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THE BOLTZMANN-GRAD LIMIT OF A STOCHASTIC LORENTZ GAS IN A FORCE FIELD

BY

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Abstract

We analyze the behavior of the stochastic Lorentz gas in the presence of a force field and we show how can be rigorously proved that certain (very smooth) fields prevent the process obtained by the Boltzmann-Grad limit from being Markovian. The markovianity of the limit can be recovered by introducing a slightly different setting which allows this difficulty to be removed.

1. Introduction

We present here an analysis of the rigorous derivation of linear kinetic transport equations from stochastic particles systems when a force field is present.

We shall describe the behavior of a linear system of particles in which a single particle moves in a random distribution of obstacles (the so called Lorentz gas) in the Boltzmann-Grad asymptotics, that is the one in which the mean free path of the particle is kept finite, and we shall show how the presence of a force field strongly modifies the stochastic properties of the limit system with respect to the case in which the force field vanishes. In particular, we shall show that when the obstacles have fixed random positions certain fields can prevent the limit process from being Markovian, and therefore the limit transport equation is not of linear Boltzmann type. In order to obtain a markovian limit, an additional stochasticity in the velocity distribution of the obstacles is needed. While the mathematical theorems

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are formulated for very simple kinds of forces and properties of the obstacles, the contexts in which the discussion can be applied is much wider. We shall refer to results obtained in [6].

The failure of the validation of the linear Boltzmann equation in the presence of a force field was illustrated at the formal level in [2], in the analysis of the motion of charged particles in a constant magnetic field: in this case, the equation which is derived heuristically in the Boltzmann-Grad asymptotics has a non–Markovian collision term, which is a consequence of the high probability of recollision with a given obstacle. This originates from the particular shape of the trajectory associated to the particle's position in the dynamics defined by the Lorentz force.

This example shows that, at variance with the case where no force field is acting on the particle, having a stochastic distribution in the positions of the obstacles does not guarantee to obtain a Markovian limit in the Boltzmann-Grad asymptotics. We recall in fact that in the absence of force fields it has been proved for different random distributions of fixed obstacles that it is possible to derive the linear Boltzmann equation from the Lorentz gas dynamics in the Boltzmann–Grad limit (see [7, 8] and [1, 11] for the continuum case; [5, 10] for obstacles with random distribution on a lattice), while this is not the case, in the same asymptotics, for a (deterministic) distribution of obstacles on a lattice (see [3, 9]).

We want here to make precise how the phenomenon described in [2] occurs and which kind of randomness we need in order to remove it.

The outline of the paper is the following: we shall first introduce the validity problem for the linear Boltzmann equation and we shall discuss the relevance of recollisions for both cases, in the absence and in the presence of a force field; then we shall introduce a simple model (absorbing obstacles) for which it can be proved that the equation associated to the limit system in the Boltzmann–Grad asymptotics is non–Markovian; we shall finally show that, considering a distribution which is random both in the positions and in the velocities of the obstacles, we can recover a Markovian limit equation for the Lorentz gas.

2. The Lorentz Gas and the Boltzmann–Grad Asymptotics

The (generalized) linear Boltzmann equation

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \nabla_v \cdot (Ff) = L(f) \\ f(0,x,v) = f_{in}(x,v) \in L^1(\mathbb{R}^d \times \mathbb{R}^d) \end{cases} \tag{1}$$

is meant to describe the dynamics of light particles moving in a medium through the evolution of their one particle density $f(t,x,v)$ in the phase space in a suitable asymptotics. Here $t \in [0,T]$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$ denote resp. time, position and velocity variables and f is the density to find a particle at the phase space point (x,v) at time t; $F = F(t,x,v)$, $F : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$, is a force field (in more general cases, like in the linear Vlasov-Boltzmann equation, F can depend on f); L is a suitable linear operator which describes the interactions of the particles with particles in the medium (which are of different species with respect to the species associated to f): for instance, $L(f) = \mu(C_d \int_{S^{d-1}} dn|v\cdot n| f(t,x,v-2(v\cdot n)n) - |v| f(t,x,v))$ for a hard–sphere type interaction and $L(f) = -\mu|v|f(t,x,v)$ for an absorbing medium, with μ constant and $|v|(C_d)^{-1} = \int_{S^{d-1}} |v\cdot n|\, dn$.

For the L considered here, equation (1) describes a Markov process in which particles move according to the equation of motion

$$\begin{cases} \dot{x} = v \\ \dot{v} = F \end{cases}$$

between interactions with the medium (*collisions*) and suffer collisions at exponentially distributed free lengths.

Our aim is to analyze the derivation from particle dynamics of (1).

A particle system which can be associated to (1) is the (generalized) Lorentz gas: a test particle, having initial position $(x,v) \in \mathbb{R}^d \times \mathbb{R}^d$, moves under the action of the force F among fixed obstacles, interacting with them through a compactly supported potential $V_\varepsilon(z) = V(|z|/\varepsilon)$, $z \in \mathbb{R}^d$, of interaction radius (the size of the support) $r = \varepsilon$. The obstacles positions $\mathbf{c} = (c_1, \ldots, c_n, \ldots)$, $c_i \in \mathbb{R}^d$, are distributed according to a given probability distribution $P(d\mathbf{c})$, with mean density μ_ε. We denote by

$$\begin{cases} \dot{x}_{\mathbf{c}} = v_{\mathbf{c}} \\ \dot{v}_{\mathbf{c}} = F_{\mathbf{c}}(t, x_{\mathbf{c}}, v_{\mathbf{c}}) \end{cases} \tag{2}$$

the equation of motion of the particle ($F_{\mathbf{c}}$ now includes the contribution of the collisions) and by $T^t_{\mathbf{c}}(x,v) = (x_{\mathbf{c}}(t), v_{\mathbf{c}}(t))$ the flow associated to (2). In what follows, we shall call trajectory of the particle the trajectory in the position space associated to the equation of motion of the particle. We can define the evolution of an initial density f_0 (induced from the previously defined stochastic process) on test functions ϕ as

$$\int dxdv f_\varepsilon(t,x,v)\phi(x,v) = \int dxdv f_0(x,v)\mathbb{E}^c\phi(T^t_{\mathbf{c}}(x,v)) \tag{3}$$

where $\mathbb{E}^c$ denotes the expectation with respect to $P(d\mathbf{c})$.

We say that equation (1) can be derived from the Lorentz gas dynamics in a suitable topology if $\lim_{\varepsilon\to 0} f_\varepsilon = f$ in that topology, where f is the solution of (1). This may happen, under suitable conditions on $P(d\mathbf{c})$ and F, in the so called Boltzmann–Grad asymptotics, which in this case means $\varepsilon \to 0$, $\mu_\varepsilon \to \infty$, $\varepsilon^{d-1}\mu_\varepsilon \to \mu > 0$. Here μ is a constant proportional to the inverse of the mean free path.

The evolved density f_ε, defined in (3), can be split into two components:

$$f_\varepsilon = f^M_\varepsilon + f^{NM}_\varepsilon$$

The first one, f^M_ε, which we shall denote as *Markovian component*, is associated to the trajectories of the stochastic process which collide at most once with each obstacles, the second one, f^{NM}_ε, the *non–Markovian component*, includes all other kinds of trajectories, which are associated to recollisions.

In general, what we wish to prove is that $f^M_\varepsilon \to f$ (and $f^{NM}_\varepsilon \to 0$).

When $F = 0$, the particle travels between collisions along straight lines. This implies that it can recollide with a given obstacle only after having suffered collisions with other different obstacles in the medium. In the Boltzmann–Grad asymptotics, for well-behaving probability distributions $P(d\mathbf{c})$ (like f. i. the Poisson distribution), the probability of having such a kind of event vanishes. This seems to indicate that a good asymptotic behavior for the stochasticity in the positions of the obstacles alone allows to obtain a Markovian limit for f_ε in the Boltzmann-Grad limit.

The first result in this direction was obtained in [7]. Here the linear Boltzmann equation is derived in the case of a Poisson distribution of obstacles and a hard–sphere cross section ($v \in S^{d-1}$): in this case, estimates are particularly simple and it is possible to prove the convergence $f^M_\varepsilon \to f$

without needing an explicit evaluation of the size of f_ε^{NM}. This result has been then improved in [S1] and [BoBuS]. The linear Boltzmann equation with hard–sphere cross section can be derived also from a stochastic distribution on a lattice (lattice gas; see [5, 10]), but, due to the discrete structure of the space of positions of the obstacles, here explicit estimates of the size of the recollisional part are needed.

On another side, in the case of a deterministic distribution on a lattice (periodic Lorentz gas), the distribution of the first impact time in the Boltzmann–Grad asymptotics is non–exponential and the limit is not described by a linear Boltzmann equation. We refer to [3, 4, 9] for details on this case.

When $F \neq 0$, the laps of the trajectory between collisions are no more straight lines: as it is obvious for the case described in [2], where F is the Lorentz force (that is $F = const B \wedge v$ and B is a constant magnetic field) and the trajectories associated to the free motion are arcs of a circle, in this case recollisions may occur even when only one single obstacle is present. In general, when the behavior of the trajectories becomes very different from the one of straight lines (in the case of the magnetic field, when $d = 2$, this happens when $t > 2\pi/const|B|$), the limit fails to be Markovian. This will be more precisely shown in the example in the following section.

3. The Boltzmann–Grad Limit of a Lorentz Gas with Fixed Absorbing Obstacles

We consider a Lorentz gas in which the scatterers are spheres of radius ε distributed according to a Poisson law with parameter $\mu_\varepsilon = \mu\,\varepsilon^{-1}$ on $\mathbb{R}^2$ (the case of $\mathbb{R}^3$ can be treated similarly). The scatterers are assumed to be absorbing (i.e., the test particle disappears when it enters an obstacle) and the probability distribution of finding exactly N obstacles in a bounded measurable set $\Lambda \subset \mathbb{R}^2$ is given by:

$$P(d\mathbf{c}_N) = e^{-\mu_\varepsilon|\Lambda|}\frac{\mu_\varepsilon^N}{N!}dc_1 \dots dc_N, \tag{4}$$

where $c_1 \dots c_N = \mathbf{c}_N$ are the positions of the centers of the scatterers and $|\Lambda|$ denotes the Lebesgue measure of Λ; $\mathbb{E}^\varepsilon$ will denote the expectation with respect to the Poisson repartition of parameter μ_ε.

We consider a fixed force, $F \equiv F(t,x)$, $F \in C(\mathbb{R}; W^{1,+\infty}(\mathbb{R}^2))$ acting on the test particle: the equation of motion of the particle, with initial position x and initial velocity v, is given by

$$\frac{d}{dt}(T_1^t(x,v)) = T_2^t(x,v), \quad \frac{d}{dt}(T_2^t(x,v)) = F(t, T_1^t(x,v)), \tag{5}$$

up to the first time $\tau_{\mathbf{c}}(x,v)$ when the particle enters an obstacle. Since F is globally Lipschitz, the flow T^t is well–defined for all t and for $t \in [0,T]$ the trajectory $T_1^{-t}(x,v)$ is included in some ball $B(0,R(T))$ depending on the initial datum (x,v).

This system has two convenient features: recollisions are absent and $f_\varepsilon = f_\varepsilon^M$ can be calculated explicitly.

For a given initial datum $f_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, the evolved density for the particle system is given by:

$$\begin{aligned} f_\varepsilon(t,x,v) &= \mathbb{E}^\varepsilon[f_{in}(T^{-t}(x,v))\, 1_{\{t \le \tau_{\mathbf{c}}(x,v)\}}] \\ = \sum_{N \ge 0} & e^{-\mu_\varepsilon\, |B(0,R(T))|} \frac{\mu_\varepsilon^N}{N!} \int_{c_1 \in B(0,R(T))} \cdots \\ & \int_{c_N \in B(0,R(T))} f_{in}(T^{-t}(x,v))\, 1_{\{T_1^{-s}(x,v) \notin B(c_i,\varepsilon), s \in [0,t], i=1..N\}}\, d\mathbf{c}. \end{aligned} \tag{6}$$

Let denote by

$$\theta_\varepsilon(t,x,v) = \{y \in \mathbb{R}^2, \exists s \in [0,t], |y - T_1^{-s}(x,v)| \le \varepsilon\} \tag{7}$$

the tube of width ε around the trajectory; θ_ε is the set of points around the particle's trajectory which has to be free of centers of obstacles in order not to have absorption in the interval $[0,t)$. Since here θ_ε does not depend on the configuration of obstacles, we get from (6) a simpler expression for f_ε:

$$f_\varepsilon(t,x,v) = e^{-\mu_\varepsilon\, |\theta_\varepsilon(t,x,v)|}\, f_{in}(T^{-t}(x,v)). \tag{8}$$

If the behavior in the Boltzmann–Grad limit of such a particle system would be Markovian, we would expect the limit density to satisfy the following evolution equation:

$$\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = -2\,\mu\, |v|\, f \tag{9}$$

whose solution is

$$f(t,x,v) = e^{-2\mu \int_0^t |T_2^{-s}(x,v)|ds} f_{in}(T^{-t}(x,v)). \tag{10}$$

But, as it is easily detected by comparing (8) and (10), this may happen only if the asymptotic behavior for the tube (7) when $\varepsilon \to 0$ is given by:

$$\varepsilon^{-1}|\theta_\varepsilon(t,x,v)| \approx 2\int_0^t |T_2^{-s}(x,v)|ds. \tag{11}$$

Now, what we can hope to get at best is actually only:

$$\varepsilon^{-1}|\theta_\varepsilon(t,x,v)| \approx 2l(\gamma) \tag{12}$$

where $l(\gamma)$ is the 1-dimensional measure of the set $\gamma = \{y \in \mathbb{R}^2, y \in \cup_{\sigma\in[0,t[}\{T_1^{-\sigma}(x,v)\}\}$, that is the length of the trajectory of the particle up to time t, while (11) involves the distance traveled by the particle along its trajectory up to time t, so that each time the two quantities do not coincide (that is the case for instance when the trajectory up to time t is a closed curve which is traveled by the particle more than once) (8) does not converge in the $\varepsilon \to 0$ limit to (10). Moreover, there are trajectories such that $\varepsilon^{-1}|\theta_\varepsilon(t,x,v)|$ may not follow at all the behavior described by (12). These two situations correspond to the fact that after a time smaller than the considered time t the particle comes sufficiently near to a space point it already visited in the past, and therefore it is possible to predict if it will suffer or not a collision, and this happens for a set of times whose closure has nonvanishing measure in the limit.

Both cases may in principle occur, and sometimes for not such unusual forces: for instance, the case analyzed in [2] and the 2-dimensional harmonic oscillator $F(x) = -x$ when $t > \pi/2$ correspond to the situation described at first, while the second one occurs whenever there is an accumulation point of self–intersections of the trajectory of the particle in the interval of time considered.

What actually can be proved for the Lorentz gas with absorbing obstacles is the following theorem ([6]):

Theorem 1. *Let* **c** *be given by a Poisson's repartition of parameter* $\mu_\varepsilon = \mu\,\varepsilon^{-1}$ *on* $\mathbb{R}^2$ *and* $F \equiv F(t,x) \in C(\mathbb{R}; W^{1,+\infty}(\mathbb{R}^2))$. *We denote by* T^t *the flow defined (for* $t \in \mathbb{R}$*) by (5) and by* f_ε *the quantity defined by*

(8). *We suppose moreover that F is such that for a.e. initial data* $(x,v) \in \mathbb{R}^2 \times \mathbb{R}^2$, $T_2^s(x,v) \neq 0$ *for* $s \in \mathbb{R}$. *Then for* $f_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, *we have, when* $\varepsilon \to 0$, $f_\varepsilon \overset{L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)}{\longrightarrow} f$ *for all* $T > 0$, *where f is the unique solution in* $L^1([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$ *of the equation*

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = -\,2\,\mu\,|v|\,f\,1_{\{x \neq T_1^{-s}(x,v), s \in]0,t[\}} \\ f(0,x,v) = f_{in}(x,v). \end{cases} \tag{13}$$

As soon as the trajectories in the space of x of the ODE (5) cross themselves for a set of times of strictly positive measure and for a non zero measure set of initial data, equation (13) is at variance with (9).

Theorem 1 is easily proved, through Lebesgue's dominated convergence theorem, after proving the following lemma :

Lemma 1. *Under the assumptions of Theorem* 1, *for all* $t \in [0,T]$ *and a.e.* x, v, *the volume of the tube* $\theta_\varepsilon(t,x,v)$ *satisfies the following asymptotic property:*

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} |\theta_\varepsilon(t,x,v)| = 2 \int_0^t |T_2^{-s}(x,v)|\, 1_{\{T_1^{-s}(x,v) \notin \cup_{\sigma \in [0,s[} \{T_1^{-\sigma}(x,v)\}\}} ds. \tag{14}$$

Lemma 1 states in fact the validity of (12) for the class of forces selected in Theorem 1 and its proof is obtained by expressing through convenient coordinates the integration variable in the definition of the flow tube.

The condition $T_2^{-s}(x,v) \neq 0$ for $s \in \mathbb{R}$ a.e. in (x,v) allows to define (a.e.) $\nu(-u)$ and $R(-u)$, resp. the normal vector to the trajectory and its (signed) radius of curvature, at the point $T_1^{-u}(x,v)$ and, together with the Lipschitz property of F, implies, for $u \in [0,t]$, $|R(-u)| > R_{min} > 0$. This bound prevents the occurrence of accumulation points of self–crossings. When $0 < \varepsilon < R_{min}/2$ we can define the following change of variable (which is only locally one-to-one) for given t, x, v

$$\begin{aligned} \zeta :\ \ [0,t] \times [-\varepsilon, \varepsilon] &\longrightarrow \mathbb{R}^2 \\ (s,z) &\mapsto \zeta(s,z) = \int_0^s T_2^{-h}(x,v)dh + \nu(-s)\, z. \end{aligned} \tag{15}$$

and obtain lower and upper bounds on the $\mathbb{R}^2$-measure of the flow tube (for enough small ε and $\delta > 0$) as follows:

$$|\theta_\varepsilon(t,x,v)| \;\le\; 2\varepsilon\,(1+\varepsilon/R_{min}) \int_{s\in B^c} |T_2^{-s}(x,v)|\,ds + \pi\,\varepsilon^2,$$
$$|\theta_\varepsilon(t,x,v)| \;\ge\; 2\,\varepsilon\,(1-\varepsilon/R_{min}) \int_{\{s\in[0,t]:\; d(s,B)\ge\delta\}} |T_2^{-s}(x,v)|ds,$$

where $B = \big\{s \in [0,t] : T_1^{-s}(x,v) \in \cup_{\sigma\in[0,s[}\{T_1^{-\sigma}(x,v)\}\big\}$ is the set of times for which a selfcrossing occurs. The bound from above is obtained by simply performing the integration in the new variables on the whole domain but the subdomain s.t. $s \in B$ (the term $\pi\,\varepsilon^2$ comes from the extremities of the trajectory, which do not belong to the image of ζ), while the lower bound is obtained by cutting out from the domain of integration the points where the change of variable ζ is in fact not one-to-one (typically, for ε small enough, the points close to some self-crossing of the trajectory).

We obtain therefore:

$$\limsup_{\varepsilon\to 0} \varepsilon^{-1}\,|\theta_\varepsilon(t,x,v)| \;\le\; 2\int_0^t |T_2^{-s}(x,v)|\,1_{\{s\in B^c\}}\,ds.$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon^{-1}\,|\theta_\varepsilon(t,x,v)| \;\ge\; 2\int_{\{s\in[0,t]:\; d(s,B)\ge\delta\}} |T_2^{-s}(x,v)|\,ds.$$

so that, since for all $s \in [0,t]$, $1_{\{s\in[0,t]:\; d(s,B)\ge\delta\}}$ converges to $1_{\bar{B}^c}$, and, as a consequence of the fact that $|R(-u)| > R_{min} > 0$, B is a closed set of $[0,t]$, we obtain (14) by letting δ go to 0, thanks to Lebesgue's dominated convergence theorem.

Of course, even though we just considered the case $F = F(t,x)$, the behavior described by Theorem 1 is more general and it is a consequence of the topological properties of the trajectories associated to the free motion (5).

4. The Lorentz Gas with Moving Absorbing Obstacles: the Markovian Limit

As we showed in the previous section, in the presence of a force field a well-behaving stochasticity in the localization in space of the collision is not sufficient to get a Markovian limit for the density describing the (generalized) Lorentz process.

We now try now to obtain an equation describing a Markovian process, in the Boltzmann-Grad asymptotics, by adding some additional stochasticity on the system.

We shall consider a new distribution of obstacles: their initial position $\mathbf{c}$ is still given by the Poisson law with parameter $\mu_\varepsilon = \mu\,\varepsilon^{-1}$, but now the obstacles also move with a (fixed) velocity $\mathbf{w} = (w_1, \ldots, w_N)$ which is distributed according to a centered Gaussian law with variance 1. The velocities of the obstacles are independent from each other and independent of $\mathbf{c}$. The obstacles are again assumed to be absorbing. We denote the expectation with respect to the measure we just described by $\mathbb{E}^{\varepsilon'}$. We still consider the force $F(t,x)$, the equation of motion (5) and τ now also depends on w.

For a given initial datum $g_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2)$, the density associated to the particle system is:

$$g_\varepsilon(t,x,v) = \mathbb{E}^{\varepsilon'}[g_{in}(T^{-t}(x,v))\, 1_{\{t \le \tau_{\mathbf{c},\mathbf{w}}(x,v)\}}] \tag{16}$$

and for this system we can prove the following theorem [6]:

Theorem 2. *Let* $\mathbf{c}, \mathbf{w}$ *be given by a repartition as described above (with independence of* $\mathbf{c}$ *and* $\mathbf{w}$*), and* $F \equiv F(t,x) \in C(\mathbb{R}; W^{1,+\infty}(\mathbb{R}^2))$*. Then for* $g_{in} \in L^1(\mathbb{R}^2 \times \mathbb{R}^2))$*, we have, when* $\varepsilon \to 0$, $g_\varepsilon \overset{L^1([0,T]\times\mathbb{R}^2\times\mathbb{R}^2)}{\longrightarrow} g$ *for all* $T > 0$*, where* g *is the unique solution in* $L^1([0,T] \times \mathbb{R}^2 \times \mathbb{R}^2)$ *of the equation*

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + F \cdot \nabla_v g = -2\mu\, g \int_{w \in \mathbb{R}^2} |v - w| \, \frac{e^{-\frac{|w|^2}{2}}}{2\pi} \, dw \\ g(0,x,v) = g_{in}(x,v). \end{cases} \tag{17}$$

Note that in this theorem, no assumption on F (or on the flow T^t) is made, apart from the smoothness assumption of the force field.

Since g_ε is given now by:

$$g_\varepsilon(t,x,v) = \lim_{R \to +\infty} \sum_{N \ge 0} e^{-\mu_\varepsilon |B(0,R)|} \frac{\mu_\varepsilon^N}{N!} \int_{c_1 \in B(0,R)} \cdots \int_{c_N \in B(0,R)} \int_{w_1 \in \mathbb{R}^2} \cdots \int_{w_N \in \mathbb{R}^2}$$
$$\times g_{in}(T^{-t}(x,v)) 1_{\{T_1^{-s}(x,v) \notin B(c_i - w_i s, \varepsilon), s \in [0,t], i=1,\ldots,N\}} e^{-\frac{|\mathbf{w}|^2}{2}} \frac{d\mathbf{w}}{(2\pi)^N} d\mathbf{c}. \tag{18}$$

we can get an explicit expression analogous to (8) by modifying our definition of the tube θ in the following way, for each $w \in \mathbb{R}^2$:

$$\theta'_\varepsilon(t,x,v,w) = \{y \in \mathbb{R}^2, \exists s \in [0,t], |y - T_1^{-s}(x,v) + w\,s| \le \varepsilon\}. \tag{19}$$

We get then:

$$g_\varepsilon(t,x,v) = \lim_{R\to+\infty} e^{-\mu_\varepsilon \int_{w\in\mathbb{R}^2} |\theta'_\varepsilon(t,x,v,w)|\, e^{-\frac{|w|^2}{2}} \frac{dw}{2\pi}} g_{in}(T^{-t}(x,v)). \tag{20}$$

and therefore, in a similar way to what we did in the case of fixed obstacles, we can get the proof of Theorem 2 through Lebesgue's dominated convergence theorem after proving the lemma :

Lemma 2. *The volume of the tube $\theta'_\varepsilon(t,x,v,w)$ satisfies the following asymptotic property : for all $(t,x,v) \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{w\in\mathbb{R}^2} |\theta'_\varepsilon(t,x,v,w)| e^{-\frac{|w|^2}{2}} \frac{dw}{2\pi} = 2\int_0^t \int_{w\in\mathbb{R}^2} |T_2^{-s}(x,v) - w| e^{-\frac{|w|^2}{2}} \frac{dw}{2\pi} ds. \tag{21}$$

The proof of Lemma 2 is analogous to the one of Lemma 1, We consider a given $(t,x,v) \in [0,T] \times \mathbb{R}^2 \times \mathbb{R}^2$ and we prove first that $\varepsilon^{-1}\,|\theta'_\varepsilon(t,x,v,w)|$ converges to $\int_0^t |T_2^{-s}(x,v) - w|\, ds$ for a.e. w.

Since for a.e. $w \in \mathbb{R}^2$, the translated velocity $T_2^{-s}(x,v) - w$ is different from 0 for all s (since $\{T_2^{-s}(x,v), s \in [0,t]\}$ is a Lipschitz curve of $\mathbb{R}^2$) we can apply the same technique as in Lemma 1. We get first the convergence of $\varepsilon^{-1}\,|\theta'_\varepsilon(t,x,v,w)|$ towards $2\int_{B_w^c} |T_2^{-s}(x,v) - w|\, ds$, where $B_w = \{s \in [0,t] : \exists \sigma < s, T_1^{-\sigma}(x,v) - w\,\sigma = T_1^{-s}(x,v) - w\,s\}$. Then, since the set $U = \left\{\left(s, \frac{T_1^{-s}(x,v) - T_1^{-\sigma}(x,v)}{s-\sigma}\right), 0 \le \sigma < s \le t\right\}$ has Lebesgue measure 0, we get (from Fubini's theorem) that for a.e. $w \in \mathbb{R}^2$, the set B_w is negligible.

This concludes the proof of Lemma 2.

As we saw, adding a stochasticity in the velocity of the obstacles allows to recover in the limit a Markovian process, described from equation (17). This happens for very general F and makes the meaning of Theorem 2 very general.

What is reasonable to think is that Theorem 2 can be extended to more general kind of interactions between the test particle and the obstacles, like the hard–spheres rebound, and to different kind of stochasticity (in the interaction or in the distribution of obstacles), whenever this is somehow equivalent to consider moving obstacles.

We notice that in our Markovian model the obstacles do not interact among themselves (i.e. they are transparent except to the test particle), so that, in the given asymptotics, effects due to overlappings disappear (even for hard–spheres interaction).

References

1. C. Boldrighini, C. Bunimovitch and Ya. G. Sinai, On the Boltzmann Equation for the Lorentz gas, *J. Statist. Phys.*, **32**(1983), 477-501.

2. A. V. Bobylev, A. Hansen, J. Piasecki and E. H. Hauge, From the Liouville equation to the generalized Boltzmann equation for magnetotransport in the 2D Lorentz model, *J. Statist. Phys.*, **102**(2001), no.5-6, 1133-1150.

3. J. Bourgain, F. Golse and B. Wennberg, On the distribution of free path lengths for the periodic Lorentz gas, *Commun. Math. Phys.*, **190**(1998), no.3, 491-508.

4. E. Caglioti and F. Golse, On the distribution of free path lengths for the periodic Lorentz gas III, *Commun. Math. Phys.*, **236**(2003), no.2, 199-221.

5. E. Caglioti, M. Pulvirenti and V. Ricci, Derivation of a linear Boltzmann equation for a lattice gas, *Markov Process. Related fields*, **6**(2000), no.3, 265-285.

6. L. Desvillettes and V. Ricci, Non-Markovianity of the Boltzmann–Grad limit of a system of random obstacles in a given force field, *Bull. Sci. Math.*, **128**(2004), no.1, 39-46.

7. G. Gallavotti, Rigorous theory of the Boltzmann equation in the Lorentz gas, Nota interna n. 358, Istituto di Fisica, Università di Roma, 1973.

8. G. Gallavotti, *Statistical Mechanics*, Springer, Berlin, 1999.

9. F. Golse and B. Wennberg, On the distribution of free path lengths for the periodic Lorentz gas II, *M2AN Modél. Math. et Anal. Numér.*, **34**(2000), no.6, 1151-1163.

10. V. Ricci and B. Wennberg, On the derivation of a linear Boltzmann equation from a periodic lattice gas, *Stochastic Process. Appl.*, **111**(2004), no.2, 281-315.

11. H. Spohn, The Lorentz flight process converges to a random flight process, *Commun. Math. Phys.*, **60**(1978), no.3, 277-290.

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