SOME INSTABILITY RESULTS ON CERTAIN THIRD ORDER NONLINEAR VECTOR DIFFERENTIAL EQUATIONS

BY

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Abstract

In this paper, we obtain some sufficient conditions under which the zero solution of a certain third order non-linear ordinary vector differential equation is unstable. Our results include and improve some well-known results exist in the literature.

1. Introduction

As known from the relevant literature in a sequence of results, till now, instability behaviors of solutions for various third-, fourth-, fifth-, sixth-, seventh and eighth order certain linear and nonlinear differential equations have been discussed extensively by many researchers. In this connection, ones can refer to the papers of Berketoğlu [1], Berketoğlu and Kart [2], Ezeilo ([3], [4], [5], [6], [7]), Kipnis [8], Krasovskii [9], Li and Yu [10], Li and Duan [11], Liao and Lu [12], Losprime [13], Lu [14], Reissig et al [15], Sadek [16], Skrapek ([17], [18]), Tejumola [19], Tiryaki [20], [21], [22]), C. Tunç([23], [24], [25], [26], [27], [28], [29]), C. Tunç and E. Tunç ([30], [31], [32]), C. Tunç and H. Şevli [33] and E. Tunç [34] and the references cited in that papers for the related works. However, according to our observations from the literature, the instability properties of linear and nonlinear scalar differential equations of third order have been discussed only by Bereketoğlu & Kart [2], Kipnis
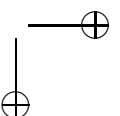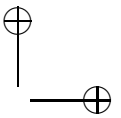
[8], Losprime [13], Lu [14] and Skrapek [18]. Now, these results can be summarized as following: First, in 1966, Losprime [13] considered the third-order scalar linear differential equation with periodic coefficients as follows

$$\dddot{x} + \ddot{x} + S(t)\dot{x} + T(t)x = 0.$$

The author found the regions of stability and instability of this differential equation by means of some expansions and using Lyapunov's second (or direct) method (see, Lyapunov [35]). Then, in 1974, Kipnis [8] discussed the instability of the scalar linear differential equation

$$\dddot{x} + p(t)x = 0.$$

The author showed that if $p(t)$ is continuous, ω-periodic, non-positive, and satisfies an inequality involving ω, then the above equation is unstable. Later, in 1980, Skrapek [18] studied the instability of the trivial solution of the scalar non-linear differential equation

$$\dddot{x} + f_1(\ddot{x}) + f_2(\dot{x}) + f_3(x) + f_4(x, \dot{x}, \ddot{x}) = 0.$$

using Lyapunov's second (or direct) method. Afterward, in 1995, Lu [14] investigated a similar problem for the third order nonlinear scalar differential equations of the form

$$\dddot{x} + f(x, \dot{x})\ddot{x} + g(x) = 0.$$

A year later, that is, in 1996, Bereketoğlu & Kart [2] obtained sufficient conditions which ensure that the trivial solution of scalar differential equation

$$\dddot{x} + f(\dot{x})\ddot{x} + g(x)\dot{x} + h(x, \dot{x}, \ddot{x}) = 0.$$

is unstable, and also that nontrivial solutions of this equation are not periodic. Besides these works, recently, E. Tunç [34] analyzed instability of zero solution of the non-linear vector differential of the form

$$\dddot{X} + F(\dot{X})\ddot{X} + G(X)\dot{X} + H(X, \dot{X}, \ddot{X}) = 0.$$

Furthermore, it has not been founded another research on the instability of solutions of certain nonlinear vector differential equations of the third order
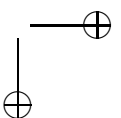
in the relevant literature. This paper is interested in the instability of the zero solution $X = 0$ of the third-order nonlinear vector differential equations of the form:

$$\dddot{X} + F(\dot{X})\ddot{X} + G(\dot{X}) + H(X) = 0. \tag{1}$$

in which $X \in \mathfrak{R}^n$; F is an $n \times n$-symmetric matrix function; $G : \mathfrak{R}^n \to \mathfrak{R}^n$, $H : \mathfrak{R}^n \to \mathfrak{R}^n$ and $G(0) = H(0) = 0$. Let F, G and H be continuous and so constructed such that the uniqueness theorem is valid. Let $J_F(\dot{X})$, $J_G(\dot{X})$ and $J_H(X)$, respectively, denote the linear operators from the matrix $F(\dot{X})$ and vectors $G(\dot{X})$, $H(X)$ to the matrices

$$J_F(\dot{X}) = \left(\frac{\partial f_{ik}}{\partial \dot{x}_j}\right), \quad J_G(\dot{X}) = \left(\frac{\partial g_i}{\partial \dot{x}_j}\right) \quad \text{and} \quad J_H(X) = \left(\frac{\partial h_i}{\partial x_j}\right),$$
$$(i, j, k = 1, 2, \ldots, n),$$

where $(x_1, x_2, \ldots, x_n)$, $(\dot{x}_1, \dot{x}_2, \ldots, \dot{x}_n)$, (f_{ik}), $(i, k = 1, 2, \ldots, n)$, $(g_1, g_2, \ldots, g_n)$ and $(h_1, h_2, \ldots, h_n)$ are components of X, $\dot{X}$, F, G and H, respectively. Other than these, it is also assumed that $G(\dot{X})$ and $H(X)$ are gradient vector fields, that is, there are scalar functions g and h such that $G = \nabla g$ and $H = \nabla h$ and the matrices $J_F(\dot{X})$, $J_G(\dot{X})$ and $J_H(X)$ exist and are continuous. Given any X, Y in $\mathfrak{R}^n$. Throughout the paper, the symbol $\langle X, Y \rangle$ is used to denote the usual scalar product in $\mathfrak{R}^n$, that is, $\langle X, Y \rangle = \sum_{i=1}^{n} x_i y_i$, thus $\|X\|^2 = \langle X, Y \rangle$. The matrix A is said to be negative-definite, when $\langle AX, X \rangle < 0$ for all non-zero X in $\mathfrak{R}^n$, and $\lambda_i(A)$, $(i = 1, 2, \ldots, n)$, are eigenvalues of the $n \times n$-matrix A.

In what follows it will be convenient to use the equivalent differential system:

$$\dot{X} = Y, \quad \dot{Y} = Z,$$
$$\dot{Z} = -F(Y)Z - G(Y) - H(X) \tag{2}$$

which is obtained from the equation (1) by setting $\dot{X} = Y$, $\ddot{X} = Z$.

It should be clarified that, nearly through all of the papers just stated above based on Krasovskii's criterion-properties (see Miller & Michel [36] or Krasovskii [9]), the Lyapunov's second (or direct) method has been used to prove the results established there. In this paper, we use this method in the proof of our main results. The motivation for the present work has
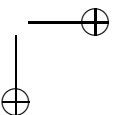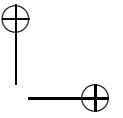
been inspired basically by the paper of Skrapek [18] and that just mentioned above. It should also be remarked that the assumptions established and the Lyapunov's function used here are completely different than that used in [34].

2. Main Results

For the first time we need the following algebraic result.

Lemma. *Let A be a real symmetric $n \times n$-matrix and*

$$a' \geq \lambda_i(A) \geq a > 0, \quad (i = 1, 2, \ldots, n),$$

where a', a are constants.

Then

$$a'\langle X, X \rangle \geq \langle AX, X \rangle \geq a\langle X, X \rangle$$

and

$$a'^2 \langle X, X \rangle \geq \langle AX, AX \rangle \geq a^2 \langle X, X \rangle.$$

Proof. See [26]. □

Next, the first main result is the following theorem.

Theorem 1. *In addition to the fundamental assumptions imposed on F, G and H, it is assumed that $J_G(Y)$ and $J_H(X)$ are symmetric matrices and there are constants a_1, a_2, $\overline{a}_2$ and a_3 such that the following conditions are satisfied:*
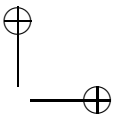
(i) $\lambda_i(F(Y)) \leq a_1$, $-\overline{a}_2 \leq \lambda_i(J_G(Y)) \leq -a_2 < 0$ and $\lambda_i(J_H(X)) \geq a_3 > 0$ for all $X, Y \in \mathfrak{R}^n$

or

(i)′ $\lambda_i(F(Y)) \leq a_1$, $-\overline{a}_2 \leq \lambda_i(J_G(Y)) \leq -a_2 < 0$ and $\lambda_i(J_H(X)) \leq -a_3 < 0$ for all $X, Y \in \mathfrak{R}^n$, $(i = 1, 2, \ldots, n)$.

Then the zero solution $X = 0$ of the system (2) is unstable.
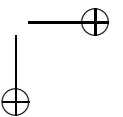
Remark 1. In respect of Krasovskii's criterion-properties the kernel of the proof of Theorem 1 will be to show that, under the conditions stated in Theorem 1, there exists a continuous function $V_0 = V_0(X, Y, Z)$ which has the following three properties:

(p_1) In every neighborhood of $(0,0,0)$ there exists a point (ξ, η, ζ) such that $V_0(\xi, \eta, \zeta) > 0$.

(p_2) The time derivative $\dot{V}_0 = \frac{d}{dt}V_0(X, Y, Z)$ along solution paths of the system (2) for Theorem 1 is positive semi definite.

(p_3) The only solution $(X(t), Y(t), Z(t))$ of the system (2) for Theorem 1 which satisfies

$$V_0(X(t), Y(t), Z(t)) = 0, \quad (t \geq 0),$$

is the trivial solution $(0,0,0)$.

The existence of a V_0 with the properties (p_1), (p_2) and (p_3) is sufficient, in view of the instability criterion of Krasovskii [9], for the instability of the zero solution of the equation (1).

Proof of Theorem 1. For the proof of first part of Theorem 1 we consider the function $V_0 = V_0(X, Y, Z)$ defined as follows:

$$2V_0 = 2\alpha \int_0^1 \langle H(\sigma X), X \rangle d\sigma + 2\alpha \langle Y, Z \rangle + 2\alpha \int_0^1 \sigma \langle F(\sigma Y)Y, Y \rangle d\sigma$$
$$+ \langle Y, Y \rangle - 2\langle X, Z \rangle, \tag{3}$$

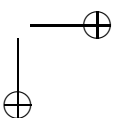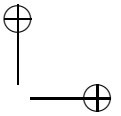where α is a positive constant. It is clear from (3) that $V_0(0,0,0) = 0$. Since

$$H(0) = 0, \quad \frac{\partial}{\partial \sigma} H(\sigma X) = J_H(\sigma X)X,$$

then

$$H(X) = \int_0^1 J_H(\sigma X)X d\sigma.$$

Hence the assumption (i) of Theorem 1 and the expression $H(X) = \int_0^1 J_H(\sigma X)$

$X d\sigma$ show that

$$\int_0^1 \langle H(\sigma X), X \rangle d\sigma = \int_0^1 \int_0^1 \langle \sigma_1 J_H(\sigma_1 \sigma_2 X) X, X \rangle d\sigma_2 d\sigma_1$$

$$\geq \int_0^1 \int_0^1 \langle \sigma_1 a_3 X, X \rangle d\sigma_2 d\sigma_1 = \frac{a_3}{2} \langle X, X \rangle = \frac{a_3}{2} \|X\|^2. \quad (4)$$

Obviously, in view of the assumption (i) of Theorem 1, (3) and (4), it follows that

$$V_0(\varepsilon, 0, 0) \geq \alpha \frac{a_3}{2} \langle \varepsilon, \varepsilon \rangle = \alpha \frac{a_3}{2} \|\varepsilon\|^2,$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \mathfrak{R}^n$. So, in every neighborhood of $(0, 0, 0)$ there exists a point (ξ, η, ζ) such that $V_0(\xi, \eta, \zeta) > 0$. Next, let $(X, Y, Z) = (X(t), Y(t), Z(t))$ be an arbitrary solution of the system (2). A straightforward calculation from (3) and (2) yields that
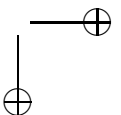
$$\begin{aligned}
\dot{V}_0 &= \frac{d}{dt} V_0(X, Y, Z) \\
&= \alpha \langle Z, Z \rangle - \alpha \langle Y, G(Y) \rangle + \langle X, H(X) \rangle \\
&\quad + \langle X, F(Y)Z \rangle - \alpha \langle F(Y)Z, Y \rangle + \langle X, G(Y) \rangle - \alpha \langle H(X), Y \rangle \\
&\quad + \alpha \frac{d}{dt} \int_0^1 \langle H(\sigma X), X \rangle d\sigma + \alpha \frac{d}{dt} \int_0^1 \sigma \langle F(\sigma Y)Y, Y \rangle d\sigma. \quad (5)
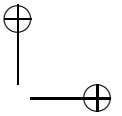\end{aligned}$$

Recall that

$$\begin{aligned}
\frac{d}{dt} \int_0^1 \langle H(\sigma X), X \rangle d\sigma &= \int_0^1 \sigma \langle J_H(\sigma X)Y, X \rangle d\sigma + \int_0^1 \langle H(\sigma X), Y \rangle d\sigma \\
&= \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle H(\sigma X), Y \rangle d\sigma + \int_0^1 \langle H(\sigma X), Y \rangle d\sigma \\
&= \sigma \langle H(\sigma X), Y \rangle \mid_0^1 = \langle H(X), Y \rangle \quad (6)
\end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt} \int_0^1 \sigma \langle F(\sigma Y), Y \rangle d\sigma &= \int_0^1 \langle \sigma F(\sigma Y)Z, Y \rangle d\sigma + \int_0^1 \sigma^2 \langle J_F(\sigma Y)YZ, Y \rangle d\sigma \\
&\quad + \int_0^1 \sigma \langle F(\sigma Y)Y, Z \rangle d\sigma
\end{aligned}$$

$$= \int_0^1 \langle \sigma F(\sigma Y)Z, Y \rangle d\sigma + \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle \sigma F(\sigma Y)Z, Y \rangle d\sigma$$

$$= \sigma^2 \langle F(\sigma Y)Z, Y \rangle \mid_0^1 = \langle F(Y)Z, Y \rangle. \tag{7}$$

By collecting the estimates (6) and (7) into (5) we obtain

$$\dot{V}_0 = \alpha \langle Z, Z \rangle - \alpha \langle Y, G(Y) \rangle + \langle X, H(X) \rangle$$
$$+ \langle X, F(Y)Z \rangle + \langle X, G(Y) \rangle. \tag{8}$$

Since

$$G(0) = 0, \quad \frac{\partial}{\partial \sigma} G(\sigma Y) = J_G(\sigma Y)Y,$$

then

$$G(Y) = \int_0^1 J_G(\sigma Y)Y \, d\sigma.$$

Thus, the assumption (i) of Theorem 1 shows that

$$\alpha \langle Y, G(Y) \rangle = \alpha \int_0^1 \langle Y, J_G(\sigma Y)Y \rangle d\sigma \leq -\alpha a_2 \int_0^1 \langle Y, Y \rangle d\sigma$$
$$= -\alpha a_2 \langle Y, Y \rangle = -\alpha a_2 \|Y\|^2 \tag{9}$$

and

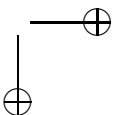$$\langle G(Y), G(Y) \rangle = \int_0^1 \langle J_G(Y)Y, J_G(\sigma Y)Y \rangle d\sigma \leq \overline{a}_2^2 \int_0^1 \langle Y, Y \rangle d\sigma$$
$$= \overline{a}_2^2 \langle Y, Y \rangle = \overline{a}_2^2 \|Y\|^2. \tag{10}$$
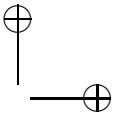
On combining the estimate (9) with (8) we can easily obtain

$$\dot{V}_0 \geq \alpha \|Z\|^2 + \alpha a_2 \|Y\|^2 + a_3 \|X\|^2$$
$$+ \langle X, F(Y)Z \rangle + \langle X, G(Y) \rangle. \tag{11}$$

For some constants k_1 and k_2, conveniently chosen later, we have

$$\langle X, G(Y) \rangle = \frac{1}{2} \|k_1 X + k_1^{-1} G(Y)\|^2 - \frac{1}{2} k_1^2 \langle X, X \rangle - \frac{1}{2} k_1^{-2} \langle G(Y), G(Y) \rangle$$
$$\geq -\frac{1}{2} k_1^2 \langle X, X \rangle - \frac{1}{2} k_1^{-2} \overline{a}_2^2 \langle Y, Y \rangle$$
$$= -\frac{1}{2} k_1^2 \|X\|^2 - \frac{1}{2} k_1^{-2} \overline{a}_2^2 \|Y\|^2 \tag{12}$$

and

$$\langle X, F(Y)Z \rangle = \frac{1}{2}\|k_2 X + k_2^{-1} F(Y)Z\|^2 - \frac{1}{2}k_2^2 \langle X, X \rangle - \frac{1}{2}k_2^{-2} \langle F(Y)Z, F(Y)Z \rangle$$

$$\geq -\frac{1}{2}k_2^2 \langle X, X \rangle - \frac{1}{2}k_2^{-2} \langle F(Y)Z, F(Y)Z \rangle$$

$$\geq -\frac{1}{2}k_2^2 \|X\|^2 - \frac{1}{4}k_2^{-2} a_1^2 \|Z\|^2. \tag{13}$$

By using the estimates (11)-(13), we deduce that

$$\dot{V}_0 \geq \left[a_3 - \frac{1}{2}k_1^2 - \frac{1}{2}k_2^2 \right] \|X\|^2 + \left[\alpha a_2 - \frac{1}{2k_1^2} \bar{a}_2^2 \right] \|Y\|^2$$

$$+ \left[\alpha - \frac{1}{4k_2^2} a_1^2 \right] \|Z\|^2.$$

Let

$$k_1^2 = \min\left\{ \frac{a_3}{2}, \frac{\bar{a}_2^2}{a_2 \alpha} \right\}, \quad k_2^2 = \min\left\{ \frac{a_3}{2}, a_1^2 \alpha^{-1} \right\}.$$

Then

$$\dot{V}_0 \geq \left(\frac{a_3}{2} \right) \|X\|^2 + \left(\frac{\alpha a_2}{2} \right) \|Y\|^2 + \left(\frac{3\alpha}{4} \right) \|Z\|^2$$

$$\geq k \left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 \right) > 0,$$

where

$$k = \min\left\{ \frac{a_3}{2}, \frac{\alpha a_2}{2}, \frac{3\alpha}{4} \right\}.$$

Thus, the assumption (i) of Theorem 1 shows that $\dot{V}_0(t) \geq 0$ for all $t \geq 0$, that is, $\dot{V}_0$ is positive semi-definite. Furthermore, $\dot{V}_0 = 0$ $(t \geq 0)$ necessarily implies that $Y = 0$ for all $t \geq 0$; and therefore also that $X = \xi$ (a constant vector), $Z = \dot{Y} = 0$ for all $t \geq 0$. Substituting the estimates
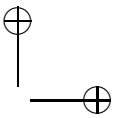
$$X = \xi, \quad Y = Z = 0$$

in (2) it follows that $H(\xi) = 0$ which necessarily implies (only) that $\xi = 0$ because of $H(0) = 0$. So

$$X = Y = Z = 0 \text{ for all } t \geq 0.$$

Therefore, the function V_0 has the entire requisite Krasovskii criterion [9] if
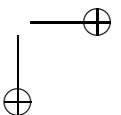
the assumption (i) of Theorem 1 holds. This proves the proof of the first part of Theorem 1. Similarly, for the proof of second part of Theorem 1, we consider the Lyapunov function $V_1 = V_1(X, Y, Z)$ defined by:

$$2V_1 = -2\overline{\alpha} \int_0^1 \langle H(\sigma X), X \rangle d\sigma + 2\overline{\alpha} \langle Y, Z \rangle + 2\overline{\alpha} \int_0^1 \sigma \langle F(\sigma Y)Y, Y \rangle d\sigma \\ - \langle Y, Y \rangle + 2\langle X, Z \rangle, \tag{14}$$

where $\overline{\alpha}$ is a positive constant.

Evidently, the assumption (i)$'$ of Theorem 1 shows that

$$V_1(0, 0, 0) = 0 \text{ and } V_1(\overline{\varepsilon}, 0, 0) \geq \frac{a_3}{2} \|\overline{\varepsilon}\|^2 > 0$$

for all arbitrary $\overline{\varepsilon} \neq 0$, $\overline{\varepsilon} \in \mathfrak{R}^n$. Thus, in every neighborhood of $(0, 0, 0)$ there exists a point $(\overline{\xi}, \overline{\eta}, \overline{\zeta})$ such that $V_1(\overline{\xi}, \overline{\eta}, \overline{\zeta}) > 0$. Next, let $(X, Y, Z) = (X(t), Y(t), Z(t))$ be an arbitrary solution of the system (2). By using (14) and (2), an elementary differentiation shows that

$$\dot{V}_1 = -\langle X, H(X) \rangle + \overline{\alpha} \langle Z, Z \rangle - \overline{\alpha} \langle Y, G(Y) \rangle \\ - 2\overline{\alpha} \langle H(X), Y \rangle - \langle X, F(Y)Z \rangle - \langle X, G(Y) \rangle \\ \geq a_3 \|X\|^2 + \overline{\alpha} a_2 \|Y\|^2 + \overline{\alpha} \|Z\|^2 \\ - 2\overline{\alpha} \langle H(X), Y \rangle - \langle X, F(Y)Z \rangle - \langle X, G(Y) \rangle.$$

Proceeding exactly along the lines indicated in the proof of first part of Theorem 1 just above we deduce that

$$\dot{V}_1 \geq \overline{k} \Big(\|X\|^2 + \|Y\|^2 + \|Z\|^2 \Big) > 0,$$

where $\overline{k}$ is a certain positive constant. Similarly, it can be shown that the function V_1 has the entire requisite Krasovskii criterion [9] if the assumption (i)$'$ of Theorem 1 holds. This proves the proof of the second part of Theorem 1. Thus, the basic properties of $V_0(X, Y, Z)$ and $V_1(X, Y, Z)$, which we have proved just above, justify that the zero solution of the system (2) is unstable.(See Theorem 1.15 in [15], see also [9], [36]). The system (2) is equivalent to the differential equation (1). It follows thus the original statement of the theorem. $\square$

The second main result is the following theorem.
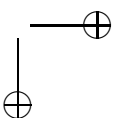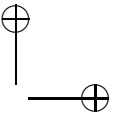
Theorem 2. *Further to the basic assumptions imposed on F, G and H, it is being tacitly assumed that $J_G(Y)$ and $J_H(X)$ are symmetric matrices and there are constants a_1, a_2 and a_3 such that the following that the following conditions are satisfied:*

(i) $\lambda_i(F(Y)) \leq -a_1 < 0$, $0 \leq \lambda_i(J_G(Y)) \leq a_2$ and $\lambda_i(J_H(X)) \geq a_3 > 0$ *for all $X, Y \in \mathfrak{R}^n$*

or

(i)$'$ $\lambda_i(F(Y)) \geq a_1 > 0$, $0 \leq \lambda_i(J_G(Y)) \leq a_2$ and $\lambda_i(J_H(X)) \leq -a_3 < 0$ *for all $X, Y \in \mathfrak{R}^n$.*

Then the zero solution $X = 0$ of the system (2) is unstable.

*Proof of Theorem* 2. Consider the function $V_2 = V_2(X, Y, Z)$ defined by:

$$
\begin{aligned}
2V_2 &= \beta\langle Z, Z\rangle + 2\beta\langle Y, H(X)\rangle + 2\beta \int_0^1 \langle G(\sigma Y), Y\rangle d\sigma \\
&\quad + \langle Y, Y\rangle - 2\langle X, Z\rangle,
\end{aligned} \tag{15}
$$

where β is a positive constant. It is clear that $V_2(0, 0, 0) = 0$. It is also clear from the assumption (i) of Theorem 2 and the above lemma that

$$
V_2(0, 0, \varepsilon) \geq \beta\langle \varepsilon, \varepsilon\rangle = \beta\|\varepsilon\|^2 > 0
$$

for all arbitrary $\varepsilon \neq 0$, $\varepsilon \in \mathfrak{R}^n$, so that in every neighborhood of $(0, 0, 0)$ there exists a point (ξ, η, ζ) such that $V_2(\xi, \eta, \zeta) > 0$. Next, let $(X, Y, Z) = (X(t), Y(t), Z(t))$ be an arbitrary solution of the system (2). An easy calculation from (15) and (2) yields that

$$
\begin{aligned}
\dot{V}_2 = \frac{d}{dt}V_2(X, Y, Z) &= -\beta\langle Z, F(Y)Z\rangle + \beta\langle Y, J_H(X)Y\rangle + \langle X, H(X)\rangle \\
&\quad + \langle X, F(Y)Z\rangle + \langle X, G(Y)\rangle - \beta\langle G(Y), Z\rangle \\
&\quad + \beta\frac{d}{dt}\int_0^1 \langle G(\sigma Y), Y\rangle d\sigma.
\end{aligned}
$$
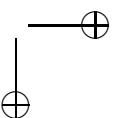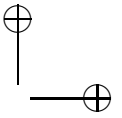
Recall that

$$
\begin{aligned}
\frac{d}{dt} \int_0^1 \langle G(\sigma Y), Y \rangle d\sigma &= \int_0^1 \sigma \langle J_G(\sigma Y) Z, Y \rangle d\sigma + \int_0^1 \langle G(\sigma Y), Z \rangle d\sigma \\
&= \int_0^1 \sigma \frac{\partial}{\partial \sigma} \langle G(\sigma Y), Z \rangle d\sigma + \int_0^1 \langle G(\sigma Y), Z \rangle d\sigma \\
&= \sigma \langle G(\sigma Y), Z \rangle \, |_0^1 = \langle G(Y), Z \rangle. \tag{16}
\end{aligned}
$$

Therefore, by using (16) and the assumption (i) of Theorem 2, we deduce that

$$
\begin{aligned}
\dot{V}_2 &= -\beta \langle Z, F(Y) Z \rangle + \beta \langle Y, J_H(X) Y \rangle + \langle X, H(X) \rangle \\
&\quad + \langle X, F(Y) Z \rangle + \langle X, G(Y) \rangle \\
&\geq \beta a_1 \|Z\|^2 + \beta a_3 \|Y\|^2 + a_3 \|X\|^2 + \langle X, F(Y) Z \rangle + \langle X, G(Y) \rangle. \tag{17}
\end{aligned}
$$

Similarly, as is shown just above for some constants $\overline{k}_1$ and $\overline{k}_2$ conveniently chosen later, we can easily obtain from (17) that

$$
\dot{V}_2 \geq \left(a_3 - \frac{1}{2}\overline{k}_1{}^2 - \frac{1}{2}\overline{k}_2{}^2 \right) \|X\|^2 + \left(\beta a_3 - \frac{1}{2}\overline{k}_1{}^{-2} a_2^2 \right) \|Y\|^2 + \left(\beta a_1 - \frac{1}{4}\overline{k}_2{}^{-2} a_1^2 \right) \|Z\|^2.
$$

Let

$$
k_1{}^2 = \min \left\{ \frac{a_3}{2}, \frac{a_2^2}{\beta a_3} \right\}, \quad \overline{k}_2{}^2 = \min \left\{ \frac{a_3}{2}, \frac{a_1}{\beta} \right\}.
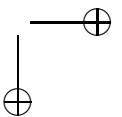$$

Hence

$$
\begin{aligned}
\dot{V}_2 &\geq \left(\frac{a_3}{2} \right) \|X\|^2 + \left(\frac{\beta a_3}{2} \right) \|Y\|^2 + \left(\frac{3\beta a_1}{4} \right) \|Z\|^2 \\
&\geq \overline{k} \left(\|X\|^2 + \|Y\|^2 + \|Z\|^2 \right) > 0,
\end{aligned}
$$

where

$$
\overline{k} = \min \left\{ \frac{a_3}{2}, \frac{\beta a_3}{2}, \frac{3\beta a_1}{4} \right\}.
$$

The rest of the proof of first part of Theorem 2 is the same as the proof of part (i) of Theorem 1 just proved above and hence it is omitted.

Finally, for the proof of second part of Theorem 2 we consider the Lya-

punov function $V_3 = V_3(X, Y, Z)$ defined as

$$V_3 = V_2(X, Y, Z) - 2\beta \int_0^1 \langle H(\sigma X), X \rangle d\sigma,$$

where $V_2(X, Y, Z)$ is defined as the same the function in (15). The remaining of the proof can be verified proceeding exactly along the lines indicated just in the proof of Theorem 1. Hence we omit the detailed proof.   □

Remark 2. If we take $f_4(x, \dot{x}, \ddot{x}) = 0$ and $f_1(\dot{x})\ddot{x}$ instead of $f_1(\ddot{x})$ in [18], then our assumptions are less restrictive then those established by Skrapek [18], and our theorems also give n-dimensional extensions for the results established in ([8], [18]).
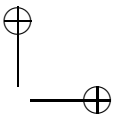
Acknowledgment

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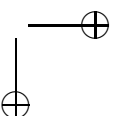
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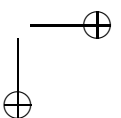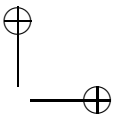
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