Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **2** (2007), No. 1, pp. 103-107

PENCILS ON COVERINGS OF A GIVEN CURVE WHOSE DEGREE IS LARGER THAN THE CASTELNUOVO-SEVERI LOWER BOUND

BY

E. BALLICO[1], C. KEEM[2] AND D. SHIN[3]

Abstract

Fix integers q, g, k, d. Set $\pi_{d,k,q} := kd - d - k + kq + 1$ and assume $q > 0$, $k \geq 2$, $d \geq 3q + 1$, $g \geq kq - k + 1$ and $\pi_{d,k,q} - ((\lfloor d/2 \rfloor + 1 - q) \cdot (\lfloor k/2 \rfloor + 1) \leq g \leq \pi_{d,k,q}$. Let Y be a smooth and connected genus q projective curve. Here we prove the existence of a smooth and connected genus g projective curve X, a degree k morphism $f : X \to Y$ and a degree d morphism $u : X \to \mathbf{P}^1$ such that the morphism $(f, u) : X \to Y \times \mathbf{P}^1$ is birational onto its image.

1. Introduction

Let X (resp. Y) be a smooth and connected curve of genus g (resp. genus q) and $f : X \to Y$ a degree k covering. Let $u : X \to \mathbf{P}^1$ be a degree d morphism. Assume $d \leq (g - kq)/(k - 1)$. By Castelnuovo-Severi inequality ([4]) the induced morphism $(f, u) : X \to Y \times \mathbf{P}^1$ is not birational onto its image. Very roughly speaking, "u factors through f". Several papers were devoted to the proof of the existence of X, Y, f, u for certain $d > (g - kq)/(k - 1)$ for which the morphism $(f, u) : X \to Y \times \mathbf{P}^1$ is birational onto its image (see [1] and references therein). In this paper we prove the following result.

Received February 20, 2006 and in revised form March 31, 2006.
AMS Subject Classification: 14H51, 14H30.
Key words and phrases: Covering of curves, Brill-Noether theory, ruled surface.
[1]The author was partially supported by MIUR and GNSAGA of INdAM (Italy).
[2,3]The author is supported by Korea Research Foundation 2005-070-C00005.

Theorem 1. *Fix integers q, g, k, d. Set $\pi_{d,k,q} := kd - d - k + kq + 1$ and assume $q > 0$, $k \geq 2$, $d \geq 3q+1$, $g \geq kq - k + 1$ and $\pi_{d,k,q} - ((\lfloor d/2 \rfloor + 1 - q) \cdot (\lfloor k/2 \rfloor + 1) \leq g \leq \pi_{d,k,q}$. Let Y be a smooth and connected genus q projective curve. Then there exist a smooth and connected genus g projective curve X, a degree k morphism $f : X \to Y$ and a degree d morphism $u : X \to \mathbf{P}^1$ such that the morphism $(f, u) : X \to Y \times \mathbf{P}^1$ is birational onto its image.*

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$.

2. Proof of Theorem 1.

Remark 1. Let $f : X \to Y$ be a finite morphism between smooth and connected projective curves and $D = \sum n_i P_i$ any divisor on X. Set $f_!(D) := \sum n_i f(P_i)$. A key property of rational equivalence say that if D and D' are linearly equivalent divisors on X, then $f_!(D)$ and $f_!(D')$ are linearly equivalent divisors on Y; here the smoothness of Y is essential, because it implies that rational equivalence and linear equivalence are the same on Y. Hence for any $d \in \mathbb{Z}$ the map $f_!$ induces a map $f_! : \mathrm{Pic}^d(X) \to \mathrm{Pic}^d(Y)$ such that $h^0(Y, f_!(L)) \geq h^0(X, L)$ for all $L \in \mathrm{Pic}^d(X)$. Furthermore, if L is base point free, then $f_!(L)$ is base point free.

In the next remark we introduce our set-up. We will use several times the notations and results proved in this remark.

Remark 2. Let Y be a smooth and connected projective curve. Set $q := p_a(Y)$ and $S := Y \times \mathbf{P}^1$. Hence $h^1(S, \mathcal{O}_S) = q$. Let $\pi_1 : S \to Y$ and $\pi_2 : S \to \mathbf{P}^1$ denote the two projections. For any $R \in \mathrm{Pic}(S)$ there are unique $M \in \mathrm{Pic}(Y)$ and $k \in \mathbb{Z}$ such that $R \cong \pi_1^*(M) \otimes \pi_2^*(\mathcal{O}_{\mathbf{P}^1}(k))$. Set $\mathcal{O}_S(M, k) := \pi_1^*(M) \otimes \pi_2^*(\mathcal{O}_{\mathbf{P}^1}(k))$. If $k < 0$, then $h^0(S, \mathcal{O}_S(M, k)) = 0$ and $h^1(S, \mathcal{O}_S(M, k)) = (-k-1) \cdot h^0(Y, M)$ (Künneth formula). If $k \geq 0$, then $h^0(S, \mathcal{O}_S(M, k)) = (k+1) \cdot h^0(Y, M)$ and $h^1(S, \mathcal{O}_S(M, k)) = (k+1) \cdot h^1(Y, M)$ (Künneth formula). Furthermore, if M is spanned and $k \geq 0$, then $\mathcal{O}_S(M, k)$ is spanned, while if M is (birationally) very ample and $k > 0$, then $\mathcal{O}_S(M, k)$ is (birationally) very ample. Fix integers $k \geq 2$ and $d > 0$ and $M \in \mathrm{Pic}^d(Y)$ such that $|M|$ has no base point. Let $C \subset S$ be an integral curve in the linear system $|\mathcal{O}_S(M, k)|$ and $\nu : X \to C$ the normalization map. For any $A \in \mathrm{Pic}(Y)$ and any integer x set $\mathcal{O}_C(A, x) := \mathcal{O}_S(A, x)|C$. Notice that $\mathcal{O}_C(A, x)$ is a line bundle of degree $k \cdot \deg(A) + x \cdot \deg(M)$. The morphism

$\pi_1 \circ \nu : X \to Y$ is a degree k covering between smooth and projective curves. Since $\omega_S \cong \mathcal{O}_S(\omega_Y, -2)$, then $\omega_C \cong \mathcal{O}_C(M \otimes \omega_S, k-2)$ (adjunction formula). Thus $p_a(C) = kd - d - k + kq + 1$. Set $\pi_{d,k,q} := kd - d - k - kq + 1$.

Proposition 1. *Fix integers $q \geq 0$, $k \geq 2$ and $d > 0$ and a smooth curve Y of genus q. Assume the existence of a base point free $M \in Pic^d(Y)$. Set $g := kd - d - k + kq + 1$. Then there exist a smooth genus g curve, a degree k covering $f : X \to Y$ and $L \in Pic^d(X)$ such that L is base point free and $f_!(L) = M$.*

Proof. Notice that $g = \pi_{d,k,q}$ and take as X the curve C described in the second part of Remark 2. Here we may take as C a smooth curve because $\mathcal{O}_S(M, k)$ is base point free and hence we may apply Bertini's theorem. $\square$

Proposition 2. *Take Y, S as in Remark 2 and integers u, v, a such that $u > 0$, $v \geq 2$, $0 < a \leq (v+1)(u+1-q) - 3$ and there is a very ample $R \in Pic^u(Y)$. Let $A \subset S$ be a general subset such that $\sharp(A) = a$. Then the linear system $|\mathcal{I}_{A,S}(R, v)|$ has no base point outside A and its scheme-theoretic base locus is exactly A.*

Proof. Since $h^0(Y, R) \geq u+1-q$, we have $h^0(S, \mathcal{I}_{A,S}(R, v)) \geq 3$. Since R is very ample, $\mathcal{O}_S(R, v)$ is very ample. Fix two general $D, D' \in |\mathcal{O}_S(R, v)|$. Hence $D \cap D'$ is the union of $2uv$ points, each of them appearing with multiplicity one. By semicontinuity it is sufficient to show the result when we take as A a subset of $D \cap D'$ with $\sharp(A) = a$. By the very ampleness of $\mathcal{O}_S(R, v)$ and hence of $\mathcal{O}_D(R, v)$ we may apply the monodromy theorem ([3], Ch. III, or [5]), and get that the result is true for one such A if and only if it is true for all $A' \subset D \cap D'$ with $\sharp(A') = a$. Furthermore, the linear span of $\{D, D'\}$ in $|\mathcal{O}_S(R, v)|$ is uniquely determined by any $E \subset D \cap D'$ such that $\sharp(E) = h^0(S, \mathcal{O}_S(R, v)) - 2$. Fix any such E, any $P \in E$ and set $F := E \backslash \{P\}$. Hence $h^0(S, \mathcal{I}_{F,S}(R, v)) = h^0(S, \mathcal{O}_S(R, v)) - 3$. First, we will show that F is scheme-theoretically the base locus $B(F)$ of $|\mathcal{I}_{F,S}(R, v)|$. Indeed, $B(F) \subseteq D \cap D'$ and hence each point of F appears with multiplicity one in $B(F)$, while other base points (if any) are contained in $D \cap D' \backslash F$. Assume there is $Q \in B(F) \backslash F$. By the monodromy theorem all $Q' \in D \cap D'$ are contained in $B(F)$. Hence $h^0(S, \mathcal{I}_{F,S}(R, v)) = h^0(S, \mathcal{I}_{E,S}(R, v)) = h^0(S, \mathcal{O}_S(R, v)) - 2$, contradiction. Notice that $\sharp(F) \geq a$. If $\sharp(F) = a$, we are done. If $\sharp(F) > a$

we apply the same trick $\sharp(F) - a$ times and get the result for $\sharp(F)-1, \dots, a$ points. □

Proof of Theorem 1. The case $k = 2$ is well-known ([1]) and the case $k = 3$ was done (in a different, but more explicit way) in [6] under different numerical assumptions. Hence to simplify the numerical computations we will assume $k \geq 4$. Set $y := \pi_{d,k,q} - g$. By assumption we have $0 \leq y \leq (\lfloor d/2 \rfloor + 1 - q) \cdot (\lfloor k/2 \rfloor + 1)$. Proposition 1 gives the result when $g = \pi_{d,k,q}$. Hence we may assume $y > 0$. Let $A \subset S$ be a general subset with $\sharp(A) = y$, say $A = \{P_1, \dots, P_y\}$. For any $P \in S$ let $2P$ denote the first infinitesimal neighborhood of P in S, i.e. the closed zero-dimensional subscheme of S with ${\mathcal{I}_{P,S}}^2$ as its ideal sheaf. Hence length$(2P) = 3$. Fix a base point free $M \in \mathrm{Pic}^d(Y)$. □

Claim. There is a reduced curve $C \in |\mathcal{I}_{2P_1 \cup \cdots \cup 2P_y}(M,k))|$ such that $\mathrm{Sing}(C) = A$ and each $P_i \in A$ is an ordinary node of C.

Proof of the Claim. By Bertini's theorem it is sufficient to show that the linear system $|\mathcal{I}_{2P_1 \cup \cdots \cup 2P_y}(M,k)|$ has no base point outside $\{P_1, \dots, P_y\}$ and that a general $C \in |\mathcal{I}_{2P_1 \cup \cdots \cup 2P_y}(M,k))|$ has an ordinary node at each point of $\{P_1, \dots, P_y\}$. Take $L, L' \in \mathrm{Pic}^{\lfloor d/2 \rfloor}(Y)$ such that either $L^{\otimes 2} \cong M$ (case d even) or $L^{\otimes 2}(Q) \cong M$ for some $Q \in Y$ (case d odd) and apply Proposition 2 taking $R := L$, $v := \lfloor k/2 \rfloor$, $a := y$ and $A = \{P_1, \dots, P_a\}$. □

The Claim shows that the image of S by the linear system $|L|$ is not weakly defective in the sense of [2] and hence it is not defective, so that the linear system of curves in $|L|$ has the expected dimension $\dim(|L|) - 3y$, i.e. $h^0(S, \mathcal{I}_{2P_1 \cup \cdots \cup 2P_y}(M,k)) = h^0(S, \mathcal{O}_S(M,k)) - 3y = (k+1)(d+1-q) - 3y$ and $h^1(S, \mathcal{I}_{2P_1 \cup \cdots \cup 2P_y}(M,k)) = 0$. By [2], Th. 1.4, the general such curve, C, has y ordinary nodes as its only singularities. Now we will check that a general $C \in |\mathcal{I}_{2P_1 \cup \cdots \cup 2P_y}(M,k))|$ is integral. Since C is nodal, it is reduced. Assume that C has $t \geq 2$ irreducible components, say $C_1, \dots, C_t$. Since $\sharp(\mathrm{Sing}(C)) = y$ and each sigular point of C is an ordinary node, this give a very strong restriction on the sum of the intersection numbers $\sum_{1 \leq i < j \leq t} C_i \cdot C_j$. Since S^y is irreducible and $A \subset S^y$ is general in S^y, moving A we also see that either each point of A is contained in exactly two irreducible components of C or each $P \in A$ is in contained a unique irreducible component of C. In the first case we get $t = 2$. Furthermore, moving again A we also see that

C_1 and C_2 are algebraically equivalent. Since $C_1 \cdot C_2 = y$, the contradiction comes from the inequality $2\lfloor d/2\rfloor\lfloor k/2\rfloor > y$. In the second case we get $t \geq y$ and $C_i \cap C_j = \emptyset$ for all $i \geq j$. Hence all C_i are fibers of one of the projections $\pi_1 : S \to Y$ or $\pi_2 : S \to \mathbf{P}^1$, contradicting the assumptions $d > 0$ and $k > 0$. Take C as in the Claim and integral. and let $\nu : X \to C$ be the normalization. The curve X satisfies the thesis of Theorem 1 for the curve Y, the genus g and the integers d, k.

Our main numerical restriction in the statement of Theorem 1 came from the use of Proposition 2.

References

1. E. Ballico, C. Keem and S. Park, Double covering of curves, *Proc. Amer. Math. Soc.*, **132** (2004), no. 11, 3153-3158.

2. L. Chiantini, C. Ciliberto, Weakly defective varieties, *Trans. Amer. Math. Soc.*, **454** (2002), no. 1, 151-178.

3. J. Harris, with the collaboration of D. Eisenbud, *Curves in Projective Space*, Les Presses de l'Université de Montréal, Montréal, 1982.

4. E. Kani, On Castelnuovo's equivalent defect, *J. Reine Angew. Math.*, 352 (1984), 24-70.

5. J. Rathmann, The uniform position principle for curves in characteristic p, *Math. Ann.*, **276** (1987), no. 4, 565-579.

6. D. Shin, Existence of triple coverings satisfying the converse of Castelnuovo-Severi inequality, preprint.

Department of Mathematics, University of Trento, 38050 Povo (TN), Italy.

E-mail: ballico@science.unitn.it

Department of Mathematics, Seoul National University, Seoul 151-742, South Korea.

E-mail: ckeem@math.snu.ac.kr

Department of Mathematics, Seoul National University, Seoul 151-742, South Korea.

E-mail: qeds@korea.com