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SOME RECENT RESULTS ON THE BOLTZMANN EQUATION NEAR VACUUM

BY

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Abstract

In this paper, we will address some recent progress on the L^1-stability of the Boltzmann equation near vacuum. We discuss a kinetic Glimm type interaction potential measuring the possible crossings of projected particle trajectories in physical space, and a nonlinear functional equivalent to the L^1-distance between two classical solutions. These functionals are employed to establish the large-time behavior and L^1-stability of classical solutions.

1. Introduction

The study of collisional kinetic equations was first started by Boltzmann [9] in 1872 and since then, there has been much progress on the existence theory and applications of Boltzmann type kinetic equations in neighboring sciences such as physics, chemistry, biology and civil engineering. In this paper, we will discuss some recent progress on the L^1-stability of the Boltzmann equation near vacuum. Since we are mainly interested in the stability of solutions, other interesting and important issues on the existence theory and convergence toward the equilibrium will not be pursued in this paper, and refer them to [2, 9, 34].

Let $f = f(x, v, t)$ be the number density function of dilute gas particles in phase space, and $x \in \mathbb{R}^3$ and $v \in \mathbb{R}^3$ denote position and velocity respectively. The Boltzmann equation in the absence of external forces is

$$\begin{cases} \partial_t f + v \cdot \nabla_x f = Q(f, f), \\ f(x, v, 0) = f_0(x, v). \end{cases} \tag{1.1}$$

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Here $Q(f,f)$ is an integral operator which takes account of binary collisions between particles.

$$Q(f,f) := \frac{1}{\kappa}\iint_{\mathbb{R}^3\times\mathbb{S}^2_+} B(v-v_*,\omega)\Big(f(v')f(v'_*)-f(v)f(v_*)\Big)d\omega dv_*, \quad (1.2)$$

Here κ denotes the knudsen number proportional to the mean free path, $\mathbb{S}^2_+ := \{\omega\in\mathbb{S}^2 : (v-v_*)\cdot\omega \ge 0\}$ and we used abbreviated notations:

$$f(v') := f(x,v',t),\ f(v'_*) := f(x,v'_*,t),\ f(v) := f(x,v,t),\ f(v_*) := f(x,v_*,t).$$

Scattered velocities (v',v'_*) are given by the incident velocities (v,v_*) and $\omega\in\mathbb{S}^2_+$:

$$v' = v - [(v-v_*)\cdot\omega]\omega \quad \text{and} \quad v'_* = v_* + [(v-v_*)\cdot\omega]\omega. \quad (1.3)$$

Throughout the paper, we use the following simplified notations:

$$f^\sharp(x,v,t) := f(x+tv,v,t) \quad \text{and} \quad Q^\sharp(f,f)(x,v,t) := Q(f,f)(x+tv,v,t).$$

We integrate (1.1) along the particle path $(x+sv,v,s)$ to get a mild form of (1.1):

$$f^\sharp(x,v,t) = f_0(x,v) + \int_0^t Q^\sharp(f,f)(x,v,s)ds, \quad (x,v,t)\in\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+. \quad (1.4)$$

We define a mild solution and a classical solution for (1.1) as follows.

Definition 1.1. 1. Let T be a given positive number. A nonnegative function $f\in C([0,T); L^1_+(\mathbb{R}^3\times\mathbb{R}^3))$ is a mild solution of (1.1) with a nonnegative initial datum f_0 if and only if for all $t\in[0,T)$ and a.e $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$, f satisfies the integral equation (1.4) pointwise.

2. A function $f = f(x,v,t)\in C(\mathbb{R}^3\times\mathbb{R}^3\times[0,T))$ is a classical solution of (1.1) with a nonnegative initial datum f_0 if and only if f is continuously differentiable with respect to (x,t) and f satisfies the equation (1.1) pointwise.

There are extensive literatures on the initial value problem for the Boltzmann equation (1.1), for example, the local and global existence of solutions, uniqueness and qualitative properties of solutions such as H-theorem, time-asymptotic behavior, etc. The local existence of mild solutions to (1.1) has been studied in [15, 24], while initial datum is a small perturbation of vacuum, the global existence of mild solutions and renormalized solutions to (1.1) was investigated in [3, 21, 23, 27, 29, 31, 32, 33].

The purpose of this paper is to present the uniform in time L^1-stability:

$$\sup_{0\le t<\infty} ||f(t)-\bar{f}(t)||_{L^1} \le G||f_0-\bar{f}_0||_{L^1}, \tag{1.5}$$

where $f(t)$ and $\bar{f}(t)$ are classical solutions to (1.1) corresponding to initial data f_0 and $\bar{f}_0$ respectively, and G is a generic positive constant independent of t, moreover we used a simplified notation for L^1-norm:

$$||f(t)||_{L^1} := ||f(\cdot,\cdot,t)||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)}.$$

The above estimate (1.5) is well known in the community of hyperbolic systems of conservation laws, and leads to the resolution of the uniqueness issue in some class of BV weak solutions by the Glimm's random choice method [25] and a front tracking method [5]. While in the community of kinetic equations, the estimate (1.5) was first addressed by Tartar [30] for one-dimensional discrete velocity models and recently Ha and Tzavaras [20] further improved Tartar's estimate (1.5) for general one-dimensional discrete velocity Boltzmann models using a nonlinear functional approach. In the case of the full Boltzmann equation, the weighted L^1-stability estimates was first obtained by Arkeryd [1] for the space homogeneous setting, and was also improved by the author and his collaborators for the space-inhomogeneous setting [11, 18, 19].

The rest of this paper is organized as follows. In Section 2, we summarize the main assumptions and some a priori estimates. In Section 3, we discuss a kinetic Glimm type interaction potential measuring the possible crossings of the projected particle trajectories in physical space for the pure transport equation, and finally Section 4 is devoted to the nonlinear functional approach for the L^1-stability. The details presented in this paper can be found in [18].

2. Assumptions and a Priori Estimates

In this section, we list main assumptions on initial datum, the collision kernel B in (1.2), and present some a priori estimates to be used in Section 4.

We introduce bounding functions decaying algebraically: For $\mu_1, \mu_2 > 0$,

$$\Phi_{\mu_1\mu_2}(x,v) := h_{\mu_1}(x)m_{\mu_2}(v), \quad h_{\mu_1}(x) := \frac{1}{(1+|x|)^{\mu_1}}$$

$$\text{and} \qquad m_{\mu_2}(v) := \frac{1}{(1+|v|)^{\mu_2}}.$$

Define a function space into set $\mathcal{S}(\varepsilon_0, \mu_1, \mu_2)$ and a norm $|||\cdot|||$:

$$|||f||| := \sup_{x,v,t} f^\sharp(x,v,t)\Phi^{-1}_{\mu_1\mu_2}(x,v),$$

$$\mathcal{S}(\varepsilon_0,\mu_1,\mu_2) := \{f \in C(\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}_+) : \text{(S1) - (S2) hold}\} :$$

(S1) f is continuously differentiable in x and t.

(S2) $|||f||| + \sum_{i=1}^{3} |||\partial_{x_i} f||| \le \varepsilon_0$.

The main assumptions of this paper are

(H1) The collision kernel satisfies an inverse power law and an angular cut-off assumption:

$$B(v-v_*,\omega) = |v-v_*|^\gamma b_\gamma(\theta), \quad -2 < \gamma \le 1 \quad \text{and} \quad \frac{b_\gamma(\theta)}{\cos\theta} \le B_* < \infty,$$

where θ is a scattering angle between $(v-v_*)$ and ω, i.e.,

$$\theta := \cos^{-1}\Big(\frac{(v-v_*)\cdot\omega}{|v-v_*|}\Big).$$

(H2) The parameters $\mathcal{S}(\varepsilon_0,\mu_1,\mu_2)$ satisfy

$$\varepsilon_0 \ll \kappa, \qquad \mu_1 > 4 \qquad \text{and} \qquad \mu_2 > 7.$$

Remark 2.1. The existence of classical solutions in $\mathcal{S}(\varepsilon_0,\mu_1,\mu_2)$ for sufficiently smooth initial data was established in [29] under rather mild decay conditions:

$$\varepsilon_0 \ll \kappa, \qquad \mu_1 > 1, \qquad \text{and} \qquad \mu_2 > 3.$$

We set

$$Q_+(f,f) := \frac{1}{\kappa}\int\int_{\mathbb{R}^3\times\mathbb{S}^2_+} B(v-v_*,\omega)f(v')f(v'_*)d\omega dv_*.$$

The key a priori estimates for the stability estimates are the *phase-space decay* of $f^\sharp$ and *time-phase space decay* of the gain operator $Q_+^\sharp(f,f)$: For $f \in \mathcal{S}(\varepsilon_0, \mu_1, \mu_2)$,

(E1) Four dimensional integral of $f^\sharp$ is finite and small; for $\gamma \in (-2, 1]$,

$$\iint_{\mathbb{R}^3\times\mathbb{R}_+} |v - v_*|^{\gamma-1} f^\sharp(x + t(v - v_*) + \tau n(v, v_*), v_*, t) d\tau dv_* = \mathcal{O}(1)\varepsilon_0,$$

where $n(v, v_*)$ ia a unit vector defined as

$$n(v, v_*) := \frac{v - v_*}{|v - v_*|}, \qquad v \neq v_*. \tag{2.1}$$

(E2) The gain operator $Q_+^\sharp(f,f)$ satisfies a decay estimate in time-phase space;

$$Q_+^\sharp(f,f) = \mathcal{O}(1)\frac{\varepsilon_0^2}{\kappa}\Big[\frac{h_{0.5\mu_1-\gamma}(x) m_{\mu_2-4}(v)}{(1+t)^{\min\{\gamma+3,2\}}}\Big],$$

where $Q_+^\sharp(f,f)$ denotes the gain part of the collision operator and $\mathcal{O}(1)$ denotes a bounded positive function depending only on μ_1, μ_2 and γ.

3. Kinetic Glimm Type Interaction Potential

In this part, we present a kinetic version of Glimm's interaction potential introduced by Feldmann and Ha for multi-dimensional discrete velocity Boltzmann models in [13]. The origin of such an interaction potential can be traced back to Glimm's fundamental paper [17], where Glimm introduced an interaction potential measuring the possible future interactions between elementary nonlinear waves. In contrast, in the context of kinetic equations, Glimm type functional was first constructed by Bony [4] for one-dimensional discrete velocity Boltzmann models (see [6, 7, 8] for the 1D-Boltzmann model). It is well known [23, 30] that the pure transport equation is close to the Boltzmann equation near vacuum time-asymptotically, hence we motivate the functional and its time-decay for the pure linear transport equation:

$$\begin{cases} \partial_t f(x,v,t) + v\cdot\nabla_x f(x,v,t) = 0, \\ f(x,v,0) = f_0(x,v). \end{cases} \tag{3.1}$$

where initial datum f_0 satisfies a decay and smoothness properties:

$$f_0(x,v) \leq \frac{\varepsilon_0}{(1+|x|)^{\mu_1}(1+|v|)^{\mu_2}}, \ \mu_1, \mu_2 > 3 \quad \text{and} \quad f_0 \in C^1(\mathbb{R}^3 \times \mathbb{R}^3).$$

Then the unique classical solution is given by

$$f(x,v,t) = f_0(x-vt,v,t) \quad \text{or equivalently} \quad f^\sharp(x,v,t) = f_0(x,v), \quad (3.2)$$

and satisfies a pointwise bound:

$$f^\sharp(x,v,t) \leq \frac{\varepsilon_0}{(1+|x|)^{\mu_1}(1+|v|)^{\mu_2}}. \quad (3.3)$$

Below, we give a hueristic construction of the kinetic interaction functional $\mathcal{D}(f)$. Unlike the one-dimensional case, it is very difficult to identify interacting pairs of particles in phase space. However, in the absence of external forces, we can identify the interacting pairs as follows. For given positive times t and τ, assume that a group A of particles with strength $f(x+tv,v,t)$ will interact with unknown group B of particles with velocities v_* at future time $s = t+\tau$. Then the physical space location y of this unknown group B at time t is

$$\begin{aligned} y &= \underbrace{x+(t+\tau)v}_{\text{the location of the group A at time } s=t+\tau} -\tau v_* \\ &= x+tv+\tau(v-v_*). \end{aligned}$$

Hence all particles located on the half ray $\{(y,v_*) \in \mathbb{R}^3 \times \mathbb{R}^3 : y = x+tv+\tau(v-v_*), \tau > 0\}$ will collide with a group A in future. Since f is invariant along the particle path, we simply define interaction potential between two groups A and B by

$$W(|v-v_*|)f(x+tv,v,t)\int_{\mathbb{R}^+} f(x+tv+\tau n(v,v_*),v_*,t)d\tau,$$

where $W(\cdot)$ is a weight to be chosen later. Once two groups of particles cross in physical space, they will not cross anymore, hence their interaction potentials are zero after they pass by so that interaction potentials decrease in time.

Based on the above observation and the choice of a weight $W(|v-v_*|) = |v-v_*|^{\gamma-1}$, we define an interaction potential $\mathcal{D}(f)$ and its production functional $\Lambda(f)$ as follows:

$$\begin{aligned}\mathcal{D}(f(t)) &:= \iint_{\mathbb{R}^3\times\mathbb{R}^3} f^\sharp(x,v,t)\\ \times\Big[\iint_{\mathbb{R}^3\times\mathbb{R}_+} &|v-v_*|^{\gamma-1} f^\sharp(x+t(v-v_*)+\tau n(v,v_*),v_*,t)d\tau dv_*\Big]dvdx;\\ \Lambda(f(t)) &:= \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |v-v_*|^\gamma f^\sharp(x,v,t) f^\sharp(x+t(v-v_*),v_*,t)dv_*dvdx,\end{aligned}$$

where $n(v,v_*)$ was defined in (2.1). Note that (E1) in Section 2 and conservation of mass imply

$$\mathcal{D}(f(t)) \le C(\varepsilon_0,\gamma,\mu_1,\mu_2) < \infty.$$

Proposition 3.1. *Let f be a classical solution of* (3.1) *satisfying a pointwise bound* (3.1) *corresponding to smooth initial datum f_0. Then $\mathcal{D}(f(t))$ satisfies a priori bound:*

$$\mathcal{D}(f(t)) + \int_0^t \Lambda(f(s))ds = \mathcal{D}(f(0)), \qquad t\ge 0.$$

Proof. It follows from (3.1) that

$$\begin{aligned}&\partial_t f^\sharp(x,v,t) = 0,\\ &\partial_t f^\sharp(x+t(v-v_*)+\tau n(v,v_*),v_*,t)\\ &\qquad = |v-v_*| n(v,v_*)\cdot\nabla_x f^\sharp(x+t(v-v_*)+\tau n(v,v_*),v_*,t)\\ &\qquad = \partial_\tau\Big(|v-v_*| f^\sharp(x+t(v-v_*)+\tau n(v,v_*),v_*,t)\Big).\end{aligned}$$

Then above two equations yield

$$\begin{aligned}&\partial_t\Big(|v-v_*|^{\gamma-1} f^\sharp(x,v,t) f^\sharp(x+t(v-v_*)+\tau n(v,v_*),v_*,t)\Big)\\ &\quad = \partial_\tau\Big(|v-v_*|^{\gamma} f^\sharp(x,v,t) f^\sharp(x+t(v-v_*)+\tau n(v,v_*),v_*,t)\Big).\end{aligned}$$

We integrate the above equation over $\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}_+$ with respect to (x,v,v_*,τ) to find

$$\frac{d}{dt}\mathcal{D}(f(t)) = -\Lambda(f(t)).$$

Finally we integrate the above equality with respect to t to get the desired result. □

Similar to the above estimates, the interaction potential $\mathcal{D}(f)$ works for the full Boltzmann equation as well.

Proposition 3.2. ([18]) *Suppose the main assumptions* (H) *in Section* 2 *hold, and let* f *be a classical solution of* (1.1) *to initial datum* f_0. *Then* $\mathcal{D}(f(t))$ *satisfies*

$$\mathcal{D}(f(t)) + C_0 \int_0^t \Lambda(f(s))ds \le \mathcal{D}(f_0),$$

where C_0 *is a positive constant independent of time* t.

Proof. We use the same argument in Proposition 3.1 and a priori estimate (E1). The details can be found in [18]. □

Remark 3.1. Above kinetic Glimm type interaction potentials can be constructed for the Boltzmann equation with quantum effects and the Vlasov-Poisson equations (see [10, 11]).

The interaction potential $\mathcal{D}(f(t))$ can be used to the study of time-asymptotic behavior toward the collision free flow. For this, we set the time-asymptotic state f_∞ as follows.

$$f_\infty(x,v) := f_0(x,v) + \int_0^\infty Q^\sharp(f,f)(x,v,t)dt.$$

Then it is easy to see that as $t \to \infty$,

$$\begin{aligned} & ||f(x,v,t) - f_\infty(x-tv,v)||_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \\ & \quad \le \int_t^\infty \iint_{\mathbb{R}^3\times\mathbb{R}^3} |Q^\sharp(f,f)| dvdxds \le C_1 \int_t^\infty \Lambda(f(s))ds \to 0, \end{aligned}$$

where C_1 is a positive constant independent of time t, and we used

$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} Q^\sharp_+(f,f)(x,v,t)dvdx = \iint_{\mathbb{R}^3\times\mathbb{R}^3} Q^\sharp_-(f,f)(x,v,t)dvdx \quad \text{and}$$
$$\iint_{\mathbb{R}^3\times\mathbb{R}^3} Q^\sharp_-(f,f)(x,v,t)dvdx \le \mathcal{O}(1)\Lambda(f(t)) \;\; \text{using angular cut-off (H1).}$$

4. Uniform L^1-Stability

In this section, we present a robust nonlinear functional approach for the uniform L^1-stability. This approach was originally motivated by the recent progress in the hyperbolic system of conservation laws in one-space dimension [5, 25].

Let f and $\bar{f}$ be two classical solutions in $\mathcal{S}(\varepsilon_0, \mu_1, \mu_2)$ of (1.1) corresponding to initial data f_0 and $\bar{f}_0$ respectively. Define a nonlinear functional $\mathcal{H}$ as the weighted linear combination of two sub-functionals $||f(t) - \bar{f}(t)||_{L^1}$, $\mathcal{D}_d(t)$ and its production functional $\Lambda_d(t)$:

$$\begin{aligned}
\mathcal{D}_d(t) &:= \iint_{\mathbb{R}^3\times\mathbb{R}^3} |f - \bar{f}|^\sharp(x, v, t) \\
&\times\Big[\iint_{\mathbb{R}^3\times\mathbb{R}_+} |v-v_*|^{\gamma-1}(f^\sharp + \bar{f}^\sharp)(x + t(v - v_*) + \tau n(v, v_*), v_*, t)d\tau dv_*\Big]dvdx, \\
\Lambda_d(t) &:= \iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} |v-v_*|^\gamma |f-\bar{f}|^\sharp(x, v, t)(f^\sharp+\bar{f}^\sharp)(x+t(v-v_*), v_*, t)dv_*dvdx, \\
\mathcal{H}(t) &:= ||f(t) - \bar{f}(t)||_{L^1} + K\mathcal{D}_d(t),
\end{aligned}$$

where K is a positive constant to be determined later.

The functional $\mathcal{D}_d$ measures potential interactions between $|f-\bar{f}|$, f and $\bar{f}$. On the other hand, note that the functional $\mathcal{H}$ can be rewritten as

$$\begin{aligned}
\mathcal{H}(t) \;&=\; \iint_{\mathbb{R}^3\times\mathbb{R}^3} |f - \bar{f}|^\sharp(x, v, t) \\
&\times\Big[1+K\underbrace{\iint_{\mathbb{R}^3\times\mathbb{R}_+}|v-v_*|^{\gamma-1}(f^\sharp+\bar{f}^\sharp)(x+t(v-v_*)+\tau n(v, v_*), v_*, t)d\tau dv_*}_{\mathcal{W}_1(x,v,t)}\Big]dvdx.
\end{aligned}$$

It follows from a priori estimates (E1) that

$$\mathcal{W}_1(x, v, t) \le \bar{C}_1(\gamma, \varepsilon_0, \mu_1, \mu_2) < \infty.$$

Now we set

$$C_2 := 1 + K\bar{C}_1$$

to see the equivalence between $\mathcal{H}(t)$ and $||f(t) - \bar{f}(t)||_{L^1}$:

$$||f(t) - \bar{f}(t)||_{L^1} \le \mathcal{H}(t) \le C_2||f(t) - \bar{f}(t)||_{L^1}.$$

Notice that the difference $|f-\bar{f}|^\sharp(x, v, t)$ satisfies a differential inequality.

$$\partial_t\Big(|f - \bar{f}|^\sharp(x, v, t)\Big) \le \mathcal{R}_2^\sharp(f, \bar{f})(x, v, t),$$

where

$$
\begin{aligned}
\mathcal{R}_2^\sharp(f,\bar f)(x,v,t):=\frac{1}{2\varepsilon}\iint_{\mathbb{R}^3\times\mathbb{S}^2_+}&|v-v_*|^\gamma b_\gamma(\theta)\Big(|f-\bar f|^\sharp(x+t(v-v'),v',t)\\
&\times(f^\sharp+\bar f^\sharp)(x+t(v-v'_*),v'_*,t)\\
&+|f-\bar f|^\sharp(x+t(v-v'_*),v'_*,t)(f^\sharp+\bar f^\sharp)(x+t(v-v'),v',t)\\
&+|f-\bar f|^\sharp(x,v,t)(f^\sharp+\bar f^\sharp)(x+t(v-v_*),v_*,t)\\
&+|f-\bar f|^\sharp(x+t(v-v_*),v_*,t)(f^\sharp+\bar f^\sharp)(x,v,t)\Big)d\omega dv_*.
\end{aligned}
$$

We set

$$
\mathcal{E}(f(t)):=\sup_{x,v}\iint_{\mathbb{R}^3\times\mathbb{R}_+}|v-v_*|^{\gamma-1}Q_+^\sharp(f,f)(x+t(v-v_*)+\tau n(v,v_*),v_*,t)d\tau dv_*.
$$

Below, we present some lemmas without proofs. Details can be found in [18].

Lemma 4.1.([18]) *Suppose that the main assumption* (H) *in Section* 2 *hold. Let* f *be a classical solution in* $\mathcal{S}(\varepsilon_0,\mu_1,\mu_2)$. *Then* $\mathcal{E}(f(t))$ *is integrable, i.e.,*

$$
\int_0^\infty \mathcal{E}(f(t))dt\le \mathcal{O}(1)\frac{\varepsilon_0^2}{\kappa},
$$

where $\mathcal{O}(1)$ *is a bounded function.*

Lemma 4.2.([18]) *Suppose the main assumptions* (H) *in Section* 2 *hold, and let* f *and* $\bar f$ *be two classical solutions corresponding to initial data* f_0 *and* $\bar f_0$ *respectively. Then* $||f(t)-\bar f(t)||_{L^1}$ *and* $\mathcal{D}_d(t)$ *satisfy*

(a) $\displaystyle \frac{d}{dt}||f(t)-\bar f(t)||_{L^1}\le \frac{\mathcal{O}(1)}{\kappa}\Lambda_d(t).$

(b) $\displaystyle \frac{d}{dt}\mathcal{D}_d(t)\le -C_3\Lambda_d(t)+\Big(\mathcal{E}(f(t))+\mathcal{E}(\bar f(t))\Big)||f(t)-\bar f(t)||_{L^1},$

where C_3 *is a positive constant independent of* t, *and* $\mathcal{O}(1)$ *is a bounded function.*

Theorem 4.1.([18]) *Assume that the main assumptions* (H) *in Section* 2 *hold, and let* f *and* $\bar f$ *be two classical solutions corresponding to initial data* f_0 *and* $\bar f_0$ *respectively. Then uniform* L^1*-stability estimate holds:*

$$
\sup_{0\le t<\infty}||f(t)-\bar f(t)||_{L^1}\le G||f_0-\bar f_0||_{L^1},
$$

where G is a positive constant independent of t.

Proof. By definition of $\mathcal{H}$ and Lemma 3.2, we have

$$\begin{aligned}\frac{d}{dt}\mathcal{H}(t) &= \frac{d}{dt}||f(t)-\bar{f}(t)||_{L^1} + K\frac{d}{dt}\mathcal{D}_d(t)\\ &\le K\Big(\mathcal{E}(f(t))+\mathcal{E}(\bar{f}(t))\Big)||f(t)-\bar{f}(t)||_{L^1} + \Big(\frac{\mathcal{O}(1)}{\kappa} - C_3K\Big)\Lambda_d(t).\end{aligned}$$

We now choose K sufficiently large so that

$$\frac{\mathcal{O}(1)}{\kappa} - C_3K < -C_4, \qquad \text{for some positive constant } C_4.$$

For such K and C_4, we have

$$\frac{d\mathcal{H}(t)}{dt} + C_4\Lambda_d(t) \le K\Big(\mathcal{E}(f(t))+\mathcal{E}(\bar{f}(t))\Big)\mathcal{H}(t).$$

Here we used $||f(t)-\bar{f}(t)||_{L^1} \le \mathcal{H}(t)$. The above differential inequality implies

$$\begin{aligned}\mathcal{H}(t) + C_4\int_0^t \Lambda_d(s)ds &\le \exp\Big(\int_0^t (\mathcal{E}(f(s))+\mathcal{E}(\bar{f}(s)))ds\Big)\mathcal{H}(0)\\ &\le \exp\Big(||\mathcal{E}(f)||_{L^1} + ||\mathcal{E}(\bar{f})||_{L^1}\Big)\mathcal{H}(0)\\ &\le \exp\Big(\mathcal{O}(1)\frac{\varepsilon_0^2}{\kappa}\Big)\mathcal{H}(0).\end{aligned}$$

For some positive constant C_5, we have

$$\exp\Big(\mathcal{O}(1)\frac{\varepsilon_0^2}{\kappa}\Big) \le C_5.$$

Then for such C_5, we have

$$\mathcal{H}(t) + C_4\int_0^t \Lambda_d(s)ds \le C_5\mathcal{H}(0).$$

The L^1 stability of classical solutions can be obtained as follows.

$$||f(t)-\bar{f}(t)||_{L^1} \le \mathcal{H}(t) \le C_5\mathcal{H}(0) \le C_2C_5||f_0-\bar{f}_0||_{L^1}.$$

Finally we set

$$G := C_2C_5$$

to obtain the desired result. □

Remark 4.1. In [26], Lu obtained the following a priori L^1-stability estimate [26] without smallness assumption on the size of initial data ε_0:

$$||f(t) - \bar{f}(t)||_{L^1} \leq G||f_0 - \bar{f}_0||^{\theta}_{L^1} \qquad \text{for some } \theta \in (0,1).$$

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