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KINETIC AND FLUID MODELS FOR SUPPLY CHAINS SUPPORTING POLICY ATTRIBUTES

BY

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Abstract

We consider a supply chain consisting of a sequence of buffer queues and processors with certain throughput times and capacities. In a previous work, we have derived a hyperbolic conservation law for the part density and flux in the supply chain. In the present paper, we introduce internal variables (named attributes: e.g. the time to due-date) and extend the previously defined model into a kinetic-like model for the evolution of the part in the phase-space (degree-of-completion, attribute). We relate this kinetic model to the hyperbolic one through the moment method and a 'monokinetic' (or single-phase) closure assumption. If instead multi-phase closure assumptions are retained, richer dynamics can take place. In a numerical section, we compare the kinetic model (solved by a particle method) and its two-phase approximation and demonstrate that both behave as expected.

1. Introduction

This paper is a follow-up of a previous work [2] where a fluid-like model for supply chains was derived. We consider a chain of suppliers or processors $S_0, \ldots, S_{M-1}$. Each of them processes a certain good (measured in units of parts) and passes it to the next supplier. A given processor is characterized by its thoughput time $T(m)$ (the time needed to process a single part) and by its capacity or release rate $\mu(m)$ (the number of parts which can be

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processed per unit of time). Each processor has a queue and the parts are processed on a 'first come first served basis'. The queues are supposed of infinite length and therefore there is no limitation in the number of parts in the queues. This leads to the following rule, for the the time $\tau(m,n)$ at which part number n passes from supplier $m-1$ to supplier m:

$$\tau(m+1,n) = \max\{\tau(m,n) + T(m), \tau(m+1,n-1) + \frac{1}{\mu(m,n-1)}\}, \\ m = 0,\dots,M-1, \quad n \geq 1, \qquad (1.1)$$

where we allow the capacity to depend on the part number as well. Formula (1.1) expresses that the time at which part n leaves processor m is at least equal to the time at which it entered its queue plus the throughput time (the first argument of the max) and also at least equal to the time the previous part $n-1$ left processor m plus the inverse of the machine capacity $\mu(m)$. That it must be equal to the max of these two times follows from the fact that the machine queue is either empty (in which case the max is equal to its first argument) or non-empty (in which case it is equal to the second one). We refer to [2] for details. In using formula (1.1), there is no room for a policy, since the parts are supposed to be processed on a 'first come first served basis'. The operator is not supposed to take a part at the end of the queue and to put it in front. Obviously, this is a shortcoming of the model which we shall try to circumvent in the present work.

In [2], the limit $M \to \infty$ of the automaton (1.1) was explored. Introducing the continuous variables $x \in [0,1]$ as a continous version of the processor index m/M, the density of parts $n(x,t)$ and the flux of parts $q(x,t)$, we showed that in this limit, the automaton (1.1) can be approximated by the following conservation law:

$$\partial_t n + \partial_x q = 0, \qquad (1.2)$$

$$q = \min\{vn, \mu\}. \qquad (1.3)$$

where $v = T^{-1}$. Note that, because the capacity and throughput time are processor dependent, v and μ are functions of x. In realistic cases, these functions are piecewise constants because the number of processors is finite and not large. The apparent paradox of taking the asymptotic limit $M \to \infty$ while keeping the total number of processor finite is waived by the method discussed in [2] which involves the decomposition of each processor into many virtual 'subprocessors'.

The interpretation of (1.2), (1.3) is as follows: the density of parts solves a linear convection equation with given velocity v as long as the queues are empty (the first argument of the min). When the particle flux vn wants to exceed the capacity μ, queues starts to build up and the flux eventually saturates to the capacity value (the second argument of the min). Note that the 'min' makes this hyperbolic problem non linear. The derivation method makes use of the concept of 'N-curve' originally developed by Newell in the context of traffic [20].

Our goal is to extend the fluid-like model (1.2), (1.3) so that it can incorporate the possibility of defining priorities in serving the parts. This is desirable in the so-called 'hot lot' situation, in which a certain lot of parts requires a faster treatment than the average. Instead of using the discrete model (1.1) once again, we shall use the fluid-like model as a starting point. To each part, we attach an attribute, which is a real variable y and defines its priority. Parts with lower values of y have larger priorities. We shall denote by $f(x, y, t)$ the quantity such that $f\,dx\,dy$ represents the number of parts with attributes in $[y, y + dy]$ currently processed by processors with index lying in $[x, x + dx]$. Our first task will be to write an evolution equation for f which supports a flux constraint of the same kind as that expressed by (1.3). We shall refer to this model as the 'kinetic model'.

Among particular solutions of the kinetic model are distributions of the form

$$f(x, y, t) = n(x, t)\delta(y - Y(x, t)). \tag{1.4}$$

Such distributions describe the case where all parts at a given location x bear the same attribute $Y(x, t)$ at time t and will be referred to as 'single-phase distributions'. Of course, $n(x, t)$ is the number density of these parts. We shall show that single-phase distributions are solutions of the kinetic model provided that n satisfies the fluid model (1.2), (1.3), which for this reason will be later on referred to as the 'single-phase fluid model' or 'SP' fluid model in short. In this case, the equation for $Y(x)$ is decoupled from that of n and simply translates the evolution of the part attributes in the absence of any policy.

Of course, single-phase solutions such as (1.4) are of limited interested but we would like to retain the idea that instead of being continuous, the

attribute distribution is best represented by a sum of delta distributions, i.e. by

$$f(x,y,t) = \sum_{k=1}^{K} n_k(x,t)\delta(y - Y_k(x,t)). \tag{1.5}$$

where $n_k(x,t)$ and $Y_k(x,t)$ represents the density and attributes of lot k at point x and time t. For instance, in the case of two lots, the normal one and the 'hot lot', we would have $k = 2$. We shall derive a system of fluid equations for the multiphase case which will later be referred to as the 'multiphase fluid model or 'MP' fluid model in short.

In a last part, we shall develop particle approximations of the kinetic model and of the MP fluid model. These simulations are based on the particle method.

We conclude this introduction by some references. The time discrete system (1.1) is an example of a 'Discrete Event Simulator' (see [8] for an overview). Fluid models of the type (1.2), (1.3) have been previously proposed and investigated in [1], [10] and in [3, 4, 5, 6]. These models bear strong analogies with traffic flow fluid models, which are quite extensively used (see e.g. [7, 9, 11, 19, 24]). Kinetic models have been fruitfully used in the context of supply chain modeling as well as in traffic flow (see e.g. [5, 6, 13, 14, 17, 21, 22, 23]).

In particular, the relations between the SP fluid model and the traffic model of [9] should be pointed out. The SP fluid model encompasses a flux constraint (the flux cannot exceed the upper bound μ), while the traffic flow model of [9] imposes a density constraint (the car density cannot exceed that corresponding to a bumper to bumper situation). These two types of constraints are kind of dual to each other. The flux constraint produces queues which in a certain sense can be viewed as concentrations of the solution (even if initially the solution is smooth), while the density constraint prevents concentrations but instead produces jams, i.e. stretches of space where the density coincides with the maximal allowed density. We could imagine supply chain models incorporating a density constraint (that could be for instance a limitation of the size of the queue in front of each processor). In this case, the model would exhibit simultaneously a flux and a density constraint and would combine the features of the SP model and that of [9].

The paper is organized as follows: in Section 2, the kinetic model is introduced. The multiphase model is derived in Section 3. Section 4 is

devoted to the presentation of the particle method which solves the kinetic model. Numerical results are discussed in Section 5. Lastly, a conclusion is drawn in Section 6.

2. The Kinetic Model of Supply Chain with Policy Attributes

In order to motivate the introduction of the kinetic model, we first give a particle interpretation of the SP fluid model (1.2), (1.3). We can define the characteristics of this first order system separately in each of the regions $vn < \mu$ and $vn > \mu$. They are defined by:

$$\begin{cases} \dot{X} = v, & \text{if } vn < \mu, \\ \dot{X} = 0, & \text{if } vn > \mu, \end{cases}$$

where the dots indicate time derivatives. Now, supposing for a while that v and μ are smooth functions, an easy computation shows that

$$\frac{d}{dt} n(X(t), t) = \begin{cases} -(n\partial_x v)(X(t), t), & \text{if } vn < \mu, \\ -\partial_x \mu(X(t)), & \text{if } vn > \mu. \end{cases} \tag{2.1}$$

Obviously, this dynamics is quite singular and the kinetic model can be viewed in a first instance as a way to express it in a less singular way. Let us suppose that parts located at time t and point x have different attributes y. For instance, attributes can be arrival times in the queue ; we shall come back to this point later on. Then, a way to achieve this dynamics is to say that all parts are moving with an actual velocity $V(x, y, t)$ below the processor defined velocity $v(x)$ (i.e. $0 \leq V(x, y, t) \leq v(x)$) and $V(x, y, t)$ is chosen such that the total flux does not exceed $\mu(x)$. We now develop such a possible choice of $V(x, y, t)$.

First, we define $f(x, y, t)$ the density of parts which at time t are found at position x with attribute y (i.e. $f(x, y, t)\, dx\, dy$ is the number of such particles in a volume of size $dx\, dy$ about the point (x, y)). Obviously, the density $n(x, t)$ and flux $q(x, t)$ in the sense of the SP fluid model are related with the distribution function f by integration w.r.t. y:

$$n(x, t) = \int_{\mathbb{R}} f(x, y, t)\, dy, \quad q(x, t) = \int_{\mathbb{R}} V(x, y, t) f(x, y, t)\, dy. \tag{2.2}$$

We also define

$$Q = \int_{\mathbb{R}} v(x) f(x,y,t)\, dy = vn,$$

the value of the flux if there would be no capacity limitation.

Our aim is at modeling a policy giving higher priority to parts with lower attributes y and allowing them to be processed faster. The most simple policy consists in processing parts by increasing attribute number and moving them along the processor chain with the processor speed v until the machine capacity μ is reached. The number of parts having attributes below α at position x and time t is given by $\int_{-\infty}^{\alpha} f(x,y,t)\, dy$ and accordingly the flux of such parts, i.e. the number of such parts crossing x per unit of time is:

$$\beta(x,\alpha,t) = v(x) \int_{-\infty}^{\alpha} f(x,y,t)\, dy. \tag{2.3}$$

In other words, β is the y-antiderivative of vf. β is a non-negative non-decreasing function of α from $\mathbb{R}$ onto $[0, Q(x,t)]$. We temporarily assume that f is a locally integrable function of y (note that this excludes solutions of the form (1.5)). In this case, β is also a continuous function of α. It may be non-strictly increasing but we can still define its functional inverse β^{-1} as an increasing, possibly discontinuous function from $[0, Q(x,t)]$ onto $\mathbb{R}$. To fix the ideas, we suppose that β^{-1} is left-continuous at any discontinuity point (it can easily be checked that the model is independent of this particular choice). We have $u = \beta(x,y,t) \Leftrightarrow y = \beta^{-1}(x,u,t)$.

According to the above-defined policy, the processor standing at position x will process all parts with attributes y below the critical attribute value $\alpha(x,t)$ such that the flux of such parts exactly equals $\mu(x)$ or in other words, such that

$$\beta(x,\alpha,t) = \mu(x) \quad \text{i.e.} \quad \alpha(x,t) = \beta^{-1}(x,\mu(x),t). \tag{2.4}$$

Of course, if $\mu(x) \geq Q(x,t)$, all parts can be processed without any limitation and in this case, we can set up $\alpha = +\infty$. the critical value α is thus given by:

$$\begin{cases} \alpha(x,t) = \beta^{-1}(x,\mu(x),t), & \text{if } Q(x,t) \leq \mu(x), \\ \alpha(x,t) = \infty, & \text{if } Q(x,t) \geq \mu(x). \end{cases} \tag{2.5}$$

From (2.5), it is readily checked that

$$\beta(x, \alpha(x,t), t) = \min\{\mu(x), Q(x,t)\}. \tag{2.6}$$

With this expression at hand, we can easily extend our theory to the case where the nominal processor velocity v also depends on the attribute $v = v(x,y)$, in which case, β and Q must be defined according to

$$\beta(x,\alpha,t) = \int_{-\infty}^{\alpha} v(x,y) f(x,y,t)\, dy, \quad Q = \int_{\mathbb{R}} v(x,y) f(x,y,t)\, dy.$$

With the rule given above, the actual part velocity $V(x,y,t)$ at point (x,y,t) is given either by the nominal processor velocity $v(x,y)$ if the part moves, i.e. if $y < \alpha(x,t)$, or by 0 if the part stays blocked i.e. if $y > \alpha(x,t)$. Therefore, denoting by $H(y)$ the Heaviside step function:

$$H(y) = \begin{cases} 0, & \text{if } y < 0, \\ 1, & \text{if } y > 0, \end{cases}$$

we can write

$$V(x,y,t) = v(x,y) H(\alpha(x,t) - y). \tag{2.7}$$

Since β is a non-decreasing function of α, we can equivalently write in view of (2.6):

$$\begin{aligned} V(x,y,t) &= v(x,y)\, H(\beta(x,\alpha(x,t),t) - \beta(x,y,t)) \\ &= v(x,y)\, H(\min\{\mu(x), Q(x,t)\} - \beta(x,y,t)). \end{aligned} \tag{2.8}$$

Note that this expression also simply equals

$$V(x,y,t) = v(x,y)\, H(\mu(x) - \beta(x,y,t)). \tag{2.9}$$

Now, we consider the dynamics associated with $V(x,y,t)$:

$$\dot{X} = V(X,y,t) = v(X,y) H(\mu(X) - \beta(X,y,t)). \tag{2.10}$$

By analogy with the SP fluid model, we suppose that f varies along the characteristics (2.10) in a way defined by the first equation (2.1), i.e. $f(X(t),y,t)$ satisfies

$$\frac{d}{dt} f(X(t),y,t) = -(f \partial_x (V(x,y,t)))(X(t),y,t). \tag{2.11}$$

Obviously, f must then satisfy the following equation:

$$\partial_t f + \partial_x(Vf) = 0. \tag{2.12}$$

However, it might be desirable to make the part attributes vary with time as this obviously allows a much broader range of possible policies. Denoting by $r(x, y, t)$ the attribute variation rate, we finally end up with the following kinetic model:

$$\partial_t f + \partial_x(Vf) + \partial_y(rf) = 0, \tag{2.13}$$

with V given by (2.9). It is a simple matter to show that, in the case where $v(x, y) = v(x)$ does not depend on y, the density n and flux q obey the SP fluid model (1.2), (1.3). Indeed, integrating (2.13) w.r.t. y leads to (1.2) while multiplying (2.9) by f, integrating w.r.t. y and using the change of variables $u = \beta(x, y, t)$ leads to (1.3). On the other-hand, when $v(x, y)$ depends on y, the moment model for n, q is not closed as we cannot express Q in terms of a closed expression involving n and q.

Let us now propose a possible definition of part attributes and a possible policy for varying it. Suppose each part enters the supply chain at $x = 0$ and initial time t_I with a tagged due-date t_D (hopefully larger than t_I). The due-date is the latest date at which the part should be delivered, i.e. the exit time t_E at which the part exits the supply chain at $x = 1$ should ideally be less than t_D. Then let us define the attribute as the time left to due-date i.e. $t_D - t$. As time proceeds, the time to due-date diminishes (it may even become negative if the part is late), thus increasing its priority level in the chain. In this case, the attribute variation rate is obviously $r = -1$ and the initial value of y at the entry of the supply chain is $t_D - t_I$. However, we can also think of other possible policies such as increasing the priority level faster when time approaches due-date.

To complete the model, we need initial and boundary conditions. We make no assumption on the initial state of the supply chain. On the other hand, since $V > 0$, we only need to specify boundary conditions at $x = 0$. Therefore, we specify:

$$f(x, y, 0) = f_I(x, y), \quad f(0, y, t) = g(y, t),$$

where f_I and g are given. Finally, we suppose that there are no parts with arbirtrary large attributes, hence:

$$\lim_{|y|\to\infty} f(x,y,t) = 0.$$

3. The Multi-Phase Model

From the kinetic model (2.13), we would like to deduce a model for distributions of the form (1.5). This would indeed lead to a reduction of the problem from a 2-dimensional one (in x and y) plus time into a 1-dimensional one (in x only). However, we cannot just insert solutions of the form (1.5) into (2.13) because the product Vf would be undefined (it would involve products of discontinuous functions at the points Y_k with delta functions $\delta(y - Y_K)$ which is undefined).

Rather, we take another route. We first write the system satisfied by the moments $\int y^j f\, dy$ for a convenient set of power functions y^j. As often in kinetic theory (see e.g. [18], the moment system is not closed. To express the various unknown fluxes in terms of the moments, we close the expressions by a smoothed version of (1.5), where the delta functions are replaced by smoothed approximations. We show that in the limit of vanishing smoothing, well-defined closed expressions of the moment fluxes can be recovered, which gives rise to what we shall call the Multi-Phase fluid model, or MP fluid model.

According to the previous section, the kinetic model can be written in the form

$$\partial_t f + \partial_x[H(\mu(x) - \beta(x,y,t))v(x,y)f] + \partial_y[r(x,y,t)f] = 0, \quad (3.1)$$

$$\beta(x,y,t) = \int H(y - y')v(x,y',t)f(x,y',t)\, dy'. \quad (3.2)$$

To define the moment equations, we integrate (3.1) against y^j, $j = 0, \dots J-1$. This gives the following set of moment equations:

$$\partial_t m_j + \partial_x F_j - jR_{j-1} = 0,\ j = 0, \dots J-1 \quad (3.3)$$

where the moments m_j, moment fluxes F_j and acceleration terms R_j are given by

$$m_j(x,t) = \int y^j \, f \, dy, \tag{3.4}$$

$$F_j(x,t) = \int y^j \, [H(\mu(x) - \beta(x,y,t)]v(x,y,t)f(x,y,t) \, dy, \tag{3.5}$$

$$R_j(x,t) = \int y^j \, r(x,y,t)f(x,y,t) \, dy. \tag{3.6}$$

This gives J equations for the $3J$ unknowns m_j, F_j, R_j, $j = 0, \dots J-1$. Some Ansatz must be made to find $2J$ relations among these $3J$ data.

For that purpose, we are going to close the expressions in (3.4)-(3.6) by an Ansatz of the form

$$f^\varepsilon(x,y,t) = \sum_{k=1}^{K} n_k(x,t) \, \frac{1}{\varepsilon}\phi'(\frac{y - Y_k(x,t)}{\varepsilon}), \tag{3.7}$$

where $\varepsilon^{-1}\phi'(y/\varepsilon)$ is a smoothed out version of $\delta(y)$, i.e. $\phi(y)$ is a strictly monotone function such that $\phi(-\infty) = 0$, $\phi(\infty) = 1$ holds, and $\phi'(y)$ is compactly supported. In the limit $\varepsilon \to 0$, we find back a multi-phase Ansatz of the form (1.5). Note that this method is somehow similar to that of [15] for closing the semi-classical limit of the Schrödinger equation.

Remark. As we already pointed out, we need to smooth out the δ-function, since we actually will integrate δ-functions against Heaviside functions, which is ill defined. The question is whether the evaluation of the δ-function at the discontinuity of the Heaviside functions happens on a set of measure 0 (and therefore it does not matter) or not. The answer to this question will be given by whether our final result depends on the choice of the function ϕ or not. We now are going to see that this result is actually independent of ϕ.

Proposition 3.1. *Using the Ansatz* (3.7)*, the moments fluxes and acceleration terms in* (3.4)-(3.6) *are given asymptotically by*

$$m_j(x,t) \;=\; \sum_{k=1}^{K} n_k \, Y_k^j + O(\varepsilon), \tag{3.8}$$

$$R_j(x,t) \;=\; \sum_{k=1}^{K} n_k \, Y_k^j \, r(x,Y_k,t) + O(\varepsilon), \tag{3.9}$$

$$F_j(x,t) \;=\; \sum_{k=1}^{K} n_k\, Y_k^j\, v_k\, Z_k + O(\varepsilon), \quad v_k(x,t) := v(x,Y_k,t) \tag{3.10}$$

$$Z_k \;=\; \max\{0, \min\{1, \frac{\mu(x) - \sum_{Y_s \neq Y_k} n_s v_s H(Y_k - Y_s)}{v_k \sum_{Y_s = Y_k} n_s}\}\} + O(\varepsilon). \tag{3.11}$$

Proof. Eq. (3.8) and (3.9) are immediately obtained by substituting $y \to Y_k + \varepsilon y,\ dy \to \varepsilon dy$ in the integrals. For instance, for (3.8), this gives

$$\begin{aligned} m_j(x,t) \;&=\; \sum_{k=1}^{K} n_k \int (Y_k + \varepsilon y)^j \phi'(y)\; dy \\ &=\; \sum_{k=1}^{K} n_k \int Y_k^j \phi'(y)\; dy + O(\varepsilon) = \sum_{k=1}^{K} n_k\, Y_k^j + O(\varepsilon). \end{aligned}$$

Eq. (3.9)(b) can be obtained in the same way. To prove (3.10) we start similarly:

$$\begin{aligned} F_j(x,t) &= \sum_{k=1}^{K} n_k \int y^j H(\mu(x) - \beta(x,y,t))\, v(x,y,t)\, \frac{1}{\varepsilon}\phi'(\frac{y - Y_k}{\varepsilon}) dy \\ &= \sum_{k=1}^{K} n_k \int (Y_k + \varepsilon y)^j H(\mu(x) - \beta(x, Y_k + \varepsilon y, t)) v(x, Y_k + \varepsilon y, t) \phi'(y) dy. \end{aligned}$$

The dependence of the terms $(Y_k + \varepsilon y)^j$ and $v(x, Y_k + \varepsilon y, t)$ on y can be neglected again because they are smooth functions. The dependence of $\beta(x, Y_k + \varepsilon y, t)$ on y cannot be neglected, since β is actually discontinuous at Y_k. This gives:

$$F_j(x,t) \;=\; \sum_{k=1}^{K} n_k\, Y_k^j\, v(x,Y_k,t)\, Z_k^{\varepsilon} + O(\varepsilon), \tag{3.12}$$

with

$$Z_k^{\varepsilon}(x,t) \;:=\; \int H(\mu(x) - \beta(x, Y_k + \varepsilon y, t)) \phi'(y)\; dy. \tag{3.13}$$

This yields (3.10) with $Z_k = \lim_{\varepsilon \to 0} Z_k^{\varepsilon}$. It remains to compute the limiting expression Z_k. Computing β we obtain

$$\beta(x, Y_k + \varepsilon y, t) \;=\; \sum_{s=1}^{K} n_s \int H(Y_k + \varepsilon y - z) v(x,z,t) \frac{1}{\varepsilon} \phi'(\frac{z - Y_s}{\varepsilon})\; dz$$

$$= \sum_{s=1}^{K} n_s \int H(Y_k + \varepsilon y - Y_s - \varepsilon z) v(x, Y_s + \varepsilon z, t) \phi'(z)\, dz$$
$$= \sum_{s=1}^{K} n_s v(x, Y_s, t) \int H(Y_k + \varepsilon y - Y_s - \varepsilon z) \phi'(z)\, dz + O(\varepsilon).$$

Now for $Y_s \neq Y_k$ the dependence on $(y - z)$ disappears when $\varepsilon \to 0$. For $Y_s = Y_k$ however, the ε scales out because of the scaling invariance of the Heaviside function. This gives for β:

$$\beta(x, Y_k + \varepsilon y, t) = \sum_{Y_s \neq Y_k} n_s\, v(x, Y_s, t) \int H(Y_k - Y_s)\, \phi'(z)\, dz + \sum_{Y_s = Y_k} n_s\, v(x, Y_s, t) \int H(y - z)\, \phi'(z)\, dz + O(\varepsilon),$$

or, integrating out ϕ:

$$\beta(x, Y_k + \varepsilon y, t) = \sum_{Y_s \neq Y_k} n_s\, v(x, Y_s, t)\, H(Y_k - Y_s) + \sum_{Y_s = Y_k} n_s\, v(x, Y_s, t) \phi(y) + O(\varepsilon). \tag{3.14}$$

The terms Z_k^ε in (3.12) are therefore given by

$$Z_k^\varepsilon = \int H(\mu(x) - \beta(x, Y_k + \varepsilon y, t)) \phi'(y)\, dy$$
$$= \int H\Bigg(\mu(x) - \sum_{Y_s \neq Y_k} n_s\, v(x, Y_s, t)\, H(Y_k - Y_s) - \sum_{Y_s = Y_k} n_s\, v(x, Y_s, t)\, \phi(y) + O(\varepsilon) \Bigg) \phi'(y)\, dy.$$

By the change of variables $\phi(y) = u$, we obtain:

$$Z_k^\varepsilon = \int_0^1 H\Bigg(\mu(x) - \sum_{Y_s \neq Y_k} n_s\, v(x, Y_s, t)\, H(Y_k - Y_s) - u \sum_{Y_s = Y_k} n_s\, v(x, Y_s, t) + O(\varepsilon) \Bigg)\, du$$

Now, let $b \geq 0$ hold. Then we have the relation

$$\int_0^1 H(a-bu)\, du = \left(\begin{array}{ll} 0 & \text{for } \frac{a}{b} < 0 \\ \frac{a}{b} & \text{for } 0 < \frac{a}{b} < 1 \\ 1 & \text{for } 1 < \frac{a}{b} \end{array} \right) = \max\{0, \min\{1, \frac{a}{b}\}\}. \quad (3.15)$$

(3.15) holds for $b = 0$ formally as well if we define $\frac{a}{b} = \text{sign}(a)\infty$ in this case. Thus we obtain in the limit $\varepsilon \to 0$:

$$Z_k = \max\{0, \min\{1, \frac{\mu(x) - \sum_{Y_s \neq Y_k} n_s\, v_s\, H(Y_k - Y_s)}{v_k \sum_{Y_s = Y_k} n_s}\}\}$$

and therefore (3.11), which ends the proof. □

Now, we use the closure (3.8)-(3.11) (with $\varepsilon = 0$) to close the moment system (3.3). By using n_k, Y_k, $k = 1, \ldots, K$, we have introduced $2K$ additional unknowns, making the total count of unknowns to $3J+2K$. Additionally, we have obtained $3J$ additional equations (3.8)-(3.10), making, together with (3.3), the total count of equations to $4J$. In order to get the same number of equations as unknowns, we obviously need $J = 2K$. We review the cases $K = 1$ (single-phase closure) and $K = 2$ (two-phase closure).

In the case $K = 1$, the unknowns are m_0, F_0, n_1 and Y_1. We obtain:

$$Z_1 = \max\{0, \min\{1, \frac{\mu(x)}{v_1 n_1}\}\},$$

with $v_1(x,t) = v(x, Y_1(x,t))$. However, the second argument of the 'max' is always non-negative and therefore Z_1 is always equal to it. It follows that

$$n_1 v_1 Z_1 = \min\{\mu(x), n_1 v_1\},$$

and

$$\begin{aligned} m_0 &= n_1, \quad m_1 = n_1 Y_1, \\ F_0 &= \min\{\mu(x), n_1 v_1\}, \quad F_1 = F_0 Y_1. \end{aligned}$$

In this case, denoting $n := m_0 = n_1$, $q := F_0$, $Y = Y_1$, $v = v_1 = v(x, Y(x,t))$, the moment system leads to

$$\partial_t n + \partial_x q = 0, \quad (3.16)$$

$$q = \min\{\mu(x), nv(x,Y)\}, \quad (3.17)$$

$$\partial_t (nY) + \partial_x (qY) = nr(x, Y, t). \quad (3.18)$$

If $v = v(x)$ is independent of y, the system for n, q decouples from the equation for Y: we get on the one-hand

$$\partial_t n + \partial_x q = 0, \tag{3.19}$$

$$q = \min\{\mu(x), nv(x)\}, \tag{3.20}$$

which is nothing but the Single-Phase fluid model (1.2), (1.3), followed by (3.18) for the determination of Y. If however $v = v(x, y)$ is truely dependent on y, the evolution of (n, q) cannot be decoupled from that of Y.

Now, we investigate the case $K = 2$, i.e. $J = 4$. In that case, eliminating m_j, F_j and r_j, $j = 0, \ldots, 3$ by using (3.8)-(3.11), we obtain the following set of equations for n_k, $k = 1, 2$:

$$\partial_t(n_1 Y_1^j + n_2 Y_2^j) + \partial_x(n_1 v_1 Z_1 Y_1^j + n_2 v_2 Z_2 Y_2^j) = n_1 r_1 Y_1^j + n_2 r_2 Y_2^j, \tag{3.21}$$

with $v_k(x,t) = v(x, Y_k(x,t))$ and $r_k(x,t) = r(x, Y_k(x,t), t)$, $k = 1, 2$. The issue is now the computation of Z_k, $k = 1, 2$. Let us suppose that $Y_1 < Y_2$ to fix the ideas. Then, (3.11) leads to the following discussion :

$$\text{(i)} \quad \text{if } \mu < n_1 v_1 \quad \text{then} \quad n_1 v_1 Z_1 = \mu \quad \text{and} \quad n_2 v_2 Z_2 = 0, \tag{3.22}$$

$$\text{(ii)} \quad \text{if } n_1 v_1 < \mu < n_1 v_1 + n_2 v_2 \quad \text{then } n_1 v_1 Z_1 = n_1 v_1$$
$$\text{and} \quad n_2 v_2 Z_2 = \mu - n_1 v_1, \tag{3.23}$$

$$\text{(iii)} \quad \text{if } n_1 v_1 + n_2 v_2 < \mu \quad \text{then} \quad n_1 v_1 Z_1 = n_1 v_1$$
$$\text{and} \quad n_2 v_2 Z_2 = n_2 v_2. \tag{3.24}$$

Of course, the roles of 1 and 2 must be exchanged in the case $Y_1 > Y_2$. When $Y_1 = Y_2$, then

$$n_1 v_1 Z_1 = \min\{n_1 v_1, \mu \frac{n_1}{n_1 + n_2}\}, \quad n_2 v_2 Z_2 = \min\{n_2 v_2, \mu \frac{n_2}{n_1 + n_2}\}. \tag{3.25}$$

What formulas (3.22)-(3.24) express is very simple. $n_k v_k$ is the 'free flux' of parts k and $n_1 v_1 + n_2 v_2$ is the total 'free flux' (we call 'free fluxes' the fluxes if there would be no flux limitation). In the first case, the flux limitation μ is already below the free flux of parts 1 and therefore, the actual flux of these parts is equal to the flux constraint and parts 2 simply do not move. In the second case, the flux constraint μ is larger than the free flux of parts 1 but below the total free flux. Therefore, the flux constraint does not apply to parts 1 which move with actual flux equal to their free flux. The actual flux constraint which applies to parts 2 is the total flux constraint μ

diminished by the flux of parts 1 and therefore, parts 2 move under this flux constraint. In the last case, there is no flux constraint at all because the flux constraint is above the total free flux and each part actually moves according to its own free flux. Clearly, this is consistent with the policy consisting in processing parts with lower attributes first. Again, the role of 1 and 2 must be exchanged in the case $Y_2 < Y_1$.

In the case $Y_1 = Y_2$, (3.25) expresses that the flux constraint is dispatched onto each part according to the ratio of their part number to the total number of parts. Then, each part moves independently according to the same rule as in the single-phase case.

We expect system (3.21) to be hyperbolic, i.e. to have all its characteristic velocities real and the corresponding jacobian diagonalizable. In fact, we have a more general result, valid for any system of the form (3.3) with fluxes of the form (3.10). More precisely, we have the following:

Proposition 3.2. *We consider the following system of unknowns* $\{n_k, Y_k\}$ *for* $k = 1, \dots, K$*:*

$$\partial_t m_j + \partial_x F_j = 0, \ j = 0, \dots J-1, \tag{3.26}$$

$$m_j(x,t) = \sum_{k=1}^{K} n_k Y_k^j, \quad F_j(x,t) = \sum_{k=1}^{K} q_k Y_k^j, \tag{3.27}$$

with $J = 2K$ *and* $q_k = q_k(\{n_{k'}, Y_{k'}\}_{k'=1,\dots K})$*. Then, as long as the phases* Y_k *are mutually distinct, this system is equivalent* (*at least for smooth solutions*) *to the following system:*

$$\partial_t n_k + \partial_x q_k = 0, \ k = 1, \dots K, \tag{3.28}$$

$$\partial_t (n_k Y_k) + \partial_x (q_k Y_k) = 0, \ k = 1, \dots K, \tag{3.29}$$

An example of such a system is system (3.3), (3.10), where $q_k = n_k v_h Z_k$ and Z_k is given by (3.11).

Corollary 3.3. *System* (3.26), (3.27) *is hyperbolic about a state* $\{n_k, Y_k\}_{k=1,\dots,K}$ *such that the* Y_k*'s are all distinct if and only if system* (3.28) *alone (with frozen* Y_k*'s) is hyperbolic. The characteristic velocities of* (3.26), (3.27) *are those of* (3.28) *on the one hand and the quantities* $u_k = q_k/n_k$ *for* $k = 1, \dots, K$ *on the other hand.*

Proof of Corollary 3.3. By combining it with (3.28), eq. (3.29) is equivalent (at least for smooth solutions) to the following transport equation:

$$\partial_t Y_k + u_k \partial_x Y_k = 0, \quad u_k = \frac{q_k}{n_k}, \qquad k = 1, \dots K. \tag{3.30}$$

An easy computation shows that the characteristic velocities of system (3.28), (3.30) are those of the system (3.28) alone (considering that the Y_k's are frozen), supplemented with the characteristic velocities of system (3.30), which are nothing but the u_k's. □

Remark 3.1. This result generalizes the hyperbolicity result proven in the appendix of [15] by extending it to a large class of systems. Indeed, the hyperbolicity of (3.28) is just an hypothesis on the fluxes q_k's which depends on the considered model. We shall prove below that for the fluxes given by (3.10), the model is hyperbolic.

Proof of Proposition 3.2. We shall restrict ourselves to the case $J = 4$, $K = 2$, the general case being an easy extension of it. We first show that if $\{(n_k, Y_k)\}_{k=1,2}$ is a solution of (3.28), (3.30), it is a solution of (3.26), (3.27). Indeed, multiplying (3.30) by Y_k^{j-1}, $j = 1, \dots J-2$, we find

$$\partial_t Y_k^j + u_k \partial_x Y_k^j = 0, \quad j = 0, \dots J-1,$$

and consequently, using (3.28),

$$\partial_t (n_k Y_k^j) + \partial_x (q_k Y_k^j) = 0, \quad j = 0, \dots J-1.$$

Then adding the equations for the two phases, we find

$$\partial_t (\sum_{k=1}^{2} n_k Y_k^j) + \partial_x (\sum_{k=1}^{2} q_k Y_k^j) = 0, \quad j = 0, \dots J-1,$$

which is nothing but system (3.26), (3.27).

Conversely, let $\{(n_k, Y_k)\}_{k=1,2}$ be a solution of (3.26), (3.27). Then, we can write:

$$\partial_t n_k + \partial_x q_k = S_k, \quad k = 1, 2, \tag{3.31}$$

$$\partial_t Y_k + u_k \partial_x Y_k = T_k, \quad k = 1, 2. \tag{3.32}$$

with appropriate definitions of S_k, T_k. We wish to prove that necessarily,

$$S_k = T_k = 0, \quad k = 1, 2. \tag{3.33}$$

First, adding (3.31) for the two phases and using (3.26) for $j = 0$, we get

$$S_1 + S_2 = 0. \tag{3.34}$$

Then, combining (3.31) with (3.32), we get,

$$\partial_t(n_k Y_k) + \partial_x(q_k Y_k) = n_k T_k + Y_k S_k, \quad k = 1, 2. \tag{3.35}$$

Addding (3.35) for the two phases and using (3.26) for $j = 1$, we get:

$$n_1 T_1 + n_2 T_2 + Y_1 S_1 + Y_2 S_2 = 0, \tag{3.36}$$

Now, multiplying (3.32) by Y_k, we obtain:

$$\partial_t Y_k^2 + u_k \partial_x Y_k^2 = 2 Y_k T_k, \quad k = 1, 2. \tag{3.37}$$

Proceeding like in the previous case, we deduce that

$$2n_1 Y_1 T_1 + 2n_2 Y_2 T_2 + Y_1^2 S_1 + Y_2^2 S_2 = 0. \tag{3.38}$$

Finally, multiplying (3.32) by Y_k^2 and proceeding as previously leads to

$$3n_1 Y_1^2 T_1 + 3n_2 Y_2^2 T_2 + Y_1^3 S_1 + Y_2^3 S_2 = 0. \tag{3.39}$$

Collecting (3.34), (3.36), (3.38) and (3.39), we deduce that the vector (S_1, S_2, T_1, T_2) is a solution of a homogeneous linear system the matrix of which is given by:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ Y_1 & Y_2 & n_1 & n_2 \\ Y_1^2 & Y_2^2 & 2n_1 Y_1 & 2n_2 Y_2 \\ Y_1^3 & Y_2^3 & 3n_1 Y_1^2 & 3n_2 Y_2^2 \end{pmatrix}.$$

It is a matter of elementary algebra to show that this matrix is non singular as soon as $Y_1 \neq Y_2$. Therefore, in this case, we deduce (3.33), which shows the equivalence of the two systems. This result is easily extended to an arbitrary number of phases. □

Lemma 3.4. *System* (3.28) *with the specific form* (3.10) *of the fluxes in the two-phase cases* (*i.e. with* Z_k *given by* (3.22)-(3.24)) *is hyperbolic.*

Proof. We denote the two characteristic velocities of system (3.28) by λ_1, λ_2. An easy computation shows that:

$$\text{(i)} \quad \text{if } \mu < n_1 v_1 \quad \text{then} \quad \lambda_1 = \lambda_2 = 0, \tag{3.40}$$

$$\text{(ii)} \quad \text{if } n_1 v_1 < \mu < n_1 v_1 + n_2 v_2 \quad \text{then } \lambda_1 = v_1, \quad \lambda_2 = 0, \tag{3.41}$$

$$\text{(iii)} \quad \text{if } n_1 v_1 + n_2 v_2 < \mu \quad \text{then} \quad \lambda_1 = v_1, \quad \lambda_2 = v_2. \tag{3.42}$$

This proves that all the eigenvalues are real and ends the proof. □

The lemma can be easily extended to an arbitrary number of phases. Of course, all this discussion is dependent on the hypotheses that the phases do not meet. When some phases are equal, the fluxes have discontinuous derivatives and are also space dependent through the threshold μ which can itself be space (and time) dependent. A more detailed study of this system requires a theory for hyperbolic systems with space-dependent, non-differentiable fluxes. Such a theory is still in progress (see [12, 16]).

4. Particle Discretization of the Kinetic Model

In this section we derive a particle model for the kinetic equation (2.13) (alternatively we could start from this particle model and derive the corresponding kinetic equation). We suppose that part number n enters the chain at time a_n with attribute b_n. The particle trajectory $x = \xi_n,\ y = \eta_n$ for this part is then given by

$$\frac{d}{dt}\xi_n = V(\xi_n, \eta_n, t),\ \frac{d}{dt}\eta_n = r(\xi_n, \eta_n, t),\ \xi_n(a_n) = 0,\ \eta_n(a_n) = b_n \tag{4.1}$$

where $V(x, y, t)$ is given by (2.7), i.e.

$$V(x, y, t) = v(x, y)H(\alpha(x, t) - y), \tag{4.2}$$

$v(x, y)$ being the nominal processor velocity and r is the attribute rate-of-change. Indeed, an expression of the form

$$f(x, y, t) = \sum_n \delta(x - \xi_n(t))\delta(y - \eta_n(t))H(t - a_n), \tag{4.3}$$

(where again, δ stands for the Dirac delta measure and H for the Heaviside step function) provides an **exact** measure solution of (2.13) if and only if (4.1) is satisfied.

The density and flux corresponding to (4.1) are given by

$$n(x,t) = \int f(x,y,t)\ dy = \sum_n \delta(x-\xi_n)\delta(y-\eta_n)H(t-a_n),$$
$$q(x,t) = \int V(x,y,t)f(x,y,t)\ dy = \sum_n V(\xi_n,\eta_n,t)\delta(x-\xi_n)H(t-a_n),$$

while the total flux in the absence of flux constraint $Q(x,t)$ would be equal to

$$Q(x,t) = \int v(x,y)f(x,y,t)\ dy = \sum_n v(\xi_n,\eta_n)\delta(x-\xi_n)H(t-a_n).$$

The threshold value α is determined such that q is the maximal flux satisfying the flux constraint $q(x,t) \le \mu(x)$. This gives

$$q(x,t) = \int H(\alpha(x,t)-y)v(x,y)f(x,y,t)\ dy = \min\{\mu(x), Q(x,t)\}. \quad (4.4)$$

However, as pointed out in Section 2, α is well-defined by (4.4) only under some smoothness assumptions on f which are not satisfied by particle distributions like (4.3). Indeed, for a particle model (4.4) cannot hold pointwise, since the flux q will be a superposition of $\delta-$ functions. We have

$$\begin{aligned} q(x,t) &= \sum_n H(\alpha(\xi_n,t)-\eta_n)\, v(\xi_n,\eta_n)\, \delta(x-\xi_n) \\ &= \sum_n \delta(x-\xi_n)\frac{d}{dt}\xi_n = \frac{d}{dt}\sum_n H(\xi_n - x), \end{aligned}$$

where we dropped the $H(t-a_n)$ terms (for the sake of simplicity, we suppose that there is no part entering the supply chain between t and $t+\Delta t$). We replace (4.4) by an integrated constraint of the form

$$\int_t^{t+\Delta t} q(x,s)\ ds \le \Delta t\, \mu(x)$$

or

$$\sum_n [H(\xi_n(t+\Delta t)-x) - H(\xi_n(t)-x)] \le \Delta t \mu(x) \quad (4.5)$$

and such that the l.h.s. is 'maximal' in a sense to make precise later on. Next we make $\alpha(x,t)$ a piecewise constant function in space. We define a

mesh by $0 = x_0 < \ldots < x_M = 1$ and

$$\alpha(x,t) = \sum_m \chi_m(x)\alpha_m(t),$$

where χ_m is the indicator function on $[x_m, x_{m+1}]$. The particle motion (4.1) is then replaced by

$$\frac{d}{dt}\xi_n = \sum_m \chi_m(\xi_n) H(\alpha_m(t) - \eta_n) v(\xi_n, \eta_n)$$

and we enforce the flux constraint (4.5) at the discrete points x_m.

To compute the flux constraint, we make the particle motion linear in between discrete times t_k. So we get

$$\xi_n(t_k + t) = \xi_n^k + t \sum_m \chi_m(\xi_n^k) H(\alpha_m^k - \eta_n^k) v_n^k, \quad v_n^k := v(\xi_n^k, \eta_n^k), \tag{4.6}$$

$$\eta_n(t_k + t) = \eta_n^k + t r_n^k, \quad r_n^k := r(\xi_n^k, \eta_n^k, t_k), \tag{4.7}$$

and the values α_m^k have to be chosen such that

$$\sum_n [H(\xi_n^k + \Delta t \sum_s \chi_s(\xi_n^k) H(\alpha_s^k - \eta_n^k) v_n^k - x_m) - H(\xi_n^k - x_m)] \le \Delta t\, \mu_m, \tag{4.8}$$

holds for all $m = 0, \ldots, M$, where we let $\mu_m := \mu(x_m)$. If $\xi_n^k > x_m$ holds, the argument of both Heaviside functions in (4.8) will be positive and there is no contribution to the sum. If $\xi_n^k < x_{m-1}$ holds, and we assume that $v_n^k \Delta t \le \Delta x_m := x_m - x_{m-1}\ \forall m$ holds (CFL condition) then the arguments are both negative and, again, there is no contribution. So we can add the indicator function of the interval (x_{m-1}, x_m) without changing the value of the sum and write (4.8) as

$$\sum_n \chi_{m-1}(\xi_n^k)[H(\xi_n^k + \Delta t \sum_s \chi_s(\xi_n^k) H(\alpha_s^k - \eta_n^k) v_n^k - x_m) - H(\xi_n^k - x_m)] \le \Delta t \mu_m,$$

which means that now only the term for $s = m-1$ in the inner sum remains and the second Heaviside function drops out, giving

$$\sum_n \chi_{m-1}(\xi_n^k)\, H(\xi_n^k + \Delta t H(\alpha_{m-1}^k - \eta_n^k) v_n^k - x_m) \le \Delta t\, \mu_m, \tag{4.9}$$

(4.9) has the following interpretation: we count only the particles which are in the interval (x_{m-1}, x_m) and which would cross x_m within the next time step, thus contributing to the flux at x_m.

Let us now assume that the η_n^k are ordered in ascending order, i.e. $s < l \Rightarrow \eta_s^k < \eta_l^k$ holds. This implies that we sort the particles at each time step but this can very quickly be realized with a fast-sorting procedure. Doing so, we will lose the identification of the particles. To maintain the identity we can always add an additional attribute to each part (i.e. an additional component of y and η), which is the original part number and which does not change in time. So assume that

$$\eta^k_{\nu(m,k)} < \alpha^k_m < \eta^k_{\nu(m,k)+1}$$

holds. For $n \geq \nu(m-1,k)+1$ we have $H(\alpha^k_{m-1} - \eta^k_n) = 0$ and there is no contribution to the sum. So we get

$$\sum_{n \leq \nu(m-1,k)} \chi_{m-1}(\xi^k_n) H(\xi^k_n + \Delta t v^k_n - x_m) \leq \Delta t \mu_m. \tag{4.10}$$

and the flux (the l.h.s. of (4.10)) is the maximal one if

$$\sum_{n \leq \nu(m-1,k)+1} \chi_{m-1}(\xi^k_n) H(\xi^k_n + \Delta t v^k_n - x_m) > \Delta t \mu_m.$$

Therefore, we define

$$\nu(m,k) = \max\{\omega : \sum_{n \leq \omega} \chi_m(\xi^k_n) H(\xi^k_n + \Delta t v^k_n - x_{m+1}) \leq \Delta t \mu_{m+1}\} \tag{4.11}$$

and if

$$\sum_n \chi_m(\xi^k_n) H(\xi^k_n + \Delta t v^k_n - x_{m+1}) \leq \Delta t \mu_{m+1},$$

we let

$$\nu(m,k) = \infty$$

With the definition (4.11) for ν we now can define the α^k_m and, more importantly, the term $H(\alpha^k_m - \eta^k_n)$ in (4.6) according to:

$$\alpha^k_m = \frac{1}{2}[\eta^k_{\nu(m,k)} + \eta^k_{\nu(m,k)+1}], \quad H(\alpha^k_m - \eta^k_n) = H(\nu(m,k) - n + \frac{1}{2}). \tag{4.12}$$

Our particle discretization of the kinetic equation thus consists of (4.1), (4.2) with the approximation (4.12) of α.

5. Numerical Results

We have implemented the particle model of Section 4 and the 2-phase model ($K = 2$) of Section 3. Our test example is intended to highlight the features of the two models. In our example, the attribute is identified as the due-date (see the discussion at the end of Section 2).

We consider a chain of 20 stations, all with throughput time =1. So the total minimal throughput time is 20. They all have a capacity of $\mu = 160$ parts per unit time, except for number 5, which has $\mu = 80$ and number 15 which has $\mu = 40$ (two bottlenecks). We consider a constant influx of 'low priority' parts, i.e. with a due date far in the future, of 60 parts per unit time. At time $t = 40$ 'hot lots' (parts with a much closer due date) arrive at a rate of 60 parts per unit time. With these data, the first bottleneck with $\mu = 80$ can accommodate the flow of one of either parts (hot or low) but not both together. The second bottleneck ($\mu = 40$) cannot even accommodate one single flow. Within the low priority lot and the hot lot population the due dates are chosen randomly in a given interval.

The phenomena we expect to see are the following. The low priority lots pass freely through the first bottleneck but start to pile up at station 15. This is the picture until the hot lots arrive at $t = 40$. Once the hot lots arrive, they pass freely through the first bottleneck, but constrict the flow of the low priority lots there. As soon as they reach the second bottleneck, they start to pile up and strangle the low priority flow there completely. Once the hot lots have passed through, the queues start to dissolve. The simulation runs from $t = 0$ to $t = 140$ using 8000 particles for the particle model.

To compare the 2-phase simulations with the particle ones, we have artificially generated particles from the solutions of the 2-phase model. Indeed, as an output of the 2-phase model, we have the values of the attributes Y_1 and Y_2 and of the densities n_1, n_2. For each of the phases, we generate $\Delta x n_k(x_m)$ particles in the interval (x_m, x_{m+1}) with attribute values randomly set around $y = Y_k$. We insist on the fact that these are not real computational particles but only an artifact which is aimed at facilitating the comparisons with the particle model. Figure 1 shows a comparison between the particle and 2-phase models on snapshots of the particle locations for different times. In the first snapshot (at $t = 50$) the hot lots have already constricted the flow of the regular lots at the first bottleneck, but have not

reached the second bottleneck yet. Regular lots are already piling up at the second bottleneck since it cannot even accommodate the flow of the regular lots. In the second snapshot (at $t = 70$) the hot lots have strangled the flow of regular lots at the station 15 completely. Finally in the third snapshot (at $t = 90$) all the hot lots have passed through the first bottleneck and the regular lots start to flow again through station 5. Note, that the bottlenecks at station 5 and 15 have the effect of ordering the particles. That is, once particles have been held back at a bottleneck they will leave strictly in the order of their due date. Therefore, the 'cold' particles in the left panel of Figure 1 reduce to a straight line (one particle per $x-$ value) as soon as they leave the bottleneck. This effect is not visible, of course, in the right panel since the artificial particles, generated from the two phase model, are always generated with a certain bandwidth.

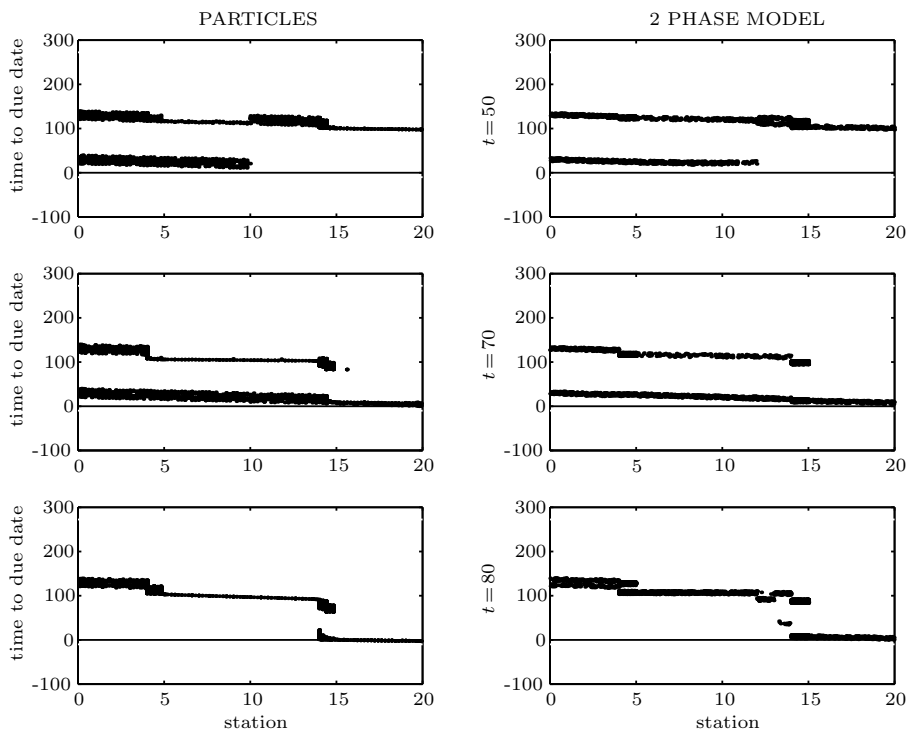

Figure 1. Comparison of the particle picture, left panel: particle model, right panel: pseudo particles generated from the 2-phase model.

To give a more quantitative comparison, we have also performed the reverse transformation, i.e. generate 2-phase solutions out of particle solu-

tion. Given the particle solution, we first compute its first four moments $m_0, \ldots, m_3$ and compute a corresponding phase and density according to (3.8). The corresponding result is compared with the solution of the 2-phase model in Figure 2 for different times. The solid and the dashed lines denote the hot and the cold phase of the 2-phase model. The triangles and $\times$'s denote the data points for the corresponding phases extracted from the particle model. (Note, that, numerically, there will always be two phases!). The left panel shows the values of the attributes Y_1 and Y_2, and the right panel shows the densities n_1 and n_2. The densities are plotted on a logarithmic scale. So, for perfect agreement, the $\times$ symbols, the values for the 'cold' phase of the particle model, should be on top of the dashed line, the 'cold' phase of the two phase model. The triangles, the values for the 'hot' phase of the particle model, should be on top of the solid line, the 'hot' phase of the two phase model.

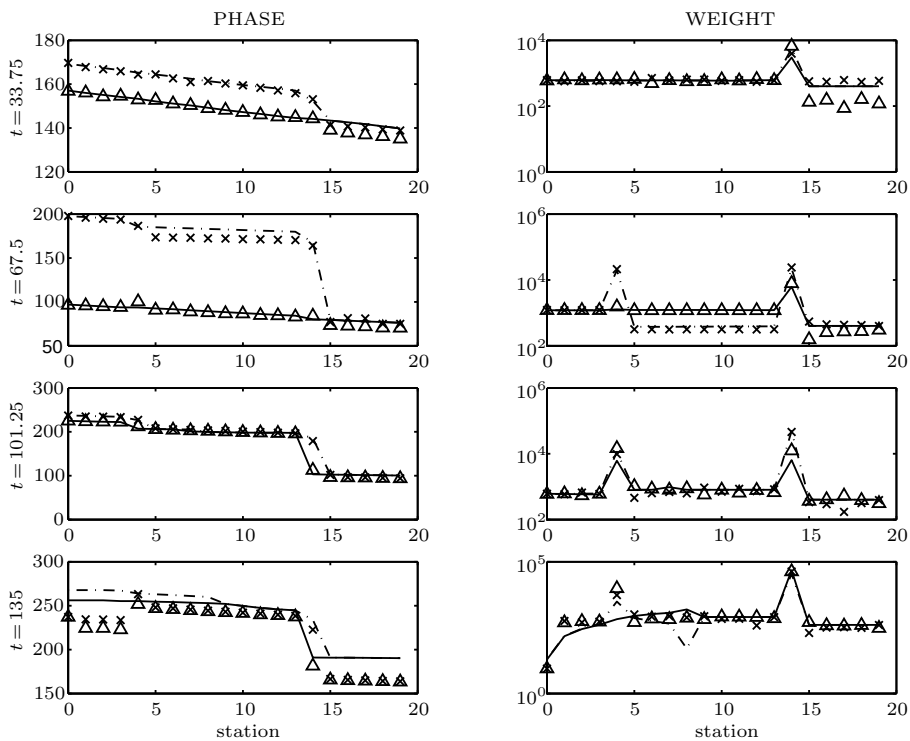

Figure 2. 2-Phase picture, left panel = attributes Y_1, Y_2, right panel = densities, $\times, \triangle$ = particles, '-,-.' = 2phase model.

Finally, we compare the expectation of the time to due date in the last cell (i.e. $\frac{m_1}{m_0}$) in Figure 3. Again, the dots are the particle solution and the solid lines are the 2-phase model.

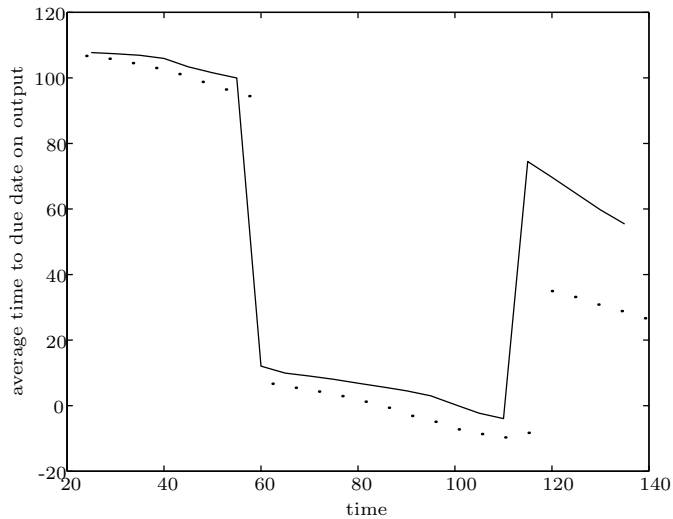

Figure 3. Expected time to due date in the last station (on time performance) '.'=particles, '-'=2phase model.

Figures 1, 2 and 3 show a reasonable agreement between the particle solution and the 2-phase model, although the latter underestimates the throughput time somewhat (see Figure 3). The obvious question arises why there is any discrepancy between the models at all. Since there are basically two phases (hot lots and regular lots), and there is no passing within the two groups, the 2-phase model should actually be exact. The reason for this paradox can be found in the boundary conditions and, as a matter of fact, points to a fundamental difficulty in comparing multi-phase closures to particle based solutions of kinetic equations. In order to obtain a meaningful quantitative comparison the influx data for the particle solution, which consist of a superposition of $\delta-$ functions in time concentrated at the discrete arrival times, have to be smoothed out to provide a smooth influx density for any differential equation model. Because of this smoothing, the resulting kinetic equations will not have an exact 2-phase solution, even at the left boundary point, as can be seen in Figure 2. In the left upper panel, for $t = 33.75$, there are two phases at station 1 although at this point no hot lots have arrived yet. In order to obtain an exact 2-phase solution the time

scale of the intervals between individual arrivals would have to be resolved, which would result in an unacceptably small time step.

Now, we comment on the computing efficiencies of the various models. The particle scheme requires about the same CPU time as the Discrete Event Simulator (1.1). More importantly, the CPU time for the kinetic model scales at least linearly with the number of parts, which is comparable with a DES simulator. On the other hand, the numerical complexity of the multiphase fluid model is independent of the number of parts, which is an enormous advantage for the simulation of large systems. Typically, the kinetic simulations which have been described in this section take a few hours of CPU time on a current size PC, while the multiphase models give almost instantaneous answers. In both cases, the codes have been developed using MATLAB®.

6. Conclusion

In this paper, we have presented several models of a supply chain. The distinctive feature of these models is that they incorporate part attribute numbers (such as time to due-date) which allow to define processing policies. In this paper, we have considered a policy consisting in processing parts by increasing attribute number. We have derived a first model of kinetic type and have proposed a particle discretization of it. We also have derived fluid-type models from a moment expansion of the kinetic model. The moment models are closed by by a multiphase ansatz which has been shown to behave satisfactorily on a typical test problem.

One main deficiency of these models are their fully deterministic character, while in practice, many parameters are incompletly known, and the characteristics of the processors themselve involve some statistical fluctuations (some may undergo breakdown, or scheduled maintenance, and so on). In future work, we shall propose probabilistic versions of the present models which, to some extent, remedy to the deficiencies of the present model.

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