Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **2** (2007), No. 3, pp. 713-729

THE FRAILTY AND THE ARCHIMEDEAN STRUCTURE OF THE GENERAL MULTIVARIATE PARETO DISTRIBUTIONS

BY

HSIAW-CHAN YEH

Abstract

The mixture property of the general multivariate Pareto $\mathrm{MP}^{(k)}$ distributions has been studied by Yeh (2004a). Arnold (1996) mentioned that any mixing distribution with support $(0, \infty)$ is a candidate for a frailty model. This fact drives Yeh to study the frailty structure of the $\mathrm{MP}^{(k)}$ distributions. It is discerned that the $\mathrm{MP}^{(k)}$ distributions is in the one-parameter k-variate Clayton family with k-variate Archimedean survival copulas and thus the $\mathrm{MP}^{(k)}$ can be treated as a marginally specified multivariate distribution. Several properties of the k-variate survival copulas and the limiting special cases of the $\mathrm{MP}^{(k)}$ Archimedean survival copulas are studied in this paper.

1. Introduction and Motivation

The importance of the four general multivariate Pareto distributions (denoted as $\mathrm{MP}^{(k)}(\mathrm{I})$, $\mathrm{MP}^{(k)}(\mathrm{II})$, $\mathrm{MP}^{(k)}(\mathrm{III})$, and $\mathrm{MP}^{(k)}(\mathrm{IV})$) proposed by Arnold (1983) are discussed in many literatures such as Kotz et al. (2000, Chapter 52 and the references therein) and Yeh (1994, 2000, 2004a, b). These four general multivariate Pareto distributions are candidate models for the multivariate continuous income data and some other socio-economic multivariate variables. Yeh (2004a) has also studied several distributional

Received September 28, 2005 and in revised form August 2, 2006.

Key words and phrases: General multivariate Pareto distributions, $\mathrm{MP}^{(k)}$, mixture, Laplace transform, frailty model, Archimedean generator, k-variate survival copulas, Clayton family, Kendall's τ.

properties and characterizations of them. It is found that the four general $\mathrm{MP}^{(k)}$(I), (II), (III), and (IV) distributions have many mixture properties, they are mixed either by geometric, Weibull, or exponential variables. Arnold (1996) mentioned that any mixing distributions with support $(0, \infty)$ may be used to construct a frailty model and this result can be extended to k-variate ($k \geq 2$) case by assuming the exis-tence of a common stress on conditionally independent components. This fact drives Yeh to study the frailty structure of the four $\mathrm{MP}^{(k)}$ distributions. Owing to the hierarchy of the four $\mathrm{MP}^{(k)}$ distributions, it suffices to study the $\mathrm{MP}^{(k)}$(IV) distribution.

The original reference of frailty model is Oakes (1989), Oakes claimed that any bivariate distributions generated by frailty models are subclasses of the Archimedean distributions studied by Genest and Mackay (1986). Most frailty models and copulas developed in the literature such as Genest and Mackay (1986), Genest and Rivest (1993), and the two books by Nelson (1998) and Joe (1997) are emphasized on the bivariate case. Moreover, their results deal mostly with the joint distribution function instead of the joint survival function and the k-variate survival copulas are only briefly mentioned in their papers.

As we know the recent research has focused on the subclass of the Archimedean copula (AC) class. It is fortunate that the general multivariate Pareto, $\mathrm{MP}^{(k)}$(IV) is in the one-parameter k-variate Clayton family with Archimedean k-variate survival copulas $\{\hat{C}_\alpha(\cdot)|\alpha \in \mathfrak{R}^+\}$.

In this article, the frailty structure and the Archimedean distributional property of the general $\mathrm{MP}^{(k)}$(IV) distribution are studied in Section 2. Several properties between the $\mathrm{MP}^{(k)}$(IV) and its corresponding survival copulas are proved in Section 3. It will be discerned that the Archimedean generator of the $\mathrm{MP}^{(k)}$(IV) distribution is the Laplace transform of a Gamma $(\alpha, 1)$ variable, therefore, the corresponding k-variate Archimedean survival copulas can be used to generate one-parameter family of the general $\mathrm{MP}^{(k)}$ distributions with specified univariate Pareto P(IV) (not necessarily identically distributed) as marginals. This one-parameter is just the shape parameter α in the $\mathrm{MP}^{(k)}(\mathrm{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ distribution which plays an important role in the pairwise association measure, such as Kendall's τ between any pairs of the random vector of $\mathrm{MP}^{(k)}$ distribution. Finally, the limiting and special cases of $\{\hat{C}_\alpha(\cdot)|\alpha \in \mathfrak{R}^+\}$ and their corresponding generators are studied in Section 4.

2. The Frailty Structure of the MP$^{(k)}$ Distributions

Let $\underline{X} = (X_1, X_2, \ldots, X_k)$ denote k-dimensional general multivariate Pareto distributions MP$^{(k)}$(I), MP$^{(k)}$(II), MP$^{(k)}$(III), and MP$^{(k)}$(IV). Some of their properties are studied by Yeh (1994), (2000), (2004a,b). As mentioned in Property 2.3 of Yeh (2004a), it is stated again in the following.

Property 2.1. *Suppose that $Z \sim$ Gamma $(\alpha, 1)$ and given $Z = z$ in $\underline{X} = (X_1, X_2, \ldots, X_k)$, each $X_i|_{Z=z} \overset{independent}{\sim}$ Weibull variable with conditional survival function $P(X_i > x_i \mid_{Z=z}) = e^{-z(\frac{x_i-\mu_i}{\sigma_i})^{1/\gamma_i}}$, $\forall\ X_i > \mu_i$, then $\underline{X} \sim$MP$^{(k)}$(IV)$(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$.*

According to the definition of Oakes (1989), the MP$^{(k)}$(IV) distribution is a multivariate survival model induced by frailties. The k coordinates of $\underline{X}$, $\{X_i\}_1^k$, can be viewed as k observed survival times depending via a proportional hazard model on the same variable, this common dependence induces an association between the observed times.

In general, a k-variate frailty model is defined as follows:

Definition 2.1. Let $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_k) \in \Re^k$, a frailty representation of the distribution of a random vector $\underline{X} = (X_1, X_2, \ldots, X_k)$ with support $\prod_{i=1}^k (\mu_i, \infty)$ is of the form

$$P(\underline{X} > \underline{x}) = \int_0^\infty \Big[\prod_{i=1}^k \overline{F}_0(x_i)\Big]^z dM(z), \tag{2.1}$$

where $\overline{F}_0(\cdot)$ is a univariate survival function with support (μ_i, ∞) and $M(\cdot)$ is a mixing distribution reflecting environmental stress. The k-variate frailty model (eq. (2.1)) assumes the existence of a common stress on the conditionally independent components.

Using the terminology of Hougaard(1984) and Vaupel et al. (1979), the function $\overline{F}_0(\cdot)$ is a continuous baseline survival function, it is the survival function of each X_i given $Z = z$, i.e.,

$$P(X_i > x_i) \mid Z = z) = \{\overline{F}_0(x_i)\}^z,$$

and the common stress random variable Z is called a frailty.

Let $\psi_Z(\cdot)$ be the Laplace transform of Z, i.e.,

$$\psi_Z(t) = E(e^{-tz}) = \int_0^\infty e^{-tz} dM(z).$$

Oakes (1989)'s result can be extended to the multivariate $k \geq 3$ case and the resulting frailty-based multivariate model assumes the form

$$p(\underline{X} > \underline{x}) = \int_0^\infty e^{-z\{-\sum_{i=1}^k \ln(\overline{F}_0(x_i))\}} dM(z) = \psi_Z\Big(-\sum_{i=1}^k \ln\Big(\overline{F}_0(x_i)\Big)\Big). \tag{2.2}$$

Arnold (1996) remarked that any mixing distribution $M(\cdot)$ can be used in such a frailty model. This fact motivates Yeh to study the frailty property of the general multivariate Pareto distributions.

Property 2.2. *Let* $\underline{X} = (X_1, X_2, \ldots, X_k) \sim \mathrm{MP}^{(k)}(\mathrm{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$, *the frailty variable* Z *follows a Gamma* $(\alpha, 1)$ *variable. Suppose the survival function of each* X_i *in* $\underline{X}$, *given* $Z = z$ *is conditionally independently distributed as a Weibull variable with the conditional survival function*

$$P(X_i > x_i \mid Z = z) = \{\overline{F}_0(x_i)\}^z = \begin{cases} e^{-(\frac{x_i-\mu_i}{\sigma_i})^{1/\gamma_i} z}, & \forall \quad x_i > \mu_i. \\ 1, & \text{o.w.} \end{cases}$$

Then the joint survival function of $\underline{X}$ *can be calculated by the following two representations:*

$$P(\underline{X} > \underline{x}) = \psi_Z\Big(-\sum_{i=1}^k \ln\Big(\overline{F}_0(x_i)\Big)\Big) = \psi_Z\Big(\sum_{i=1}^k \psi_Z^{-1}\Big(\overline{F}_{X_i}(x_i)\Big)\Big), \tag{2.3}$$

where $\psi_Z(\cdot)$ *is the Laplace transform of* Z, $\psi_Z^{-1}(\cdot)$ *is its inverse function,*

$$\overline{F}_0(x_i) = \begin{cases} e^{-(\frac{x_i-\mu_i}{\sigma_i})^{1/\gamma_i}}, & \forall \quad x_i > \mu_i \\ 1, & \text{o.w.} \end{cases}$$

and $\overline{F}_{X_i}(x_i)$ *is the marginal survival function of each* $X_i \sim \mathrm{P(IV)}(\mu_i, \sigma_i, \gamma_i, \alpha)$ *variable with*

$$\overline{F}_{X_i}(x_i) = \begin{cases} \Big(1 + (\frac{x_i-\mu_i}{\sigma_i})^{1/\gamma_i}\Big)^{-\alpha}, & \forall \quad x_i > \mu_i. \\ 1, & \text{o.w.} \end{cases} \tag{2.4}$$

Proof. The Laplace transform of $Z \sim$ Gamma $(\alpha, 1)$ is

$$\psi_Z(t) = E(e^{-tz}) = (1+t)^{-\alpha}, \quad \forall\, t > 0,$$

its inverse function is $\psi_Z^{-1}(u) = u^{-1/\alpha} - 1$, $\forall\ 0 \le u \le 1$, then the first representation is

$$\begin{aligned}\psi_Z\Big(-\sum_{i=1}^{k} \ln\Big(\overline{F}_0(x_i)\Big)\Big) &= \psi_Z\Big(\sum_{i=1}^{k}\Big(\frac{x_i-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big)\\ &= \Big(1+\sum_{i=1}^{k}\Big(\frac{x_i-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big)^{-\alpha} = P(\underline{X} > \underline{x}).\end{aligned}$$

The second representation is

$$\begin{aligned}\psi_Z\Big(\sum_{i=1}^{k}\psi_Z^{-1}\Big(\overline{F}_{X_i}(x_i)\Big)\Big) &= \psi_Z\Big(\sum_{i=1}^{k}\Big\{\Big(\overline{F}_{X_i}(x_i)\Big)^{-1/\alpha} - 1\Big\}\Big)\\ &= \psi_Z\Big(\sum_{i=1}^{k}\Big\{\Big[\Big(1+\Big(\frac{x_i-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big)^{-\alpha}\Big]^{-1/\alpha} - 1\Big\}\Big)\\ &= \psi_Z\Big(\sum_{i=1}^{k}\Big\{\Big[1+\Big(\frac{x_i-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big] - 1\Big\}\Big)\\ &= \psi_Z\Big(\sum_{i=1}^{k}\Big(\frac{x_i-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big)\\ &= \Big(1+\sum_{i=1}^{k}\Big(\frac{x_i-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big)^{-\alpha} = P(\underline{X} > \underline{x}). \qquad \square\end{aligned}$$

Oakes (1989) mentioned that bivariate distributions generated by frailty models are a subclass of the Archimedean distributions studied by Genest and Mackay(1986). The definition of the general Archimedean distribution is given below

Definition 2.2. Let $\overline{F}_{\underline{X}}(\cdot)$ be the joint survival of $\underline{X}$ with support $\prod_{i=1}^{k}(\mu_i, \infty)$ and there exists a nonnegative decreasing function $\varphi(\cdot)$ with

$\varphi(0) = 1$, $\varphi'(\cdot) < 0$, $\varphi''(\cdot) \geq 0$ such that $\overline{F}_{\underline{X}}(\cdot)$ can be written as

$$\overline{F}_{\underline{X}}(\underline{x}) = P(\underline{X} > \underline{x}) = \varphi\Big(\sum_{i=1}^{k} \varphi^{-1}\Big(\overline{F}_{X_i}(x_i)\Big)\Big) \tag{2.5}$$

for all $\underline{x} > \underline{\mu}$, where $\varphi^{-1}(\cdot)$ is the inverse function of $\varphi(\cdot)$ and $\overline{F}_{X_i}(\cdot)$, $1 \leq i \leq k$, are the marginal survival functions of X_i, respectively. Then $\underline{X}$ is said to have the Archimedean distribution.

Note that the definition of Archimedean distribution is more general than Genest and Mackay (1986), because the support of $\underline{X}$ is allowed to be any k-variate real vector $\underline{\mu} = (\mu_1, \mu_2, \ldots, \mu_k) \in \Re^k$.

3. Survival Copulas of the $MP^{(k)}$ Distributions

Some more general bivariate and multivariate Pareto distributions than Hutchinson and Lai (1990) are given by Arnold (1983) and Yeh (2004a, b). The k-variate ($k \geq 2$) of the survival copulas can be defined as

$$\hat{C}_\alpha(\underline{u}) = \overline{F}_{\underline{X}}\Big(\overline{F}_{X_1}^{-1}(u_1), \overline{F}_{X_2}^{-1}(u_2), \ldots, \overline{F}_{X_k}^{-1}(u_k)\Big), \tag{3.1}$$

where $\underline{X} = (X_1, X_2, \ldots, X_k) \sim \mathrm{MP}^{(k)}(\mathrm{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ and $\overline{F}_{X_i}^{-1}(\cdot)$ is the quasi-inverse of $\overline{F}_{X_i}(\cdot)$, the marginal survival function of each X_i, for $1 \leq i \leq k$. Note that each X_i is marginally distributed as a univariate $P(\mathrm{IV})(\mu_i, \sigma_i, \gamma_i, \alpha)$ with $\mu_i \in \Re$, $\sigma_i > 0$, $\gamma_i > 0$, and $\alpha > 0$ thus X_i's are not identically distributed and each X_i is allowed to have different support sets (μ_i, ∞) for all $i = 1, 2, \ldots, k$. Hence the definition of the survival copula given by eq.(3.1) is more general than what Nelson (1998) defined.

Schweizer and Sklar (1983) proved a theorem that elucidates the role of copulas played in the relationship between multivariate distribution functions and their univariate margins. Their result can be analogously extended to the k-variate survival copulas version and is stated in the following theorem.

Theorem 3.1. *Let $\underline{X} = (X_1, X_2, \ldots, X_k)$ be any k-variate random vector and let $\overline{F}_{\underline{X}}(\cdot)$ be its k-variate survival function with k marginal survival* *functions* $\overline{F}_{X_1}(\cdot), \overline{F}_{X_2}(\cdot), \ldots, \overline{F}_{X_k}(\cdot)$. *Then there exists a k-variate survival copula* $\hat{C}(\cdot)$ *such that for all* $\underline{x} = (x_1, x_2, \ldots, x_k) \in \prod_{i=1}^k (\mu_i, \infty)$,

$$\overline{F}_{\underline{X}}(\underline{x}) = \hat{C}\Big(\overline{F}_{X_1}(x_1), \overline{F}_{X_2}(x_2), \ldots, \overline{F}_{X_k}(x_k)\Big). \tag{3.2}$$

If $\overline{F}_{X_i}(\cdot)$, $1 \le i \le k$, *are all continuous, then the copula* $\hat{C}(\cdot)$ *is unique; otherwise,* $\hat{C}(\cdot)$ *is uniquely determined on* $\prod_{i=1}^k \mathrm{Ran}(\overline{F}_{X_i}(\cdot))$, *where* $\mathrm{Ran}(\overline{F}_{X_i}(\cdot))$ *is the range of* $\overline{F}_{X_i}(\cdot)$. *Conversely, if* $\hat{C}(\cdot)$ *is a k-survival copula and* $\overline{F}_{X_i}(\cdot)$ *are k's univariate survival functions, then the function defined by eq.*(3.2) *is a k-variate survival function with marginal survival functions* $\overline{F}_{X_i}(\cdot)$, $1 \le i \le k$.

Corollary 3.1.1. *Let* $\overline{F}_{\underline{X}}(\cdot)$, $\hat{C}(\cdot)$, $\overline{F}_{X_i}(\cdot)$, $1 \le i \le k$, *be as in Theorem* 3.1, *and let* $\overline{F}_{X_i}^{-1}(\cdot)$ *be quasi-inverses of* $\overline{F}_{X_i}(\cdot)$, *respectively. Then for any*

$$\hat{C}(\underline{u}) = \overline{F}_{\underline{X}}\Big(\overline{F}_{X_1}^{-1}(u_1), \overline{F}_{X_2}^{-1}(u_2), \ldots, \overline{F}_{X_k}^{-1}(u_k)\Big). \tag{3.3}$$

The proofs of Theorem 3.1 and Corollary 3.1.1 are analogous to that of Nelson (1998) and hence is omitted.

Referring back to the general multivariate Pareto distributions, each X_i in $\underline{X} = (X_1, X_2, \ldots, X_k)$ is marginally distributed as the univariate $P(\mathrm{IV})(\mu_i, \sigma_i, \gamma_i, \alpha)$ variable, so $\overline{F}_{X_i}(x_i) = \{1 + (\frac{x_i - \mu_i}{\sigma_i})^{1/\gamma_i}\}^{-\alpha}$ with support $x_i \in (\mu_i, \infty)$, the quasi-inverse of each $\overline{F}_{X_i}(\cdot)$ is derived from the relation that for any $u_i \in [0,1]$, $u_i = \overline{F}_{X_i}(\overline{F}_{X_i}^{-1}(u_i)) = \{1 + (\frac{\overline{F}_{X_i}^{-1}(u_i) - \mu_i}{\sigma_i})^{1/\gamma_i}\}^{-\alpha}$, and thus $\overline{F}_{X_i}^{-1}(u_i)$ is solved as $\overline{F}_{X_i}^{-1}(u_i) = \mu_i + \sigma_i(u_i^{-1/\alpha} - 1)^{\gamma_i}$, for each $1 \le i \le k$.

According to Corollary 3.1.1, the survival copula of the $\mathrm{MP}^{(k)}(\mathrm{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ is

$$\begin{aligned} \hat{C}(\underline{u}) &= \overline{F}_{\underline{X}}\Big(\mu_1 + \sigma_1(u_1^{-1/\alpha} - 1)^{\gamma_1}, \ldots, \mu_k + \sigma_k(u_k^{-1/\alpha} - 1)^{\gamma_k}\Big) \\ &= \Big\{1 + \sum_{i=1}^k (u_i^{-1/\alpha} - 1)\Big\}^{-\alpha} = \Big\{\sum_{i=1}^k u_i^{-1/\alpha} - (k-1)\Big\}^{-\alpha}, \end{aligned} \tag{3.4}$$

for all $\underline{u} = (u_1, u_2, \ldots, u_k) \in [0,1]^k$, i.e., all $0 \le u_i \le 1$, $i = 1, \ldots, k$.

From eq.(3.4), it is discerned that the survival copula of the $\mathrm{MP}^{(k)}(\mathrm{IV})$ $(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ is dependent only on the parameter α and is independent of the other three parameter vectors $\underline{\mu}$, $\underline{\sigma}$, $\underline{\gamma}$. This is a fact of k-copula proposed by Schweizer and Sklar (1983). For any k-survival copula, it has the following analogous theorem.

Theorem 3.2. *For $k \geq 2$, let $X_1, X_2, \ldots, X_k$ be random variables with continuous survival functions $\overline{F}_1(\cdot), \overline{F}_2(\cdot), \ldots, \overline{F}_k(\cdot)$, and the joint survival function of $\underline{X} = (X_1, X_2, \ldots, X_k)$ is $\overline{F}_{\underline{X}}(\cdot)$, and its survival k-copula is $\hat{C}(\cdot)$. Let $g_1(\cdot), g_2(\cdot), \ldots, g_k(\cdot)$ be k's strictly increasing functions from real numbers $\mathfrak{R}$ into $\mathfrak{R}$. Then $g_1(X_1), g_2(X_2), \ldots, g_k(X_k)$ are random variables (on the same probability space as $X_1, X_2, \ldots, X_k$) with continuous survival function and survival k-copula $\hat{C}(\cdot)$. Thus $\hat{C}(\cdot)$ is invariant under strictly increasing transformations of $X_1, X_2, \ldots, X_k$.*

In the $\mathrm{MP}^{(k)}(\mathrm{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ case, for each random variable X_i, $1 \leq i \leq k$, the strictly increasing function $g_i(\cdot)$ is chosen as $g_i(X_i) = (\frac{X_i - \mu_i}{\sigma_i})^{1/\gamma_i} \stackrel{\triangle}{=} Z_i$ with the location parameter $\mu_i \in \mathfrak{R}$, the scale parameter $\sigma_i > 0$, and the inequality parameter $\gamma_i > 0$.

If Theorem 3.2 is applied to the random vector $\underline{Z} = (Z_1, Z_2, \ldots, Z_k)$, then the survival k-copula of $\underline{Z}$ is the same as eq.(3.4). It is stated as the following corollary.

Corollary 3.2.1. *Let $\underline{Z}$ $\sim\mathrm{MP}^{(k)}(IV)(\underline{0}, \underline{1}, \underline{1}, \alpha)$ and $\underline{X}$ $\sim\mathrm{MP}^{(k)}(IV)$ $(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$, then the survival k-copulas of $\underline{Z}$ and $\underline{X}$ are the same, which is eq.(3.4), i.e., $\hat{C}(\underline{u}) = \{\sum_{i=1}^{k} u_i^{-1/\alpha} - (k-1)\}^{-\alpha}$, for all $\underline{u} = (u_1, u_2, \ldots, u_k) \in [0,1]^k$.*

Proof. It is straightforward to check that the relation between $\underline{Z}$ and $\underline{X}$ is coordinatewisely $Z_i \stackrel{d}{=} (\frac{X_i - \mu_i}{\sigma_i})^{1/\gamma_i}$ $1 \leq i \leq k$, so each Z_i is just a strictly increasing function of X_i, thus, this corollary follows. □

From eq.(2.3) and eq.(3.4), it is discerned that there is another expression for the survival copula of the $\mathrm{MP}^{(k)}(\mathrm{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ distribution with the following property.

Property 3.1. *Let $\underline{X} = (X_1, X_2, \ldots, X_k)$ $\sim\mathrm{MP}^{(k)}(IV)(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$, then the survival copula of $\underline{X}$ given in eq.(3.4) is an Archimedean copula, i.e.,* *there exists a nonnegative decreasing function with* $\varphi(0) = 1$ *and* $\varphi'(\cdot) < 0$, $\varphi''(\cdot) \geq 0$, *such that*

$$\hat{C}(\underline{u}) = \varphi\Big(\sum_{i=1}^{k} \varphi^{-1}(u_i)\Big), \tag{3.5}$$

for any $\underline{u} = (u_1, u_2, \ldots, u_k) \in [0,1]^k$.

Proof. According to eq.(3.4), if the function $\varphi(\cdot)$ is chosen to be the Laplace transform of a Gamma$(\alpha, 1)$ variable Z, i.e., $\varphi(t) \stackrel{\triangle}{=} \psi_Z(t) = (1+t)^{-\alpha}$, $\forall\, t > 0$, then its inverse $\varphi^{-1}(t) = t^{-1/\alpha} - 1$, $\forall\, 0 \leq t \leq 1$. From eq.(3.4), the survival copula of $\underline{X}$ is

$$\begin{aligned}
\hat{C}(\underline{u}) &= \overline{F}_{\underline{X}}\Big(\mu_1 + \sigma_1(u_1^{-1/\alpha} - 1)^{\gamma_1}, \ldots, \mu_k + \sigma_k(u_k^{-1/\alpha} - 1)^{\gamma_k}\Big), \quad \text{by eq.(2.5)} \\
&= \psi_Z\Big(\sum_{i=1}^{k} \psi_Z^{-1}\Big(\overline{F}_{X_i}(\mu_i + \sigma_i(u_i^{-1/\alpha} - 1)^{\gamma_i})\Big)\Big) \\
&= \psi_Z\Big(\sum_{i=1}^{k} \psi_Z^{-1}\{1 + (\mu_i^{-1/\alpha} - 1)\}^{-\alpha}\Big) \\
&= \psi_Z\Big(\sum_{i=1}^{k} \psi_Z^{-1}(u_i)\Big) = \psi_Z\Big(\sum_{i=1}^{k} (u_i^{-1/\alpha} - 1)\Big) \\
&= \Big(1 + \sum_{i=1}^{k} (u_i^{-1/\alpha} - 1)\Big)^{-\alpha},
\end{aligned}$$

for any $\underline{u} \in [0,1]^k$, where $\overline{F}_{X_i}(\cdot)$ is the marginal survival function of $\underline{X}$ with eq.(2.4) as its survival function. Thus, $\hat{C}(\underline{u})$ can be expressed as

$$\hat{C}(\underline{u}) = \psi_Z\Big(\sum_{i=1}^{k} \psi_Z^{-1}(u_i)\Big), \tag{3.6}$$

according to the definition of Nelson (1998), $\hat{C}(\underline{u})$ is an Archimedean copula, hence this property follows. □

From eq.(3.6), we know that the general MP$^{(k)}$(IV)$(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ distribution is in the one-parameter Clayton (1978) family and its survival copula is an Archimedean copula which is parameterized by the new notation $\hat{C}_\alpha(\underline{u})$ for the shape parameter $\alpha \in (0, \infty)$.

The function $\psi_Z(\cdot)$ is called the Archimedean generator of the general $\text{MP}^{(k)}(\text{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ distribution. It is straightforward to check that $\psi_Z(\cdot)$ has derivatives of all orders which alternate in sign, i.e., $(-1)^\ell \frac{d^\ell}{dt^\ell}\psi_Z(t) \geq 0$, $\forall\, t > 0$ and $\ell = 0, 1, 2, \ldots$, hence according to the definition of Widder (1941), the function $\psi_Z(t)$ is completely monotonic on $\mathfrak{R}^+$ and since $\psi_Z(0) = 1$, and $\lim_{t\to\infty} \psi_Z(t) = 0$, so the generator $\psi_Z(\cdot)$ is strict.

Remark

(1) From eq.(2.3) and eq.(3.6), we know that the general $\text{MP}^{(k)}(\text{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ distribution is a marginally specified multivariate survival model and its survival copula has the desirable feature that the dependence structure is modeled separately from the marginal distributions. The shape parameter $\alpha(> 0)$ measures pairwise association and is related to Kendall's τ by

$$\tau = 4\int_0^1\int_0^1 \hat{C}_\alpha(u_i, u_j)du_i du_j - 1 = 4\int_0^1 \frac{\psi_\alpha^{-1}(t)}{(\psi_\alpha^{-1}(t))'}dt + 1 = \frac{1}{1+2\alpha}, \qquad (3.7)$$

for any pair u_i, u_j in $\underline{u} = (u_1, \ldots, u_k)$, $i \neq j$, where $\psi_\alpha^{-1}(t) = t^{-1/\alpha} - 1$, $0 < t < 1$, and eq.(3.7) is the result from Theorem 2 of Genest and Mackay (1986).

(2) From eq.(3.7), it is clear that as $\alpha \to 0^+$, $\tau \to 1$ and as $\alpha \to \infty$, $\tau \to 0$. In general, $0 < \tau < 1$, i.e., the general $\text{MP}^{(k)}(\text{IV})$ distribution has pairwise positive association.

(3) The general $\text{MP}^{(k)}(\text{IV})(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$ has an Archimedean copula, some tractable analytical properties of the AC class have been studied heavily since Genest and Mackay (1986) and Oakes (1989).

(4) The inference about α had been studied in Yeh (2000).

4. Limiting and Speical Cases of $\hat{C}_\alpha(\underline{u})$

The Frechet-Hoeffding bounds of the general multivariate Pareto distributions are derived in Yeh (2002) as for any $\underline{X} = (X_1, X_2, \ldots, X_k)$ with univariate Pareto $P(\text{IV})(\mu_i, \sigma_i, \gamma_i, \alpha)$ marginals, denoted by $\overline{F}_{X_i}(\cdot)$, then the joint survival function of $\underline{X}$, $\overline{F}_{\underline{X}}(\underline{x})$, is within the following two bounds

$$\max\left\{0, \sum_{i=1}^{k} \overline{F}_{X_i}(x_i) - (k-1)\right\} \leq \overline{F}_{\underline{X}}(\underline{x}) \leq \min_{1\leq i\leq k}\left\{\overline{F}_{X_i}(x_i)\right\}, \tag{4.1}$$

for any $\underline{x} > \underline{\mu}$. These bounds are satisfied analogously for the survival copula of $\underline{X}$. Before this section is studied, there are some notations needed to be introduced.

Notations. Let $\underline{u} = (u_1, u_2, \ldots, u_k) \in [0,1]^k$, define $L_k(\underline{u}) = \max\{0, \sum_{i=1}^{k} u_i - (k-1)\}$, $U_k(\underline{u}) = \min_{1\leq i\leq k}\{u_i\}$, $\pi_k(\underline{u}) = u_1, u_2, \ldots, u_k$.

Property 4.1. *For all* $k \geq 2$

(i) $U_k(\underline{u})$ *and* $\pi_k(\underline{u})$ *are survival* k*-copulas.*

(ii) $L_2(\underline{u})$ *is a survival* 2*-copula only, and* $L_k(\underline{u})$ *fails to be a survival* k*-copula whenever* $k > 2$.

Proof. Let $\underline{a} = (a_1, \ldots, a_k) \in [0,1]^k$, $\underline{b} = (b_1, \ldots, b_k) \in [0,1]^k$ such that $[\underline{a}, \underline{b}]$ be any k-box in I^k, it can be shown that the U_k-volumn of $[\underline{a}, \underline{b}]$ is $V_{U_k}([\underline{a}, \underline{b}]) = \max\{\min_{1\leq i\leq k}\{b_i\} - \min_{1\leq i\leq k}\{a_i\}, 0\} \geq 0$, and the π_k-volumn of $[\underline{a}, \underline{b}]$ is $V_{\pi_k}([\underline{a}, \underline{b}]) = \prod_{i=1}^{k}(b_i - a_i) \geq 0$, hence we conclude that $U_k(\cdot)$ and $\pi_k(\cdot)$ are survival k-copulas for all $k \geq 2$. However, for $L_k(\cdot)$, if choose the k-variate box as $[1/2, 1]^k$, then the L_k-volumn of $[1/2, 1]^k$ can be shown to be $V_{L_k}([1/2, 1]^k) = 1 - k/2 < 0$, whenever $k > 2$. Thus $L_k(\cdot)$ is not a survival k-copula for $k > 2$. For $k = 2$, $L_2(\cdot)$ is indeed a survival 2-copula.

The k-variate ($k \geq 2$) survival copulas $U_k(\cdot)$ and $\pi_k(\cdot)$ have characterizations similar to the k-copulas M and π given in Nelson (1998).

Theorem 4.1. *Let* $X_1, X_2, \ldots, X_k$ *be continuous random variables,* $k \geq 2$. *Then*

(a) $X_1, X_2, \ldots, X_k$ *are independent if and only if the survival* k*-copula of* $X_1, X_2, \ldots, X_k$ *is* $\pi_k(\cdot)$.

(b) *each of the random variables* $X_1, X_2, \ldots, X_k$ *is almost surely a strictly increasing function of any of the others if and only if the survival* n*-copula of* $X_1, X_2, \ldots, X_k$ *is* $U_k(\cdot)$.

Proof. Let $\underline{X} = (X_1, X_2, \ldots, X_k)$, be the k-variate continuous random vector composed by the X_i's, $1 \le i \le k$ and let $\overline{F}_{\underline{X}}(\cdot)$ be the joint survival function of $\underline{X}$, and $\overline{F}_i(\cdot)$ be the survival function of X_i, $1 \le i \le k$,

(a) $\Rightarrow$: By independence of X_i's, we have for $\forall\, \underline{x} > \underline{\mu}$, $\overline{F}_{\underline{X}}(\underline{x}) = \prod_{i=1}^k \overline{F}_i(x_i)$, also, by eq.(3.3), the survival k-copula of $\underline{X}$ is

$$\begin{aligned} \hat{C}(\underline{u}) &= \overline{F}_{\underline{X}}\Big(\overline{F}_1^{-1}(u_1), \overline{F}_2^{-1}(u_2), \ldots, \overline{F}_k^{-1}(u_k)\Big), \quad \text{by independence} \\ &= \prod_{i=1}^k \overline{F}_i\Big(\overline{F}_i^{-1}(u_i)\Big) = \prod_{i=1}^k u_i = \pi_k(\underline{u}), \quad \text{for all } \underline{u} \in [0,1]^k \end{aligned}$$

(a) $\Leftarrow$: If the survival k-coupula of $\underline{X} = (X_1, X_2, \ldots, X_k)$ is $\pi_k(\cdot)$, i.e., $\hat{C}(\underline{u}) = \prod_{i=1}^k u_i$ for all $\underline{u} \in [0,1]^k$, on the other hand, by eq.(3.2), the relation between the joint survival function and the survival copula of $\underline{X}$ is

$\overline{F}_{\underline{X}}(\underline{x}) = \hat{C}(\overline{F}_1(x_1), \overline{F}_2(x_2), \ldots, \overline{F}_k(x_k)) = \prod_{i=1}^k \overline{F}_i(x_i)$ for all $\underline{x} > \underline{\mu}$, hence the independence of $X_1, X_2, \ldots, X_k$ follows.

To prove:

(b) $\Rightarrow$: Let $\underline{S} = \{\underline{x} \mid \underline{x} > \underline{\mu}\} \in \Re^k$ denote the support of $\overline{F}_{\underline{X}}(\cdot)$ and let $\underline{x} \in \underline{S}$.

As in (a), let $\overline{F}_i(\cdot)$ denote the marginal survival function of $\overline{F}_{\underline{X}}(\cdot)$. Then for all x_i in $\underline{x} = (x_1, x_2, \ldots, x_k)$, $x_i > \mu_i$,

$$\begin{aligned} \overline{F}_i(x_i) &= P(X_i > x_i) = P(X_1 > x_1, \ldots, X_{i-1} > x_{i-1}, X_i > x_i, \ldots, X_k > x_k) \\ &\quad + \sum_{j \ne i} P\Big(X_i > x_i \cap \Big[\text{at least one } X_j \in \{X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_k\}, \\ &\quad X_j \le x_j\Big]\Big) \\ &= \overline{F}_{\underline{X}}(\underline{x}) + P(X_i > x_i, \text{ the other coordinates of } \underline{X} \text{ are disconcordant} \\ &\quad \text{with } X_i). \end{aligned} \tag{4.2}$$

Since each of the random variables $X_1, X_2, \ldots, X_k$ is almost surely a strictly increasing function of any of the others, hence $\min\limits_{1 \le i \le k} \{P(X_i > x_i$, the other coordinates of $\underline{X}$ are disconcordant with $X_i\} = 0$, then in this case $\min\limits_{1 \le i \le k} \{\overline{F}_i(\underline{x})\}$

$= \overline{F}_{\underline{X}}(\underline{x})$, and the survival copula of $\underline{X}$ is $\hat{C}(\underline{u}) = \overline{F}_{\underline{X}}(\overline{F}_1^{-1}(u_1), \ldots, \overline{F}_k^{-1}(u_k))$ $= \min_{1 \le i \le k} \{\overline{F}_i(\overline{F}_i^{-1}(u_i))\} = \min_{1 \le i \le k} \{u_i\} = U_k(\underline{u})$ for any $\underline{u} \in [0,1]^k$.

(b) $\Leftarrow$: If the survival k-copula of $\underline{X}$ is $U_k(\cdot)$, i.e., $\hat{C}(\underline{u}) = \min_{1 \le i \le k} \{u_i\}$, for all $\underline{u} \in [0,1]^k$, then the corresponding joint survival function of $\underline{X}$ is

$$\overline{F}_{\underline{X}}(\underline{x}) = \hat{C}\Big(\overline{F}_1(x_1), \ldots, \overline{F}_k(x_k)\Big) = \min_{1 \le i \le k} \Big\{\overline{F}_i(x_i)\Big\},$$

by eq.(4.2), we obtain that $\min_{1 \le i \le k} \{P(X_i > x_i$, and the other X_j in $\underline{X}$, $j \neq i$, are disconcordant with $X_i\} = 0$, hence the necessity part of (b) follows. □

Note: Since the definition for the Archimedean generator of the general multivariate $\mathrm{MP}^{(k)}$(IV) distribution is different from that given in Alsina, Frank and Schweizer (1998) and their results are mainly focused on bivariate case, hence there are some new properties about the k-variate Clayton family developed as follows.

Theorem 4.2. *Let $\{\hat{C}_\alpha^{(k)}(\cdot) \mid \alpha \in \Re^+\}$ be a family of Archimedean survival copulas with $\hat{C}_\alpha(\underline{u}) = (\sum_{i=1}^k u_i^{-1/\alpha} - (k-1))^\alpha$ and let $\Omega = \{\varphi_\alpha(\cdot) \mid \alpha \in \Re^+\}$ be the set of corresponding generators for the Archimedean survival copulas, then the k-variate Clayton subfamily is*

(a) *negatively ordered.*

(b) *the family $\{\hat{C}_\alpha(\cdot) \mid \alpha \in \Re^+\}$ contains only survival copulas which are larger than $\pi_k(\cdot)$ and its limiting cases are respectively*

  (i) $\lim_{\alpha \to 0^+} \hat{C}(\underline{u}) = U_k(\underline{u}) = \min_{1 \le i \le k} \{u_i\}$.

  (ii) $\lim_{\alpha \to \infty} \hat{C}_\alpha^{(k)}(\underline{u}) = \pi_k(\underline{u}) = u_1 u_2 \cdots u_k$ *for all* $\underline{u} \in [0,1]^k$.

Proof.

(a) For any $0 < \alpha_1 < \alpha_2$, $\hat{C}_{\alpha_1}^{(k)}(\cdot)$, $\hat{C}_{\alpha_2}^{(k)}(\cdot)$ both are in the k-variate Clayton family, and let $\varphi_{\alpha_1}(\cdot)$, $\varphi_{\alpha_2}(\cdot) \in \Omega$ and $\varphi_{\alpha_1}(t) = (1+t)^{-\alpha_1}$, $\varphi_{\alpha_2}(t) = (1+t)^{-\alpha_2}$, for $t \ge 0$, then $\varphi_{\alpha_1}^{-1}(t) = t^{-1/\alpha_1} - 1$, $\varphi_{\alpha_2}^{-1}(t) = t^{-1/\alpha_2} - 1$, for $0 < t \le 1$,
let $f(t) = \frac{(\varphi_{\alpha_1}^{-1}(t))'}{(\varphi_{\alpha_2}^{-1}(t))'} = \frac{(-\frac{1}{\alpha_1}t)^{-1/\alpha_1 - 1}}{(-\frac{1}{\alpha_2})t^{-1/\alpha_2 - 1}} = \frac{\alpha_2}{\alpha_1} t^{1/\alpha_2 - 1/\alpha_1}$, to check the function

$f(t)$ is non-increasing by computing

$$f'(t) = \frac{\alpha_2}{\alpha_1}\Big(\frac{1}{\alpha_2} - \frac{1}{\alpha_1}\Big)t^{1/\alpha_2 - 1/\alpha_1 - 1} < 0 \quad \text{for } 0 < t \leq 1,$$

by Corollary 4.4.6 of Nelson (1998), then $\hat{C}^{(k)}_{\alpha_1}(\underline{u}) > \hat{C}^{(k)}_{\alpha_2}(\underline{u})$, i.e., this family $\{\hat{C}^{(k)}_{\alpha}(\cdot) \mid \alpha \in \mathfrak{R}^+\}$ is negatively ordered.

(b) For $\forall\ \ 0 < \alpha < \infty$, by the result of (a), the survival copulas with $\hat{C}^{(k)}_{\alpha}(\underline{u}) = (\sum_{i=1}^k u_i^{-1/\alpha} - (k-1))^{-\alpha}$ have two limiting special cases

$$\hat{C}^{(k)}_{\infty}(\underline{u}) < \hat{C}^{(k)}_{\alpha}(\underline{u}) < \hat{C}^{(k)}_{0}(\underline{u}) \tag{4.3}$$

for all $\underline{u} \in [0,1]^k$.

(b)(i): If $\alpha \to 0^+$, by eq.(3.7), Kendall' $\tau = \frac{1}{1+2\alpha} \to 1$, and also, by the representation $\tau = 4\int_0^1 \frac{\psi_\alpha^{-1}(t)}{(\psi_\alpha^{-1}(t))'}dt + 1 \to 1$ as $\alpha \to 0^+$, and hence $\lim_{\alpha\to 0^+} \dfrac{\psi_\alpha^{-1}(t)}{(\psi_\alpha^{-1}(t))'} = 0$ for $0 < t \leq 1$, by Theorem 4.4.8 of Nelson (1998), we get the limiting Fréchet's upper bound $\lim_{\alpha\to 0^+} \hat{C}_\alpha(u_i,u_j) = \min\{u_i,u_j\} \stackrel{\triangle}{=} U_2(u_i,u_j)$ for any pair u_i, u_j in $\underline{u}$, $i \neq j$.
Since $\min(\cdot)$ is a binary operation and its serial iteration can be defined recursively via $\min(u_1,u_2,u_3) = \min(\min(u_1,u_2),u_3)$, by induction it follows that $\min_{1\leq i\leq k}\{u_i\} = \min(\min_{1\leq i\leq k-1}\{u_i\}, u_k)$, and thus if $\lim_{\alpha\to 0^+} \dfrac{\psi_\alpha^{-1}(t)}{(\psi_\alpha^{-1}(t))'} = 0$, then $\min_{\alpha\to 0^+} \hat{C}^{(k)}_\alpha(\underline{u}) = U_k(\underline{u}) = \min_{1\leq i\leq k}\{u_i\}$ follows.

(b)(ii) : If $\alpha \to \infty$, then $\tau \to 0$, by eq.(3.7), it means $\lim_{\alpha\to\infty}\int_0^1\int_0^1 \hat{C}_\alpha(u_i,u_j)$ $du_i du_j = \frac{1}{4}$ and hence $\lim_{\alpha\to\infty} \hat{C}_\alpha(u_i,u_j) = u_i u_j \stackrel{\triangle}{=} \pi_2(u_i,u_j)$. Since the product $\pi(\cdot)$ is also a binary operation and its serial iteration can be defined recursively via $\pi_j(u_1,u_2,\ldots,u_j) = \pi_2(\pi_{j-1}(u_1,u_2,\ldots,$ $u_{j-1}),u_j)$, for $j \geq 3$, by induction, it follows that $\pi_k(\underline{u}) = u_1u_2\cdots u_k$, hence $\lim_{\alpha\to\infty} \hat{C}_\alpha(\underline{u}) = \pi_k(\underline{u}) = u_1u_2\cdots u_k$, also, by the relation (4.2), we obtain the lower and upper limits of $\{\hat{C}^{(k)}_\alpha(\cdot)|\alpha\in\mathfrak{R}^+\}$ which are $\pi_k(\underline{u}) < \hat{C}^{(k)}_\alpha(\underline{u}) < \min_{1\leq i\leq k}\{u_i\}$ for all $\underline{u} \in [0,1]^k$, thus (b) follows. □

Referring back to the relation (4.1), the Fréchet-Hoeffding upper and lower bounds for the survival copula of $\underline{X}$ are respectively $\min_{1\le i\le k}\{u_i\}$ and $\max\{0, \sum_{i=1}^k u_i - (k-1)\}$ for all $\underline{u} \in [0,1]^k$. These two bounds can be expressed in terms of the Archimedean generators and stated as the following property:

Property 4.2. *Let* $\underline{X} \sim \mathrm{MP}^{(k)}(IV)(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$, *for any* $\alpha \in \Re^+$, *let* $\{\hat{C}_\alpha^{(k)}(\cdot) \mid \alpha \in \Re^+\}$ *be its corresponding family of Archimedean survival copulas with* $\hat{C}_\alpha(\underline{u}) = (\sum_{i=1}^k u_i^{-1/\alpha} - (k-1))^{-\alpha}$ *and let* $\Omega = \{\varphi_\alpha(\cdot) \mid \alpha \in \Re^+\}$ *be the set of its Archimedean generators, then*

(a) *the Fréchet-Hoeffding lower bound for* $\hat{C}_\alpha(\underline{u})$ *is*
  (i) $L_k(\underline{u}) = \max\{0, \sum_{i=1}^k u_i - (k-1)\}$, *and the generator of* $L_k(\underline{u})$ *is* $\varphi_1(t) = 1 - t$.
  (ii) $L_k(\underline{u})$ *is never a k-variate survival copula for* $k > 2$.

(b) *for general* $\alpha \in \Re^+$, *the generator of* $\hat{C}_\alpha(\cdot)$ *is* $\varphi_\alpha(t) = (1+t)^{-\alpha}$,

(c) *the lower limit of* $\{\hat{C}_\alpha^{(k)}(\cdot) \mid \alpha \in R^+\}$ *which is a k-variate survival copula is* $\pi_k(\underline{u}) = u_1 u_2 \cdots u_k$, *and the generator of* $\pi_k(\underline{u})$ *is* $\varphi_2(t) = e^{-t}$.

Proof (a)(i).

The inverse function of $\varphi_1(\cdot)$ is $\varphi_1^{-1}(t) = 1 - t$, for $0 < t < 1$, and it is easy to check that $L_k(\underline{u})$ can be expressed as

$$\begin{aligned} L_k(\underline{u}) &= \varphi_1\Big(\sum_{i=1}^k \varphi_1^{-1}(u_i)\Big) = \varphi_1\Big(\sum_{i=1}^k (1-u_i)\Big) = 1 - \Big\{\sum_{i=1}^k (1-u_i)\Big\} \\ &= \max\Big\{0, \sum_{i=1}^k u_i - (k-1)\Big\} \text{ for all } \underline{u} \in [0,1]^k. \end{aligned}$$

(a) (ii) is followed by Property 4.1 (ii).

(b) is followed by Property 3.1.

(c) The inverse of $\varphi_2(\cdot)$ is $\varphi_2^{-1}(t) = -\ln(t)$, for $0 < t < 1$. It is straightforward to check that $\pi_k(\underline{u})$ can be written as

$$\begin{aligned} \pi_k(\underline{u}) &= \varphi_2\Big(\sum_{i=1}^k \varphi_2^{-1}(u_i)\Big) = \varphi_2\Big(\sum_{i=1}^k (-\ln(u_i))\Big) = e^{-\sum_{i=1}^k(-\ln(u_i))} \\ &= e^{\ln(\prod_{i=1}^k u_i)} = u_1 u_2 \cdots u_k, \quad \text{for all } \underline{u} \in [0,1]^k. \end{aligned}$$

$\pi_k(\underline{u})$ is a k-variate survival copula followed by Property 4.1 (i) for all $k \geq 2$. □

Acknowledgements

The author would like to thank the anonymous referee for some helpful comments to improve this paper.

References

1. C. Alsina, M. J. Frank and B. Schweizer, *Associative Functions on Intervals*, Working Paper, 1998.
2. B. C. Arnold, *Pareto Distributions*, International Cooperative Publ. House, Fairland, MD, 1983.
3. B. C. Arnold, Marginally and conditionally specified multivariate survival models. In *Statistics of Quality*, (Edited by S. Ghosh, W. Schucany and W. Smith, W.), 233-252. Marcel Dekker, New York, 1996.
4. D. G. Clayton, A model for association in bivariate Life tables and its application in epidemiological studies of familial tendency in chronic disease incidence, *Biometrika*, **65** (1978), 141-151.
5. C. Genest and J. Mackay, The joy of copulas: bivariate distributions with uni-form marginals, *Amer. Statist.*, **40** (1986), 280-283.
6. C. Genest and L. P. Rivest, Statistical inference for bivariate archimedean copulas, *J. Amer. Statist. Assoc.*, **88** (1993), 1034-1043.
7. P. Hougaard, A class of multivariate failure time distributions, *Biometrika*, **73** (1986), 671-678.
8. T. P. Hutchinson and C. D. Lai, *Continuous Bivariate Distributions, Emphasizing Applications*, Rumsby Scientific Publishing, Adelaide, 1990.
9. H. Joe, *Multivariate Models and Dependence Concepts*, Chapman and Hall, 1997.
10. S. Kotz, N. Balakrishnan and N. L. Johnson, *Continuous Multivariate Distributions*, V.1, John Wiley and Sons, Inc. (2000), 577-619.
11. R. B. Nelson, *An Introduction to Copulas*, Lecture Notes in Statistics, Springer, 1998.
12. D. Oakes, Bivariate survival models induced by frailties, *J. Amer. Statist. Assoc.*, **84** (1989), 487-493.
13. B. Schweizer and A. Sklar, *Probabilistic Metric Spaces*, North-Holland, New York, 1983.
14. D. V. Widder, *The Laplace Transform*, Princeton University Press, Princeton, 1941.
15. J. W. Vaupel, K. G. Manton and E. Stallard, The imapct of hetergeneity in individual frailty and the dynamics of mortality, *Demography*, **16** (1979), 439-454.

16. H. C. Yeh, Some properties of the homogeneous multivariate Pareto (IV) distributions, *J. Multivariate Anal.*, **51**(1994), No.1, 46-53.

17. H. C. Yeh, Two multivariate Pareto distributions and their related inferences, *Bull. Inst. Math. Acad. Sinica*, **28** (2000), No.2, 71-86.

18. H. C. Yeh, Some alternative multivariate Pareto distribution, Technical Report, NSC92-2118-M-002-006, 2002.

19. H. C. Yeh, Some properties and characterizations for generalized multivariate Pareto distributions, *J. Multivariate Anal.*, **88**(2004a), 47-60.

20. H. C. Yeh, The generalized Marshall-Olkin type multivariate Pareto distribu-tions, *Comm. Statist. Theory Methods*, **33**(2004b), No.5, 1053-1068.

Department of Finance, College of Management, National Taiwan University, Rm. 1002, No.85, Roosevelt Road, Section 4, Taipei, Taiwan, R.O.C. 106

E-mail: Yeh12345@management.ntu.edu.tw