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THE SECTIONS OF UNIVALENT FUNCTIONS

BY

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Abstract

Let f∈S and $f_k(z) = f^{\frac{1}{k}}(z^k)$. The radius of convexity of the partial sums of Taylor series expansion of $f_k(z)$ is investigated. We obtain the sharp radius.

Let S be the class of functions $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ regular and univalent in $|z| < 1$. Let f∈S, the part sums of Taylor series expansion $s_n(z) = z + \sum_{\nu=2}^{n} a_\nu z^\nu$. Let $f_k(z) = f^{\frac{1}{k}}(z^k) = \sum_{\nu=0}^{\infty} b_\nu^{(k)} z^{k\nu+1}$, $k = 2, 3, \ldots$, $b_0^{(k)} = 1$ and $s_{n,k}(z) = \sum_{\nu=0}^{n} b_\nu^{(k)} z^{k\nu+1}$. The property of the sections $s_n(z)$ and $s_{n,k}(z)$ is an interesting question. Szegö (see [1]) discovered that $s_n(z)$ is univalent in $|z| < \frac{1}{4}$ and conjectured that $s_{n,k}(z)$ is univalent in $|z| < \sqrt[k]{\frac{k}{2(k+1)}}$. This question remains open. Huke and Pan ([2]) proved that $s_n(z)$ is starlike in $|z| < \frac{1}{4}$. In this paper, we prove that $s_{n,k}(z)$ are convex in $|z| < \sqrt[k]{\frac{k}{2(k+1)^2}}$ and the radii of convexity are best. Our main results are

Theorem 1. *Let* $f(z) = z+\sum_{n=2}^{\infty} a_n z^n \in S$. *Then* $s_n(z) = z+\sum_{k=2}^{n} a_k z^k$ $(n = 2, 3, \ldots)$ *are convex in* $|z| < \frac{1}{8}$. *The radius* $\frac{1}{8}$ *is sharp.*

Theorem 2. *Let* $f \in S$, $f_k(z) = f^{\frac{1}{k}}(z^k) = \sum_{\nu=0}^{\infty} b_\nu^{(k)} z^{k\nu+1}$, $k = 2, 3, \ldots$, $b_0^{(k)} = 1$. *Then* $s_{n,k}(z) = \sum_{\nu=0}^{n} b_\nu^{(k)} z^{k\nu+1}$ *are convex in* $|z| < \sqrt[k]{\frac{k}{2(k+1)^2}}$. *The radii of convexity are sharp.*

To prove the theorems, we need following lemmas.

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Lemma 1.(see [3]) *Let $f(z) \in S$. Then for $|z| \le r < 1$*

$$\Big|\frac{zf''(z)}{f'(z)}\Big| \le \frac{2r^2+4r}{1-r^2}. \tag{1}$$

Lemma 2.(see [3]) *Let $f(z) \in S$. Then for $|z| \le r < 1$*

$$1 + Re\Big\{\frac{zf''(z)}{f'(z)}\Big\} \ge \frac{1-4r+r^2}{1-r^2}. \tag{2}$$

Lemma 3. *Let $f(z) \in S$, $R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k$. Then for $|z| \le r \le \frac{1}{8}$, $n \ge 2$*

$$|R_n'(z)| \le \frac{(n+1)^2 r^n}{(1-r)^2} = G_n(r), \tag{3}$$

$$|R_n''(z)| \le G_n'(r) = \frac{n(n+1)^2 r^{n-1}}{(1-r)^2} + \frac{2(n+1)^2 r^n}{(1-r)^3}. \tag{4}$$

Proof. By de Branges inequalities $|a_n| \le n$ $(n = 1, 2, \ldots)$, it is clear that

$$|R_n(z)| \le g(r) = \sum_{k=n+1}^{\infty} kr^k = \frac{r^{n+1}(n+1-nr)}{(1-r)^2} \tag{5}$$

and

$$|R_n'(z)| \le g'(r) = \frac{(n+1)^2 r^n}{(1-r)^2} + \frac{r^{n+1}[2-n^2(1-r)]}{(1-r)^3} = G_n(r) + t_n(r).$$

It is clear for $r \le \frac{1}{8}$ and $n \ge 2$ that $t_n(r) \le 0$ and

$$t_n'(r) = \frac{r^n[(2n+1)-(n+1)n^2(1-r)+n^2 r]}{(1-r)^2} + \frac{3r^{n+1}[2-n^2(1-r)]}{(1-r)^4} < 0.$$

Hence we obtain that $|R_n''(z)| \le g''(r) \le G_n'(r)$. It is easy to see that $\{G_n(r)\}$ and $\{G_n'(r)\}$ are a monotone decreasing sequences. It follows for $n \ge 3$ that $G_n(r) \le G_3(r)$ and $G_n'(r) \le G_3'(r)$. □

Lemma 4. *Let $f(z) \in S$, $f_k(z) = f^{\frac{1}{k}}(z^k) = \sum_{\nu=0}^{\infty} b_\nu^{(k)} z^{k\nu+1}$, $k = 2, 3, \ldots$, $b_0^{(k)} = 1$. Then for $n \ge 4$ one has $k|b_n^{(k)}| < 4$.*

Proof. Milin proved ([5]) for $k=2$ that $|b_\nu^{(2)}|<1.17$, $\nu=1,2,\ldots$. We see that $2|b_\nu^{(2)}|<4$. Now we assume that $k\ge 3$. Define the logarithmic coefficients, as usual, by the expansion

$$\log\frac{f(z)}{z}=2\sum_{\nu=1}^{\infty}\gamma_\nu z^\nu.$$

We have the equalities

$$zf_k'(z)=\frac{zf_k'(z)}{f_k(z)}f_k(z)=\frac{z^kf'(z^k)}{f(z^k)}f_k(z). \tag{6}$$

Comparing the coefficients of the same power of z in (6), we obtain that

$$(kn+1)b_n^{(k)}=b_n^{(k)}+2\sum_{\nu=1}^{n}\nu\gamma_\nu b_{n-\nu}^{(k)}. \tag{7}$$

Applying Cauchy inequality, we obtain from (7) that

$$k|b_n^{(k)}|\le 2n^{-\frac{1}{2}}\Big(\sum_{\nu=1}^{n}\nu|\gamma_\nu|^2\Big)^{\frac{1}{2}}\Big(\sum_{\nu=0}^{n-1}|b_\nu^{(k)}|^2\Big)^{\frac{1}{2}}. \tag{8}$$

Milin proved ([5]) that for $n=1,2,\ldots$

$$\sum_{\nu=1}^{n}\nu|\gamma_\nu|^2\le\sum_{\nu=1}^{n}\frac{1}{\nu}+\delta \tag{9}$$

where $\delta=0.312$. And proved that

$$\sum_{\nu=0}^{n-1}|b_\nu^{(k)}|^2\le e^{\frac{2\delta}{k}}\sum_{\nu=0}^{n-1}d_\nu^2(\frac{2}{k})\le e^{\frac{2\delta}{3}}\sum_{\nu=0}^{n-1}d_\nu^2(\frac{2}{3}) \tag{10}$$

where $d_\nu(x)$ are Taylor coefficients of the function $(1-z)^{-x}$. It is known ([6]) that $d_\nu(\frac{2}{3})\le\frac{2}{3}e^{\frac{2}{3}C}\nu^{-\frac{1}{3}}<\nu^{-\frac{1}{3}}$ (C is Euler constant). Hence it follows that

$$\sum_{\nu=0}^{n-1}d_\nu^2(\frac{2}{3})\le 1+\sum_{\nu=1}^{n-1}\nu^{-\frac{2}{3}}\le 1+\int_0^{n-1}\nu^{-\frac{2}{3}}d\nu=1+3(n-1)^{\frac{1}{3}}. \tag{11}$$

We obtain from (9), (10), (11) and (8) that

$$k|b_n^{(k)}| \le 2n^{-\frac{1}{2}}\Big(\sum_{\nu=1}^{n}\frac{1}{\nu}+\delta\Big)^{\frac{1}{2}}(1+3(n-1)^{\frac{1}{3}})^{\frac{1}{2}}. \tag{12}$$

For $n \ge 4$, the right-hand of (12) is decreasing. This gives that

$$k|b_n^{(k)}| \le \Big(\sum_{\nu=1}^{4}\frac{1}{\nu}+\delta\Big)^{\frac{1}{2}}(1+3^{\frac{4}{3}})^{\frac{1}{2}} < 4. \qquad \square$$

Lemma 5. *Let $f(z) \in S$, $f_k(z) = f^{\frac{1}{k}}(z^k) = \sum_{\nu=0}^{\infty} b_\nu^{(k)} z^{k\nu+1}$, $k = 2,3,\ldots$, $b_0^{(k)} = 1$ and $R_{n,k}(z) = \sum_{\nu=n+1}^{\infty} b_\nu^{(k)} z^{k\nu+1}$. Let $|z_k| = \sqrt[k]{\frac{k}{2(k+1)^2}}$. Then for n$\ge 3$ one has $|R'_{n,k}(z_k)| < 0.004$ and $|z_k R''_{n,k}(z_k)| < 0.07$.*

Proof. Write $t = t_k = |z_k|^k$. We see for $k = 2,3,\ldots$ that $t_k \le t_2 = \frac{1}{9}$. It is clear that

$$\sum_{\nu=4}^{\infty}(\nu+1)t^\nu = \frac{d}{dt}\frac{t^5}{1-t} = \frac{5t^4-4t^5}{(1-t)^2}.$$

By Lemma 4, we obtain for $n = 3$ that

$$|R'_{n,k}(z_k)| \le \sum_{\nu=4}^{\infty}(k\nu+1)|b_\nu^{(k)}|t^\nu \le 4\sum_{\nu=4}^{\infty}(\nu+1)t^\nu < \frac{20t_2^4}{(1-t_2)^2} < 0.004. \tag{13}$$

It is clear for $k = 2,3,\ldots$ that $kt_k < \frac{1}{2}$. By Lemma 4, we obtain for $n = 3$ that

$$\begin{aligned} |z_k R''_{n,k}(z_k)| &\le \sum_{\nu=4}^{\infty} k\nu(k\nu+1)|b_\nu^{(k)}|t^\nu \le 4kt\frac{d}{dt}\sum_{\nu=4}^{\infty}(\nu+1)t^\nu \\ &\le 2[\frac{20t^3}{1-t}+\frac{10t^4-8t^5}{(1-t)^3}] \le \frac{40t_2^3}{1-t_2}+\frac{20t_2^4}{(1-t_2)^3} < 0.07. \end{aligned} \tag{14}$$

From the proof of (13) and (14), we see easy that the conclusion is true for $n > 3$. $\square$

Proof of Theorem 1. It is enough to prove the inequalities

$$I = 1 + Re\{\frac{zs''_n(z)}{s'_n(z)}\} \ge 0 \tag{15}$$

in $|z|=\frac{1}{8}$. The inequalities (15) are identical with the inequalities

$$J = Re\{zs_n''(z)\overline{s_n'(z)}\} + |s_n'(z)|^2 \geq 0. \tag{16}$$

We consider two cases respectively. (A) Case $n=2$. In this case, we obtain that $s_2(z) = z + a_2z^2$. It follows that

$$J = Re\{(1+2\overline{a_2z})2a_2z\} + |1+2a_2z|^2 = 1 + 2|2a_2z|^2 + 3Re(2a_2z). \tag{17}$$

Write $2a_2z = x+iy$. Since $|a_2| \leq 2$, $|z|=\frac{1}{8}$, we obtain that $x^2+y^2 \leq \frac{1}{4}$. It follows from (17) that

$$J = 1 + 2(x^2+y^2) + 3x \geq 1 + 3x + 2x^2 = J_1(x).$$

It is clear that $J_1(x) \geq J_1(-\frac{1}{2}) = 0$ for $|x| \leq \frac{1}{2}$. This gives that $s_2(z)$ is convex in $|z| < \frac{1}{8}$. For Koebe function, we obtain that $s_2(z) = z + 2z^2$. The $J=0$ when $z = -\frac{1}{8}$. Hence we see that the radius $\frac{1}{8}$ is sharp.

(B) Case n≥ 3. Since $f(z) = s_n(z) + R_n(z)$, by (15), we have

$$\begin{aligned} I &= 1 + Re\{z\frac{f''-R_n''}{f'-R_n'}\} = 1 + Re\{\frac{zf''}{f'}\} + Re\Big\{z\frac{f''R_n' - R_n''f'}{f'(f'-R_n')}\Big\} \\ &\geq 1 + Re\{\frac{zf''}{f'}\} - \frac{|\frac{zf''}{f'}||R_n'| + |zR_n''|}{||f'|-|R_n'||}. \end{aligned} \tag{18}$$

We shall prove that $I \geq 0$ for $|z| = \frac{1}{8}$. By (1)-(4), by a simple calculation, we obtain following inequalities for $|z|=\frac{1}{8}$ that

$$|\frac{zf''}{f'}| \leq (\frac{1}{32} + \frac{1}{2})\frac{64}{63} = \frac{34}{63} < 0.6, \tag{19}$$

$$1 + Re\{\frac{zf''}{f'}\} \geq (\frac{1}{2} + \frac{1}{32})\frac{64}{63} \geq \frac{33}{63} \geq 0.5, \tag{20}$$

$$|R_n'| \leq G_3(\frac{1}{8}) = 16(\frac{1}{8})^3(\frac{8}{7})^2 = \frac{2}{49} \leq 0.05, \tag{21}$$

$$|R_n''| \leq G_3'(\frac{1}{8}) = (\frac{8}{7})^2 \times \frac{3}{4} + (\frac{1}{7})^3 \times 32 = \frac{48}{49} + \frac{32}{343} \leq 1.1. \tag{22}$$

By the distortion theorem, we get that for $|z| = r = \frac{1}{8}$

$$|f'(z)| \geq \frac{1-r}{(1+r)^3} = \frac{7}{8}(\frac{8}{9})^3 \geq 0.6. \tag{23}$$

Hence it follows from (21) and (23) that

$$|f'(z)| - |R'_n| > 0.6 - 0.05 = 0.55. \tag{24}$$

Combining (19)-(24), we obtain from (18) that

$$I \geq 0.5 - \frac{0.6 \times 0.05 + 0.125 \times 1.1}{0.55} > 0.5 - 0.31 > 0. \tag{25}$$

Combining (A) and (B), we have proved Theorem. □

Proof of Theorem 2. First we consider the case $n = 2$. We assume, without loss generality, that $a_2 = a \geq 0$. We see easy that $s_{2,k} = z + \frac{az^{k+1}}{k}$. Write $s = z^k$, $Re(s) = x$ and $t = t_k = \frac{k}{2(k+1)^2}$. It follows from (16) that

$$\begin{aligned} J &= Re\Big[a(k+1)z^k(1 + \frac{a(k+1)}{k}\overline{z}^k)\Big] + \Big|1 + \frac{a(k+1)^2}{k}z^k\Big|^2 \\ &= \frac{a^2(k+1)^2}{k}(\frac{1}{k}+1)x^2 + a(k+1)(\frac{2}{k}+1)x + 1 = Ax^2 + Bx + 1. \end{aligned}$$

It is clear that

$$-t \geq -\frac{B}{2A} = -\frac{k(k+2)}{2a(k+1)^2}$$

and

$$\begin{aligned} At^2 - Bt + 1 &= \frac{a^2k}{4(k+1)^2}(\frac{1}{k}+1) - \frac{a}{k+1}(\frac{2}{k}+1) + 1 \\ &= \frac{2-a}{4(k+1)^2}[2k^2 + (4-a)k + (2-a)] \geq 0. \end{aligned}$$

Hence we obtain for $|s| < t$ that $J > 0$ and $J = 0$ if and only if $a = 2$. It follows that t_k is best. Now we consider the case $n \geq 3$. By a simple calculation, we get

$$1 + \frac{zf'_k(z)}{f_k(z)} = (k-1)\frac{sf'(s)}{f(s)} + k(1 + \frac{sf''(s)}{f'(s)}). \tag{26}$$

We write

$$\frac{|R'_{n,k}(z)|}{||f'_k(z)|-|R'_{n,k}(z)||}=a_{nk}(z),\qquad \frac{|zR''_{n,k}(z)|}{||f'_k(z)|-|R'_{n,k}(z)||}=b_{nk}(z).$$

From (19), we obtain for $f=f_k$ that

$$\begin{aligned} I \;\geq\; & kRe[1+\frac{sf''(s)}{f'(s)}]+(k-1)Re\frac{sf'(s)}{f(s)} \\ & -a_{nk}(z)\Big[(k-1)|\frac{sf'(s)}{f(s)}|+k|1+\frac{sf''(s)}{f'(s)}|+1\Big]-b_{nk}(z). \end{aligned} \tag{27}$$

Write

$$\begin{aligned} I_1 &= Re\frac{sf'(s)}{f(s)}-a_{nk}|\frac{sf'(s)}{f(s)}|, \\ I_2 &= k\Big[Re(1+\frac{sf''(s)}{f'(s)})-a_{nk}|1+\frac{sf''(s)}{f'(s)}|\Big]-a_{nk}-b_{nk}. \end{aligned}$$

We estimate I_1 and I_2 respectively. It is well know that

$$\frac{1-|s|}{1+|s|}\leq|\frac{sf'(s)}{f(s)}|\leq\frac{1+|s|}{1-|s|}.$$

Since $t_k\leq t_2=\frac{1}{9}$ and $k=2,3,\ldots$, we obtain for $|s|=t_k$ that

$$|f'_k(z)|=|\frac{sf'(s)}{f(s)}||\frac{f_k(z)}{z}|\geq\frac{1-|s|}{1+|s|}(\frac{1}{1+|s|})^{\frac{2}{k}}\geq\frac{1-t_2}{(1+t_2)^2}>0.72. \tag{28}$$

By Lemma 5 and (28), we get for $|z|^k=t_k$ that

$$a_{nk}(z) \;\leq\; \frac{0.004}{0.72-0.004}<0.006, \tag{29}$$

$$b_{nk}(z) \;\leq\; \frac{0.11}{0.72-0.004}<0.16 \tag{30}$$

where $k\geq 2$, $n\geq 4$. Golusin proved ([3]) that

$$|\arg\frac{sf'(s)}{f(s)}|\leq\log\frac{1+|s|}{1-|s|}.$$

By this inequality, it follows for $|s| = t_k$ that

$$\begin{aligned} I_1 &\geq \frac{1-|s|}{1+|s|}\cos\log\frac{1+|s|}{1-|s|} - 0.006\frac{1+|s|}{1-|s|} \\ &\geq \frac{1-t_2}{1+t_2}\cos\log\frac{1+t_2}{1-t_2} - 0.006\frac{1+t_2}{1-t_2} \\ &\geq \frac{8}{10}\cos\log\frac{10}{8} - 0.006\times\frac{10}{8} > 0.79 - 0.007 > 0. \end{aligned} \tag{31}$$

By Lemma 1, Lemma 2, (29) and (30), it follows for $|s| = t_k$ that

$$\begin{aligned} I_2 &\geq k\Big(\frac{1-4|s|+|s|^2}{1-|s|^2} - 0.006\frac{1+4|s|+|s|^2}{1-|s|^2}\Big) - 0.006 - 0.16 \\ &\geq k\Big(\frac{1-4t_2+t_2^2}{1-t_2^2} - 0.006\frac{1+4t_2+t_2^2}{1-t_2^2}\Big) - 0.166 \\ &= k\Big(\frac{46}{80} - 0.006\times\frac{118}{80}\Big) - 0.166 > 0. \end{aligned} \tag{32}$$

Combining (31), (32) and (27), we get that $I > 0$ in $|z| < \sqrt[k]{\frac{k}{2(k+1)^2}}$. □

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