\mathbf{Z}_+)^n$,
2. $r_1/r_2 \notin \mathcal{Q}(J_n(t))$.

We may then conclude $f \equiv 0$. *In case* r_1, r_2 *do not satisfy conditions* 1 *and* 2, *then there exists* $f \not\equiv 0$ *satisfying the integral conditions.*

Theorem 5.4. *Let* $\mathbf{a}, \mathbf{d} \in \mathbf{R}^n$, *each* $a_j, d_j > 0$ *and* $0 < r_1 < r_2 \in \mathbf{R}$. *Let* $f \in L^\infty(\mathbf{H}^n_{\mathbf{a}}) \cap C(\mathbf{H}^n_{\mathbf{a}})$. *Assume that for* $i = 1, 2$, f *satisfies the integral conditions* (4), *that is*

$$\int_{E_{\mathbf{d},r_i}} L_{\mathbf{g}} f(\mathbf{z}, 0) d\mu_{\mathbf{d},r_j}(\mathbf{z}) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^n_{\mathbf{a}}.$$

Assume further that r_1, r_2 *satisfy the conditions*

(1) $(r_1/r_2)^2 \notin \mathcal{Q}(\Phi_{\mathbf{ad}^2,\mathbf{k}}(t))$, *for all* $\mathbf{k} \in (\mathbf{Z}_+)^n$,
(2) $r_1/r_2 \notin \mathcal{Q}(J_n(t))$.

We may then conclude $f \equiv 0$. *In case* r_1, r_2 *do not satisfy conditions* 1 *and* 2, *then there exists* $f \not\equiv 0$ *satisfying the integral conditions.*

We now make a brief explanation of Theorems 5.3 and 5.4. The proof is just like that of Theorem 5.1, but here we use the Gelfand transform for $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$. For Theorem 5.3, we find

$$\widetilde{T}_{r_i}(\lambda, \mathbf{k}) = \int_{\mathrm{B}_{r_i}} \psi^\lambda_{\mathbf{k},\mathbf{a}}(\mathbf{z}, 0) d\mu_{r_i}(\mathbf{z}), \quad \text{and} \quad \widetilde{T}_{r_i}(\mathbf{0}; \rho) = \int_{\mathrm{B}_{r_i}} \mathcal{J}_\rho(\mathbf{z}) d\mu_{r_i}(\mathbf{z}).$$

We have seen that

$$\widetilde{T}_{r_i}(\lambda,\mathbf{k})=\frac{1}{(4|\lambda|)^n}\Phi_{\mathbf{a},\mathbf{k}}(4\pi|\lambda|r_i^2),\ \text{and}\ \widetilde{T}_{r_i}(\mathbf{0};\rho)=c'\frac{J_n(r_i\sqrt{a_1\rho_1^2+\cdots+a_n\rho_n^2})}{(r_i\sqrt{a_1\rho_1^2+\cdots+a_n\rho_n^2})^n}.$$

Condition 1 of the Tauberian theorem is the same as the condition for no common zeros of $\widetilde{T}_{r_1}(\lambda,\mathbf{k})$ and $\widetilde{T}_{r_2}(\lambda,\mathbf{k})$. Likewise condition 2 is the same as the condition for no common zeros of $\tilde{T}_{r_1}(\mathbf{0};\rho)$ and $\tilde{T}_{r_2}(\mathbf{0};\rho)$. The Tauberian theorem then implies the results of Theorem 5.3. In Theorem 5.4 we apply the Gelfand transform for $L^1_{\mathbf{0}}(\mathbf{H}^n_{\mathbf{a}})$ instead to $T_{\mathbf{d},r_i}$. We consider

$$\widetilde{T}_{\mathbf{d},r_i}(\lambda,\mathbf{k}) = \int_{E_{\mathbf{d}},r_i}\psi^\lambda_{\mathbf{k},\mathbf{a}}(\mathbf{z},0)d\mu_{\mathbf{d},r_i}(\mathbf{z}),$$

and

$$\widetilde{T}_{\mathbf{d},r_i}(\mathbf{0};\rho) = \int_{E_{\mathbf{d}},r_i}\mathcal{J}_\rho(\mathbf{z})d\mu_{\mathbf{d},r_i}(\mathbf{z}).$$

These give

$$\widetilde{T}_{\mathbf{d},r_i}(\lambda,\mathbf{k}) = \frac{1}{(4|\lambda|)^n}\Phi_{\mathbf{a}\mathbf{d}^2,\mathbf{k}}(4\pi|\lambda|r_i^2),$$

and

$$\widetilde{T}_{\mathbf{d},r_i}(\mathbf{0};\rho) = c'\frac{J_n(r_i\sqrt{a_1d_1^2\rho_a^2+\cdots+a_nd_n^2\rho_n^2})}{(r_i\sqrt{a_1d_1^2\rho_1^2+\cdots+a_nd_n^2\rho_n^2})^n}.$$

Likewise, an application of the Tauberian theorem gives the results of Theorem 5.4. Note that in the case where $\mathbf{d}^2=\gamma/\mathbf{a}$, these integrals reduce to

$$\widetilde{T}_{\mathbf{d},r_i}(\lambda,\mathbf{k}) = \gamma^{-n}\int_{|\mathbf{z}|\le\gamma r_i}\psi^\lambda_{\mathbf{k}}(\mathbf{z},0)d\mu_{\gamma r_i}(\mathbf{z}),$$

and

$$\widetilde{T}_{\mathbf{d},r_i}(\mathbf{0};\rho) = c\gamma^{-n}\int_{|\mathbf{z}|\le\gamma r_i}\mathcal{J}_\rho d\mu_{\gamma r_i}(\mathbf{z}).$$

These integrals are then easily evaluated as in previous papers [3], leading to the same exceptional set as in [3], Theorem 6.2.

6. Issue of Rotations of Complex Ellipsoids

One of the early advances in the Pompeiu problem on Euclidean space was the paper of Brown, Schreiber and Taylor [10]. Using these methods the authors were able to classify a large class of regions which, when considered with their rotations, are shown to possess the Pompeiu property. These include regions which possess a corner. However the methods also show that an ellipse, which has a boundary that is real-analytic, together with its rotations, possesses the Pompeiu property. This is the result which we will generalize to $\mathbf{H}^n$ in this section. In particular, we will demonstrate that a solid complex ellipsoid, $E_{\mathbf{b}} = \{\mathbf{z} \in \mathbf{C}^n : |z_1/b_1|^2 + \cdots + |z_n/b_n|^2 \leq 1\}$, together with its rotations $\mathrm{U}E_{\mathbf{b}}$ for $\mathrm{U} \in \mathrm{U}(n)$, possesses the Pompeiu property. This is to say, for $f \in L^\infty(\mathbf{H}^n)$, the integral conditions

$$\int_{\mathrm{U}\cdot E_{\mathbf{b}}} L_{\mathbf{g}} f(\mathbf{z},0)\,d\mu_{\mathbf{b}}(\mathbf{z}) = 0 \quad \text{for all } \mathbf{g} \in \mathbf{H}^n \text{ and all } \mathrm{U} \in \mathrm{U}(n)$$

will imply that $f \equiv 0$. This improves upon the results in Section 4 by demonstrating that we may reduce to complex ellipsoid by including its rotations. A closer examination of the Euclidean case, treated in [10], shows that the same is also true in that case (if we restrict to $f \in L^\infty$). The analysis is based on avoiding common zeros of

$$\mathcal{F}\chi_E = c j_{n/2}(|\sqrt{\mathbf{a}} \cdot \xi|) = c \frac{J_{n/2}(\sqrt{a_1\xi_1^2 + \cdots + a_n\xi_n^2})}{(\sqrt{a_1\xi_1^2 + \cdots + a_n\xi_n^2})^{n/2}}$$

and its rotations $\mathcal{F}\chi_{\mathrm{U}\cdot E} = j_{n/2}(\sqrt{\mathbf{a}} \cdot \mathrm{U}\xi)$. Since $j_{n/2}$ is real-analytic, its zeros must be isolated. Assume the ellipse E is not a circle, and we may write $a_1 \geq \cdots \geq a_n$ and $a_1 > a_n$. Consider any $\xi \in \mathbf{R}^n$. If $\xi = \mathbf{0}$, then $\mathcal{F}\chi_{\mathrm{U}\cdot E}(\mathbf{0}) = j_{n/2}(0) \neq 0$ for all U. Otherwise, say $|\xi| = \rho$. Then, exist rotations $\mathrm{U} \in \mathrm{U}(n)$ such that $|\sqrt{\mathbf{a}} \cdot \mathrm{U}\xi| = \sqrt{a_1\xi_1^2 + \cdots + a_n\xi_n^2}$ takes on all values between $\sqrt{a_n}\rho$ and $\sqrt{a_1}\rho$. Because of the isolated zeros, then there exist (many) $\mathrm{U} \in \mathrm{U}(n)$ such that $\mathcal{F}\chi_{\mathrm{U}\cdot E}(\xi) = j_{n/2}(\sqrt{\mathbf{a}} \cdot \mathrm{U}\xi) \neq 0$. With the Wiener Tauberian theorem, this is enough to give the Pompeiu property in the Euclidean case $\mathbf{R}^n$. We expand on this idea in moving to $\mathbf{H}^n$.

We now show that in $L^\infty(\mathbf{H}^n)$ a complex ellipsoid $E_{\mathbf{b}}$, considered together with its rotations, will possess the Pompeiu property.

Theorem 6.1. *Let $\mathbf{b} \in (\mathbf{R}_+)^n$ such that $b_1 \geq \cdots \geq b_n$ and $b_1 > b_n$. Consider $f \in L^\infty(\mathbf{H}^n)$ satisfying the integral conditions*

$$\int_{\mathrm{U}\cdot E_{\mathbf{b}}} L_{\mathbf{g}} f(\mathbf{z},0)\,d\mu_{\mathbf{b}}(\mathbf{z}) = 0 \quad \textit{for all } \mathbf{g} \in \mathbf{H}^n \textit{ and } \mathrm{U} \in \mathrm{U}(n). \tag{6}$$

Then $f \equiv 0$.

Proof. Let $T_{\mathrm{U},\mathbf{b}}$ be the distribution given by $\langle \phi, T_{\mathrm{U},\mathbf{b}} \rangle = \int_{\mathrm{U}E_{\mathbf{b}}} f(\mathbf{z}, 0)\, d\mu_{\mathbf{b}}(\mathbf{z})$. We apply the Gelfand transform on $L^1_{\mathbf{0}}(\mathbf{H}^n)$ of Section 5 to each of the sets $\{T_{\mathrm{U},\mathbf{b}} : \mathrm{U} \in \mathrm{U}(n)\}$. In particular, we are concerned with the closed ideal $\mathcal{I}$ generated by the set $\{T_{\mathrm{U},\mathbf{b}} * g : \mathrm{U} \in \mathrm{U}(n), g \in L^1_{\mathbf{0}}(\mathbf{H}^n)\}$, and we know from the integral conditions that $f * \mathcal{I} \equiv 0$. Then apply the Tauberian theorem to show $\mathcal{I} = L^1_{\mathbf{0}}(\mathbf{H}^n)$. Since we know $f * \mathcal{I} \equiv 0$, it will then follow that $f \equiv 0$. Thus it is sufficient to verify the conditions of the Tauberian theorem in order to prove this theorem. These conditions amount to showing that the generators of the closed ideal $\mathcal{I}$ do not have a common zero among any of the $(\lambda, \mathbf{k})$ or ρ of the spectrum of the Gelfand transform. For each $(\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$ we find a generator $T_{\mathrm{U},\mathbf{b}} * g$ of the closed ideal $\mathcal{I}$ such that $\widetilde{T}_{\mathrm{U},\mathbf{b}}(\lambda, \mathbf{k}) \cdot \tilde{g}(\lambda, \mathbf{k}) \neq 0$. Likewise, for each ρ we find a generator $T_{\mathrm{U},\mathbf{b}} * g$ of the closed ideal $\mathcal{I}$ such that $\widetilde{T}_{\mathrm{U},\mathbf{b}}(\mathbf{0}, \rho) \cdot \tilde{g}(\mathbf{0}, \rho) \neq 0$. However, we note that the Gelfand transform for the Bessel part of the spectrum reduces to the Euclidean Fourier transform:

$$\widetilde{T}_{\mathrm{U},\mathbf{b}}(0; \rho) = \int_{\mathrm{U}E_{\mathbf{b}}} \mathcal{J}_\rho(\mathbf{z}) dm(\mathbf{z}) = c \frac{J_n(\mathbf{b} \cdot \mathrm{U}\rho)}{(\mathbf{b} \cdot \mathrm{U}\rho)^n}.$$

And we have already described in the beginning of the section why there cannot be a common zero ρ for all such $\widetilde{T}_{\mathrm{U},\mathbf{b}}(\mathbf{0}, \rho)$ for all $\mathrm{U} \in \mathrm{U}(n)$. So it only remains to show there is no common zero $(\lambda, \mathbf{k})$ for all such $\widetilde{T}_{\mathrm{U},\mathbf{b}}(\lambda, \mathbf{k})$ for all $\mathrm{U} \in \mathrm{U}(n)$. It turns out that for this purpose it is sufficient to consider the rotations in one specific direction, a subset of all the rotations. Thus we consider the distributions $T_{\phi,\mathbf{b}}$ for $\phi \in [0, 2\pi)$, where ϕ represents the rotation $\phi = \begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix}$ of $\mathbf{C}^2 \subset \mathbf{C}^n$, where writing (z_1, z_2) as (x_1, x_2, y_1, y_2). Without loss of generality, we assume $b_1 > b_2$. It is certainly true that $b_1 > b_n$, as the ellipsoid is not spherical, and we can interchange the indices for z_2 and z_n without affecting the proof. For each fixed $(\lambda, \mathbf{k}) \in \mathbf{R}^* \times (\mathbf{Z}_+)^n$, we will consider the values of $\widetilde{T}_{\phi,\mathbf{b}}(\lambda, \mathbf{k}) = h_{\lambda,\mathbf{k}}(\phi)$ as a function of this angle ϕ. It will be sufficient for us to show that $h_{\lambda,\mathbf{k}}(\phi)$ is real-analytic in the variable ϕ. As a consequence, it will follow that $h_{\lambda,\mathbf{k}}(\phi)$ has isolated zeros for $\phi \in [0, 2\pi)$. This will be enough for us to conclude there exists ϕ such that $\widetilde{T}_{\phi,\mathbf{b}}(\lambda, \mathbf{k}) \neq 0$, since we know $\widetilde{T}_{\phi,\mathbf{b}} \not\equiv 0$ as a function of ϕ or of λ. Since this is valid for each $(\lambda, \mathbf{k})$, it is enough for us to apply the Tauberian theorem and reach the conclusions of the theorem. Thus it only remains to prove the real-analyticity of $h_{\lambda,\mathbf{k}}(\phi)$ claimed above. This is the content of the following lemma. □

Lemma 6.2. *Let ω_ϕ represent the rotation*

$$\begin{pmatrix} \cos\phi & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi & 0 & 0 \\ 0 & 0 & \cos\phi & -\sin\phi \\ 0 & 0 & \sin\phi & \cos\phi \end{pmatrix},$$

for $\phi \in [0, 2\pi)$, and $T_{\phi,\mathbf{b}}$ be the distribution associated to the Radon measure on $\omega_\phi E_{\mathbf{b}}$. Consider $f(\phi) = \widetilde{T}_{\phi,\mathbf{b}}(\lambda, \mathbf{k})$, the Gelfand transform of $T_{\phi,\mathbf{b}}$ as a function of ϕ. We claim that $f(\phi)$ is real-analytic in the variable ϕ.

Proof. We prove this result by use of power series to verify $\widetilde{T}_{\phi,\mathbf{b}}(\lambda, \mathbf{k})$ is real-analytic in the variable ϕ.

$$\begin{aligned} \widetilde{T}_{\phi,\mathbf{b}}(\lambda;\mathbf{k}) &= \int_{E_{\phi,\mathbf{b}}} \psi^\lambda_{\mathbf{k}}(\mathbf{z},0)d\mu_{\mathbf{b}}(\mathbf{z}) \\ &= \int_{\|\omega_\phi \mathbf{z}\|_{\mathbf{b}}\le 1} e^{-2\pi|\lambda||\mathbf{z}|^2} \prod_{j=1}^n L^{(0)}_{k_j}(4\pi|\lambda||z_j|^2)d\mu_{\mathbf{b}}(\mathbf{z}) \\ &= \int_{\|\mathbf{z}\|_{\mathbf{b}}\le 1} e^{-2\pi|\lambda||\mathbf{z}|^2}\mathcal{W}^\lambda_{\mathbf{l}}(\omega_{-\phi}\mathbf{z})d\mu_{\mathbf{b}}(\mathbf{z}) \\ &= c\int_0^{2\pi}\cdots\int_0^{2\pi}\Big(\int_{\sum r_j^2\le 4\pi|\lambda|} e^{-\sum b_j^2r_j^2/2}L^{(0)}_{k_1}(u_1(r,\theta))L^{(0)}_{k_2}(u_2(r,\theta)) \\ &\qquad\qquad \times\prod_{j=3}^n L^{(0)}_{k_j}(b_j^2r_j^2)r_1dr_1\ldots r_ndr_n\Big)d\theta_1\ldots d\theta_n \end{aligned}$$

where

$$u_1(b_1,r_1,r_2,\theta_1,\theta_2,\phi)=b_1^2\big(r_1^2\cos^2\phi+r_2^2\sin^2\phi+2r_1r_2\sin\phi\cos\phi\cos(\theta_1-\theta_2)\big)$$

and

$$u_2(b_2,r_1,r_2,\theta_1,\theta_2,\phi)=b_2^2\big(r_1^2\sin^2\phi+r_2^2\cos^2\phi-2r_1r_2\sin\phi\cos\phi\cos(\theta_1-\theta_2)\big).$$

To reduce, we integrate in the variables $\theta_1, \ldots, \theta_n$. Also make the change of variables $r_j^2 = x_j$. Thus,

$$\begin{aligned}
&\widetilde{T}_{\phi,\mathbf{b}}(\lambda;\mathbf{k}) \\
&= c\frac{(\pi)^{n-2}}{4}\int_{\sum x_j \leq 4\pi|\lambda|}\Big(\int_0^{2\pi}\int_0^{2\pi} L_{k_1}^{(0)}(u_1(\theta_1,\theta_2))L_{k_2}^{(0)}(u_2(\theta_1,\theta_2))d\theta_1 d\theta_2\Big) \\
&\quad \times e^{-\frac{1}{2}\sum a_j x_j}\prod_{j=3}^{n} L_{k_j}^{(0)}(a_j x_j)dx_n \ldots dx_1.
\end{aligned}$$

Let us first consider the case when $n = 2$, where this integral reduces to

$$\widetilde{T}_{\phi,\mathbf{b}}(\lambda,\mathbf{k}) = \frac{c}{4}\int_{x_1+x_2\leq 4\pi|\lambda|} e^{-\frac{1}{2}(a_1x_1+a_2x_2)}\Big(\int_0^{2\pi}\int_0^{2\pi} L_{k_1}^{(0)}(u_1)L_{k_2}^{(0)}(u_2)d\theta_1 d\theta_2\Big) dx_2 dx_1,$$

and later move to the case of general n. Complete integration in the θ variables by evaluating the inner integral

$$\int_0^{2\pi}\int_0^{2\pi} L_{k_1}^{(0)}(u_1)L_{k_2}^{(0)}(u_2)d\theta_1 d\theta_2,$$

where u_1 and u_2 are as above. After further simplification to isolate the variables of integration, this inner integral now becomes

$$\sum_{j=0}^{k_1+k_2} P_{k_1+k_2-j}(x_1\cos^2\phi, x_2\sin^2\phi, x_1\sin^2\phi, x_2\cos^2\phi)\Big(\sqrt{x_1x_2}\sin\phi\cos\phi\Big)^j \int_0^{2\pi}\int_0^{2\pi}\cos^j(\theta_1-\theta_2)d\theta_1 d\theta_2,$$

where $P_{k_1+k_2-j}$ is a polynomial of degree k_1+k_2-j. We let $K_j = \int_0^{2\pi}\int_0^{2\pi}\cos^j(\theta_1-\theta_2)d\theta_1 d\theta_2$, observing that $K_j = 0$ for j odd and $K_j \leq 2\pi^2$. Thus in the even cases, when the integral does not vanish, we are left with

$$\frac{c}{4}\sum_{j=0}^{[(k_1+k_2)/2]} K_{2j}\int_{x_1+x_2\leq 4\pi|\lambda|} e^{-\frac{a_1x_1+a_2x_2}{2}} P_{k_1+k_2-2j} (x_1\cos^2\phi, x_2\sin^2\phi, x_1\sin^2\phi, x_2\cos^2\phi)\Big(x_1x_2\sin^2\phi\cos^2\phi\Big)^j dx_2 dx_1.$$

After further reduction, we observe we are left with a finite sum of terms of the form

$$\frac{c}{4}\alpha \int_{x_1+x_2=4\pi|\lambda|} e^{-\frac{a_1x_1+a_2x_2}{2}} x_1^{\alpha}(\cos^2\phi)^{\alpha-r}(\sin^2\phi)^r x_2^{\beta}(\cos^2\phi)^{\beta-s}(\sin^2\phi)^s dx_2dx_1$$
$$= \frac{c}{4}\gamma\Big(\int_0^{4\pi|\lambda|} e^{-\frac{a_1x_1}{2}} x_1^{\alpha}\int_0^{4\pi|\lambda|-x_1} e^{-\frac{a_2x_2}{2}} x_2^{\beta} dx_2dx_1\Big)\Big(\cos^2\phi\Big)^{\alpha+\beta-(r+s)}\Big(\sin^2\phi\Big)^{r+s}.$$

Thus the integral of $\widetilde{T}_{\phi,\mathbf{b}}(\lambda,\mathbf{k})$ has been reduced to

$$\sum_{0\le j\le k\le k_1+k_2}\Big[\sum_{\alpha+\beta=k}\frac{c}{4}\gamma\int_0^{4\pi|\lambda|} e^{-\frac{a_1x_1}{2}} x_1^{\alpha}\int_0^{4\pi|\lambda|-x_1} e^{-\frac{a_2x_2}{2}} x_2^{\beta} dx_2dx_1\Big](\cos^2\phi)^{k-j}(\sin^2\phi)^j$$

where the second sum also requires $\alpha,\beta\ge 0$. We will next evaluate the double integrals in the inner sum. Focusing first on the integral in x_2, we have

$$\int_0^{4\pi|\lambda|-x_1} e^{-\frac{a_2x_2}{2}} x_2^{\beta}dx_2$$
$$= \Big(\frac{2}{a_2}\Big)^{\beta+1}\int_0^{a_2(4\pi|\lambda|-x_1)/2} e^{-x_2}x_2^{\beta}dx_2$$
$$= \Big(\frac{2}{a_2}\Big)^{\beta+1}\Big[\Big(1-e^{-a_2(4\pi|\lambda|-x_1)/2}\Big) - e^{-a_2(4\pi|\lambda|-x_1)/2}\Big(\sum_{n=0}^{\beta}\frac{\beta!}{n!}\Big(\frac{a_2}{2}(4\pi|\lambda|-x_1)^n\Big)^n\Big)\Big].$$

Now the larger integral becomes

$$\Big(\frac{2}{a_2}\Big)^{\beta+1}\frac{c}{4}\int_0^{4\pi|\lambda|} e^{-a_1x_1/2}x_1^{\alpha}\Big[\Big(1-e^{-a_2(4\pi|\lambda|-x_1)/2}\Big) - e^{-a_2(4\pi|\lambda|-x_1)/2}\Big(\sum_{n=0}^{\infty}\frac{\beta!}{n!}\Big(\frac{a_2}{2}(4\pi|\lambda|-x_1)\Big)^n\Big)\Big]dx_1.$$

This integral may then be broken up as a finite sum of $(\beta+1)$ integrals of the following form, for $0 \le t \le \beta$,

$$\Big(\frac{a_2}{2}4\pi|\lambda|\Big)^{\beta-t} c_{\beta,t}\int_0^{4\pi|\lambda|} e^{-a_1x_1/2}e^{-a_2(4\pi|\lambda|-x_1)/2}x_1^{\alpha+t}dx_1,$$

plus one additional integral of the form

$$c_\beta\int_0^{4\pi|\lambda|} e^{-a_1x_1/2}\left(e^{-a_2(4\pi|\lambda|-x_1)/2}\right)x_1^{\alpha}dx_1.$$

These first $(\beta+1)$ integrals evaluate as follows:

$$\begin{aligned}
&\int_0^{4\pi|\lambda|} e^{-a_1x_1/2}e^{-a_2(4\pi|\lambda|-x_1)/2}x_1^{\alpha+t}dx_1\\
&= e^{-a_2(4\pi|\lambda|)/2}\int_0^{4\pi|\lambda|} e^{-(a_1-a_2)x_1/2}x_1^{\alpha+t}dx_1\\
&= e^{-2\pi|\lambda|a_2}\left(\frac{2}{a_1-a_2}\right)^{\alpha+t+1}\int_0^{2\pi|\lambda|(a_1-a_2)} e^{-x_1}x_1^{\alpha+t}dx_1\\
&= e^{-2\pi|\lambda|a_2}\Big(\frac{2}{a_1-a_2}\Big)^{\alpha+t+1}\Big[\Big(1-e^{-2\pi|\lambda|(a_1-a_2)}\Big)\\
&\qquad -e^{-2\pi|\lambda|(a_1-a_2)}\Big(\sum_{n=0}^{\alpha+t}\frac{(\alpha+t)!}{n!}\Big(2\pi|\lambda|(a_1-a_2)\Big)^n\Big)\Big].
\end{aligned}$$

The additional term also evaluates to give

$$\begin{aligned}
&\int_0^{4\pi|\lambda|} e^{-a_1x_1/2}\left(1-e^{-a_2(4\pi|\lambda|-x_1)/2}\right)x_1^{\alpha}dx_1\\
&= \int_0^{4\pi|\lambda|} e^{-a_1x_1/2}x_1^{\alpha}dx_1 - e^{-2\pi|\lambda|a_2}\int_0^{4\pi|\lambda|} e^{-a_1x_2/2}e^{-(a_1-a_2)x_1/2}x_1^{\alpha}dx_1\\
&= \left(\frac{2}{a_1}\right)^{\alpha+1}\int_0^{2\pi|\lambda|a_1} e^{-x_1}x_1^{\alpha}dx_1\\
&\qquad -\left(\frac{2}{a_1-a_2}\right)^{\alpha+1} e^{-2\pi|\lambda|a_2}\int_0^{2\pi|\lambda|(a_1-a_2)} e^{-x_1}x_1^{\alpha}dx_1\\
&= \left(\frac{2}{a_1}\right)^{\alpha+1}\Big[(1-e^{2\pi|\lambda|a_1}) - e^{-2\pi|\lambda|a_1}\sum_{n=0}^{\alpha}\frac{\alpha!}{n!}\left(2\pi|\lambda|a_1\right)^n\Big]\\
&\quad -\left(\frac{2}{a_1-a_2}\right)^{\alpha+1} e^{-2\pi|\lambda|a_2}\Big[(1-e^{-2\pi|\lambda|(a_1-a_2)})
\end{aligned}$$

$$-e^{-2\pi|\lambda|(a_1-a_2)}\sum_{n=0}^{\alpha}\frac{\alpha!}{n!}\,(2\pi|\lambda|(a_1-a_2))^n\Big].$$

Thus each of the integrals of the form $\frac{c}{4}\int_0^{4\pi|\lambda|}e^{-a_1x_1/2}x_1^{\alpha}(\int_0^{4\pi|\lambda|-x_1}e^{-a_2x_2/2}x_2^{\beta}dx_2)dx_1$, which was the coefficient of the term $[\cos^2\phi]^{\alpha+\beta-(r+2)}[\sin^2\phi]^{r+s}$, may be written as

$$\begin{aligned}&\Big(\frac{2}{a_1}\Big)^{\beta+1}\frac{c}{4}\gamma\sum_{m=0}^{\beta}(2\pi|\lambda|a_2)^{\beta-m}\,c_{\beta,m}\Big(\frac{2}{a_1-a_2}\Big)^{\alpha+m+1}\\&\quad\times\Big[(1-e^{-2\pi|\lambda|(a_1-a_2)})-e^{-2\pi|\lambda|(a_1-a_2)}\sum_{j=0}^{\alpha+m}\frac{(\alpha+m)!}{j!}\,(2\pi|\lambda|(a_1-a_2))^m\Big]\\&\quad+\Big(\frac{2}{a_1}\Big)^{\alpha+1}\Big[(1-e^{-2\pi|\lambda|a_1})-e^{-2\pi|\lambda|a_1}\sum_{j=0}^{\alpha}\frac{\alpha!}{j!}(2\pi|\lambda|a_1)^j\Big]\\&\quad-\Big(\frac{2}{a_1-a_2}\Big)^{\alpha+1}e^{-2\pi|\lambda|a_2}\Big[(1-e^{-2\pi|\lambda|(a_1-a_2)})\\&\qquad\qquad-e^{-2\pi|\lambda|(a_1-a_2)}\sum_{j=0}^{\alpha}\frac{\alpha!}{j!}\,(2\pi|\lambda|(a_2-a_1))^j\Big].\end{aligned}$$

We notice that this is an exponential polynomial, which will be denoted by $EP_{\alpha,\beta}(\mathbf{k},|\lambda|,\mathbf{a})$.

Thus as a function of ϕ, we have determined that

$$\widetilde{T}_{\phi,\mathbf{b}}(\lambda,\mathbf{k})=h_{\lambda,\mathbf{k}}(\phi)=\sum_{0\le j\le k\le k_1+k_2}\Big[\sum_{\alpha+\beta=k}EP_{\alpha,\beta}(\mathbf{k},|\lambda|,\mathbf{a})\Big](\cos^2\phi)^{k-j}(\sin^2\phi)^j.$$

As a function of ϕ, we have a finite sum of functions which are powers of $\cos^2\phi$ and $\sin^2\phi$, and each of the coefficients are exponential polynomials in $\mathbf{a},\lambda$. Thus each of these terms in the finite sum is real-analytic in ϕ, and $h_{\lambda,\mathbf{k}}(\phi)$ is real-analytic in ϕ. This completes the case $n=2$.

The general case is not considerably more difficult because the dependence on ϕ is confined to two variables. In the general case, we need to evaluate

$$\int_0^{4\pi|\lambda|}e^{-\frac{a_1x_1}{2}}L_{k_1}^{(0)}(u_1)\int_0^{4\pi|\lambda|-x_1}e^{-\frac{a_2x_2}{2}}L_{k_2}^{(0)}(u_2)$$

$$\times\Big(\int_0^{4\pi|\lambda|-x_1-x_2} e^{-\frac{a_3x_3}{2}} L_{k_3}^{(0)}(a_3x_3)\dots \int_0^{4\pi|\lambda|-x_1-\cdots-x_{n-1}} e^{-\frac{a_nx_n}{2}} L_{k_n}^{(0)}(a_nx_n)dx_n\dots dx_3\Big)dx_2dx_1. \quad (7)$$

We will first observe why the integrals in $x_3,\dots,x_n$ inside the parentheses yield an exponential polynomial in $4\pi|\lambda|, x_1, x_2$ of a certain form. Then we will observe why the total integral yields the same form of result as in the case $n=2$ above. We begin with the innermost integral

$$\int_0^{4\pi|\lambda|-x_1-\cdots-x_{n-1}} e^{-\frac{a_nx_n}{2}} L_{k_n}^{(0)}(a_nx_n)dx_n$$
$$= \frac{2}{a_n}\sum_{j=0}^{k_n} c_{k_n,j}2^j j!\Big[\big(1-e^{-\frac{a_nl_u}{2}}\big)-e^{-\frac{a_nl_u}{2}}\sum_{i=0}^{j}\frac{j!}{i!}\big(\frac{a_nl_u}{2}\big)^i\Big],$$

where $c_{k_n,j}$ are the coefficients of the polynomial $L_{k_n}^{(0)}$ and $l_u = 4\pi|\lambda|-x_1-\cdots-x_{n-1}$ is the upper limit of integration. Clearly, this is an exponential polynomial in $l_u = 4\pi|\lambda|-x_1-\cdots-x_{n-1}$, which can be written as an exponential polynomial in x_{n-1} with coefficients which are exponential polynomials in $4\pi|\lambda|-x_1-\cdots-x_{n-2}$. We also want to describe the form of the exponential polynomials which arise from this integral. In particular, the above exponential polynomial is of the form

$$\sum_{\text{fin}} EP(4\pi|\lambda|-x_1-\cdots-x_{n-2})e^{\phi_{n-1}(\mathbf{a})x_{n-1}}x_{n-1}^{r_{n-1}},$$

where $\sum_{\text{fin}}$ means a finite number of terms and EP is an exponential polynomial that is also a finite sum of terms of the form $e^{\phi_1(\mathbf{a})}\cdots e^{\phi_{n-2}(\mathbf{a})}(4\pi|\lambda|-x_1-\cdots-x_{n-2})^r$. The ϕ_j for $j=1,\dots,n$ are polynomials in $\mathbf{a}$. The integral $\int_0^{4\pi|\lambda|-x_1-\cdots-x_{n-1}} e^{-a_nx_n/2}L_{k_n}^{(0)}(a_nx_n)dx_n$ may then also be expressed as a finite sum of terms of the form $e^{\phi_1(\mathbf{a})x_1}\cdots e^{\phi_{n-1}(\mathbf{a})x_{n-1}}(4\pi|\lambda|)^s x_1^{r_1}\cdots x_{n-1}^{r_{n-1}}$. We now observe that in each of the $n-2$ integrations required to evaluate inside the parentheses in (7), the result is this same form of exponential polynomial. For example, the next integral

$$\int_0^{4\pi|\lambda|-x_1-\cdots-x_{n-2}} e^{-\frac{a_{n-1}x_{n-1}}{2}} L_{k_{n-1}}^{(0)}(a_{n-1}x_{n-1}) \times\Big(\sum_{\text{fin}} e^{\phi_1(\mathbf{a})x_1}\cdots e^{\phi_{n-1}(\mathbf{a})x_{n-1}}x_1^{r_1}\cdots x_{n-1}^{r_{n-1}}(4\pi|\lambda|)^s\Big)dx_{n-1}$$

is rewritten as

$$\sum_{\text{fin}}(4\pi|\lambda|)^s e^{\phi_1(\mathbf{a})x_1}\cdots e^{\phi_{n-2}(\mathbf{a})x_{n-2}}x_1^{r_1}\cdots x_{n-2}^{r_{n-2}}$$
$$\times\int_0^{4\pi|\lambda|-x_1-\cdots-x_{n-2}} e^{-\frac{a_{n-1}x_{n-1}}{2}}x_{n-1}^{r_{n-1}}L_{k_{n-1}}^{(0)}(a_{n-1}x_{n-1})dx_{n-1},$$

which again evaluates to an exponential polynomial, of the above form, in variables $x_1,\ldots,x_{n-2}$. However the ϕ_j are now polynomials in $\mathbf{a}$. Thus we conclude that in (7), when evaluating the integrals in $x_3,\ldots,x_n$ inside the parentheses, the result is of the form

$$\sum_{\text{fin}} ce^{\phi_1(\mathbf{a})x_1}e^{\phi_2(\mathbf{a})x_2}x_1^{r_1}x_2^{r_2}(4\pi|\lambda|)^s$$

Thus (7) becomes a finite sum of integrals, expressed as

$$c(4\pi|\lambda|)^s\sum_{\text{fin}}\int_0^{4\pi|\lambda|} e^{-\frac{a_1x_1}{2}}e^{\phi_1(\mathbf{a})x_1}x_1^{r_1}L_{k_1}^{(0)}(u_1)$$
$$\times\Big(\int_0^{4\pi|\lambda|-x_1} e^{-\frac{a_2x_2}{2}}e^{\phi_2(\mathbf{a})x_2}x_2^{r_2}L_{k_2}^{(0)}(u_2)dx_2\Big)dx_1,$$

where u_1 and u_2 are as above. The arguement for the case $n=2$ then applies to this case, where a_1 becomes $\phi_1(\mathbf{a})-a_1/2$, a_2 becomes $\phi_2(\mathbf{a})-a_2/2$, and the degree of the terms in x_1 and x_2 are shifted upward by r_1 and r_2, respectively. Nevertheless, the result is of the same form as in the case of $n=2$. Thus the integral (7) evaluates to an expression of the form

$$\sum_{\text{fin}}\sum_{0\le j\le k\le k_1+k_2}\Big[\sum_{\alpha+\beta=k} EP_{\alpha,\beta}(\mathbf{k},|\lambda|,\mathbf{a})\Big](\cos^2\phi)^{k-j}(\sin^2\phi)^j.$$

In particular, as observed above in the case of $n=2$, it must be real-analytic in the variable ϕ. This completes the proof of Lemma 6.2, thus also completing the proof of Theorem 6.1. □

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