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A LONG WAVE APPROXIMATION FOR CAPILLARY-GRAVITY WAVES AND THE KAWAHARA EQUATION

BY

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Abstract

The Kawahara equation is a higher-order Korteweg-de Vries equation with an additional fifth order derivative term. It was derived by Hasimoto as a model of capillary-gravity waves in an infinitely long canal over a flat bottom in a long wave regime when the Bond number is nearly one third. In this paper, we give a mathematically rigorous justification of this modeling and show that the solution of the Kawahara equation approximates that of the full problem of capillary-gravity waves in an appropriate sense for a long time interval. We also consider the case where the bottom is not flat and derive coupled Kawahara type equations whose solution approximates that of the full problem in that case.

1. Introduction

We are concerned with a two-dimensional, irrotational flow of an incompressible ideal fluid with a free surface under the gravitational field. The domain occupied by the fluid is bounded from below by a solid bottom and above by an atmosphere of constant pressure. The upper surface is a free boundary and we take the influence of the surface tension into account on the free surface. Our main interest is the motion of the free surface, which

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is called a capillary-gravity wave. In the case without the surface tension, it is called a gravity wave or a water wave.

Mathematically, the problem is formulated as a free boundary problem for the incompressible Euler equation with the irrotational condition. After rewriting the equations in an appropriate non-dimensional form, we have two non-dimensional parameters δ and ε the ratio of the water depth h to the wave length λ and the ratio of the amplitude of the free surface a to the water depth h, respectively, and another non-dimensional parameter μ called the Bond number, which comes from the surface tension on the free surface. The waves characterized by the physical condition $\delta \ll 1$ are called long waves or shallow water waves, but there are several long wave approximations according to relations between ε and δ. For example, we have the following three long wave regimes.

(I) The shallow water regime: $\varepsilon = 1$ and $\delta \ll 1$.

(II) The Korteweg-de Vries regime: $\varepsilon = \delta^2 \ll 1$ and $\mu \neq \frac{1}{3}$.

(III) The Kawahara regime: $\varepsilon = \delta^4 \ll 1$ and $\mu = \frac{1}{3} + \nu\varepsilon^{1/2}$.

In the shallow water regime we obtain the so-called shallow water equations as the limit $\delta \to 0$. The shallow water equations have the same form as one-dimensional compressible Euler equation for an isentropic flow of a gas of the adiabatic index 2 and its solution generally has a singularity in finite time even if the initial data are sufficiently smooth. Therefore, this long wave regime is used to explain breaks of water waves. In the long wave regime (II), Korteweg and de Vries [24] derived a very notable equation, which is called the KdV equation, from the equations for capillary-gravity waves. The KdV equation takes of the form

$$\pm 2u_t + 3uu_x + \Big(\frac{1}{3} - \mu\Big)u_{xxx} = 0.$$

When $\mu = \frac{1}{3}$, this equation degenerates to the inviscid Burgers equation. In connection with this critical Bond number, Hasimoto [10] derived a higher-order KdV equation of the form

$$\pm 2u_t + 3uu_x - \nu u_{xxx} + \frac{1}{45}u_{xxxxx} = 0$$

in the long wave regime (III), which is nowadays called the Kawahara equation. Historically, this type of equation was first found by Kakutani and

Ono [18] in an analysis of magnet-acoustic waves in a cold collision free plasma. Then, Hasimoto [10] derived the above equation from capillary-gravity waves. Kawahara [23] studied this type of equation numerically and observed that the equation has both oscillatory and monotone solitary wave solutions.

There are several results giving a mathematically rigorous justification for the long wave approximations. The justification of the shallow water equations was given by Ovsjannikov [35] under the periodic boundary condition with respect to the horizontal spatial variable, and then by Kano and Nishida [20]. In order to guarantee the existence of the solution for the full equations, they used an abstract Cauchy-Kowalevski theorem in a scaled Banach space due to Ovsjannikov [33, 34] and its modified version by Nirenberg [29] and Nishida [30], so that the analyticity of the initial data is required. Kano and Nishida [21] also established the Friedrichs expansion, which is an expansion of the solution with respect to δ^2. Similar arguments in the three-dimensional case are found in Kano [19]. Here, it should be mentioned that these results were given in a class of analytic functions and that the justification in the framework of Sobolev spaces is still untouched.

Concerning the KdV approximation for water waves, Kano and Nishida [22] gave the justification in a class of analytic functions. Based on the existence theorem due to Nalimov [27] and Yosihara [46], Craig [6] gave the justification in the framework of Sobolev spaces. In the long wave regime (II), the dynamics of the free surface is approximately translation of two waves without change of the shape, one moving to the right and the other to the left, for a short time interval $0 \le t \le O(1)$. The dynamics of each waves is very slow so that it is invisible for the short time interval. By introducing a slow time scale $\tau = \varepsilon t$, the dynamics can be visible and described by the KdV equation for a long time interval $0 \le t \le O(1/\varepsilon)$. One of the difficulties in the justification is to obtain a uniform estimate with respect to ε of the solution of the initial value problem for the full water waves for the long time interval. Craig established well the estimate under a restriction on the initial data which emphasizes that the wave is almost one-directional. Then, Schneider and Wayne gave the justification without assuming the one-directional motion of the wave in [38] and extended it to the capillary-gravity waves in [40]. They showed that the interactions between two waves are negligible so that the solution of the full water wave problem is approximated by a sum of the solutions, which are appropriately scaled, of the decoupled KdV equations for the long time interval. However, they treated the problem in unscaled variables, whereas Craig treated it in the scaled variables called Boussinesq ones. Let $\tilde{\eta}(\tilde{x},\tilde{t})$ and $\eta(x,t)$ be the wave heights measured from an undisturbed wave level in unscaled and the scaled variables, respectively. In the long wave regime (II), these are related by the formula $\tilde{\eta}(\tilde{x},\tilde{t}) = \varepsilon\eta(\varepsilon^{1/2}\tilde{x},\varepsilon^{1/2}\tilde{t})$, so that their norms in the Sobolev space H^m are related by

$$\begin{aligned}\|\tilde{\eta}(\cdot,\tilde{t})\|_m^2 &= \|\tilde{\eta}(\cdot,\tilde{t})\|^2 + \|\partial_{\tilde{x}}\tilde{\eta}(\cdot,\tilde{t})\|^2 + \cdots + \|\partial_{\tilde{x}}^m\tilde{\eta}(\cdot,\tilde{t})\|^2 \\ &= \varepsilon^{3/2}\bigl(\|\eta(\cdot,\varepsilon^{1/2}\tilde{t})\|^2 + \varepsilon\|\partial_x\eta(\cdot,\varepsilon^{1/2}\tilde{t})\|^2 + \cdots + \varepsilon^m\|\partial_x^m\eta(\cdot,\varepsilon^{1/2}\tilde{t})\|^2\bigr).\end{aligned}$$

This means that the norm of the Sobolev space in unscaled variables corresponds to the weighted (with respect to ε) norm of the Sobolev space in the Boussinesq variables and that uniform estimates of the solution in unscaled variables do not give those in the Boussinesq ones. The latter uniform estimates are very important when one tries to justify the formal derivation of the KdV equation by nonlinear perturbation methods, because in the methods one treats the equations in the Boussinesq variables and assumes tacitly the uniform boundedness with respect to ε of the solution and its derivatives. Therefore, the estimates obtained by Schneider and Wayne are somewhat weak and do not recover those by Craig even if the initial data are restricted in order that the wave is almost one-directional. Moreover, from the viewpoint of well-posedness of the initial value problem, one should estimate the solution in the same class (Sobolev space) and in the same variables (Boussinesq ones) as those of the initial data. Note that they assumed the uniform boundedness in the Sobolev space of the initial data in the Boussinesq variables. Recently, in [14] the author gave the justification of this KdV approximation for capillary-gravity waves together with uniform estimates of the solution and the estimate of the error term in the Boussinesq variables. It was shown that the norm of the error term in the Sobolev space is of order ε, which is the optimal rate, while in Schneider and Wayne [40] the L^∞-norm of the error term is of order $\varepsilon^{1/6}$. Moreover, the author considered this KdV approximation in the case where the bottom is not flat and studied an effect of the bottom to the approximation, and then coupled KdV type equations were derived.

The motion of the free surface in the long wave regime (III) is somewhat similar to that in the regime (II), that is, the dynamics of the free surface is approximately translation of two waves without change of the shape, one moving to the right and the other to the left, for a short time interval $0 \leq t \leq O(1)$. By introducing a slow time scale $\tau = \varepsilon t$, we see that the dynamics of each waves can be described by the Kawahara equation for a long time interval $0 \leq t \leq O(1/\varepsilon)$. In [40] Schneider and Wayne also discussed the validity of this Kawahara approximation. However, they treated the problem in unscaled variables so that their estimates do not give uniform estimates of derivatives of the solution in the scaled variables. One of the main purpose of this paper is to refine the result due to Schneider and Wayne by giving uniform estimates of the solution in the scaled variables. Another purpose is to analyze this long wave approximation in the case where the bottom is not flat and to derive simple equations (coupled Kawahara type equations) whose solution approximates that of the original equations for a long time interval $0 \leq t \leq O(1/\varepsilon)$. We remark that the time interval of existence is the same order as that in [40] but estimates of the solution, especially those of its derivatives, are much stronger than those in [40] and that the norm of the error term in the Sobolev space is of order $\varepsilon^{1/2}$, which is the optimal rate, while in [40] the L^∞-norm of the error term is of order $\varepsilon^{1/8}$. See Remark 4.2 for the details.

There are many papers studying the effect of an uneven bottom on the long wave approximations for water waves. By introducing appropriate coordinate-stretching and applying a reductive perturbation method, Kakutani [17] derived a KdV type equation with variable coefficients and a lowest order term. Rosales and Papanicolaou [37] studied the effect of periodic and random bottom topography. In the former case they derived the KdV equation with effective coefficients, which are constants, by using a perturbation method combined with a multiscale ansatz. Moreover, they showed that the waves are delayed with respect to a flat bottom because of an uneven periodic bottom. Recently, Craig, Guyenne, Nicholls, and Sulem [7] extended the work of Rosales and Papanicolaou [37] to the case of bottom topography with multiple spatial scales and the three dimensional case. They adopted a Hamiltonian formulation of the problem, which goes back to the work of Zakharov [48] in the case of deep water, and derived systematically the KdV and the KP equations with effective coefficients by using an asymptotic analysis of multiple scale operators and the homogenisation. In these results they treated the case where the amplitude of the bottom variations is of order 1, whereas in this paper we assume that it is of order ε. Therefore, we treat the problem in an easier case. However, we give the existence theorem of the initial value problem for the full problem together with uniform estimates of the solution for an appropriately long time interval, which is very important in order to give a mathematically rigorous justification of the approximation.

This part was untouched in the above papers, and we also leave it open in the case where no assumptions is posed on the bottom variations.

The contents of this paper are as follows. In section 2 we formulate the problem, rewrite it in a non-dimensional form and transform it into an equivalent problem according to [13, 14]. In section 3 we give several properties of the Dirichlet-to-Dirichlet map K for the Cauchy-Riemann operator which occurs in the transformed problem. In section 4 we formally derive coupled Kawahara type equations as a long wave approximation and give the statements of our main theorems. In section 5 we reduce the system derived in section 2 to a quasi-linear system of equations. In section 6 we give several estimates for remainder terms appearing in the quasi-linearization. Finally, in section 7, by applying the energy estimates established in [14] to the quasi-linear equations derived in section 5 we prove the main theorems. In Appendix we give a priori estimates for the Kawahara equation.

Notation. For $s \in \mathbf{R}$, we denote by H^s the Sobolev space of order s on $\mathbf{R}$ equipped with the inner product $(u,v)_s = \frac{1}{2\pi}\int_{\mathbf{R}}(1+\xi^2)^s\hat{u}(\xi)\overline{\hat{v}(\xi)}d\xi$, where $\hat{u}$ is the Fourier transform of u, that is, $\hat{u}(\xi) = \int_{\mathbf{R}} u(x)\mathrm{e}^{-ix\xi}dx$. We put $\|u\|_s = \sqrt{(u,u)_s}$, $(u,v) = (u,v)_0$, and $\|u\| = \|u\|_0$. For a non-negative integer m and a real γ, we denote by $H^{m,\gamma}$ the weighted Sobolev space on $\mathbf{R}$ equipped with the norm $\|u\|_{m,\gamma} = \big(\sum_{l=0}^m \|\langle x\rangle^\gamma \big(\frac{d}{dx}\big)^l u\|^2\big)^{1/2}$, where $\langle x\rangle = (1+x^2)^{1/2}$. For $1 \le p \le \infty$, we denote by $|\cdot|_p$ the norm of the Lebesgue space $L^p = L^p(\mathbf{R})$. For a non-negative integer m, we denote by $W^{m,\infty}$ the Banach space of all functions $u = u(x)$ on $\mathbf{R}$ such that $\big(\frac{d}{dx}\big)^l u \in L^\infty$ for $0 \le l \le m$ with the norm $\|u\|_{W^{m,\infty}} = \max_{0\le l\le m}|\big(\frac{d}{dx}\big)^l u|_\infty$. For $0 < T < \infty$, a non-negative integer j, and a Banach space X, we denote by $C^j([0,T];X)$ the Banach space of all functions of C^j-class on the interval $[0,T]$ with the value in X. A pseudo-differential operator $P(D)$, $D = -i\frac{d}{dx}$, with a symbol $P(\xi)$ is defined by $P(D)u(x) = \frac{1}{2\pi}\int_{\mathbf{R}} P(\xi)\hat{u}(\xi)\mathrm{e}^{ix\xi}d\xi$. For operators A and B, we denote by $[A,B] = AB-BA$ the commutator. Throughout this paper, we denote inessential constants by the same symbol C.

2. Formulation of the Problem

We assume that the domain $\Omega(t)$ occupied by the fluid at time $t \ge 0$, the free surface $\Gamma(t)$, and the bottom Σ are of the forms

$$\Omega(t) \;=\; \big\{(x,y) \in \mathbf{R}^2\,;\, b(x) < y < h + \eta(x,t)\big\},$$

$$\begin{aligned}\Gamma(t) &= \{(x,y)\in\mathbf{R}^2\,;\ y=h+\eta(x,t)\},\\ \Sigma &= \{(x,y)\in\mathbf{R}^2\,;\ y=b(x)\},\end{aligned}$$

where h is the mean depth of the fluid. In this paper b is a given function, while η is the unknown. The motion of the fluid is described by the velocity $v=(v_1,v_2)$ and the pressure p satisfying the equations

$$\begin{cases}\rho\big(v_t+(v\cdot\nabla)v\big)+\nabla p=-\rho(0,g),\\ \nabla\cdot v=0,\quad \nabla^{\perp}\cdot v=0 \qquad \text{in}\quad \Omega(t),\quad t>0,\end{cases}\tag{2.1}$$

where ρ is the constant density and g is the gravitational constant. It is assumed that both ρ and g are positive constants. The dynamical and the kinematical boundary conditions on the free surface are given by

$$\begin{cases}p=p_0-\sigma H,\\ \big(\partial_t+v\cdot\nabla\big)\big(y-\eta(x,t)\big)=0 \qquad \text{on}\quad \Gamma(t),\quad t>0,\end{cases}\tag{2.2}$$

where p_0 is the atmospheric pressure, σ is the surface tension coefficient, and H is the curvature of the free surface. It is assumed that p_0 is a constant and σ is a positive constant. In our parametrization of the free surface the curvature H at the point $\big(x,h+\eta(x,t)\big)$ is written as

$$H(x,t)=\Big(\big(1+\big(\eta_x(x,t)\big)^2\big)^{-1/2}\eta_x(x,t)\Big)_x.$$

The boundary condition on the bottom is given by

$$v\cdot N=0 \qquad \text{on}\quad \Sigma,\quad t>0,\tag{2.3}$$

where N is the unit normal vector to Σ. Finally, we impose the initial conditions

$$\eta(x,0)=\eta_0(x),\quad v(x,y,0)=v_0(x,y).\tag{2.4}$$

It is assumed that the initial data satisfy the compatibility conditions, that is,

$$\begin{cases}\nabla\cdot v_0=0,\quad \nabla^{\perp}\cdot v_0=0\quad \text{in}\quad \Omega(0),\\ v_0\cdot N=0\quad \text{on}\quad \Sigma.\end{cases}$$

We proceed to rewrite the equations (2.1)–(2.4) in an appropriate non-dimensional form. Let λ be a horizontal characteristic length of the wave and a the maximum vertical amplitude of the free surface. We introduce three non-dimensional parameters δ, ε, and μ by

$$\delta = \frac{h}{\lambda}, \quad \varepsilon = \frac{a}{h}, \quad \text{and} \quad \mu = \frac{\sigma}{\rho g h^2}$$

respectively. In this paper we will consider an asymptotic behavior of capillary-gravity waves when δ and ε tend to zero keeping the relations

$$\delta^4 = \varepsilon \quad \text{and} \quad \mu = \frac{1}{3} + \nu \varepsilon^{1/2},$$

where ν is a constant. We rescale the independent and dependent variables by

$$\begin{cases} x = \lambda \tilde{x}, \quad y = h\tilde{y}, \quad t = \dfrac{\lambda}{\sqrt{gh}}\,\tilde{t}, \\ v_1 = \dfrac{a}{h}\sqrt{gh}\,\tilde{v}_1, \quad v_2 = \dfrac{a}{\lambda}\sqrt{gh}\,\tilde{v}_2, \\ p = p_0 + \rho g h \tilde{p}, \quad \eta = a\tilde{\eta}, \quad b = a\tilde{b}. \end{cases} \tag{2.5}$$

We call these new variables Boussinesq ones. Here, we note that the function b of the bottom is rescaled by a the maximum vertical amplitude of the free surface. Putting these into (2.1)–(2.4) and dropping the tilde sign in the notation we obtain

$$\begin{cases} \varepsilon v_{1t} + \varepsilon^2 (v_1 v_{1x} + v_2 v_{1y}) + p_x = 0, \\ \varepsilon^{3/2} v_{2t} + \varepsilon^{5/2}(v_1 v_{2x} + v_2 v_{2y}) + p_y + 1 = 0, \\ v_{1x} + v_{2y} = 0, \quad v_{1y} - \varepsilon^{1/2} v_{2x} = 0 \qquad \text{in} \quad \Omega^\varepsilon(t), \quad t > 0, \end{cases} \tag{2.6}$$

$$\begin{cases} p = -\varepsilon^{3/2}\mu\big((1+\varepsilon^{5/2}\eta_x^2)^{-1/2}\eta_x\big)_x, \\ \eta_t + \varepsilon v_1 \eta_x - v_2 = 0 \qquad \text{on} \quad \Gamma^\varepsilon(t), \quad t > 0, \end{cases} \tag{2.7}$$

$$\varepsilon b' v_1 - v_2 = 0 \qquad \text{on} \quad \Sigma^\varepsilon, \quad t > 0, \tag{2.8}$$

$$\eta(x,0) = \eta_0(x), \quad v(x,y,0) = v_0(x,y), \tag{2.9}$$

where

$$\begin{aligned} \Omega^\varepsilon(t) &= \big\{(x,y) \in \mathbf{R}^2\,;\, \varepsilon b(x) < y < 1 + \varepsilon\eta(x,t)\big\}, \\ \Gamma^\varepsilon(t) &= \big\{(x,y) \in \mathbf{R}^2\,;\, y = 1 + \varepsilon\eta(x,t)\big\}, \end{aligned}$$

$$\Sigma^\varepsilon = \{(x,y) \in \mathbf{R}^2\,;\ y = \varepsilon b(x)\}.$$

The function b and the initial data η_0 and v_0 may depend on ε.

According to [13, 14], we reformulate the initial value problem (2.6)–(2.9) as a problem on the free surface. Put

$$u(x,t) = v(x, 1+\varepsilon\eta(x,t), t),$$

which is the boundary value of the velocity on the free surface. Then, we see that the unknowns η and $u = (u_1, u_2)$ are governed by the equations

$$\begin{cases} u_{1t} + \eta_x + \varepsilon u_1 u_{1x} + \varepsilon^{3/2}\eta_x(u_{2t} + \varepsilon u_1 u_{2x}) \\ \quad = \varepsilon^{1/2}\mu\big((1+\varepsilon^{5/2}\eta_x^2)^{-1/2}\eta_x\big)_{xx}, \\ \eta_t + \varepsilon u_1 \eta_x - u_2 = 0, \\ u_2 = K(\eta, b, \varepsilon)u_1 \qquad \text{for} \quad t > 0, \end{cases} \tag{2.10}$$

$$\eta = \eta_0, \quad u_1 = u_0 \qquad \text{at} \quad t = 0, \tag{2.11}$$

where $K = K(\eta, b, \varepsilon)$ is a linear operator depending on η, b, and ε, whose explicit form will be given in the next section. We refer to [14] for the derivation of these equations. This is the initial value problem that we are going to investigate in this paper. Here, we emphasize that the initial value problem (2.10) and (2.11) is equivalent to (2.6)–(2.9) and that we did not neglect any terms in the derivation of (2.10).

3. The Operator K

The Dirichlet-to-Dirichlet map K for the Cauchy-Riemann equations can be written explicitly in terms of integral operators as

$$K = -\varepsilon^{-1/4}\Big(\frac{1}{2} - B_2\Big)^{-1} B_1, \tag{3.1}$$

where

$$\begin{cases} B_1 = A_2 + (\varepsilon^{5/4}A_5 b' - A_6)\Big(\dfrac{1}{2} + A_3 + \varepsilon^{5/4}A_4 b'\Big)^{-1} A_7, \\ B_2 = A_1 - (\varepsilon^{5/4}A_5 b' - A_6)\Big(\dfrac{1}{2} + A_3 + \varepsilon^{5/4}A_4 b'\Big)^{-1} A_8. \end{cases} \tag{3.2}$$

Here, $A_1, \dots, A_8$ are integral operators, which map real valued functions to real valued ones, defined by

$$
\begin{cases}
(A_1 + iA_2)f(x) = \dfrac{i}{2}(i \operatorname{sgn} D)f(x) \\
\qquad\qquad + \dfrac{1}{2\pi i}\displaystyle\int_{\mathbf{R}} \log\Big(1 + i\varepsilon^{5/4}\frac{\eta(y,t)-\eta(x,t)}{y-x}\Big)\frac{df}{dy}(y)dy, \\
(A_3 + iA_4)f(x) = \dfrac{i}{2}(i \operatorname{sgn} D)f(x) \\
\qquad\qquad + \dfrac{1}{2\pi i}\displaystyle\int_{\mathbf{R}} \log\Big(1 + i\varepsilon^{5/4}\frac{b(y)-b(x)}{y-x}\Big)\frac{df}{dy}(y)dy, \\
(A_5 + iA_6)f(x) = \dfrac{1}{2}\mathrm{e}^{-\varepsilon^{1/4}|D|}\big(-1 + i(i \operatorname{sgn} D)\big)f(x) \\
\qquad\qquad + \dfrac{1}{2\pi i}\displaystyle\int_{\mathbf{R}} \log\Big(1 + i\varepsilon^{5/4}\frac{b(y)-\eta(x,t)}{y-x-i\varepsilon^{1/4}}\Big)\frac{df}{dy}(y)dy, \\
(A_7 + iA_8)f(x) = \dfrac{1}{2}\mathrm{e}^{-\varepsilon^{1/4}|D|}\big(1 + i(i \operatorname{sgn} D)\big)f(x) \\
\qquad\qquad + \dfrac{1}{2\pi i}\displaystyle\int_{\mathbf{R}} \log\Big(1 + i\varepsilon^{5/4}\frac{\eta(y,t)-b(x)}{y-x+i\varepsilon^{1/4}}\Big)\frac{df}{dy}(y)dy.
\end{cases}
\tag{3.3}
$$

We can expand this operator $K = K(\eta, b, \varepsilon)$ in terms of (η, b) as

$$
K = \sum_{k=0}^{n-1} K_k + \widetilde{K}_n, \tag{3.4}
$$

where the linear operator K_k is homogeneous of degree k in (η, b) and can be written in terms of pseudo-differential operators. In fact, it holds that

$$
\begin{cases}
K_0 = -\varepsilon^{-1/4} i \tanh(\varepsilon^{1/4}D), \\
K_1 = -\varepsilon\big(\eta + i\tanh(\varepsilon^{1/4}D)\eta\, i\tanh(\varepsilon^{1/4}D)\big)(iD) \\
\qquad + \varepsilon \operatorname{sech}(\varepsilon^{1/4}D)(iD)b \operatorname{sech}(\varepsilon^{1/4}D).
\end{cases}
\tag{3.5}
$$

Remark 3.1. Under suitable assumptions on η and b, for each positive ε the operator K_1 possesses a smoothing property so that we do not need the expression of K_1 when we fix ε. However, in order to get uniform estimates of the solution for the initial value problem (2.10) and (2.11) with respect to ε the above explicit formula for K_1 plays an important role.

In the next lemma, the time t is arbitrarily fixed, so that $\eta(x,t)$ is simply denoted by $\eta(x)$. We refer to [14] for the proof.

Lemma 3.1. *Let m, m_0, and n be positive integers satisfying $m, m_0 \geq 2$ and $n + m \geq m_0$. Put $m_1 = \max\{m, m_0 - 1\}$ and $m_2 = \max\{m, m_0\} + 1$. There exist constants $C > 0$ and $\delta_1 > 0$ such that for any $\eta \in H^{m_1}$, $b \in W^{m_2,\infty}$, and $\varepsilon \in (0, 1]$ satisfying $\varepsilon(\|\eta\|_{m_1} + \|b\|_{W^{m_2,\infty}}) \leq \delta_1$ we have*

$$\|\widetilde{K}_n f\|_m \leq C\varepsilon^{-(m-m_0+1)/4}\big(\varepsilon(\|\eta\|_{m_1} + \|b\|_{W^{m_2,\infty}})\big)^n \|f\|_{m_0}.$$

Remark 3.2. This estimate says that $\widetilde{K}_n$ has a smoothing property, which is very important to the existence theory for the initial value problem (2.10) and (2.11). But, if we use the smoothing property, then we lose a power of ε and we shall face a difficulty when we try to get uniform estimates of the solution with respect to ε. However, taking n sufficiently large, we gain a power of ε. As we will see later, it is sufficient to expand the operator K up to $n = 2$.

Remark 3.3. By virtue of Taylor's formula we have $\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + O(x^7)$ and $\operatorname{sech} x = 1 + O(x^2)$ so that (3.5) implies

$$\begin{cases} K_0 = -(iD) - \frac{\varepsilon^{1/2}}{3}(iD)^3 - \frac{2}{15}\varepsilon(iD)^5 + O(\varepsilon^{3/2}), \\ K_1 = -\varepsilon\eta(iD) + \varepsilon(iD)b + O(\varepsilon^{3/2}). \end{cases}$$

By Lemma 3.1 $\widetilde{K}_2 = O(\varepsilon^2)$, so that we obtain

$$K = -(1 + \varepsilon\eta)(iD) + \varepsilon(iD)b - \frac{\varepsilon^{1/2}}{3}(iD)^3 - \frac{2}{15}\varepsilon(iD)^5 + O(\varepsilon^{3/2}).$$

Here, we should note that the remainder term $O(\varepsilon^{3/2})$ contains higher-order derivatives. This is one of the reason why we require much differentiability on the data.

It is easy to see that the commutators of the operator $\widetilde{K}_n$ and the differential operators $\partial_x = \frac{\partial}{\partial x}$ and $\partial_t^k = \big(\frac{\partial}{\partial t}\big)^k$ have similar forms as $\widetilde{K}_n$, so that we can show the following lemmas.

Lemma 3.2. *Let m, m_0, and n be positive integers satisfying $m, m_0 \geq 2$ and $n + m \geq m_0$. Put $m_1 = \max\{m, m_0 - 1\}$ and $m_2 = \max\{m, m_0\} + 1$. There exist constants $C > 0$ and $\delta_1 > 0$ such that for any $\eta \in H^{m_1+1}$, $b \in W^{m_2+1,\infty}$, and $\varepsilon \in (0, 1]$ satisfying $\varepsilon(\|\eta\|_{m_1} + \|b\|_{W^{m_2,\infty}}) \leq \delta_1$ we have*

$$\|\big[\partial_x, \widetilde{K}_n\big] f\|_m \leq C\varepsilon^{-(m-m_0+1)/4}\big(\varepsilon(\|\eta\|_{m_1+1} + \|b\|_{W^{m_2+1,\infty}})\big)$$

$$\times \big(\varepsilon(\|\eta\|_{m_1} + \|b\|_{W^{m_2,\infty}})\big)^{n-1}\|f\|_{m_0}.$$

Lemma 3.3. *Let m and m_0 be positive integers satisfying $m, m_0 \geq 2$ and $m+1 \geq m_0$. Put $m_2 = \max\{m, m_0\} + 1$. There exist constants $C > 0$ and $\delta_1 > 0$ such that for any $\eta \in C^2([0,T]; H^m)$, $b \in W^{m_2,\infty}$, and $\varepsilon \in (0,1]$ satisfying $\varepsilon(\|\eta\|_m + \|b\|_{W^{m_2,\infty}}) \leq \delta_1$ we have*

$$\begin{cases} \|\big[\partial_t, \widetilde{K}_1\big] f\|_m \leq C\varepsilon^{-(m-m_0+1)/4}\big(\varepsilon\|\eta_t\|_m\big)\|f\|_{m_0}, \\ \|\big[\partial_t, \big[\partial_t, \widetilde{K}_1\big]\big] f\|_m \leq C\varepsilon^{-(m-m_0+1)/4}\big(\varepsilon\|\eta_{tt}\|_m + (\varepsilon\|\eta_t\|_m)^2\big)\|f\|_{m_0}. \end{cases}$$

4. Formal Asymptotic Analysis and Main Results

In this section we begin to study formally an asymptotic behavior of the solution $(\eta^\varepsilon, u^\varepsilon)$ to the initial value problem (2.10) and (2.11) when ε tends to 0 and derive the Kawahara equation, whose solution approximates $(\eta^\varepsilon, u^\varepsilon)$ in a suitable sense. Then, we state our main results.

It follows from (2.10) and Remark 3.3 that

$$\begin{cases} u_{1t} + \eta_x + \varepsilon u_1 u_{1x} - \varepsilon^{1/2}\mu\eta_{xxx} = O(\varepsilon^{3/2}), \\ \eta_t + u_{1x} + \varepsilon\big((\eta - b)u_1\big)_x + \dfrac{\varepsilon^{1/2}}{3}u_{1xxx} + \dfrac{2}{15}\varepsilon u_{1xxxxx} = O(\varepsilon^{3/2}), \end{cases} \tag{4.1}$$

which approximate the equations in (2.10) up to order $O(\varepsilon^{3/2})$.

Now, let us consider the limiting case $\varepsilon = 0$. Then, the equations in (4.1) become

$$\begin{cases} u_{1t} + \eta_x = 0, \\ \eta_t + u_{1x} = 0. \end{cases}$$

Under the initial condition (2.11) this system can be easily solved and the solution has the form

$$\begin{pmatrix} u_1(x,t) \\ \eta(x,t) \end{pmatrix} = \begin{pmatrix} \alpha_1(x-t) - \alpha_2(x+t) \\ \alpha_1(x-t) + \alpha_2(x+t) \end{pmatrix},$$

where the functions α_1 and α_2 are determined from the initial data η_0 and u_0 by

$$\alpha_1(x) = \frac{1}{2}\big(\eta_0(x) + u_0(x)\big), \quad \alpha_2(x) = \frac{1}{2}\big(\eta_0(x) - u_0(x)\big). \tag{4.2}$$

For the case $0 < \varepsilon \ll 1$ we can show that under suitable assumptions on the data the initial value problem (2.10) and (2.11) has a unique solution $(\eta, u) = (\eta^\varepsilon, u^\varepsilon)$ on some time interval and that the solution satisfies

$$\begin{pmatrix} u_1^\varepsilon(x,t) \\ \eta^\varepsilon(x,t) \end{pmatrix} \simeq \begin{pmatrix} \alpha_1(x-t) - \alpha_2(x+t) \\ \alpha_1(x-t) + \alpha_2(x+t) \end{pmatrix} \tag{4.3}$$

in an appropriate sense. Therefore, the dynamics of the free surface is approximately as follows: the free surface divides into two waves, one moving to the right and the other to the left with the same speed 1 without changing their shapes. Here we should note that the approximation (4.3) is valid only for the short time interval $0 \le t \le O(1)$. Roughly speaking, this means that the dynamics is only translation for such a time interval.

In order to study the dynamics for a long time interval $0 \le t \le O(1/\varepsilon)$ we have to take account of dynamics of the shapes of the two waves. Since the dynamics is very slow, it is convenient to use a slow time scale $\tau = \varepsilon t$ in order to make the dynamics to be visible. It is natural to expect that the shapes of the two waves shall change in this time scale τ so that the functions $\alpha_1(x)$ and $\alpha_2(x)$, which describe the shapes of the waves in moving coordinates, should be replaced by the functions $\alpha_1(x,\tau)$ and $\alpha_2(x,\tau)$. These considerations lead the ansatz

$$\begin{cases} u_1(x,t) = \alpha_1(x-t,\varepsilon t) - \alpha_2(x+t,\varepsilon t) \\ \qquad\qquad - \varepsilon^{1/2}\big(\beta_1(x-t,\varepsilon t) - \beta_2(x+t,\varepsilon t)\big) \\ \qquad\qquad - \varepsilon\big(\gamma_1(x-t,\varepsilon t) - \gamma_2(x+t,\varepsilon t)\big) + \varepsilon\bar{\phi}_1(x,t), \\ \eta(x,t) = \alpha_1(x-t,\varepsilon t) + \alpha_2(x+t,\varepsilon t) + \varepsilon\bar{\phi}_2(x,t). \end{cases} \tag{4.4}$$

Putting these into (4.1) and using the relation $\mu = \frac{1}{3} + \nu\varepsilon^{1/2}$ we obtain

$$\begin{aligned} &(\alpha_{1\tau} + \alpha_1\alpha_{1x} - \nu\alpha_{1xxx} + \gamma_{1x}) - (\alpha_{2\tau} - \alpha_2\alpha_{2x} + \nu\alpha_{2xxx} - \gamma_{2x}) \\ &\quad + \varepsilon^{-1/2}\Big\{\Big(\beta_1 - \frac{1}{3}\alpha_{1xx}\Big) + \Big(\beta_2 - \frac{1}{3}\alpha_{2xx}\Big)\Big\}_x - (\alpha_1\alpha_2)_x + \bar{\phi}_{1t} + \bar{\phi}_{2x} = O(\varepsilon^{1/2}) \end{aligned}$$

and

$$\begin{aligned} &\Big(\alpha_{1\tau} + 2\alpha_1\alpha_{1x} + \frac{2}{15}\alpha_{1xxxxx} - \frac{1}{3}\beta_{1xxx} - \gamma_{1x}\Big) \\ &\quad + \Big(\alpha_{2\tau} - 2\alpha_2\alpha_{2x} - \frac{2}{15}\alpha_{2xxxxx} + \frac{1}{3}\beta_{2xxx} + \gamma_{2x}\Big) \\ &\quad - \varepsilon^{-1/2}\Big\{\Big(\beta_1 - \frac{1}{3}\alpha_{1xx}\Big) - \Big(\beta_2 - \frac{1}{3}\alpha_{2xx}\Big)\Big\}_x - \big(b(\alpha_1 - \alpha_2)\big)_x + \bar{\phi}_{2t} + \bar{\phi}_{1x} = O(\varepsilon^{1/2}), \end{aligned}$$

which are equivalent to the equations

$$
\begin{aligned}
&2\alpha_{1\tau} + 3\alpha_1\alpha_{1x} - \nu\alpha_{1xxx} + \frac{2}{15}\alpha_{1xxxxx} - \frac{1}{3}\beta_{1xxx} \\
&\quad + \Bigl(2\gamma_2 - \frac{1}{2}\alpha_2^2 - \nu\alpha_{2xx} - \frac{2}{15}\alpha_{2xxxx} + \frac{1}{3}\beta_{2xx}\Bigr)_x + 2\varepsilon^{-1/2}\Bigl(\beta_2 - \frac{1}{3}\alpha_{2xx}\Bigr)_x \\
&\quad - \bigl(\alpha_1\alpha_2 + b(\alpha_1 - \alpha_2)\bigr)_x + (\bar\phi_1 + \bar\phi_2)_t + (\bar\phi_1 + \bar\phi_2)_x = O(\varepsilon^{1/2})
\end{aligned}
$$

and

$$
\begin{aligned}
&2\alpha_{2\tau} - 3\alpha_2\alpha_{2x} + \nu\alpha_{2xxx} - \frac{2}{15}\alpha_{2xxxxx} + \frac{1}{3}\beta_{2xxx} \\
&\quad - \Bigl(2\gamma_1 - \frac{1}{2}\alpha_1^2 - \nu\alpha_{1xx} - \frac{2}{15}\alpha_{1xxxx} + \frac{1}{3}\beta_{1xx}\Bigr)_x - 2\varepsilon^{-1/2}\Bigl(\beta_1 - \frac{1}{3}\alpha_{1xx}\Bigr)_x \\
&\quad + \bigl(\alpha_1\alpha_2 - b(\alpha_1 - \alpha_2)\bigr)_x - (\bar\phi_1 - \bar\phi_2)_t + (\bar\phi_1 - \bar\phi_2)_x = O(\varepsilon^{1/2}).
\end{aligned}
$$

Here, we define the corrective terms $\beta = (\beta_1, \beta_2)$, $\gamma = (\gamma_1, \gamma_2)$, and $\bar\phi = (\bar\phi_1, \bar\phi_2)$ by

$$
\beta_1(x,\tau) = \frac{1}{3}\alpha_{1xx}(x,\tau), \quad \beta_2(x,\tau) = \frac{1}{3}\alpha_{2xx}(x,\tau), \tag{4.5}
$$

$$
\begin{cases}
\gamma_1(x,\tau) = \dfrac{1}{4}\alpha_1(x,\tau)^2 + \dfrac{\nu}{2}\alpha_{1xx}(x,\tau) + \dfrac{1}{90}\alpha_{1xxxx}(x,\tau), \\
\gamma_2(x,\tau) = \dfrac{1}{4}\alpha_2(x,\tau)^2 + \dfrac{\nu}{2}\alpha_{2xx}(x,\tau) + \dfrac{1}{90}\alpha_{2xxxx}(x,\tau),
\end{cases} \tag{4.6}
$$

and

$$
\begin{cases}
\bar\phi_1(x,t) + \bar\phi_2(x,t) = b(x)\alpha_1(x-t,\varepsilon t) - \dfrac{1}{2}b(x)\alpha_2(x+t,\varepsilon t) \\
\qquad\qquad + \dfrac{1}{2}\alpha_1(x-t,\varepsilon t)\alpha_2(x+t,\varepsilon t), \\
\bar\phi_1(x,t) - \bar\phi_2(x,t) = \dfrac{1}{2}b(x)\alpha_1(x-t,\varepsilon t) - b(x)\alpha_2(x+t,\varepsilon t) \\
\qquad\qquad - \dfrac{1}{2}\alpha_1(x-t,\varepsilon t)\alpha_2(x+t,\varepsilon t).
\end{cases} \tag{4.7}
$$

Then, the above equations become

$$
\begin{aligned}
&\Bigl(2\alpha_{1\tau} + 3\alpha_1\alpha_{1x} - \nu\alpha_{1xxx} + \frac{1}{45}\alpha_{1xxxxx}\Bigr)(x-t,\varepsilon t) \\
&\quad - \bigl(b(x) + \alpha_2(x+t,\varepsilon t)\bigr)\alpha_{1x}(x-t,\varepsilon t) + \frac{1}{2}b'(x)\alpha_2(x+t,\varepsilon t) = O(\varepsilon^{1/2})
\end{aligned}
$$

and

$$
\begin{aligned}
&\Bigl(2\alpha_{2\tau} - 3\alpha_2\alpha_{2x} + \nu\alpha_{2xxx} - \frac{1}{45}\alpha_{2xxxxx}\Bigr)(x+t,\varepsilon t) \\
&\quad + \bigl(b(x) + \alpha_1(x-t,\varepsilon t)\bigr)\alpha_{2x}(x+t,\varepsilon t) - \frac{1}{2}b'(x)\alpha_1(x-t,\varepsilon t) = O(\varepsilon^{1/2}).
\end{aligned}
$$

Neglecting the terms $O(\varepsilon^{1/2})$ in the above equations we arrive at the following coupled Kawahara type equations

$$\begin{cases} 2\alpha_{1\tau} + 3\alpha_1\alpha_{1x} - \nu\alpha_{1xxx} + \dfrac{1}{45}\alpha_{1xxxxx} \\ -\big((T_{\tau/\varepsilon}b) + (T_{2\tau/\varepsilon}\alpha_2)\big)\alpha_{1x} + \dfrac{1}{2}(T_{\tau/\varepsilon}b')(T_{2\tau/\varepsilon}\alpha_2) = 0, \\ 2\alpha_{2\tau} - 3\alpha_2\alpha_{2x} + \nu\alpha_{2xxx} - \dfrac{1}{45}\alpha_{2xxxxx} \\ +\big((T_{-\tau/\varepsilon}b) + (T_{-2\tau/\varepsilon}\alpha_1)\big)\alpha_{2x} - \dfrac{1}{2}(T_{-\tau/\varepsilon}b')(T_{-2\tau/\varepsilon}\alpha_1) = 0, \end{cases} \tag{4.8}$$

where T_θ is the translation operator with respect to the spatial variable defined by $(T_\theta\alpha)(x,\tau) = \alpha(x+\theta,\tau)$. If the functions α_1, α_2, and b decay at spatial infinity, then we can expect that the coupling terms in the above equations converge to zero when ε tends to zero and that the equations in (4.8) are reduced to the Kawahara equation

$$\begin{cases} 2\alpha_{1\tau} + 3\alpha_1\alpha_{1x} - \nu\alpha_{1xxx} + \dfrac{1}{45}\alpha_{1xxxxx} = 0, \\ 2\alpha_{2\tau} - 3\alpha_2\alpha_{2x} + \nu\alpha_{2xxx} - \dfrac{1}{45}\alpha_{2xxxxx} = 0. \end{cases} \tag{4.9}$$

In view of (4.2) it is natural to specify the initial conditions in the form

$$\alpha_1 = \frac{1}{2}(\eta_0 + u_0), \quad \alpha_2 = \frac{1}{2}(\eta_0 - u_0) \qquad \text{at} \quad \tau = 0. \tag{4.10}$$

Now, we are ready to give our main theorems in this paper.

Theorem 4.1. *Let M be a positive constant, ν a constant, and m an integer such that $m \ge 4$. There exist positive constants T, C, and ε_0 such that the following holds. For any $\varepsilon \in (0,\varepsilon_0]$, $\eta_0, u_0 \in H^{m+17}$, and $b \in W^{m+13,\infty}$ satisfying*

$$\|(\eta_0,u_0)\|_{m+17} + \|b\|_{W^{m+13,\infty}} \le M,$$

the initial value problem (2.10) *and* (2.11) *with $\mu = \frac{1}{3} + \nu\varepsilon^{1/2}$ has a unique solution $(\eta,u) = (\eta^\varepsilon,u^\varepsilon)$ on the time interval $[0,T/\varepsilon]$ such that*

$$\begin{cases} \eta^\varepsilon \in C([0,T/\varepsilon];H^{m+2}) \cap C^1([0,T/\varepsilon];H^{m+1}), \\ u^\varepsilon \in C([0,T/\varepsilon];H^{m+1}) \cap C^1([0,T/\varepsilon];H^{m}). \end{cases} \tag{4.11}$$

Moreover, the solution satisfies

$$\sup_{0\le t\le T/\varepsilon}\Big(\big\|\eta^\varepsilon(t)-\big(\alpha_1^\varepsilon(\cdot-t,\varepsilon t)+\alpha_2^\varepsilon(\cdot+t,\varepsilon t)\big)\big\|_{m+2} \\ +\big\|u_1^\varepsilon(t)-\big(\alpha_1^\varepsilon(\cdot-t,\varepsilon t)-\alpha_2^\varepsilon(\cdot+t,\varepsilon t)\big)\big\|_{m+1}\Big)\le C\varepsilon^{1/2}, \qquad (4.12)$$

where $\alpha^\varepsilon=(\alpha_1^\varepsilon,\alpha_2^\varepsilon)$ *is a unique solution of the initial value problem for coupled Kawahara type equations* (4.8) *and* (4.10).

Theorem 4.2. *Let* T *and* M *be positive constants,* ν *a constant, and* m *an integer such that* $m\ge 4$. *There exist positive constants* C *and* ε_0 *such that the following holds. For any* $\varepsilon\in(0,\varepsilon_0]$, $\eta_0,u_0\in H^{m+17}\cap H^{m+3,2}$, *and* $b\in W^{m+13,\infty}\cap H^{m+2,2}$ *satisfying*

$$\|(\eta_0,u_0)\|_{m+17}+\|(\eta_0,u_0)\|_{m+3,2}+\|b\|_{W^{m+13,\infty}}+\|b\|_{m+2,2}\le M,$$

the initial value problem (2.10) *and* (2.11) *with* $\mu=\frac{1}{3}+\nu\varepsilon^{1/2}$ *has a unique solution* $(\eta,u)=(\eta^\varepsilon,u^\varepsilon)$ *on the time interval* $[0,T/\varepsilon]$ *satisfying* (4.11) *and*

$$\sup_{0\le t\le T/\varepsilon}\Big(\big\|\eta^\varepsilon(t)-\big(\alpha_1(\cdot-t,\varepsilon t)+\alpha_2(\cdot+t,\varepsilon t)\big)\big\|_{m+2} \\ +\big\|u_1^\varepsilon(t)-\big(\alpha_1(\cdot-t,\varepsilon t)-\alpha_2(\cdot+t,\varepsilon t)\big)\big\|_{m+1}\Big)\le C\varepsilon^{1/2}, \qquad (4.13)$$

where $\alpha=(\alpha_1,\alpha_2)$ *is a unique solution of the initial value problem for the Kawahara equation* (4.9) *and* (4.10).

Theorem 4.3. *Let* T *and* M *be positive constants,* ν *a constant, and* m *an integer such that* $m\ge 4$. *There exist positive constants* C *and* ε_0 *such that the following holds. For any* $\varepsilon\in(0,\varepsilon_0]$, $\eta_0,u_0\in H^{m+17}$, *and* $b\in W^{m+13,\infty}$ *satisfying*

$$\|(\eta_0,u_0)\|_{m+17}+\|b\|_{W^{m+13,\infty}}+\varepsilon^{-1/2}\|\eta_0-u_0\|_{m+17}\ \le\ M$$

or

$$\|(\eta_0,u_0)\|_{m+17}+\|b\|_{W^{m+13,\infty}}+\varepsilon^{-1/2}\|\eta_0+u_0\|_{m+17}\ \le\ M,$$

the initial value problem (2.10) *and* (2.11) *with* $\mu=\frac{1}{3}+\nu\varepsilon^{1/2}$ *has a unique solution* $(\eta,u)=(\eta^\varepsilon,u^\varepsilon)$ *on the time interval* $[0,T/\varepsilon]$ *satisfying* (4.11) *and*

$$\sup_{0\le t\le T/\varepsilon}\Big(\|\eta^\varepsilon(t)-\alpha_1(\cdot-t,\varepsilon t)\|_{m+2}+\|u_1^\varepsilon(t)-\alpha_1(\cdot-t,\varepsilon t)\|_{m+1}\Big)\le C\varepsilon^{1/2}$$

or

$$\sup_{0\le t\le T/\varepsilon}\Big(\|\eta^{\varepsilon}(t)-\alpha_2(\cdot+t,\varepsilon t)\|_{m+2}+\|u_1^{\varepsilon}(t)+\alpha_2(\cdot+t,\varepsilon t)\|_{m+1}\Big)\le C\varepsilon^{1/2},$$

respectively, where $\alpha=(\alpha_1,\alpha_2)$ *is a unique solution of the initial value problem for the Kawahara equation* (4.9) *and* (4.10).

Remark 4.1. Concerning the initial value problem (4.8) and (4.10), we merely know a local existence theorem in time of solution, so that in Theorem 4.1 the time T may be small. On the contrary, the initial value problem for the Kawahara equation (4.9) and (4.10) has a global solution in time, so that in Theorems 4.2 and 4.3 we can take T as an arbitrarily large constant. It seems that there are not any literature discussing the global existence of the solution for the initial value problem. Therefore, we give a global existence theorem for the problem in Appendix.

Remark 4.2. Theorem 4.2 is a refined version of the result of Schneider and Wayne [40], where they studied the equations (2.6)–(2.9) in the case $\varepsilon=1$ with the initial data of the forms $\eta_0(x)=\varepsilon\Phi_1(\varepsilon^{1/4}x)$ and $u_0(x)=\varepsilon\Phi_2(\varepsilon^{1/4}x)$. Note that the solutions (η,u) of (2.10) for general $\varepsilon>0$ are related to the solutions $(\tilde\eta,\tilde u)$ of (2.10) for $\varepsilon=1$ by the formulas $\tilde\eta(x,t)=\varepsilon\eta(\varepsilon^{1/4}x,\varepsilon^{1/4}t)$, $\tilde u_1(x,t)=\varepsilon u_1(\varepsilon^{1/4}x,\varepsilon^{1/4}t)$, and $\tilde u_2(x,t)=\varepsilon^{5/4}u_2(\varepsilon^{1/4}x,\varepsilon^{1/4}t)$ in the case $b=0$. In [40], the following estimate was obtained:

$$\sup_{0\le t\le T/\varepsilon^{5/4}}\Big(\big\|\tilde\eta^{\varepsilon}(t)-\varepsilon\big(\alpha_1(\varepsilon^{1/4}(\cdot-t),\varepsilon^{5/4}t)+\alpha_2(\varepsilon^{1/4}(\cdot+t),\varepsilon^{5/4}t)\big)\big\|_{W^{l,\infty}}$$
$$+\big\|\tilde u_1^{\varepsilon}(t)-\varepsilon\big(\alpha_1(\varepsilon^{1/4}(\cdot-t),\varepsilon^{5/4}t)-\alpha_2(\varepsilon^{1/4}(\cdot+t),\varepsilon^{5/4}t)\big)\big\|_{W^{l,\infty}}\Big)\le C\varepsilon^{9/8}.$$

In the Boussinesq variables this estimate can be written as

$$\sup_{0\le t\le T/\varepsilon}\sum_{j=0}^{l}\varepsilon^{j/4}\Big(\big|\partial_x^j\eta^{\varepsilon}(t)-\big(\partial_x^j\alpha_1(\cdot-t,\varepsilon t)+\partial_x^j\alpha_2(\cdot+t,\varepsilon t)\big)\big|_{\infty}$$
$$+\big|\partial_x^j u_1^{\varepsilon}(t)-\big(\partial_x^j\alpha_1(\cdot-t,\varepsilon t)-\partial_x^j\alpha_2(\cdot+t,\varepsilon t)\big)\big|_{\infty}\Big)\le C\varepsilon^{1/8}.$$

From this estimate one can not obtain any uniform estimates for derivatives of the error terms in the Boussinesq variables. On the contrary, it follows from our estimate (4.13) that

$$\sup_{0\le t\le T/\varepsilon}\sum_{j=0}^{m}\Big(\big|\partial_x^j\eta^\varepsilon(t)-\big(\partial_x^j\alpha_1(\cdot-t,\varepsilon t)+\partial_x^j\alpha_2(\cdot+t,\varepsilon t)\big)\big|_\infty$$
$$+\big|\partial_x^j u_1^\varepsilon(t)-\big(\partial_x^j\alpha_1(\cdot-t,\varepsilon t)-\partial_x^j\alpha_2(\cdot+t,\varepsilon t)\big)\big|_\infty\Big)\le C\varepsilon^{1/2}.$$

Remark 4.3. The conditions $\|\eta_0+u_0\|_{m+17}\le M\varepsilon^{1/2}$ and $\|\eta_0-u_0\|_{m+17}\le M\varepsilon^{1/2}$ in Theorem 4.3 imply that there exists a positive constant C_1 depending only on μ, m, M, and T such that the solution $\alpha=(\alpha_1,\alpha_2)$ of (4.9) and (4.10) satisfies $\|\alpha_1(\tau)\|_{m+17}\le C_1\varepsilon^{1/2}$ and $\|\alpha_2(\tau)\|_{m+17}\le C_1\varepsilon^{1/2}$ for $0\le\tau\le T$ and $0<\varepsilon\le 1$, respectively. For the proof we refer to Appendix. Therefore, the conditions in Theorem 4.3 assure that the wave is approximately one directional.

5. Reduction to a Quasi-Linear System

In this section we reduce the system (2.10) to a quasi-linear system of equations, which leads long time ($0\le t\le O(1/\varepsilon)$) existence of the solution. Although this reduction will be carried out in almost the same way as in [14], we will show it for the completeness. Throughout this and next sections we assume that (η,u) is a solution of the system (2.10) and sufficiently smooth. Putting $\zeta=\eta_x$, we are going to derive quasi-linear equations for u_1 and ζ. We differentiate the first equation in (2.10) with respect to t and obtain

$$\begin{aligned}&u_{1tt}+\zeta_t+\varepsilon u_1u_{1tx}+\varepsilon u_{1t}u_{1x}+\varepsilon^{3/2}\zeta(u_{2tt}+\varepsilon u_1u_{2tx})\\&\quad=\ \varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\zeta_{txx}+2\varepsilon^{1/2}\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x\zeta_{tx}+\varepsilon^{3/2}f_1,\end{aligned}\tag{5.1}$$

where

$$f_1=\mu\varepsilon^{-1}\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_{xx}\zeta_t-\zeta_t(u_{2t}+\varepsilon u_1u_{2x})-\varepsilon\zeta u_{1t}u_{2x}.$$

It follows from the second and the third equations in (2.10) that

$$\eta_t=(K-\varepsilon\zeta)u_1.\tag{5.2}$$

Differentiation of this gives

$$\begin{cases}\zeta_{tx}=(K_0-\varepsilon\zeta)u_{1xx}+(\widetilde{K}_1u_1)_{xx}-\varepsilon(2\zeta_xu_{1x}+\zeta_{xx}u_1),\\ \zeta_{txx}=(K_0-\varepsilon\zeta)u_{1xxx}+(\widetilde{K}_1u_1)_{xxx}-\varepsilon(3\zeta_xu_{1xx}+3\zeta_{xx}u_{1x}+\zeta_{xxx}u_1).\end{cases}$$

In order to express ζ_{xxx} in terms of derivatives of u_1 we use the first equation in (2.10). Differentiation of it with respect to x gives

$$\begin{aligned} &u_{1tx} + \zeta_x + \varepsilon u_1 u_{1xx} + \varepsilon u_{1x} u_{1x} + \varepsilon^{3/2}\zeta(u_{2tx} + \varepsilon u_1 u_{2xx}) + \varepsilon^{3/2}\zeta_x u_{2t} \\ &\quad = \varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\zeta_{xxx} + \varepsilon^{3/2} f_2, \end{aligned} \tag{5.3}$$

where

$$f_2 = 2\mu\varepsilon^{-1}\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x \zeta_{xx} + \mu\varepsilon^{-1}\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_{xx}\zeta_x - \varepsilon(\zeta u_1)_x u_{2x}.$$

Therefore, we can rewrite (5.1) as

$$\begin{aligned} &u_{1tt} + 2\varepsilon u_1 u_{1tx} + \varepsilon^2 u_1^2 u_{1xx} + \varepsilon^{3/2}\zeta(u_{2tt} + 2\varepsilon u_1 u_{2tx} + \varepsilon^2 u_1^2 u_{2xx}) \\ &\quad + \varepsilon u_{1t} u_{1x} + \zeta_t + \varepsilon u_1 \zeta_x + 3\mu\varepsilon^{3/2}(1+\varepsilon^{5/2}\zeta^2)^{-3/2}(\zeta_x u_{1xx} + \zeta_{xx} u_{1x}) \\ &\quad - \varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big((K_0 - \varepsilon\zeta)u_{1xxx} + (\widetilde{K}_1 u_1)_{xxx}\big) \\ &\quad - 2\varepsilon^{1/2}\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x (K_0 - \varepsilon\zeta)u_{1xx} = \varepsilon^{3/2} f_3, \end{aligned}$$

where

$$\begin{aligned} f_3 &= f_1 + u_1(\varepsilon f_2 - \varepsilon^{1/2} u_{1x} u_{1x} - \varepsilon\zeta_x u_{2t}) \\ &\quad + 2\mu\varepsilon^{-1}\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x \big((\widetilde{K}_1 u_1)_{xx} - \varepsilon(2\zeta_x u_{1x} + \zeta_{xx} u_1)\big). \end{aligned}$$

Differentiation of the third equation in (2.10) gives

$$\begin{cases} u_{2tt} = K u_{1tt} + 2\big[\partial_t, \widetilde{K}_1\big]u_{1t} + \big[\partial_t, \big[\partial_t, \widetilde{K}_1\big]\big]u_1, \\ u_{2tx} = K_0 u_{1tx} + \big(\widetilde{K}_1 u_{1t} + \big[\partial_t, \widetilde{K}_1\big]u_1\big)_x, \\ u_{2xx} = K_0 u_{1xx} + (\widetilde{K}_1 u_1)_{xx}, \end{cases} \tag{5.4}$$

so that we obtain

$$\begin{aligned} &(1+\varepsilon^{3/2}\zeta K)(u_{1tt} + 2\varepsilon u_1 u_{1tx} + \varepsilon^2 u_1^2 u_{1xx}) \\ &\quad + \varepsilon u_{1t} u_{1x} + \zeta_t + \varepsilon u_1 \zeta_x + 3\mu\varepsilon^2(1+\varepsilon^3\zeta^2)^{-3/2}(\zeta_x u_{1xx} + \zeta_{xx} u_{1x}) \\ &\quad - \varepsilon\mu(1+\varepsilon^3\zeta^2)^{-3/2}\big((K_0 - \varepsilon\zeta)u_{1xxx} + (\widetilde{K}_1 u_1)_{xxx}\big) \\ &\quad - 2\varepsilon\mu\big((1+\varepsilon^3\zeta^2)^{-3/2}\big)_x (K_0 - \varepsilon\zeta)u_{1xx} = \varepsilon^{3/2} f_4, \end{aligned}$$

where

$$\begin{aligned} f_4 &= f_3 - \zeta\big\{2\big[\partial_t, \widetilde{K}_1\big]u_{1t} + \big[\partial_t, \big[\partial_t, \widetilde{K}_1\big]\big]u_1 \\ &\qquad + 2\varepsilon u_1\big(\widetilde{K}_1 u_{1t} + \big[\partial_t, \widetilde{K}_1\big]u_1\big)_x + \varepsilon^2 u_1^2(\widetilde{K}_1 u_1)_{xx}\big\} \\ &\quad + 2\varepsilon\zeta\big(\widetilde{K}_1 u_1 u_{1tx} + \big[K_0, u_1\big]u_{1tx}\big) + \varepsilon^2\zeta\big(\widetilde{K}_1 u_1^2 u_{1xx} + \big[K_0, u_1^2\big]u_{1xx}\big). \end{aligned}$$

It is easy to see that

$$(1+\varepsilon^{3/2}\zeta K)^{-1} = (1+\varepsilon^{5/2}\zeta^2)^{-1}(1-\varepsilon^{3/2}\zeta K_0) - \varepsilon^{3/2}P_1,$$

where

$$\begin{aligned} P_1 \;=\; & (1+\varepsilon^{5/2}\zeta^2)^{-1}\zeta\big\{(1-\varepsilon^{3/2}K_0\zeta)\widetilde{K}_1 - \varepsilon\zeta(\varepsilon^{1/2}K_0^2+1) \\ & -\varepsilon^{3/2}\big[K_0,\zeta\big]K_0\big\}(1+\varepsilon^{3/2}\zeta K)^{-1}. \end{aligned} \tag{5.5}$$

Therefore, we arrive at the equation

$$\begin{aligned} & u_{1tt} + 2\varepsilon u_1 u_{1tx} + \varepsilon\big(\varepsilon u_1^2 + 3\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-5/2}\zeta_x\big)u_{1xx} \\ & \quad -\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}K_0 u_{1xxx} - \varepsilon^{1/2}\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x K_0 u_{1xx} \\ & \quad -\varepsilon^{1/2}\mu(\widetilde{K}_1 u_1)_{xxx} + \varepsilon^{3/2}\mu\zeta(\varepsilon^{1/2}K_0^2+1)u_{1xxx} \\ & \quad +(1-\varepsilon^{3/2}\zeta K_0)\zeta_t + \varepsilon u_{1t}u_{1x} + \varepsilon\zeta_x u_1 + 3\varepsilon^{3/2}\mu\zeta_{xx}u_{1x} = \varepsilon^{3/2}f_5, \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} f_5 \;=\; & (1+\varepsilon^{3/2}\zeta K)^{-1}f_4 \\ & +P_1\big\{\varepsilon u_{1t}u_{1x} + \zeta_t + \varepsilon u_1\zeta_x + 3\mu\varepsilon^{3/2}(1+\varepsilon^{5/2}\zeta^2)^{-3/2}(\zeta_x u_{1xx} + \zeta_{xx}u_{1x}) \\ & \qquad - \varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big((K_0-\varepsilon\zeta)u_{1xxx} + (\widetilde{K}_1 u_1)_{xxx}\big) \\ & \qquad - 2\varepsilon^{1/2}\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x(K_0-\varepsilon\zeta)u_{1xx}\big\} \\ & +(1+\varepsilon^{5/2}\zeta^2)^{-1}\big\{ \\ & \qquad \varepsilon\zeta(\varepsilon\zeta+K_0)(u_{1t}u_{1x} + \zeta_x u_1 + 3\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\zeta_{xx}u_{1x}) \\ & \qquad + \varepsilon\zeta^2(1-\varepsilon^{3/2}\zeta K_0)\zeta_t + 3\varepsilon^{3/2}\mu\zeta\big[K_0,(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\zeta_x\big]u_{1xx} \\ & \qquad + \varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\zeta\big(\varepsilon\big[K_0,\zeta\big]u_{1xxx} - K_0(\widetilde{K}_1 u_1)_{xxx}\big) \\ & \qquad - \varepsilon^{1/2}\mu\zeta\big[K_0,(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big]\big((K_0-\varepsilon\zeta)u_{1xxx} + (\widetilde{K}_1 u_1)_{xxx}\big) \\ & \qquad - 2\mu\varepsilon^{-1}\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x\zeta\big(\varepsilon(\varepsilon^{1/2}K_0^2+1) - \varepsilon^{5/2}\big[K_0,\zeta\big]\big)u_{1xx} \\ & \qquad - 2\varepsilon^{1/2}\mu\zeta\big[K_0,\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x\big](K_0-\varepsilon\zeta)u_{1xx}\big\} \\ & -3\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}-1\big)\zeta_{xx}u_{1x} \\ & -\mu\varepsilon^{-1}\big((1+\varepsilon^{5/2}\zeta^2)^{-5/2}-1\big)\big(\varepsilon\zeta(\varepsilon^{1/2}K_0^2+1)u_{1xxx} - (\widetilde{K}_1 u_1)_{xxx}\big). \end{aligned}$$

In the next section we will show that f_5 contains lower order terms only and that an appropriate norm of f_5 is uniformly bounded with respect to small ε.

We proceed to drive a quasi-linear equation for ζ. It follows from (5.3) and (5.4) that

$$u_{1tx}+\varepsilon u_1 u_{1xx}+(1+\varepsilon^{3/2}\zeta K_0)^{-1}\big\{\zeta_x+\varepsilon u_{1x}u_{1x}-\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\zeta_{xxx} +\varepsilon^{3/2}\zeta(K_1u_{1t})_x+\varepsilon^{3/2}\zeta_x K_0 u_{1t}\big\}=\varepsilon^{3/2}f_6, \tag{5.7}$$

where

$$f_6 = (1+\varepsilon^{3/2}\zeta K_0)^{-1}\big\{f_2+\varepsilon\zeta\big[K_0,u_1\big]u_{1xx} -\zeta\big((\widetilde{K}_2u_{1t}+\big[\partial_t,\widetilde{K}_1\big]u_1)_x+\varepsilon u_1(\widetilde{K}_1u_1)_{xx}\big)-\zeta_x\big(\widetilde{K}_1u_{1t}+[\partial_t,\widetilde{K}_1]u_1\big)\big\}.$$

Differentiation of (5.2) gives

$$\begin{cases}\zeta_{tx}=(K-\varepsilon\zeta)u_{1xx}+2([\partial_x,\widetilde{K}_1]-\varepsilon\zeta_x)u_{1x}+([\partial_x,[\partial_x,\widetilde{K}_1]]-\varepsilon\zeta_{xx})u_1,\\ \zeta_{tt}=(K-\varepsilon\zeta)u_{1tx}+([\partial_x,\widetilde{K}_1]-\varepsilon\zeta_x)u_{1t}\\ \qquad +([\partial_t,\widetilde{K}_1]-\varepsilon\zeta_t)u_{1x}+([\partial_x,[\partial_t,\widetilde{K}_1]]-\varepsilon\zeta_{tx})u_1.\end{cases}$$

We also have

$$(K-\varepsilon\zeta)(1+\varepsilon^{3/2}\zeta K_0)^{-1}=K-\varepsilon\zeta(1+\varepsilon^{1/2}K_0^2)-\varepsilon^{3/2}[K_0,\zeta]K_0-\varepsilon^{3/2}P_2,$$

where

$$P_2 = \varepsilon\big(\widetilde{K}_1-\varepsilon\zeta(1+\varepsilon^{1/2}K_0^2)-\varepsilon^{3/2}[K_0,\zeta]K_0\big)\zeta^2(1+\varepsilon^{5/2}\zeta^2)^{-1} +(\varepsilon[K_0,\zeta^2]+\widetilde{K}_1\zeta K_0)(1+\varepsilon^{5/2}\zeta^2)^{-1}-(K-\varepsilon\zeta)(1+\varepsilon^{3/2}\zeta K_0)^{-1} \times\big(\varepsilon\zeta^2(1+\varepsilon^{1/2}K_0^2)+\varepsilon^{3/2}\zeta[K_0,\zeta]K_0\big)(1+\varepsilon^{5/2}\zeta^2)^{-1}. \tag{5.8}$$

Applying the operator $K-\varepsilon\zeta$ on both side of the equation (5.7) we see that

$$\zeta_{tt}+2\varepsilon u_1\zeta_{tx}-\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}K_0\zeta_{xxx}-\varepsilon^{1/2}\mu K_1\zeta_{xxx} +K(\zeta_x+\varepsilon u_{1x}u_{1x})+\varepsilon(\zeta_x u_{1t}+\zeta_t u_{1x})+\varepsilon[K,u_1]u_{1xx}-\varepsilon^{1/2}\mu\widetilde{K}_2\zeta_{xxx} -\big([\partial_x,\widetilde{K}_1]u_{1t}+[\partial_t,\widetilde{K}_1]u_{1x}+[\partial_x,[\partial_t,\widetilde{K}_1]]u_1\big) -\varepsilon u_1\big(2[\partial_x,\widetilde{K}_1]u_{1x}+[\partial_x,[\partial_x,\widetilde{K}_1]]u_1\big)+\varepsilon^{3/2}K_0\big(\zeta(K_1u_{1t})_x+\zeta_xK_0u_{1t}\big) -\big(\varepsilon\zeta(1+\varepsilon^{1/2}K_0^2)+\varepsilon^{3/2}[K_0,\zeta]K_0\big)(\zeta_x-\varepsilon^{1/2}\mu\zeta_{xxx})=\varepsilon^{3/2}f_7, \tag{5.9}$$

where

$$f_7 = (K-\varepsilon\zeta)f_6-\varepsilon^{1/2}u_1^2\zeta_{xx}-2\varepsilon^{1/2}\zeta_xu_1u_{1x}+\mu\varepsilon^{-1}[K_0,(1+\varepsilon^{5/2}\zeta^2)^{-3/2}]\zeta_{xxx} +\mu\varepsilon^{-1}\widetilde{K}_1\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}-1\big)\zeta_{xxx}-\widetilde{K}_1\big(\zeta(K_1u_{1t})_x+\zeta_xK_0u_{1t}\big) +\big(\zeta(1+\varepsilon^{1/2}K_0^2)+\varepsilon^{1/2}[K_0,\zeta]K_0\big)\big\{\varepsilon^{1/2}u_{1x}u_{1x}$$

$$
\begin{aligned}
&\quad -\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}-1\big)\zeta_{xxx}+\varepsilon\zeta(K_1u_{1t})_x+\varepsilon\zeta_xK_0u_{1t}\big\} \\
&+P_2\big\{\zeta_x+\varepsilon u_{1x}u_{1x}-\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}\zeta_{xxx} \\
&\quad +\varepsilon^{3/2}\zeta(K_1u_{1t})_x+\varepsilon^{3/2}\zeta_xK_0u_{1t}\big\}.
\end{aligned}
$$

Here, f_7 has the same property as f_5.

Next, we derive equations for an approximate solution $\phi=(\phi_1,\phi_2)$, which is defined as follows. Let $\alpha=(\alpha_1,\alpha_2)$ be the solution of the initial value problem for coupled Kawahara type equations (4.8) and (4.10) and define $\beta=(\beta_1,\beta_2)$, $\gamma=(\gamma_1,\gamma_2)$, and $\bar\phi=(\bar\phi_1,\bar\phi_2)$ by (4.5), (4.6), and (4.7), respectively. In view of the ansatz (4.4) we define an approximate solution $\phi=(\phi_1,\phi_2)$ by

$$
\begin{cases}
\phi_1(x,t)=\alpha_1(x-t,\varepsilon t)-\alpha_2(x+t,\varepsilon t) \\
\qquad\qquad -\varepsilon^{1/2}\big(\beta_1(x-t,\varepsilon t)-\beta_2(x+t,\varepsilon t)\big) \\
\qquad\qquad -\varepsilon\big(\gamma_1(x-t,\varepsilon t)-\gamma_2(x+t,\varepsilon t)\big)+\varepsilon\bar\phi_1(x,t), \\
\phi_2(x,t)=\alpha_1(x-t,\varepsilon t)+\alpha_2(x+t,\varepsilon t)+\varepsilon\bar\phi_2(x,t).
\end{cases}
\tag{5.10}
$$

Then, we have

$$
\begin{cases}
\phi_{1t}+\phi_{2x}+\varepsilon\phi_1\phi_{1x}-\varepsilon^{1/2}\mu\phi_{2xxx}=\varepsilon^{3/2}g_1, \\
\phi_{2t}+\phi_{1x}+\varepsilon\big((\phi_2-b)\phi_1\big)_x+\dfrac{\varepsilon^{1/2}}{3}\phi_{1xxx}+\dfrac{2}{15}\varepsilon\phi_{1xxxxx}=\varepsilon^{3/2}g_2,
\end{cases}
\tag{5.11}
$$

where

$$
\begin{aligned}
g_1 \;=\;& \beta_{2\tau}-\beta_{1\tau}+\varepsilon^{1/2}\Big(\gamma_{2\tau}-\gamma_{1\tau}+\frac{3}{4}b(\alpha_{1\tau}-\alpha_{2\tau})\Big)-\mu\bar\phi_{2xxx} \\
&+\Big((\alpha_1-\alpha_2)\big(\beta_2-\beta_1+\varepsilon^{1/2}(\gamma_2-\gamma_1+\bar\phi_1)\big) \\
&\qquad+\frac{\varepsilon^{1/2}}{2}\big(\beta_2-\beta_1+\varepsilon^{1/2}(\gamma_2-\gamma_1+\bar\phi_1)\big)^2\Big)_x,
\end{aligned}
$$

$$
\begin{aligned}
g_2 \;=\;& \frac{\varepsilon^{1/2}}{4}\big((b+2\alpha_2)\alpha_{1\tau}+(b+2\alpha_1)\alpha_{2\tau}\big)+\frac{1}{3}(\gamma_2-\gamma_1+\bar\phi_1)_{xxx} \\
&+\frac{2}{15}\big(\beta_2-\beta_1+\varepsilon^{1/2}(\gamma_2-\gamma_1+\bar\phi_1)\big)_{xxxxx} \\
&+\Big((\alpha_1+\alpha_2-b+\varepsilon\bar\phi_2)\big(\beta_2-\beta_1+\varepsilon^{1/2}(\gamma_2-\gamma_1+\bar\phi_1)\big)\Big)_x \\
&+\Big(\varepsilon^{1/2}\bar\phi_2(\alpha_1-\alpha_2)\Big)_x.
\end{aligned}
$$

Here, $\alpha_1 = \alpha_1(x-t, \varepsilon t)$, $\alpha_2 = \alpha_2(x+t, \varepsilon t)$, etc. By Taylor's formula we have

$$\begin{aligned} \tanh x &= xG_0(x) = x + x^3 G_1(x) = x - \frac{1}{3}x^3 + x^5 G_2(x) \\ &= x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + x^7 G_3(x) \end{aligned} \tag{5.12}$$

and $\operatorname{sech} x = 1 + x^2 G_4(x)$, where $G_0, \ldots, G_4$ are bounded and smooth functions on $\mathbf{R}$. Therefore, it holds that

$$\big(K_0+K_1(\phi_2,b)\big)\phi_1 = -\phi_{1x} - \frac{\varepsilon^{1/2}}{3}\phi_{1xxx} - \frac{2}{15}\varepsilon\phi_{1xxxxx} - \varepsilon\phi_2\phi_{1x} + \varepsilon(b\phi_1)_x + \varepsilon^{3/2} g_3,$$

where

$$\begin{aligned} K_1(\phi_2, b) &= -\varepsilon\big(\phi_2 + i\tanh(\varepsilon^{1/4}D)\phi_2\, i\tanh(\varepsilon^{1/4}D)\big)(iD) \\ &\quad + \varepsilon \operatorname{sech}(\varepsilon^{1/4}D)(iD)b \operatorname{sech}(\varepsilon^{1/4}D), \\ g_3 &= \Big\{G_3(\varepsilon^{1/4}D)(iD)^7 - G_0(\varepsilon^{1/4}D)(iD)\phi_2 G_0(\varepsilon^{1/4}D)(iD)^2 \\ &\quad - G_4(\varepsilon^{1/4}D)(iD)^3 b \operatorname{sech}(\varepsilon^{1/4}D) - (iD)bG_4(\varepsilon^{1/4}D)(iD)^2\Big\}\phi_1. \end{aligned}$$

Therefore, we obtain

$$\begin{cases} \phi_{1t} + \phi_{2x} + \varepsilon\phi_1\phi_{1x} - \varepsilon^{1/2}\mu\phi_{2xxx} = \varepsilon^{3/2} g_1, \\ \phi_{2t} = \big(K_0 + K_1(\phi_2, b) - \varepsilon\phi_{2x}\big)\phi_1 + \varepsilon^{3/2} g_4 = K_0\phi_1 + \varepsilon g_5, \end{cases} \tag{5.13}$$

where $g_4 = g_2 - g_3$ and $g_5 = \big(\varepsilon^{-1}K_1(\phi_2, b) - \phi_{2x}\big)\phi_1 + \varepsilon^{1/2} g_4$. This system is an approximate version of the first equation in (2.10) and (5.2). By the same way as the derivation of equations (5.6) and (5.9) we get

$$\begin{cases} \phi_{1tt} + 2\varepsilon\phi_1\phi_{1tx} - \varepsilon^{1/2}\mu K_0\phi_{1xxx} + \phi_{2tx} + \varepsilon(\phi_{1t}\phi_{1x} + \phi_{2xx}\phi_1) = \varepsilon^{3/2} g_6, \\ \phi_{2xtt} + 2\varepsilon\phi_1\phi_{2xxt} - \varepsilon^{1/2}\mu K_0\phi_{2xxxx} + \big(K_0 + K_1(\phi_2, b)\big)\phi_{2xx} \\ \quad + \varepsilon K_0\phi_{1x}\phi_{1x} + \varepsilon(\phi_{2xx}\phi_{1t} + \phi_{2xt}\phi_{1x}) + \varepsilon[K_0, \phi_1]\phi_{1xx} \\ \quad - \big(K_1(\phi_{2x}, b')\phi_{1t} + K_1(\phi_{2t}, 0)\phi_{1x} + K_1(\phi_{2xt}, 0)\phi_1\big) - \varepsilon\phi_{2x}\phi_{2xx} = \varepsilon^{3/2} g_7, \end{cases} \tag{5.14}$$

where

$$\begin{cases} g_6 = g_{1t} + \mu g_{5xxx} + (\mu\phi_{2xxx} - \varepsilon^{1/2}\phi_1\phi_{1x} + \varepsilon g_1)_x\phi_1, \\ g_7 = \big(K_0 + K_1(\phi_2, b) - \varepsilon\phi_{2x}\big)g_{1x} + g_{4xt} + \varepsilon^{1/2}\phi_1 g_{5xx} \\ \qquad + \big(\varepsilon^{-1}K_1(\phi_2, b) - \phi_{2x}\big)(\mu\phi_{2xxx} - \varepsilon^{1/2}\phi_1\phi_{1x})_x. \end{cases}$$

Here, an appropriate norm of each g_j, $j = 1, \ldots, 7$, is uniformly bounded with respect to small ε.

By using this approximate solution ϕ we define remainder functions $\bar{\eta}$ and $\bar{u}_1$ by

$$\begin{cases} \eta(x,t) = \phi_2(x,t) + \varepsilon^{1/2}\bar{\eta}(x,t), \\ u_1(x,t) = \phi_1(x,t) + \varepsilon^{1/2}\bar{u}_1(x,t), \end{cases} \tag{5.15}$$

and put $\bar{\zeta} = \bar{\eta}_x$. Main task in this paper is to derive uniform estimates of these remainder functions $\bar{\eta}$ and $\bar{u}_1$ with respect to small ε for the long time interval $0 \le t \le O(1/\varepsilon)$. To this end, we derive quasi-linear equations for these remainder functions. Substituting (5.15) for (5.6) and (5.9), and using (5.14) we see that

$$\begin{cases} \bar{u}_{1tt} + 2\varepsilon u_1 \bar{u}_{1tx} + \varepsilon\big(\varepsilon u_1^2 + 3\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-5/2}\zeta_x\big)\bar{u}_{1xx} \\ \quad -\varepsilon^{1/2}\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}K_0\bar{u}_{1xx}\big)_x - \varepsilon^{1/2}\mu(K_1\bar{u}_1)_{xxx} + \bar{\zeta}_t = \varepsilon h_1, \\ \bar{\zeta}_{tt} + 2\varepsilon u_1\bar{\zeta}_{tx} - \varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}K_0\bar{\zeta}_{xxx} - \varepsilon^{1/2}\mu K_1\bar{\zeta}_{xxx} \\ \quad + (K_0+K_1)\bar{\zeta}_x + \varepsilon[K_0,u_1]\bar{u}_{1xx} - \big(K_1(\zeta,b')\bar{u}_{1t} + K_1(\eta_t,0)\bar{u}_{1x}\big) = \varepsilon h_2, \end{cases} \tag{5.16}$$

where

$$\begin{aligned} h_1 \;=\; & f_5 - g_6 - 2\bar{u}_1\phi_{1tx} - \big(\varepsilon^{1/2}u_1^2 + 3\mu(1+\varepsilon^{5/2}\zeta^2)^{-5/2}\zeta_x\big)\phi_{1xx} \\ & +\mu\big(\varepsilon^{-1}((1+\varepsilon^{5/2}\zeta^2)^{-3/2}-1)K_0\phi_{1xx}\big)_x \\ & +\mu(\varepsilon^{-1}K_1\phi_1 + \varepsilon^{-1}\widetilde{K}_2\phi_1 + \varepsilon^{-1/2}\widetilde{K}_2\bar{u}_1)_{xxx} - \zeta K_0(\phi_{2xt} + \varepsilon^{1/2}\bar{\zeta}_t) \\ & -(\bar{u}_{1t}\phi_{1x} + u_{1t}\bar{u}_{1x} + \bar{\zeta}_x\phi_1 + \zeta_x\bar{u}_1) - 3\mu(\phi_{2xxx} + \varepsilon^{1/2}\bar{\zeta}_{xx})u_{1x} \\ & -\mu\zeta(\varepsilon^{1/2}K_0^2 + 1)(\phi_{1xxx} + \varepsilon^{1/2}\bar{u}_{1xxx}), \end{aligned}$$

$$\begin{aligned} h_2 \;=\; & f_7 - g_7 - 2\bar{u}_1\phi_{2xxt} + \mu\varepsilon^{-1}\big((1+\varepsilon^3\zeta^2)^{-3/2}-1\big)K_0\phi_{2xxxxx} \\ & +\varepsilon^{-1}\big(\mu K_1\phi_{2xxxxx} - (K_1(\bar{\eta},0) + \widetilde{K}_2)\phi_{2xx} - \widetilde{K}_2\bar{\zeta}_x\big) \\ & -\varepsilon^{-1}\widetilde{K}_1\big(\phi_{1x}\phi_{1x} + \varepsilon^{1/2}(\phi_{1x}+u_{1x})\bar{u}_{1x}\big) \\ & -(\phi_{2xx}\bar{u}_{1t} + \bar{\zeta}_x u_{1t} + \phi_{2xt}\bar{u}_{1x} + \bar{\zeta}_t u_{1x}) \\ & -\varepsilon^{-1/2}[\widetilde{K}_1,u_1](\phi_{1xx} + \varepsilon^{1/2}\bar{u}_{1xx}) - [K_0,\bar{u}_1]\phi_{1xx} \\ & -K_0\big((\phi_{1x}+u_{1x})\bar{u}_{1x}\big) + \mu\varepsilon^{-1}\widetilde{K}_2(\phi_{2xxxx} + \varepsilon^{1/2}\bar{\zeta}_{xxx}) \\ & +\varepsilon^{-1}\big(K_1(\bar{\zeta},0)\phi_{1t} + K_1(\bar{\eta}_t,0)\phi_{1x} + K_1(\bar{\zeta}_t,0)\phi_1 + K_1(\zeta_t,0)\bar{u}_1\big) \\ & +\varepsilon^{-3/2}\big([\partial_x,\widetilde{K}_2](\phi_{1t} + \varepsilon^{1/2}\bar{u}_{1t}) \\ & \qquad + [\partial_t,\widetilde{K}_2](\phi_{1x} + \varepsilon^{1/2}\bar{u}_{1x}) + [\partial_x,[\partial_t,\widetilde{K}_2]]u_1\big) \\ & +u_1\varepsilon^{-1/2}\big(2[\partial_x,\widetilde{K}_1](\phi_{1x} + \varepsilon^{1/2}\bar{u}_{1x}) + [\partial_x,[\partial_x,\widetilde{K}_1]](\phi_1 + \varepsilon^{1/2}\bar{u}_1)\big) \end{aligned}$$

$$
\begin{aligned}
&-K_0\big(\zeta(K_1(\phi_{1t}+\varepsilon^{1/2}\bar{u}_{1t}))_x+\zeta_x K_0(\phi_{1t}+\varepsilon^{1/2}\bar{u}_{1t})\big)\\
&+\phi_{2x}K_0^2\phi_{2xx}+\bar{\zeta}(\varepsilon^{1/2}K_0^2+1)\phi_{2xx}+\zeta(\varepsilon^{1/2}K_0^2+1)\bar{\zeta}_x\\
&-\mu\zeta(\varepsilon^{1/2}K_0^2+1)(\phi_{2xxxx}+\varepsilon^{1/2}\bar{\zeta}_{xxx})\\
&+[K_0,\zeta]K_0\big(\phi_{2xx}-\varepsilon^{1/2}\mu\phi_{2xxxx}+\varepsilon^{1/2}(\bar{\zeta}_x-\varepsilon^{1/2}\mu\bar{\zeta}_{xxx})\big).
\end{aligned}
$$

It follows from (5.2) that $\zeta_t = Ku_{1x} + [\partial_x, \widetilde{K}_1]u_1 - \varepsilon(\zeta u_1)_x$. Substituting (5.15) for this equation and the first equation in (2.10), and using (5.13) we obtain

$$
\begin{cases}
\bar{\zeta}_t = (K_0+K_1)\bar{u}_{1x} + \varepsilon h_3,\\
\bar{u}_{1t} = -\bar{\zeta} + \varepsilon^{1/2}\mu\bar{\zeta}_{xx} + \varepsilon h_4,
\end{cases}
\tag{5.17}
$$

where

$$
\begin{aligned}
h_3 &= -g_{4x} - (\phi_{2x}\bar{u}_1 + \bar{\zeta}u_1)_x + \big(\varepsilon^{-1}K_1(\bar{\eta},0) + \varepsilon^{-3/2}\widetilde{K}_2\big)\phi_{1x} + \varepsilon^{-1}\widetilde{K}_2\bar{u}_{1x}\\
&\quad + \big(\varepsilon^{-1}K_1(\bar{\zeta},0) + \varepsilon^{-3/2}[\partial_x,\widetilde{K}_2]\big)\phi_1 + \varepsilon^{-1}\big(K_1(\zeta,b') + [\partial_x,\widetilde{K}_2]\big)\bar{u}_1,\\
h_4 &= -g_1 - (\bar{u}_1\phi_{1x} + u_1\bar{u}_{1x}) - \zeta\big(K(\phi_{1t}+\varepsilon^{1/2}\bar{u}_{1t}) + [\partial_t,\widetilde{K}_1]u_1 + \varepsilon u_1 u_{2x}\big)\\
&\quad + \mu\big(\varepsilon^{-1}((1+\varepsilon^{5/2}\zeta^2)^{-1/2}-1)\zeta\big)_{xx}.
\end{aligned}
$$

In view of (3.5) we introduce a linear operator $L_1 = L_1(\eta, b)$ depending on η and b by

$$
\begin{aligned}
L_1(\eta,b)f &= -\big(\eta + i\tanh(\varepsilon^{1/4}D)\eta\, i\tanh(\varepsilon^{1/4}D)\big)f\\
&\quad + \operatorname{sech}(\varepsilon^{1/4}D)b\operatorname{sech}(\varepsilon^{1/4}D)f.
\end{aligned}
\tag{5.18}
$$

Then, it holds that $K_1(\eta,b)f = \varepsilon\big(L_1(\eta,b)f_x + L_1(0,b')f\big)$. By this relation, (5.16), and (5.17) we finally obtain

$$
\begin{cases}
\bar{u}_{1tt} + 2\varepsilon u_1\bar{u}_{1tx} + \varepsilon\big(\varepsilon u_1^2 + 3\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-5/2}\zeta_x\big)\bar{u}_{1xx}\\
\qquad - \varepsilon^{1/2}\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}K_0\bar{u}_{1xx}\big)_x + K_0\bar{u}_{1x}\\
\qquad - \varepsilon^{3/2}\mu L_1(\eta,b)\bar{u}_{1xxxx} + \varepsilon L_1(\eta,b)\bar{u}_{1xx} = \varepsilon h_5,\\
\bar{\zeta}_{tt} + 2\varepsilon u_1\bar{\zeta}_{tx} - \varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}K_0\bar{\zeta}_{xxx} + K_0\bar{\zeta}_x\\
\qquad - \varepsilon^{3/2}\mu L_1(\eta,b)\bar{\zeta}_{xxxx} + \varepsilon L_1(\eta,b)\bar{\zeta}_{xx} = \varepsilon h_6,
\end{cases}
\tag{5.19}
$$

where

$$
\begin{aligned}
h_5 &= h_1 - h_3 - L_1(0,b')\bar{u}_{1x} + \varepsilon^{1/2}\mu L_1(0,b')\bar{u}_{1xxx}\\
&\quad + \varepsilon^{-1/2}\mu\big(3K_1(\zeta,b')\bar{u}_{1xx} + 3K_1(\zeta_x,b'')\bar{u}_{1x} + K_1(\zeta_{xx},b''')\bar{u}_1\big),
\end{aligned}
$$

$$h_6 = h_2 + \varepsilon^{1/2}\mu L_1(0,b')\bar{\zeta}_{xxx} - L_1(0,b')\bar{\zeta}_x + [K_0, u_1]K_0^{-1}(iD)(\varepsilon h_3 + K_1\bar{u}_{1x} - \bar{\zeta}_t)$$
$$+\varepsilon^{-1}K_1(\zeta,b')(\varepsilon h_4 + \varepsilon^{1/2}\mu\bar{\zeta}_{xx} - \bar{\zeta}) - \varepsilon^{-1}K_1(\eta_t,0)K_0^{-1}(\varepsilon h_3 + K_1\bar{u}_{1x} - \bar{\zeta}_t).$$

The quasi-linear system (5.19) leads uniform estimates of $(\bar{u}_{1t}, \bar{u}_{1x}, \bar{\zeta}_t, \bar{\zeta}_x)$ with respect to small ε in appropriate Sobolev spaces. In order to obtain uniform L^2 estimate of $(\bar{u}_1, \bar{\eta})$ we have to derive another system. In deriving the system we do not have to care the order of derivatives. It follows from (2.10) that

$$\begin{cases} u_{1t} + \eta_x + \varepsilon u_1 u_{1x} - \varepsilon^{1/2}\mu\eta_{xxx} = \varepsilon^{3/2} f_8, \\ \eta_t + u_{1x} + \varepsilon\big((\eta - b)u_1\big)_x + \dfrac{\varepsilon^{1/2}}{3}u_{1xxx} + \dfrac{2}{15}\varepsilon u_{1xxxxx} = \varepsilon^{3/2} f_9, \end{cases}$$

where

$$\begin{cases} f_8 = \mu\big(\varepsilon^{-1}((1+\varepsilon^{5/2}\zeta^2)^{-3/2} - 1)\zeta_x\big)_x - \zeta(u_{2t} + \varepsilon u_1 u_{2x}), \\ f_9 = \Big\{\varepsilon^{-3/2}\widetilde{K}_2 - G_3(\varepsilon^{1/4}D)(iD)^7 - G_0(\varepsilon^{1/4}D)(iD)\eta G_0(\varepsilon^{1/4}D)(iD)^2 \\ \qquad - G_4(\varepsilon^{1/4}D)(iD)^3 b\,\mathrm{sech}\,(\varepsilon^{1/4}D) - (iD)bG_4(\varepsilon^{1/4}D)(iD)^2\Big\}u_1. \end{cases}$$

Substituting (5.15) for the above system and using (5.11) we obtain

$$\begin{cases} \bar{u}_{1t} + \bar{\eta}_x - \varepsilon^{1/2}\mu\bar{\eta}_{xxx} = \varepsilon h_7, \\ \bar{\eta}_t + \bar{u}_{1x} + \dfrac{\varepsilon^{1/2}}{3}\bar{u}_{1xxx} = \varepsilon h_8, \end{cases}$$

where

$$\begin{cases} h_7 = f_8 - g_1 - (\bar{u}_1\phi_{1x} + u_1\bar{u}_{1x}), \\ h_8 = f_9 - g_2 - \big(\bar{\eta}\phi_1 + (\eta - b)\bar{u}_1\big)_x - \dfrac{2}{15}\bar{u}_{1xxxxx}. \end{cases}$$

It is better to rewrite this system in the form

$$\begin{cases} \bar{u}_{1t} + \bar{\eta}_x + \varepsilon^{1/2}\mu\bar{u}_{1txx} = \varepsilon h_9, \\ \bar{\eta}_t + \bar{u}_{1x} - \dfrac{\varepsilon^{1/2}}{3}\bar{\eta}_{txx} = \varepsilon h_{10}, \end{cases} \tag{5.20}$$

where

$$\begin{cases} h_9 = \mu^2\bar{\eta}_{xxxxx} + h_7 + \varepsilon^{1/2}\mu h_{7xx}, \\ h_{10} = \dfrac{1}{9}\bar{u}_{1xxxxx} + h_8 - \dfrac{\varepsilon^{1/2}}{3}h_{8xx}. \end{cases}$$

6. Estimates for Remainder Terms

In this section we give uniform estimates for remainder terms f_j, g_j, and h_j, which were introduced in the previous section, with respect to small ε in appropriate norms.

We begin to give several estimates for Fourier multipliers and commutators. We refer to [14] (see also Nalimov [27] and Yosihara [46]) for proofs of the following lemmas.

Lemma 6.1. *Let $s \geq s_0$. There exists a positive constant C such that*

$$\begin{cases} \|K_0 u\|_s \leq \min\{\varepsilon^{-1/4}\|u\|_s, \|u_x\|_s\}, \\ \|(\varepsilon^{1/2}K_0^2 + 1)u\|_s \leq C(1+\varepsilon^{-(s-s_0)/4})\|u\|_{s_0}. \end{cases}$$

Lemma 6.2. *Let $s > 1/2$. There exists a positive constant C such that*

$$\|K_0(uv)\|_s \leq C(\|K_0 u\|_s \|v\|_s + \|u\|_s \|K_0 v\|_s).$$

Lemma 6.3. *Let $s \geq s_0 > 1/2$. There exists a positive constant C such that*

$$\begin{cases} \|[K_0, a]u\|_s \leq C(1+\varepsilon^{-(s-s_0+1)/4})\|a\|_s \|u\|_{s_0}, \\ \|[K_0, a]u\|_s \leq C(1+\varepsilon^{-(s-s_0)/4})\|a_x\|_s \|u\|_{s_0}, \\ \|[K_0, a](iD)u\|_s \leq C(1+\varepsilon^{-(s-s_0+1)/4})\|a_x\|_s \|u\|_{s_0}, \\ \|[K_0, a]K_0^{-1}(iD)u\|_s \leq C(1+\varepsilon^{-(s-s_0)/4})\|a_x\|_s \|u\|_{s_0}. \end{cases}$$

Lemma 6.4. *Let $s > 1/2$. There exists a positive constant C such that*

$$\|K_1(\eta, 0)K_0^{-1}u\|_s \leq C\varepsilon \|\eta\|_{s+1}\|u\|_s.$$

Lemma 6.5. *Let m be an integer such that $m \geq 2$ and δ_1 the constant in Lemma 3.1. There exists a small constant $\delta_2 \in (0, \delta_1]$ such that if $\eta \in H^{m+1}$, $b \in W^{m+1,\infty}$, and $\varepsilon \in (0,1]$ satisfy $\varepsilon(\|\eta\|_{m+1} + \|b\|_{W^{m+1,\infty}}) \leq \delta_2$, then we have*

$$\|(1+\varepsilon^{3/2}\zeta K)^{-1} f\|_m \leq 2\|f\|_m.$$

Proof. By Lemmas 3.1 and 6.1 we have $\|\varepsilon^{3/2}\zeta K f\|_m \leq C\varepsilon\|\eta\|_{m+1}\|f\|_m$ if $\varepsilon(\|\eta\|_m + \|b\|_{W^{m+1,\infty}}) \leq \delta_1$. Therefore, taking $\delta_2 > 0$ suitably small we obtain the desired estimate by the Neumann series expansion. □

For the operators P_1 and P_2 defined by (5.5) and (5.8) we have the following lemma.

Lemma 6.6. *Let $M > 0$, m and m_0 be integers such that $m \geq m_0 \geq 2$. There exist positive constants ε_1 and C such that if $\|\eta\|_{m+1} + \|b\|_{W^{m+1,\infty}} \leq M$ and $0 < \varepsilon \leq \varepsilon_1$, then we have*

$$\|P_1 f\|_m \leq C\varepsilon^{(3-(m-m_0))/4}\|f\|_{m_0}, \quad \|P_2 f\|_m \leq C\varepsilon^{(2-(m-m_0))/4}\|f\|_{m_0}.$$

Proof. These are simple consequences of Lemmas 3.1, 6.1, 6.3 and 6.5. □

Lemma 6.7. *Let $M_1 > 0$, m be an integer such that $m \geq 4$, and $b \in W^{m+4,\infty}$. There exist positive constants ε_2 and C such that if (η, u) is a solution of (2.10) satisfying*

$$\begin{cases} \|\eta(t)\|_{m+2} + \|\eta_t(t)\|_{m+1} + \|u_1(t)\|_{m+1} + \|u_{1t}(t)\|_m \leq M_1, \\ \|K_0 u_1(t)\|_{m+1} + \|K_0 u_{1t}(t)\|_m \leq M_1 \end{cases} \tag{6.1}$$

for $0 \leq t \leq T$, then we have

$$\|f_5(t)\|_m + \|f_7(t)\|_m + \|f_8(t)\|_2 + \|f_9(t)\|_2 \leq C \tag{6.2}$$

for $0 \leq t \leq T$ and $0 < \varepsilon \leq \varepsilon_2$.

Proof. By assumption, we have $\|\zeta(t)\|_{m+1} + \|\zeta_t(t)\|_m \leq M_1$. Since $u_2 = K_0 u_1 + \widetilde{K}_1 u_1$, Lemmas 3.1 and 3.3 imply $\|u_2(t)\|_{m+1} + \|u_{2t}(t)\|_m \leq C$. Since $\eta_t = u_2 - \varepsilon\zeta u_1$, we also have $\|\eta_{tt}(t)\|_m \leq C$. It follows from (2.10) that

$$\begin{aligned} \varepsilon^{1/2}\zeta_{xx} &= \mu^{-1}(1 + \varepsilon^{5/2}\zeta^2)^{3/2}\{u_{1t} + \zeta + \varepsilon u_1 u_{1x} \\ &\quad + \varepsilon^{3/2}\zeta(u_{2t} + \varepsilon u_1 u_{2x}) - \varepsilon^{1/2}\mu\big((1 + \varepsilon^{5/2}\zeta^2)^{-3/2}\big)_x \zeta_x\}, \end{aligned}$$

so that we obtain $\varepsilon^{1/2}\|\zeta_{xx}(t)\|_m \leq C$. Using these uniform estimates and previous lemmas, we can prove $\|f_2(t)\|_m \leq C\varepsilon$, $\|f_6(t)\|_m \leq C\varepsilon^{3/4}$, and the desired estimates. □

Lemma 6.8. *Let $M_2 > 0$, m be a positive integer, and $b \in W^{m+13,\infty}$. There exists a positive constant C such that if $\alpha = (\alpha_1, \alpha_2)$ is a solution of*

(4.8) *satisfying* $\|\alpha(\tau)\|_{m+17} \le M_2$ *for* $0 \le \tau \le T$, *then we have*

$$\|g_6(t)\|_m + \|g_7(t)\|_m + \|g_1(t)\|_2 + \|g_2(t)\|_2 \le C \tag{6.3}$$

for $0 \le t \le T/\varepsilon$ *and* $0 < \varepsilon \le 1$.

Proof. By (4.8) we have $\|\alpha_\tau(\tau)\|_{m+12} + \varepsilon\|\alpha_{\tau\tau}(\tau)\|_{m+7} \le C$, which together with the definitions (4.5)–(4.7) and (5.10) implies in turn that $\|\beta(\tau)\|_{m+15} + \|\beta_\tau(\tau)\|_{m+10} + \varepsilon\|\beta_{\tau\tau}(\tau)\|_{m+5} \le C$, $\|\gamma(\tau)\|_{m+13} + \|\gamma_\tau(\tau)\|_{m+8} + \varepsilon\|\gamma_{\tau\tau}(\tau)\|_{m+3} \le C$, $\|\bar{\phi}(t)\|_{m+13} + \|\bar{\phi}_t(t)\|_{m+12} \le C$, and that

$$\|\phi(t)\|_{m+13} + \|\phi_t(t)\|_{m+8} \le C. \tag{6.4}$$

These follows easily the desired estimates. □

Lemma 6.9. *Let* $M_1, M_2 > 0$, m *be an integer such that* $m \ge 4$, *and* $b \in W^{m+13,\infty}$. *There exist positive constants* ε_3 *and* C *such that if* $\|\alpha(\tau)\|_{m+17} \le M_2$ *for* $0 \le \tau \le T$ *and* (6.1) *is valid for* $0 \le t \le T/\varepsilon$ *and* $0 < \varepsilon \le \varepsilon_3$, *then we have*

$$\|h_5(t)\|_m^2 + \|h_6(t)\|_m^2 + \|h_9(t)\|^2 + \|h_{10}(t)\|^2 \le C\big(1 + \mathscr{E}(t)\big)$$

for $0 \le t \le T/\varepsilon$ *and* $0 < \varepsilon \le \varepsilon_3$, *where*

$$\mathscr{E}(t) = \|\bar{\eta}(t)\|_{m+2}^2 + \|\bar{\eta}_t(t)\|_{m+1}^2 + \|\bar{u}_1(t)\|_{m+1}^2 + \|\bar{u}_{1t}(t)\|_m^2.$$

Proof. Since the assumptions of Lemmas 6.7 and 6.8 are satisfied, we have (6.2), (6.3), and (6.4). By the second equation in (5.17) we obtain $\varepsilon^{1/2}\|\bar{\zeta}_{xx}(t)\|_m \le C\big(1 + \mathscr{E}(t)\big)^{1/2}$. These estimates together with Lemmas 3.1–3.3 and 6.1–6.4 yield the desired estimates. □

7. Proof of Main Theorems

Since a local existence theorem in time of solution for the initial value problem (2.10) and (2.11) for fixed $\varepsilon > 0$ was already given in [13], it is sufficient to derive a priori estimates of the solution $(\eta^\varepsilon, u^\varepsilon)$ for the long time interval $0 \le t \le O(1/\varepsilon)$. In view of the quasi-linear equations in (5.19) we consider the linear equation

$$u_{tt} + \varepsilon p_1 u_{tx} + \varepsilon p_2 u_{xx} - \varepsilon^{1/2} a K_0 u_{xxx} + \varepsilon^{1/2}\gamma a_x K_0 u_{xx} + K_0 u_x$$

$$+\,\varepsilon^{3/2}L_1(q_1,b_1)u_{xxxx}+\varepsilon L_1(q_2,b_2)u_{xx}=F_1+\varepsilon F_2, \tag{7.1}$$

where $\varepsilon>0$ is a parameter, a, p_1, p_2, q_1, q_2, b_1, b_2, F_1, and F_2 are given functions of (x,t) and may depend on ε, γ is a real constant, and K_0 and $L_1(q,b)$ are linear operators defined in (3.5) and (5.18), respectively.

Remark 7.1. Since the operator $L_1(q,b)$ has a smoothing property, we can regard the last two terms of the left hand side of (7.1) as lower order and put them into the right hand side if we fix $\varepsilon>0$. However, if we use the smoothing property, then we lose a power of ε. Therefore, in order to obtain uniform estimates of the solution u with respect to ε, especially, for the long time interval $0\le t\le O(1/\varepsilon)$, we have to deal them as one of principal terms.

This linear equation was investigated in [14] with slight change of notation and we have the following lemma.

Lemma 7.1. *Let $M>0$, $r>1$, and m be an integer such that $m\ge 4$. There exist positive constants ε_5 and C such that if*

$$\begin{cases} \varepsilon^{-1}\|a_x(t)\|_m+\|\big(p_1(t),p_2(t),q_1(t),q_2(t)\big)\|_m+\|\big(b_1(t),b_2(t)\big)\|_{W^{m,\infty}}\le M, \\ \varepsilon^{-1}\|a_t(t)\|_3+\|q_{1t}(t)\|_3+\|q_{2t}(t)\|_1+|\big(p_{2t}(t),b_{1t}(t),b_{2t}(t)\big)|_\infty\le M, \\ M^{-1}\le a(x,t)\le M \quad \textit{for} \quad (x,t)\in\mathbf{R}\times[0,T], \end{cases}$$

and $u\in C^j([0,T];H^{m+3-3j/2})$, $j=0,1,2$, is a solution of (7.1), then we have

$$E_m(t)\le C\bigg(\mathrm{e}^{C\varepsilon t}E_m(0)+\int_0^t \mathrm{e}^{C\varepsilon(t-\tau)}\big((1+\tau)^r\|F_1(\tau)\|_m^2+\varepsilon\|F_2(\tau)\|_m^2\big)d\tau\bigg) \tag{7.2}$$

for $0\le t\le T$ and $0<\varepsilon\le\varepsilon_5$, where

$$\begin{aligned} E_m(t) &= \|u_t(t)\|_m^2+\Big\|\sqrt{\frac{D\tanh(\varepsilon^{1/4}D)}{\varepsilon^{1/4}}}u(t)\Big\|_m^2 \\ &\quad +\Big\|\sqrt{\varepsilon^{1/4}D^3\tanh(\varepsilon^{1/4}D)}u(t)\Big\|_m^2. \end{aligned} \tag{7.3}$$

Remark 7.2. The energy function $E_m(t)$ satisfies

$$\|u_t(t)\|_m^2+4^{-1}\|u_x(t)\|_m^2\le E_m(t)\le\|u_t(t)\|_m^2+\|u(t)\|_{m+2}^2.$$

We proceed to prove Theorem 4.1. By standard energy method and appropriate approximation argument of the system it is not difficult to show that under the assumption of Theorem 4.1 there exist constants $T, M_2 > 0$, which depend only on μ, m, and M, such that the initial value problem (4.8) and (4.10) has a unique solution $\alpha = \alpha^\varepsilon \in C([0,T]; H^{m+17})$ satisfying

$$\|\alpha^\varepsilon(\tau)\|_{m+17} \le M_2 \quad \text{for} \quad 0 \le \tau \le T,\ \varepsilon > 0.$$

Therefore, by the proof of Lemma 6.8 there exists a constant $M_3 > 0$ such that the approximate solution $\phi = \phi^\varepsilon$ defined by (5.10) satisfies

$$\|\phi^\varepsilon(t)\|_{m+13}^2 + \|\phi_t^\varepsilon(t)\|_{m+8}^2 \le M_3^2 \quad \text{for} \quad 0 \le t \le T/\varepsilon,\ 0 < \varepsilon \le 1.$$

Now, we assume that

$$\mathscr{E}(t) = \|\bar\eta^\varepsilon(t)\|_{m+2}^2 + \|\bar\eta_t^\varepsilon(t)\|_{m+1}^2 + \|\bar u_1^\varepsilon(t)\|_{m+1}^2 + \|\bar u_{1t}^\varepsilon(t)\|_m^2 \le N_1^2 \tag{7.4}$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_0$, where the constants N_1 and ε_0 will be determined later. Then, by (5.15) we have

$$\begin{cases} \|\eta^\varepsilon(t)\|_{m+2}^2 + \|\eta_t^\varepsilon(t)\|_{m+1}^2 + \|u_1^\varepsilon(t)\|_{m+1}^2 + \|u_{1t}^\varepsilon(t)\|_m^2 \le (2M_3)^2, \\ \|K_0 u_1^\varepsilon(t)\|_{m+1}^2 + \|K_0 u_{1t}^\varepsilon(t)\|_m^2 \le (2M_3)^2 \end{cases}$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_4$, if we take $\varepsilon_4 \in (0,1]$ so small that $\varepsilon_4 \le \varepsilon_0$ and $\varepsilon_4^{1/2} N_1 \le M_3$. Thanks of these estimates and Lemma 6.9 we see that there exist constants $C_1 > 0$ independent of N_1 and $\varepsilon_3 \in (0, \varepsilon_4]$ such that

$$\|h_5(t)\|_m^2 + \|h_6(t)\|_m^2 + \|h_9(t)\|^2 + \|h_{10}(t)\|^2 \le C_1\big(1 + \mathscr{E}(t)\big)$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_3$. It follows from (5.10) and (5.15) that $\bar\eta^\varepsilon(x,0) = -\varepsilon^{1/2}\bar\phi_2^\varepsilon(x,0)$ and $\bar u_1^\varepsilon(x,0) = \beta_1^\varepsilon(x,0) - \beta_2^\varepsilon(x,0) + \varepsilon^{1/2}\big(\gamma_1^\varepsilon(x,0) - \gamma_2^\varepsilon(x,0) - \bar\phi_1^\varepsilon(x,0)\big)$, which together with (5.17) imply that there exists a constant $C_2 > 0$ independent of N_1 such that

$$\|\bar\eta^\varepsilon(0)\|_{m+3}^2 + \|\bar u_1^\varepsilon(0)\|_{m+2}^2 + \|\bar\eta_t^\varepsilon(0)\|_{m+1}^2 + \|\bar u_{1t}^\varepsilon(0)\|_m^2 \le C_2$$

for $0 < \varepsilon \le \varepsilon_3$. (See also the proof of Lemma 6.8.)

Since $\bar\zeta$ and $\bar u$ satisfy (5.19), by Lemma 7.1 and Remark 7.2 it holds that there exist constants $C_3 > 0$ independent of N_1 and $\varepsilon_5 \in (0, \varepsilon_4]$ such that

$$\|\big(\bar\zeta_t^\varepsilon(t), \bar\zeta_x^\varepsilon(t), \bar u_{1t}^\varepsilon(t), \bar u_{1x}^\varepsilon(t)\big)\|_m^2$$

$$
\begin{aligned}
&\leq\ C_3 \mathrm{e}^{C_3\varepsilon t}\big(\|\big(\bar\zeta^\varepsilon_t(0),\bar u^\varepsilon_{1t}(0)\big)\|_m^2+\|\big(\bar\zeta^\varepsilon(0),\bar u^\varepsilon_1(0)\big)\|_{m+2}^2\big)\\
&\qquad + C_3\varepsilon\int_0^t \mathrm{e}^{C_3\varepsilon(t-\tau)}\big(\|h_5(\tau)\|_m^2+\|h_6(\tau)\|_m^2\big)d\tau\\
&\leq\ C_3C_2\mathrm{e}^{C_3\varepsilon t}+C_3C_1\varepsilon\int_0^t \mathrm{e}^{C_3\varepsilon(t-\tau)}\big(1+\mathscr{E}(\tau)\big)d\tau \qquad (7.5)
\end{aligned}
$$

for $0\leq t\leq T/\varepsilon$ and $0<\varepsilon\leq\varepsilon_5$. Since $\bar\eta$ and $\bar u$ satisfy also (5.20), we see that

$$
\begin{aligned}
&\frac{d}{dt}\Big(\|\bar\eta^\varepsilon(t)\|^2+\|\bar u^\varepsilon_1(t)\|^2+\frac{\varepsilon^{1/2}}{3}\|\bar\eta^\varepsilon_x(t)\|^2-\varepsilon^{1/2}\mu\|\bar u^\varepsilon_{1x}(t)\|^2\Big)\\
&\quad=\ 2\varepsilon\big((\bar\eta^\varepsilon(t),h_{10}(t))+(\bar u^\varepsilon_1(t),h_9(t))\big)\\
&\quad\leq\ \varepsilon\big(\|\bar\eta^\varepsilon(t)\|^2+\|\bar u^\varepsilon_1(t)\|^2+\|h_9(t)\|^2+\|h_{10}(t)\|^2\big).
\end{aligned}
$$

Therefore, Gronwall's inequality implies that

$$
\begin{aligned}
&\|\bar\eta^\varepsilon(t)\|^2+\|\bar u^\varepsilon_1(t)\|^2\\
&\leq\ \varepsilon^{1/2}\mu\|\bar u^\varepsilon_{1x}(t)\|^2+\mathrm{e}^{\varepsilon t}\Big(\|\bar\eta^\varepsilon(0)\|^2+\|\bar u^\varepsilon_1(0)\|^2+\frac{\varepsilon^{1/2}}{3}\|\bar\eta^\varepsilon_x(0)\|^2\Big)\\
&\qquad+\varepsilon\int_0^t \mathrm{e}^{\varepsilon(t-\tau)}\big(\|h_9(\tau)\|^2+\|h_{10}(\tau)\|^2+\varepsilon^{1/2}\mu\|\bar u^\varepsilon_{1x}(\tau)\|^2\big)d\tau\\
&\leq\ \varepsilon^{1/2}\mu\|\bar u^\varepsilon_{1x}(t)\|^2+C_2\mathrm{e}^{\varepsilon t}+(C_1+1)\varepsilon\int_0^t \mathrm{e}^{\varepsilon(t-\tau)}\big(1+\mathscr{E}(\tau)\big)d\tau \qquad (7.6)
\end{aligned}
$$

for $0\leq t\leq T/\varepsilon$ and $0<\varepsilon\leq\varepsilon_5$. By (5.20) we also have

$$
\begin{aligned}
\|\bar\eta^\varepsilon_t(t)\|^2\ &\leq\ 3\|\bar u^\varepsilon_{1x}(t)\|^2+\frac{\varepsilon}{3}\|\bar\zeta^\varepsilon_{tx}(t)\|^2+3\varepsilon^2\|h_{10}(t)\|^2\\
&\leq\ 3\|\bar u^\varepsilon_{1x}(t)\|^2+\varepsilon(3C_1+1)\big(1+\mathscr{E}(t)\big) \qquad (7.7)
\end{aligned}
$$

for $0\leq t\leq T/\varepsilon$ and $0<\varepsilon\leq\varepsilon_5$. Summarizing the above estimates we see that there exists a constant C_4 depending only on ν, m, and M such that

$$
\mathscr{E}(t)\leq C_4\mathrm{e}^{C_4\varepsilon t}+C_4\varepsilon\int_0^t \mathrm{e}^{C_4\varepsilon(t-\tau)}\big(1+\mathscr{E}(\tau)\big)d\tau
$$

for $0\leq t\leq T/\varepsilon$ and $0<\varepsilon\leq\varepsilon_0$, by taking $\varepsilon_0\in(0,\varepsilon_5]$ so small that $2\varepsilon_0(3C_1+1)\leq 1$. This and Gronwall's inequality imply that

$$
\mathscr{E}(t)\leq (C_4+1)\mathrm{e}^{2C_4T}\quad\text{for}\quad 0\leq t\leq T/\varepsilon,\ 0<\varepsilon\leq\varepsilon_0.
$$

Therefore, by setting $N_1=(C_4+1)^{1/2}\mathrm{e}^{C_4T}$ we see that (7.4) holds for

$0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_0$. This easily follows (4.12). The proof of Theorem 4.1 is complete.

We proceed to prove Theorem 4.2. One of strategies for the proof is to compare the solution of (4.8) and (4.10) and that of (4.9) and (4.10). However, we do not know whether the solution of (4.8) and (4.10) exists globally in time or not, so that we can not take the time T arbitrarily large if we use the solution. In order to take T as an arbitrarily large time, we use the global existence theorem for the initial value problem of the Kawahara equation, which will be given in Appendix, and we should not use the solution of (4.8). Therefore, we have to modify the quasi-linearization carried out in section 5.

Let $\alpha = (\alpha^1, \alpha^2)$ be the solution of the initial value problem for the Kawahara equation (4.9) and (4.10) and define $\beta = (\beta_1, \beta_2)$ and $\gamma = (\gamma_1, \gamma_2)$ by (4.5) and (4.6) as before. We define an approximate solution $\phi = (\phi_1, \phi_2)$ by

$$\begin{cases} \phi_1(x,t) = \alpha_1(x-t,\varepsilon t) - \alpha_2(x+t,\varepsilon t) - \varepsilon^{1/2}\big(\beta_1(x-t,\varepsilon t) - \beta_2(x+t,\varepsilon t)\big) \\ \qquad\qquad - \varepsilon\big(\gamma_1(x-t,\varepsilon t) - \gamma_2(x+t,\varepsilon t)\big), \\ \phi_2(x,t) = \alpha_1(x-t,\varepsilon t) + \alpha_2(x+t,\varepsilon t), \end{cases}$$

in place of (5.10). Then, we have

$$\begin{cases} \phi_{1t} + \phi_{2x} + \varepsilon\phi_1\phi_{1x} - \varepsilon^{1/2}\mu\phi_{2xxx} = -\varepsilon(\alpha_1\alpha_2)_x + \varepsilon^{3/2}\tilde{g}_1, \\ \phi_{2t} + \phi_{1x} + \varepsilon\big((\phi_2 - b)\phi_1\big)_x + \frac{\varepsilon^{1/2}}{3}\phi_{1xxx} + \frac{2}{15}\varepsilon\phi_{1xxxxx} \\ \quad = -\varepsilon\big(b(\alpha_1 - \alpha_2)\big)_x + \varepsilon^{3/2}\tilde{g}_2, \end{cases}$$

where

$$\begin{cases} \tilde{g}_1 = \beta_{2\tau} - \beta_{1\tau} + \varepsilon^{1/2}(\gamma_{2\tau} - \gamma_{1\tau}) \\ \qquad + \big((\alpha_1 - \alpha_2)\big(\beta_2 - \beta_1 + \varepsilon^{1/2}(\gamma_2 - \gamma_1)\big)\big)_x \\ \qquad + \frac{\varepsilon^{1/2}}{2}\big(\big(\beta_2 - \beta_1 + \varepsilon^{1/2}(\gamma_2 - \gamma_1)\big)^2\big)_x, \\ \tilde{g}_2 = \big((\alpha_1 + \alpha_2 - b)\big(\beta_2 - \beta_1 + \varepsilon^{1/2}(\gamma_2 - \gamma_1)\big)\big)_x \\ \qquad + \frac{1}{3}(\gamma_2 - \gamma_1)_{xxx} + \frac{2}{15}\big(\beta_2 - \beta_1 + \varepsilon^{1/2}(\gamma_2 - \gamma_1)\big)_{xxxxx}, \end{cases}$$

and that

$$\begin{aligned} \phi_{2t} &= \big(K_0 + K_1(\phi_2, b) - \varepsilon\phi_{2x}\big)\phi_1 - \varepsilon\big(b(\alpha_1 - \alpha_2)\big)_x + \varepsilon^2\tilde{g}_4 \\ &= K_0\phi_1 + \varepsilon\tilde{g}_5, \end{aligned}$$

where $\tilde{g}_4 = \tilde{g}_2 - g_3$ and $\tilde{g}_5 = \big(\varepsilon^{-1}K_1(\phi_2, b) - \phi_{2x}\big)\phi_1 - \big(b(\alpha_1 - \alpha_2)\big)_x + \varepsilon\tilde{g}_4$.

Therefore, in place of (5.14) we have

$$
\begin{cases}
\phi_{1tt} + 2\varepsilon\phi_1\phi_{1tx} - \varepsilon^{1/2}\mu K_0\phi_{1xxx} + \phi_{2tx} + \varepsilon(\phi_{1t}\phi_{1x} + \phi_{2xx}\phi_1) \\
\quad = \varepsilon(\alpha_{1x}\alpha_2 - \alpha_1\alpha_{2x})_x + \varepsilon^{3/2}\tilde{g}_6, \\
\phi_{2xtt} + 2\varepsilon\phi_1\phi_{2xxt} - \varepsilon^{1/2}\mu K_0\phi_{2xxxx} + \big(K_0 + K_1(\phi_2, b)\big)\phi_{2xx} \\
\qquad + \varepsilon K_0\phi_{1x}\phi_{1x} + \varepsilon(\phi_{2xx}\phi_{1t} + \phi_{2xt}\phi_{1x}) + \varepsilon[K_0, \phi_1]\phi_{1xx} \\
\qquad - \big(K_1(\phi_{2x}, b')\phi_{1t} + K_1(\phi_{2t}, 0)\phi_{1x} + K_1(\phi_{2xt}, 0)\phi_1\big) - \varepsilon\phi_{2x}\phi_{2xx} \\
\quad = \varepsilon\big(b(\alpha_1 + \alpha_2)_x - K_0(\alpha_1\alpha_2)\big)_{xx} + \varepsilon^{3/2}\tilde{g}_7,
\end{cases}
\tag{7.8}
$$

where

$$
\begin{cases}
\tilde{g}_6 = \tilde{g}_{1t} + \mu\tilde{g}_{5xxx} - \varepsilon^{1/2}(\alpha_{1\tau}\alpha_2 + \alpha_1\alpha_{2\tau})_x \\
\qquad + \big(\mu\phi_{2xxx} - \varepsilon^{1/2}\phi_1\phi_{1x} - \varepsilon^{1/2}(\alpha_1\alpha_2)_x + \varepsilon\tilde{g}_1\big)_x\phi_1, \\
\tilde{g}_7 = \big(K_0 + K_1(\phi_2, b) - \varepsilon\phi_{2x}\big)\tilde{g}_{1x} \\
\qquad + \tilde{g}_{4xt} + \varepsilon^{1/2}\phi_1\tilde{g}_{5xx} - \varepsilon^{1/2}\big(b(\alpha_{1\tau} - \alpha_{2\tau})\big)_{xx} \\
\qquad + \big(\varepsilon^{-1}K_1(\phi_2, b) - \phi_{2x}\big)\big(\mu\phi_{2xxx} - \varepsilon^{1/2}\phi_1\phi_{1x} - \varepsilon^{1/2}(\alpha_1\alpha_2)_x\big)_x.
\end{cases}
$$

As before, we define remainder functions $\bar{\eta}$ and $\bar{u}_1$ by (5.15) and put $\bar{\zeta} = \bar{\eta}_x$.

Then, in place of (5.16) we have

$$
\begin{cases}
\bar{u}_{1tt} + 2\varepsilon u_1\bar{u}_{1tx} + \varepsilon\big(\varepsilon u_1^2 + 3\varepsilon^{1/2}\mu(1 + \varepsilon^{5/2}\zeta^2)^{-5/2}\zeta_x\big)\bar{u}_{1xx} \\
\qquad - \varepsilon^{1/2}\mu\big((1 + \varepsilon^{5/2}\zeta^2)^{-3/2}K_0\bar{u}_{1xx}\big)_x - \varepsilon^{1/2}\mu(K_1\bar{u}_1)_{xxx} + \bar{\zeta}_t \\
\quad = \varepsilon^{1/2}(\alpha_1\alpha_{2x} - \alpha_{1x}\alpha_2)_x + \varepsilon\tilde{h}_1, \\
\bar{\zeta}_{tt} + 2\varepsilon u_1\bar{\zeta}_{tx} - \varepsilon^{1/2}\mu(1 + \varepsilon^{5/2}\zeta^2)^{-3/2}K_0\bar{\zeta}_{xxx} - \varepsilon^{1/2}\mu K_1\bar{\zeta}_{xxx} \\
\qquad + (K_0 + K_1)\bar{\zeta}_x + \varepsilon[K_0, u_1]\bar{u}_{1xx} - \big(K_1(\zeta, b')\bar{u}_{1t} + K_1(\eta_t, 0)\bar{u}_{1x}\big) \\
\quad = \varepsilon^{1/2}\big(K_0(\alpha_1\alpha_2) - b(\alpha_1 + \alpha_2)_x\big)_{xx} + \varepsilon\tilde{h}_2,
\end{cases}
\tag{7.9}
$$

where $\tilde{h}_1 = h_1 + g_6 - \tilde{g}_6$ and $\tilde{h}_2 = h_2 + g_7 - \tilde{g}_7$. Moreover, in place of (5.17) we have

$$
\begin{cases}
\bar{\zeta}_t = (K_0 + K_1)\bar{u}_{1x} + \varepsilon^{1/2}\big(b(\alpha_1 - \alpha_2)\big)_{xx} + \varepsilon\tilde{h}_3, \\
\bar{u}_{1t} = -\bar{\zeta} + \varepsilon^{1/2}\mu\bar{\zeta}_{xx} + \varepsilon^{1/2}(\alpha_1\alpha_2)_x + \varepsilon\tilde{h}_4,
\end{cases}
\tag{7.10}
$$

where $\tilde{h}_3 = h_3 + g_{4x} - \tilde{g}_{4x}$ and $\tilde{h}_4 = h_4 + g_1 - \tilde{g}_1$. Therefore, in place of

(5.19) and (5.20) we obtain

$$\begin{cases} \bar{u}_{1tt} + 2\varepsilon u_1 \bar{u}_{1tx} + \varepsilon\big(\varepsilon u_1^2 + 3\varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-5/2}\zeta_x\big)\bar{u}_{1xx} \\ \qquad - \varepsilon^{1/2}\mu\big((1+\varepsilon^{5/2}\zeta^2)^{-3/2}K_0\bar{u}_{1xx}\big)_x + K_0\bar{u}_{1x} \\ \qquad - \varepsilon^{3/2}\mu L_1(\eta,b)\bar{u}_{1xxxx} + \varepsilon L_1(\eta,b)\bar{u}_{1xx} = \varepsilon^{1/2}\tilde{g}_8 + \varepsilon\tilde{h}_5, \\ \bar{\zeta}_{tt} + 2\varepsilon u_1\bar{\zeta}_{tx} - \varepsilon^{1/2}\mu(1+\varepsilon^{5/2}\zeta^2)^{-3/2}K_0\bar{\zeta}_{xxx} + K_0\bar{\zeta}_x \\ \qquad - \varepsilon^{3/2}\mu L_1(\eta,b)\bar{\zeta}_{xxxx} + \varepsilon L_1(\eta,b)\bar{\zeta}_{xx} = \varepsilon^{1/2}\tilde{g}_9 + \varepsilon\tilde{h}_6 \end{cases} \tag{7.11}$$

and

$$\begin{cases} \bar{u}_{1t} + \bar{\eta}_x + \varepsilon^{1/2}\mu\bar{u}_{1txx} = \varepsilon^{1/2}\tilde{g}_{10} + \varepsilon\tilde{h}_9, \\ \bar{\eta}_t + \bar{u}_{1x} - \frac{\varepsilon^{1/2}}{3}\bar{\eta}_{txx} = \varepsilon^{1/2}\tilde{g}_{11} + \varepsilon\tilde{h}_{10}, \end{cases} \tag{7.12}$$

respectively, where $\tilde{h}_5 = h_5 - (h_1 - \tilde{h}_1) + (h_3 - \tilde{h}_3)$,

$$\begin{aligned} \tilde{h}_6 \;=\; & \tilde{h}_2 + \varepsilon^{1/2}\mu L_1(0,b')\bar{\zeta}_{xxx} - L_1(0,b')\bar{\zeta}_x \\ & + \big([K_0,u_1]K_0^{-1}(iD) - \varepsilon^{-1}K_1(\eta_t,0)K_0^{-1}\big) \\ & \qquad \times \big\{\varepsilon\tilde{h}_3 + K_1\bar{u}_{1x} + \varepsilon^{1/2}\big(b(\alpha_1-\alpha_2)\big)_{xx} - \bar{\zeta}_t\big\} \\ & + \varepsilon^{-1}K_1(\zeta,b')\big(\varepsilon\tilde{h}_4 + \varepsilon^{1/2}\mu\bar{\zeta}_{xx} + \varepsilon^{1/2}(\alpha_1\alpha_2)_x - \bar{\zeta}\big), \end{aligned}$$

$\tilde{h}_7 = h_7 + g_1 - \tilde{g}_1$, $\tilde{h}_8 = h_8 + g_2 - \tilde{g}_2$,

$$\begin{cases} \tilde{h}_9 = \mu^2\bar{\eta}_{xxxxx} + \mu(\alpha_1\alpha_2)_{xxx} + \tilde{h}_7 + \varepsilon^{1/2}\mu\tilde{h}_{7xx}, \\ \tilde{h}_{10} = \frac{1}{9}\bar{u}_{1xxxxx} - \frac{1}{3}\big(b(\alpha_1-\alpha_2)\big)_{xx} + \tilde{h}_8 - \frac{\varepsilon^{1/2}}{3}\tilde{h}_{8xx}, \end{cases}$$

and

$$\begin{cases} \tilde{g}_8(x,t) = \big(\alpha_1(x-t,\varepsilon t)\alpha_{2x}(x+t,\varepsilon t) - \alpha_{1x}(x-t,\varepsilon t)\alpha_2(x+t,\varepsilon t)\big)_x \\ \qquad\qquad + \big(b(x)\big(\alpha_1(x-t,\varepsilon t) - \alpha_2(x+t,\varepsilon t)\big)\big)_{xx}, \\ \tilde{g}_9(x,t) = K_0\big(\alpha_1(x-t,\varepsilon t)\alpha_2(x+t,\varepsilon t)\big)_{xx} \\ \qquad\qquad - \big(b(x)\big(\alpha_1(x-t,\varepsilon t) + \alpha_2(x+t,\varepsilon t)\big)_x\big)_{xx}, \\ \tilde{g}_{10}(x,t) = \big(\alpha_1(x-t,\varepsilon t)\alpha_2(x+t,\varepsilon t)\big)_x, \\ \tilde{g}_{11}(x,t) = \big(b(x)\big(\alpha_1(x-t,\varepsilon t) - \alpha_2(x+t,\varepsilon t)\big)\big)_{xx}. \end{cases}$$

It is not difficult to check that $\tilde{h}_5,\dots,\tilde{h}_{10}$ satisfy the same estimate in Lemma 6.9 as $h_5,\dots,h_{10}$. For $\tilde{g}_8,\dots,\tilde{g}_{11}$, we have the following lemma. We refer to [14] for the proof.

Lemma 7.2. *Let m be a positive integer. There exists a positive con-*

stant C such that

$$\begin{aligned}&\|\tilde g_8(t)\|_m+\|\tilde g_9(t)\|_m+\|\tilde g_{10}(t)\|+\|\tilde g_{11}(t)\|\\&\le\ C(1+t)^{-2}\big(\|\alpha(\varepsilon t)\|_{m+3,2}+\|b\|_{m+2,2}\big)\|\alpha(\varepsilon t)\|_{m+3,2}\end{aligned}$$

for $t\ge 0$ and $\varepsilon>0$.

Under the assumption of Theorem 4.2, there exists a constant $M_1>0$ such that the initial value problem for the Kawahara equation (4.9) and (4.10) has a unique solution $\alpha\in C([0,T];H^{m+17}\cap H^{m+3,2})$ satisfying

$$\|\alpha(\tau)\|_{m+17}+\|\alpha(\tau)\|_{m+3,2}\le M_1\quad\text{for}\quad 0\le\tau\le T,\ \varepsilon>0,$$

(see Appendix for the proof) so that by Lemma 7.2 we have

$$\|\tilde g_8(t)\|_m^2+\|\tilde g_9(t)\|_m^2+\|\tilde g_{10}(t)\|^2+\|\tilde g_{11}(t)\|^2\le C_1(1+t)^{-4}$$

for $0\le t\le T/\varepsilon$ and $\varepsilon>0$. Now, we suppose (7.4) as before. In this case, in place of (7.5), (7.6), and (7.7) we obtain

$$\begin{aligned}&\|\big(\bar\zeta_t^\varepsilon(t),\bar\zeta_x^\varepsilon(t),\bar u_{1t}^\varepsilon(t),\bar u_{1x}^\varepsilon(t)\big)\|_m^2\\&\le\ C_3\mathrm{e}^{C_3\varepsilon t}\big(\|\big(\bar\zeta_t^\varepsilon(0),\bar u_{1t}^\varepsilon(0)\big)\|_m^2+\|\big(\bar\zeta^\varepsilon(0),\bar u_1^\varepsilon(0)\big)\|_{m+2}^2\big)\\&\qquad+C_3\int_0^t\mathrm{e}^{C_3\varepsilon(t-\tau)}\big\{(1+\tau)^2\big(\|\tilde g_8(\tau)\|_m^2+\|\tilde g_9(\tau)\|_m^2\big)\\&\qquad\qquad+\varepsilon\big(\|\tilde h_5(\tau)\|_m^2+\|\tilde h_6(\tau)\|_m^2\big)\big\}d\tau\\&\le\ C_3C_2\mathrm{e}^{C_3\varepsilon t}+C_3C_1\int_0^t\mathrm{e}^{C_3\varepsilon(t-\tau)}\big\{(1+\tau)^{-2}+\varepsilon\big(1+\mathscr{E}(\tau)\big)\big\}d\tau,\end{aligned}$$

$$\begin{aligned}&\|\big(\bar\eta^\varepsilon(t),\bar u_1^\varepsilon(t)\big)\|^2\\&\le\ \varepsilon^{1/2}\mu\|\bar u_{1x}^\varepsilon(t)\|^2+\mathrm{e}^{1+\varepsilon t}\Big(\|\bar\eta^\varepsilon(0)\|^2+\|\bar u_1^\varepsilon(0)\|^2+\frac{\varepsilon^{1/2}}{3}\|\bar\eta_x^\varepsilon(0)\|^2\Big)\\&\qquad+\int_0^t\mathrm{e}^{1+\varepsilon(t-\tau)}\Big((1+\tau)^2\big(\|\tilde g_{10}(\tau)\|^2+\|\tilde g_{11}(\tau)\|^2\big)\\&\qquad\qquad+\varepsilon\big(\|\tilde h_9(\tau)\|^2+\|\tilde h_{10}(\tau)\|^2+\varepsilon^{1/2}\mu\|\bar u_{1x}^\varepsilon(\tau)\|^2\big)\Big)d\tau\\&\le\ \varepsilon^{1/2}\mu\|\bar u_{1x}^\varepsilon(t)\|^2+C_2\mathrm{e}^{1+\varepsilon t}\\&\qquad+(C_1+\mu)\int_0^t\mathrm{e}^{1+\varepsilon(t-\tau)}\big\{(1+\tau)^{-2}+\varepsilon\big(1+\mathscr{E}(\tau)\big)\big\}d\tau,\end{aligned}$$

and

$$\begin{aligned}\|\bar{\eta}_t^\varepsilon(t)\|^2 &\le 4\|\bar{u}_{1x}^\varepsilon(t)\|^2 + \frac{4}{9}\varepsilon\|\bar{\zeta}_{tx}^\varepsilon(t)\|^2 + 4\|\tilde{g}_{11}(t)\|^2 + 4\varepsilon^2\|\tilde{h}_{10}(t)\|^2 \\ &\le 4\|\bar{u}_{1x}^\varepsilon(t)\|^2 + (4C_1+1)\big\{(1+\tau)^{-2} + \varepsilon\big(1+\mathscr{E}(t)\big)\big\},\end{aligned}$$

respectively. Summarizing the above estimates we see that

$$\begin{aligned}\mathscr{E}(t) &\le C_4 \mathrm{e}^{C_4\varepsilon t} + C_4\int_0^t \mathrm{e}^{C_4\varepsilon(t-\tau)}\big\{(1+\tau)^{-2} + \varepsilon\big(1+\mathscr{E}(\tau)\big)\big\}d\tau \\ &\le 2C_4\mathrm{e}^{C_4\varepsilon t} + \varepsilon C_4\int_0^t \mathrm{e}^{C_4\varepsilon(t-\tau)}\big(1+\mathscr{E}(\tau)\big)d\tau\end{aligned}$$

for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_0$. This and Gronwall's inequality imply that

$$\mathscr{E}(t) \le (2C_4+1)\mathrm{e}^{2C_4T} \quad \text{for} \quad 0 \le t \le T/\varepsilon,\ 0 < \varepsilon \le \varepsilon_0.$$

Therefore, by setting $N_1 = (2C_4+1)^{1/2}\mathrm{e}^{C_4T}$ we see that (7.4) holds for $0 \le t \le T/\varepsilon$ and $0 < \varepsilon \le \varepsilon_0$. This easily follows (4.13). The proof of Theorem 4.2 is complete.

It remains to prove Theorem 4.3. As explained in Remark 4.3, under the assumption of Theorem 4.3 the solution $\alpha = (\alpha_1, \alpha_2)$ of (4.9) and (4.10) satisfies $\|\alpha_1(\tau)\|_{m+17} \le C\varepsilon^{1/2}$ or $\|\alpha_2(\tau)\|_{m+17} \le C\varepsilon^{1/2}$, so that we have

$$\|\tilde{g}_8(t)\|_m + \|\tilde{g}_9(t)\|_m + \|\tilde{g}_{10}(t)\| + \|\tilde{g}_{11}(t)\| \le C\varepsilon^{1/2}$$

for $0 \le t \le T/\varepsilon$ and $\varepsilon > 0$. Therefore, we can show Theorem 4.3 in the same way as the proof of Theorem 4.1.

Appendix. A Priori Estimates for the Kawahara Equation

By an appropriate change of variables it is sufficient to consider the equation

$$u_t + uu_x + \nu u_{xxx} + u_{xxxxx} = 0, \tag{A.1}$$

where ν is a real constant, under the initial condition

$$u(x,0) = u_0(x). \tag{A.2}$$

Lemma A.1. *Let $u \in C([-T,T]; H^5) \cap C^1([-T,T]; H^2)$ be a solution of* (A.1)*. Then, it holds that*

$$\frac{d}{dt}\|u(t)\|^2 = 0,$$

$$\frac{d}{dt}\Big(\|u_{xx}(t)\|^2 - \nu\|u_x(t)\|^2 + \frac{1}{3}\int_{\mathbf{R}} u(x,t)^3 dx\Big) = 0.$$

Proof. By making use of the differential equation and integration by parts we can prove the desired equalities. □

Lemma A.2. *Let $s \geq 2$ and $u \in C([-T,T]; H^s) \cap C^1([-T,T]; H^2)$ be a solution of* (A.1)–(A.2)*. There exists a positive constant $C = C(s,\nu)$ such that*

$$\|u(t)\|_s \leq \|u_0\|_s \exp\big(C\|u_0\|_2(1 + \|u_0\|^{4/7})|t|\big) \tag{A.3}$$

holds for $t \in [-T,T]$.

Proof. We will derive estimate (A.3) assuming $u \in C([-T,T]; H^{s+5}) \cap C^1([-T,T]; H^s)$. This assumption will be removed by the standard technique of mollifier. By Lemma A.1, interpolation inequality $\|u_x\|^2 \leq \varepsilon\|u_{xx}\|^2 + C_\varepsilon\|u\|^2$, and the Sobolev imbedding theorem $|u|_\infty \leq C\|u\|^{1/2}\|u_x\|^{1/2}$, we obtain $\|u(t)\|_2 \leq C\|u_0\|_2(1+\|u_0\|^{4/7})$. Put $v = (1+|D|)^s u$. Multiplying the operator $(1+|D|)^s$ on both sides of equation (A.1) we have

$$v_t + uv_x + \nu v_{xxx} + v_{xxxxx} = f,$$

where $f = -[(1+|D|)^s, u]u_x$. It is easy to see that $\|f\| \leq C\|u\|_2\|v\|$, and that

$$\begin{aligned}\frac{d}{dt}\|v(t)\|^2 &= (v(t), u_x(t)v(t)) + 2(v(t), f(t)) \\ &\leq C\|u(t)\|_2\|v(t)\|^2.\end{aligned}$$

This and Gronwall's inequality imply the desired estimate. □

Lemma A.3. *Let $s > 5+1/2$ and $u \in C([-T,T]; H^s) \cap C^1([-T,T]; H^2)$ be a solution of* (A.1)–(A.2)*. There exists a positive constant $C = C(\|u_0\|_s, T, \nu, s)$ such that if $u_0 \in H^{s-4,1}$, then*

$$\|u(t)\|_{s-4,1} \leq C(\|u_0\|_{s-4,1} + \|u_0\|_s) \tag{A.4}$$

holds for $t \in [-T, T]$.

Proof. We will derive estimate (A.4) assuming $u \in C([-T,T]; H^{s+1,1}) \cap C^1([-T,T]; H^{s-4,1})$. This assumption will be removed by considering the function $u(x,t)\exp\bigl(-\frac{1}{n}\sqrt{1+x^2}\bigr)$ instead of $u(x,t)$ and letting $n \to +\infty$, and by the standard technique of mollifier. Put $w = (1+|D|)^{s-4}(xu)$. Then, by (A.1) we have $w_t + uw_x + \nu w_{xxx} + w_{xxxxx} = f$, where

$$f = (1+|D|)^{s-4}(u^2 + 3\nu u_{xx} + 5u_{xxxx}) - [(1+|D|)^{s-4}, u](1+|D|)^{-(s-4)} w_x.$$

It is easy to see that $\|f\| \le C(\|u\|_{s-4}^2 + \|u\|_s + \|u\|_{s-4}\|w\|)$, and that

$$\frac{d}{dt}\|w(t)\|^2 \le C\bigl((\|u(t)\|_{s-4} + 1)\|w(t)\|^2 + (\|u(t)\|_{s-4}^2 + \|u(t)\|_s)^2\bigr).$$

Therefore, Lemma A.2 and Gronwall's inequality yield the desired estimate. □

Similarly, we can show the following lemma.

Lemma A.4. *Let* $s > 9+1/2$ *and* $u \in C([-T,T]; H^s) \cap C^1([-T,T]; H^2)$ *be a solution of* (A.1)–(A.2). *There exists a positive constant* $C = C(\|u_0\|_s, T, \nu, s)$ *such that if* $u_0 \in H^{s-4,1} \cap H^{s-8,2}$, *then*

$$\|u(t)\|_{s-8,2} \le C(\|u_0\|_{s-8,2} + \|u_0\|_{s-4,1} + \|u_0\|_s) \qquad \text{(A.5)}$$

holds for $t \in [-T, T]$.

A local existence theorem for the initial value problem (A.1)–(A.2) can be proved by approximating equation (A.1) by an appropriate one. Since this technique is standard, we omit it. Combining the local existence theorem and a priori estimates in Lemmas A.2–A.4, we obtain the following global existence theorem.

Theorem A.1. *Let* $s \ge 7$. *For any* $u_0 \in H^s$ *the initial value problem* (A.1)–(A.2) *has a unique global solution* $u \in C(\mathbf{R}; H^s) \cap C^1(\mathbf{R}; H^{s-5})$ *satisfyig* (A.3). *If, in addition,* $s > 9+1/2$ *and* $u_0 \in H^{s-4,1} \cap H^{s-8,2}$, *then the solution* u *satisfies* (A.4) *and* (A.5).

Remark A.1. The condition $s \ge 7$ is not optimal. However, this theorem gives sufficient information for our purpose.

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