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# ON A FUNCTIONAL EQUATION ASSOCIATED WITH THE TRAPEZOIDAL RULE

BY

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**Abstract**

The present work aims to determine the solution $f, g, h, k : \mathbb{R} \to \mathbb{R}$ of the equation $g(y) - h(x) = (y - x)[f(x) + 2k(sx + ty) + 2k(tx + sy) + f(y)]$ for all real numbers $x$ and $y$. Here $s$ and $t$ are any two *a priori* chosen real parameters. This functional equation arises in connection with the trapezoidal rule for the numerical evaluation of definite integrals. In the book [9], it was an open problem to find the general solution of the functional equation $g(y) - g(x) = (y - x)[f(x) + 2k(x + 2y) + 2k(2x + y) + f(y)]$. This paper also determines the differentiable solution of this functional equation.

## 1. Introduction

Let $\mathbb{R}$ be the set of all real numbers. The trapezoidal rule is an elementary numerical method for evaluating a definite integral $\int_a^b f(t)\,dt$. The method consists of partitioning the interval $[a, b]$ into subintervals of equal lengths and then interpolating the graph of $f$ over each subinterval with a linear function. If $a = x_o < x_1 < x_2 < \cdots < x_n = b$ is a partition of $[a, b]$ into $n$ subintervals, each of length $\frac{b-a}{n}$, then

$$\int_a^b f(t)dt \simeq \frac{b-a}{2n}[f(x_o) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)].$$

This approximation formula is called the trapezoidal rule. It is well known

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that the error bound for trapezoidal rule approximation is

$$\Big|\int_a^b f(t)dt - \frac{b-a}{2n}[f(x_o)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)]\Big| \leq \frac{K(b-a)^3}{12n^2}$$

where $K = \sup\{|f^{(2)}(x)||x \in [a,b]\}$. It is easy to note from this inequality that if $f$ is two times continuously differentiable and $f^{(2)}(x) = 0$, then

$$\int_a^b f(t)dt = \frac{b-a}{2n}[f(x_o)+2f(x_1)+\cdots+2f(x_{n-1})+f(x_n)].$$

This is obviously true if $n = 3$ and it reduces to

$$\int_a^b f(t)dt = \frac{b-a}{6}[f(x_o)+2f(x_1)+2f(x_2)+f(x_3)].$$

Letting $a = x$, $b = y$, $x_1 = \frac{2x+y}{3}$ and $x_2 = \frac{x+2y}{3}$ in the above formula, we obtain

$$\int_x^y f(t)dt = \frac{y-x}{6}\Big[f(x)+2f\Big(\frac{2x+y}{3}\Big)+2f\Big(\frac{x+2y}{3}\Big)+f(y)\Big]. \quad (1)$$

This integral equation (1) holds for all $x, y \in \mathbb{R}$ if $f$ is a polynomial of degree at most one. However, it is not obvious that if (1) holds for all $x, y \in \mathbb{R}$, then the only solution $f$ is the polynomial of degree one. The integral equation (1) leads to the functional equation

$$g(y)-g(x) = \frac{y-x}{6}\Big[f(x)+2f\Big(\frac{2x+y}{3}\Big)+2f\Big(\frac{x+2y}{3}\Big)+f(y)\Big] \quad (2)$$

where $g$ is an antiderivative of $f$. The above equation is a special case of the functional equation

$$g(y)-h(x) = (y-x)[f(x)+2k(sx+ty)+2k(tx+sy)+f(y)] \quad (3)$$

where $s$, $t$ are two real *a priori* chosen parameters. If we choose $s = 1$ and $t = 2$, then we obtain

$$g(y)-g(x) = (y-x)[f(x)+2k(x+2y)+2k(2x+y)+f(y)] \quad (4)$$

for all $x, y \in \mathbb{R}$. In the book [9], it was an open problem to find the general solution of the functional equation (4).

It should be noted that if we consider $n = 2$ in the approximation formula, then the functional equation

$$g(y) - g(x) = \Big(\frac{y-x}{4}\Big)\Big[f(x) + 2f\Big(\frac{x+y}{2}\Big) + f(y)\Big]$$

arrises analogously and it is a special case of

$$g(y) - g(x) = (y-x)[\phi(x) + \psi(y) + h(sx+ty)].$$

This functional equation was treated by Kannappan, Riedel and Sahoo [6] (also see [9]); without any regularity conditions. Interested reader should see [1-5, 7-11] for related functional equations whose solutions are polynomials.

In this paper, our goal is to determine the general solution of the functional equation (3) without any regularity assumptions on the unknown functions $f$, $g$, $h$ and $k$ when $s$ and $t$ are any two *a priori* chosen real parameters with $s^2 = t^2$. In the case $s^2 \neq t^2$, we find the differentiable solutions of the functional equation (3). We also provide differentiable solutions of the functional equation (4).

## 2. Solution of the Functional Equation (3) when $s^2 = t^2$

The following theorem from [6] will be instrumental in solving the functional equation (3).

**Theorem 1.** *Let $s$ and $t$ be real parameters. The functions $f, g, h, \phi, \psi : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation*

$$f(x) - g(y) = (x-y)[h(sx+ty) + \psi(x) + \phi(y)]$$

*for all $x$, $y \in \mathbb{R}$ if and only if $g(x) = f(x)$ and*

$$f(x) = \begin{cases} ax^2 + (b+d)x + c & if \quad s = 0 = t \\ ax^2 + bx + c & if \quad s = 0, \quad t \neq 0 \\ ax^2 + bx + c & if \quad s \neq 0, \quad t = 0 \\ 3ax^4 + 2bx^3 + cx^2 + (d+2\beta)x + \alpha & if \quad s = t \neq 0 \\ 2ax^3 + cx^2 + (2\beta - d)x - A(x) + \alpha & if \quad s = -t \neq 0 \\ -2bstx^3 + \beta x^2 + (2\gamma + \alpha - d)x + \delta & if \quad 0 \neq s^2 \neq t^2 \neq 0 \end{cases}$$

$$\phi(x) = \begin{cases} ax + \frac{(b-\delta)}{2} & if \quad s = 0 = t \\ ax + \frac{(b+\delta)}{2} & if \quad s = 0, \quad t \neq 0 \\ ax + \frac{(b+\delta)}{2} & if \quad s \neq 0, \quad t = 0 \\ 2ax^3 + bx^2 + cx - A(x) + \beta + \frac{\delta}{2} & if \quad s = t \neq 0 \\ 3ax^2 + cx - \frac{1}{2}A_o(x) + \beta, & if \quad s = -t \neq 0 \\ bt(t-2s)x^2 + \beta x + A(tx) + \gamma & if \quad 0 \neq s^2 \neq t^2 \neq 0 \end{cases}$$

$$\psi(x) = \begin{cases} ax + \frac{(b+\delta)}{2} & if \quad s = 0 = t \\ ax + \frac{(b-\delta)}{2} - h(tx) & if \quad s = 0, \quad t \neq 0 \\ ax + \frac{(b-\delta)}{2} - h(sx) & if \quad s \neq 0, \quad t = 0 \\ 2ax^3 + bx^2 + cx - A(x) + \beta - \frac{\delta}{2} & if \quad s = t \neq 0 \\ 3ax^2 + cx + \frac{1}{2}A_o(x) + \beta - d & if \quad s = -t \neq 0 \\ bs(s-2t)x^2 + \beta x + A(sx) + \gamma + \alpha & if \quad 0 \neq s^2 \neq t^2 \neq 0 \end{cases}$$

$$h(x) = \begin{cases} arbitrary \ \ with \ \ h(0) = d & if \quad s = 0 = t \\ arbitrary & if \quad s = 0 \quad t \neq 0 \\ arbitrary & if \quad s \neq 0 \quad t = 0 \\ a\left(\frac{x}{s}\right)^3 + b\left(\frac{x}{s}\right)^2 + A\left(\frac{x}{s}\right) + d & if \quad s = t \neq 0 \\ -a\left(\frac{x}{s}\right)^2 - \frac{s}{x}A\left(\frac{x}{s}\right) + \frac{1}{2}A_o\left(\frac{x}{s}\right), \quad x \neq 0 & if \quad s = -t \neq 0, \\ -bx^2 - A(x) - d & if \quad 0 \neq s^2 \neq t^2 \neq 0, \end{cases}$$

*where* $A_o, A : \mathbb{R} \to \mathbb{R}$ *are additive functions and* $a, b, c, d, \alpha, \beta, \gamma, \delta$ *are arbitrary real constants.*

Using Theorem 1, we determine the general solution of the functional equation (3) without any regularity assumptions on the unknown functions $f$, $g$ and $h$ when the parameters $s$ and $t$ satisfy $s^2 = t^2$.

**Theorem 2.** *Let* $s$ *and* $t$ *be any two a priori chosen real parameters such that* $s = 0$, $t = 0$, *or* $s^2 = t^2$. *The functions* $f, g, h, k : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (3), *that is*

$$g(y) - h(x) = (y - x)[f(x) + 2k(sx + ty) + 2k(tx + sy) + f(y)]$$

*for all* $x, y \in \mathbb{R}$ *if and only if* $h(x) = g(x)$ *and*

$$g(x) = \begin{cases} ax^2 + bx + c & if \quad s = 0 = t \\ ax^2 + bx + c & if \quad s = 0, \quad t \neq 0 \\ ax^2 + bx + c & if \quad s \neq 0, \quad t = 0 \\ 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha & if \quad s = t \neq 0 \\ 2ax^3 + cx^2 + 2\beta x - A(x) + \alpha & if \quad s = -t \neq 0 \end{cases}$$

$$f(x) = \begin{cases} ax + \frac{(b-\gamma)}{2} & if \quad s = 0 = t \\ ax + \frac{b}{2} - 2k(tx) & if \quad s = 0, \quad t \neq 0 \\ ax + \frac{b}{2} - 2k(sx) & if \quad s \neq 0, \quad t = 0 \\ 2ax^3 + bx^2 + cx - A(x) + \beta & if \quad s = t \neq 0 \\ 3ax^2 + cx + \beta & if \quad s = -t \neq 0, \end{cases}$$

$$k(x) = \begin{cases} arbitrary & if \quad s = 0 = t \\ arbitrary & if \quad s = 0, \quad t \neq 0 \\ arbitrary & if \quad s \neq 0, \quad t = 0 \\ \frac{a}{4}\left(\frac{x}{s}\right)^3 + \frac{b}{4}\left(\frac{x}{s}\right)^2 + \frac{1}{4}A\left(\frac{x}{s}\right) + \frac{d}{4} & if \quad s = t \neq 0 \\ -\frac{a}{2}\left(\frac{x}{s}\right)^2 - \frac{1}{2}\left(\frac{s}{x}\right)A\left(\frac{x}{s}\right) - k(-x), \quad x \neq 0 & if \quad s = -t \neq 0, \end{cases}$$

*where* $A : \mathbb{R} \to \mathbb{R}$ *is an additive function,* $a, b, c, d, \alpha, \beta$ *are arbitrary real constants, and the constant* $\gamma$ *is given by* $\gamma = 4k(0)$.

*Proof.* Letting $y = x$ in (3), we see that $h(x) = g(x)$ for all $x \in \mathbb{R}$, and the functional equation (3) reduces to

$$g(y) - g(x) = (y - x)[f(x) + 2k(sx + ty) + 2k(tx + sy) + f(y)] \qquad (5)$$

for all $x, y \in \mathbb{R}$. To determine the solution of the functional equation (5), we consider several cases depending on parameters $s$ and $t$.

**Case 1.** Suppose $s = 0 = t$. Then the functional equation (5) reduces to

$$g(x) - g(y) = (x - y)[f(x) + f(y) + \gamma] \qquad (6)$$

where $\gamma = 4k(0)$. Using Theorem 1, we get

$$g(x) = ax^2 + bx + c \tag{7}$$
$$f(x) = ax + (b - \gamma)/2 \tag{8}$$
$$k(x) \text{ is an arbitrary function} \tag{9}$$

where $a, b, c$ are arbitrary constants.

**Case 2.** Suppose $s \neq 0$ and $t = 0$. Then (5) yields

$$g(x) - g(y) = (x - y)[f(x) + 2k(sx) + 2k(sy) + f(y)] \tag{10}$$

for all $x, y \in \mathbb{R}$. Hence by Theorem 1, we obtain

$$g(x) = ax^2 + bx + c \tag{11}$$
$$f(x) = ax + \frac{b}{2} - 2k(sx), \tag{12}$$

where $k : \mathbb{R} \to \mathbb{R}$ is an arbitrary function, and $a, b, c$ are arbitrary constants.

**Case 3.** Suppose $t \neq 0$ and $s = 0$. Then (5) yields

$$g(x) - g(y) = (x - y)[f(x) + 2k(tx) + 2k(ty) + f(y)] \tag{13}$$

for all $x, y \in \mathbb{R}$. Hence by Theorem 1, we obtain

$$g(x) = ax^2 + bx + c \tag{14}$$
$$f(x) = ax + \frac{b}{2} - 2k(tx), \tag{15}$$

where $k : \mathbb{R} \to \mathbb{R}$ is an arbitrary function, and $a, b, c$ are arbitrary constants.

**Case 4.** Suppose $s = t \neq 0$. Then (5) reduces to

$$g(x) - g(y) = (x - y)[f(x) + 4k(s(x + y)) + f(y)] \tag{16}$$

for all $x, y \in \mathbb{R}$. Hence by Theorem 1, we obtain

$$g(x) = 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha, \tag{17}$$
$$f(x) = 2ax^3 + bx^2 + cx - A(x) + \beta + \frac{\delta}{2}, \tag{18}$$

$$f(x) = 2ax^3 + bx^2 + cx - A(x) + \beta - \frac{\delta}{2}, \tag{19}$$

$$4k(x) = a\Big(\frac{x}{s}\Big)^3 + b\Big(\frac{x}{s}\Big)^2 + A\Big(\frac{x}{s}\Big) + d, \tag{20}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function, and $a, b, c, d, \beta, \delta, \alpha$ are constants. From (18) and (19), we see that $\delta = 0$. Therefore the solution of (5) is given by

$$g(x) = 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha, \tag{21}$$

$$f(x) = 2ax^3 + bx^2 + cx - A(x) + \beta, \tag{22}$$

$$f(x) = 2ax^3 + bx^2 + cx - A(x) + \beta, \tag{23}$$

$$k(x) = \frac{a}{4}\Big(\frac{x}{s}\Big)^3 + \frac{b}{4}\Big(\frac{x}{s}\Big)^2 + \frac{1}{4}A\Big(\frac{x}{s}\Big) + \frac{d}{4}, \tag{24}$$

where $A : \mathbb{R} \to \mathbb{R}$ is an additive function, and $a, b, c, d, \beta, \alpha$ are constants.

**Case 5.** Suppose $s = -t \neq 0$. Then from (5) we have

$$g(x) - g(y) = (x - y)[f(x) + 2k(s(x - y)) + 2k(-s(x - y)) + f(y)] \tag{25}$$

for all $x, y \in \mathbb{R}$. Defining

$$\ell(sx) := 2k(sx) + 2k(-sx) \tag{26}$$

for all $x \in \mathbb{R}$ and using (25), we obtain

$$g(x) - g(y) = (x - y)[f(x) + \ell(s(x - y)) + f(y)] \tag{27}$$

for all $x, y \in \mathbb{R}$. Hence by Theorem 1, we obtain

$$g(x) = 2ax^3 + cx^2 + (2\beta - d)x - A(x) + \alpha, \tag{28}$$

$$f(x) = 3ax^2 + cx - \frac{1}{2}A_0(x) + \beta, \tag{29}$$

$$f(x) = 3ax^2 + cx + \frac{1}{2}A_0(x) + \beta - d, \tag{30}$$

$$\ell(x) = -a\Big(\frac{x}{s}\Big)^2 - \frac{s}{x}A\Big(\frac{x}{s}\Big) + \frac{1}{2}A_0\Big(\frac{x}{s}\Big), \quad x \neq 0, \tag{31}$$

where $A, A_0 : \mathbb{R} \to \mathbb{R}$ are additive functions, and $a, c, d, \beta, \alpha$ are constants.

Hence from (29) - (30) we see that

$$A_0 \equiv 0, \quad \text{and} \quad d = 0. \tag{32}$$

Therefore, we have

$$g(x) = 2ax^3 + cx^2 + 2\beta x - A(x) + \alpha \tag{33}$$

$$f(x) = 3ax^2 + cx + \beta, \tag{34}$$

$$k(x) + k(-x) = -\frac{a}{2}\Big(\frac{x}{s}\Big)^2 - \frac{1}{2}\frac{s}{x}A\Big(\frac{x}{s}\Big), \quad x \neq 0, \tag{35}$$

where $a, c, \alpha, \beta$ are arbitrary constants, and $A : \mathbb{R} \to \mathbb{R}$ is an additive function.

Since no more cases are left the proof of the theorem is now complete. □

Notice that in the case $s = -t$, Theorem 2 does not find $k(x)$ but determines the function $k(x) + k(-x)$.

## 3. Differentiable Solution of Equation (3) when $s^2 \neq t^2$

In this section, we determine the differentiable solution of the functional equation (3) in the case the *a priori* chosen parameters $s$ and $t$ satisfy $s^2 \neq t^2$ with $s \neq 0$ and $t \neq 0$. Throughout this section, $\psi^{(n)}$ will denote the $n^{\text{th}}$ derivative of the function $\psi : \mathbb{R} \to \mathbb{R}$.

**Theorem 3.** *Let $s$ and $t$ be any two nonzero a priori chosen real parameters with $s^2 \neq t^2$. Suppose $g$ and $f$ are twice differentiable and $k$ is four time differentiable. The functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (3)*, that is*

$$g(y) - h(x) = (y - x)[f(x) + 2k(sx + ty) + 2k(tx + sy) + f(y)]$$

*for all* $x, y \in \mathbb{R}$ *if and only if* $h(x) = g(x)$ *and for* $(s-t)^2 - st \neq 0$

$$\begin{aligned}
g(x) &= 2\sum_{i=2}^{3} a_i ist[s^{i-2} + t^{i-2}]x^{i+1} + 2\sum_{i=0}^{1}[b_i + (s^i + t^i)a_i]x^{i+1} + 2b_0x + c_0 \\
f(x) &= 2\sum_{i=2}^{3} a_i\Big[ist(s^{i-2} + t^{i-2}) - (s^i + t^i)\Big]x^i + 2b_1x + 2b_0 \\
k(x) &= a_3x^3 + a_2x^2 + a_1x + a_0,
\end{aligned}$$

*and for* $(s-t)^2 - st = 0$

$$\begin{aligned}
g(x) &= 2\sum_{i=2}^{5} a_i ist[s^{i-2} + t^{i-2}]x^{i+1} + 2\sum_{i=0}^{1}[b_i + (s^i + t^i)a_i]x^{i+1} + 2b_0x + b_0 \\
f(x) &= 2\sum_{i=2}^{5} a_i\Big[ist(s^{i-2} + t^{i-2}) - (s^i + t^i)\Big]x^i + 2b_1x + 2b_0 \\
k(x) &= a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0,
\end{aligned}$$

*where* $a_i$ $(i = 0, 1, \ldots, 5)$, $b_i$ $(i = 0, 1)$, $c_0$ *are arbitrary real constants.*

*Proof.* As in the Theorem 2, letting $y = x$, we get $g(x) = h(x)$, and the functional equation reduces to

$$g(x) - g(y) = (x-y)[f(x) + 2k(sx+ty) + 2k(tx+sy) + f(y)] \tag{36}$$

for all $x, y \in \mathbb{R}$. Letting $y = 0$ in the above functional equation, we have

$$g(x) = x[f(x) + 2k(sx) + 2k(tx) + f(0)] + g(0) \tag{37}$$

for all $x \in \mathbb{R}$. Differentiating (36) with respect to $x$ twice, we obtain

$$\begin{aligned}
g^{(2)}(x) &= (x-y)\Big[f^{(2)}(x) + 2s^2k^{(2)}(sx+ty) + 2t^2k^{(2)}(tx+sy)\Big] \\
&\quad +2\Big[f^{(1)}(x) + 2sk^{(1)}(sx+ty) + 2tk^{(1)}(tx+sy)\Big].
\end{aligned} \tag{38}$$

Next, differentiating (38) with respect to $y$ twice and simplifying, we see that

$$\begin{aligned}
&4st(s-t)k^{(3)}(sx+ty) - 4st(s-t)k^{(3)}(tx+sy) \\
&\quad = 2s^2t^2(x-y)\Big[k^{(4)}(sx+ty) + k^{(4)}(tx+sy)\Big]
\end{aligned}$$

for all $x, y \in \mathbb{R}$. Since $s$ and $t$ are nonzero, we have

$$\begin{aligned} & k^{(3)}(sx+ty) - k^{(3)}(tx+sy) \\ &= \frac{1}{2}\frac{st}{s-t}(x-y)\Big[k^{(4)}(sx+ty) + k^{(4)}(tx+sy)\Big] \end{aligned} \tag{39}$$

for all $x, y \in \mathbb{R}$.

Let $u = sx + ty$ and $v = sy + tx$. Since $s^2 \neq t^2$, that is $det \begin{pmatrix} s & t \\ t & s \end{pmatrix} \neq 0$, therefore

$$x = \frac{vt - us}{t^2 - s^2} \quad \text{and} \quad y = \frac{ut - vs}{t^2 - s^2}.$$

Hence from (39), we have

$$k^{(3)}(u) - k^{(3)}(v) = \frac{1}{2}\frac{st}{(s-t)^2}(u-v)\Big[k^{(4)}(u) + k^{(4)}(v)\Big] \tag{40}$$

for all $u, v \in \mathbb{R}$. Defining

$$\phi(x) = k^{(3)}(x) \quad \text{and} \quad \psi(x) = \frac{1}{2}\frac{st}{(s-t)^2}k^{(4)}(x) \tag{41}$$

and using the equation (40), we obtain

$$\phi(u) - \phi(v) = (u-v)[\psi(u) + \psi(v)] \tag{42}$$

for all $u, v \in \mathbb{R}$. The solution of the functional equation (42) can be obtained from Lemma 3.2 in [9] as

$$\psi(u) = ax + b \quad \text{and} \quad \phi(x) = ax^2 + 2bx + c \tag{43}$$

where $a, b, c$ are real arbitrary constants. From (41) and (43), we obtain

$$\frac{1}{2}\frac{st}{(s-t)^2}k^{(4)}(x) = ax + b \quad \text{and} \quad k^{(3)}(x) = ax^2 + 2bx + c. \tag{44}$$

Differentiating the second expression in (44) with respect to $x$, we get

$$k^{(4)}(x) = 2ax + 2b.$$

Now we consider two cases.

**Case 1.** Suppose $(s-t)^2 - st \neq 0$. Comparing this with the first expression in (44), we see that $a = 0$ and $b = 0$. Hence $k^{(4)}(x) = 0$ for all $x \in \mathbb{R}$. Therefore

$$k(x) = a_3x^3 + a_2x^2 + a_1x + a_0, \tag{45}$$

where $a_i$'s are real constants.

Letting (45) into (36) and differentiating the resulting equation first with respect to $x$ and then with respect to $y$, we get

$$\begin{aligned} 0 \;=\; & (x-y)[12a_3st(sx+ty) + 4a_2st + 12a_3st(tx+sy) + 4a_2st] \\ & -\Big[f^{(1)}(x) + 6a_3s(sx+ty)^2 + 4a_2s(sx+ty) + 2a_1s \\ & +6a_3t(tx+sy)^2 + 4a_2t(tx+sy) + 2a_1t\Big] \\ & +\Big[6a_3t(sx+ty)^2 + 4a_2t(sx+ty) + 2a_1t \\ & +6a_3s(tx+sy)^2 + 4a_2s(tx+sy) + 2a_1s + f^{(1)}(y)\Big] \end{aligned}$$

for all $x, y \in \mathbb{R}$. Letting $y = 0$ in the last equation and simplifying, we have

$$f^{(1)}(x) = 2\sum_{i=2}^{3} ia_i\Big[ist(s^{i-2} + t^{i-2}) - (s^i + t^i)\Big]x^{i-1} + f^{(1)}(0). \tag{46}$$

Integrating (46) with respect to $x$, we obtain

$$f(x) = 2\sum_{i=2}^{3} a_i\Big[ist(s^{i-2} + t^{i-2}) - (s^i + t^i)\Big]x^i + 2b_1x + 2b_0, \tag{47}$$

where $2b_1 = f^{(1)}(0)$ and $b_0$ is a constant of integration. Using (47) and (45) in (37), we see that

$$g(x) = 2\sum_{i=2}^{3} a_iist\Big[s^{i-2}+t^{i-2}\Big]x^{i+1} + 2\sum_{i=0}^{1}[b_i+(s^i+t^i)a_i]x^{i+1} + 2b_0x + c_0, \tag{48}$$

where $c_0 = g(0)$. Letting the functions $f(x)$, $g(x)$ and $k(x)$ into the functional equation (36), we see that they are the solution of (36).

**Case 2.** Suppose $(s-t)^2 - st = 0$, that is $s = (\frac{3}{2} \pm \frac{\sqrt{5}}{2})t$. Then from

(44) we have $k^{(4)}(x) = 2ax + 2b$. Integrating with respect to $x$, we see that

$$k(x) = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0, \tag{49}$$

where $a_i$'s for $i = 0, 1, \ldots, 5$ are real constants. Letting this $k(x)$ into (36) and differentiating the resulting equation first with respect to $x$ and then with respect to $y$, and finally substituting $y = 0$ in the resulting expression, we obtain

$$f^{(1)}(x) = 2\sum_{i=2}^{5} ia_i\Big[ist(s^{i-2} + t^{i-2}) - (s^i + t^i)\Big]x^{i-1} + f^{(1)}(0). \tag{50}$$

Integrating (50) with respect to $x$, we obtain

$$f(x) = 2\sum_{i=2}^{5} a_i\Big[ist(s^{i-2} + t^{i-2}) - (s^i + t^i)\Big]x^i + 2b_1x + 2b_0, \tag{51}$$

where $2b_1 = f^{(1)}(0)$ and $b_0$ is a constant of integration. Using (51) and (49) in (37), we see that

$$g(x) = 2\sum_{i=2}^{5} a_iist\Big[s^{i-2} + t^{i-2}\Big]x^{i+1} + 2\sum_{i=0}^{1}\Big[b_i + (s^i + t^i)a_i\Big]x^{i+1} + 2b_0x + c_0, \tag{52}$$

where $c_0 = g(0)$. Letting the functions $f(x)$, $g(x)$ and $k(x)$ into the functional equation (36), we see that they are the solution of (36). Since no more cases are left, the proof of the theorem is now complete. □

The following three corollaries follow from Theorem 3.

**Corollary 1.** *Suppose $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are twice differentiable and $k : \mathbb{R} \to \mathbb{R}$ is four time differentiable. The functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (4)*, that is*

$$g(y) - h(x) = (y - x)[f(x) + 2k(x + 2y) + 2k(2x + y) + f(y)]$$

*for all* $x, y \in \mathbb{R}$ *if and only if* $h(x) = g(x)$ *and*

$$g(x) = 36Ax^4 + 16Bx^3 + (E + 6C)x^2 + 2(G + 2D)x + F, \tag{53}$$
$$f(x) = 18Ax^3 + 6Bx^2 + Ex + G, \tag{54}$$
$$k(x) = Ax^3 + Bx^2 + Cx + D, \tag{55}$$

*where* $A, B, C, D, E, F, G$ *are arbitrary real constants.*

**Corollary 2.** *Let* $s$ *and* $t$ *be any two nonzero a priori chosen real parameters with* $s^2 \neq t^2$. *Suppose* $g : \mathbb{R} \to \mathbb{R}$ *are twice differentiable and* $f : \mathbb{R} \to \mathbb{R}$ *is four time differentiable. The functions* $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation*

$$g(y) - g(x) = (y - x)[f(x) + 2f(sx + ty) + 2f(tx + sy) + f(y)]$$

*for all* $x, y \in \mathbb{R}$ *if and only if*

$$g(x) = (2s + 2t + 1)Cx^2 + 6Dx + E, \tag{56}$$
$$f(x) = Cx + D \tag{57}$$

*where* $C, D, E$ *are arbitrary real constants.*

**Corollary 3.** *Suppose* $g : \mathbb{R} \to \mathbb{R}$ *are twice differentiable and* $f : \mathbb{R} \to \mathbb{R}$ *is four time differentiable. The functions* $f, g : \mathbb{R} \to \mathbb{R}$ *satisfy the functional equation* (2), *that is*

$$g(y) - g(x) = \frac{y - x}{6}\Big[f(x) + 2f\Big(\frac{2x + y}{3}\Big) + 2f\Big(\frac{x + 2y}{3}\Big) + f(y)\Big]$$

*for all* $x, y \in \mathbb{R}$ *if and only if*

$$g(x) = 9Cx^2 + 6Dx + E, \tag{58}$$
$$f(x) = 18Cx + 6D \tag{59}$$

*where* $C, D, E$ *are arbitrary real constants.*

## 4. Main Result

Now using Theorem 2 and Theorem 3, we are ready to give the differentiable solutions of the functional equation (3).

**Theorem 4.** *Let $s$ and $t$ be any two priori chosen real parameters. Suppose $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are twice differentiable and $k : \mathbb{R} \to \mathbb{R}$ is four time differentiable. The functions $f, g, h, k : \mathbb{R} \to \mathbb{R}$ satisfy the functional equation* (3)*, that is*

$$g(y) - h(x) = (y - x)[f(x) + 2k(sx + ty) + 2k(tx + sy) + f(y)]$$

*for all $x, y \in \mathbb{R}$ if and only if $h(x) = g(x)$ and*

$$g(x) = \begin{cases} ax^2 + bx + c & if\ s = 0 = t \\ ax^2 + bx + c & if\ s = 0,\ t \neq 0 \\ ax^2 + bx + c & if\ s \neq 0,\ t = 0 \\ 3ax^4 + 2bx^3 + cx^2 + (d + 2\beta)x + \alpha & if\ s = t \neq 0 \\ 2ax^3 + cx^2 + (2\beta - d)x + \alpha & if\ s = -t \neq 0 \\ 2\sum_{i=2}^{3} a_i ist[s^{i-2} + t^{i-2}]x^{i+1} & \\ \quad +2\sum_{i=0}^{1}[b_i+(s^i+t^i)a_i]x^{i+1}+2b_0x+c_0 & if\ s^2 \neq t^2,\ (s-t)^2 \neq st \\ 2\sum_{i=2}^{5} a_i ist[s^{i-2} + t^{i-2}]x^{i+1} & \\ \quad +2\sum_{i=0}^{1}[b_i+(s^i+t^i)a_i]x^{i+1}+2b_0x+c_0 & if\ s^2 \neq t^2,\ (s-t)^2 = st \end{cases}$$

$$f(x) = \begin{cases} ax + \frac{b-4\eta(0)}{2} & if\ s = 0 = t \\ ax + \frac{b}{2} - 2\eta(tx) & if\ s = 0,\ t \neq 0 \\ ax + \frac{b}{2} - 2\eta(sx) & if\ s \neq 0,\ t = 0 \\ 2ax^3 + bx^2 + (c - d)x + \beta & if\ s = t \neq 0 \\ 3ax^2 + cx + \beta & if\ s = -t \neq 0, \\ 2\sum_{i=2}^{3} a_i \Big[ist(s^{i-2}+t^{i-2})-(s^i + t^i)\Big]x^i & \\ \quad + 2b_1x + 2b_0 & if\ s^2 \neq t^2,\ (s-t)^2 \neq st \\ 2\sum_{i=2}^{5} a_i \Big[ist(s^{i-2}+t^{i-2})-(s^i + t^i)\Big]x^i & \\ \quad + 2b_1x + 2b_0 & if\ s^2 \neq t^2,\ (s-t)^2 = st \end{cases}$$

$$k(x) = \begin{cases} \eta(x) & if\ s = 0 = t \\ \eta(x) & if\ s = 0,\ t \neq 0 \\ \eta(x) & if\ s \neq 0,\ t = 0 \\ \frac{a}{4}\left(\frac{x}{s}\right)^3 + \frac{b}{4}\left(\frac{x}{s}\right)^2 + \frac{1}{4}\delta\frac{x}{s} + \frac{d}{4} & if\ s = t \neq 0 \\ -\frac{a}{2}\left(\frac{x}{s}\right)^2 - \frac{d}{2} - k(-x) & if\ s = -t \neq 0 \\ \sum_{i=0}^{3} a_i x^i & if\ s^2 \neq t^2,\ (s-t)^2 \neq st \\ \sum_{i=0}^{5} a_i x^i & if\ s^2 \neq t^2,\ (s-t)^2 = st \end{cases}$$

*where* $\eta : \mathbb{R} \to \mathbb{R}$ *is an arbitrary function, and* $a_i$ $(i = 0, 1, 2, \ldots, 5)$, $b_i$ $(i = 0, 1)$, $a, b, c, d, c_0, \alpha, \beta, \delta$ *are arbitrary real constants.*

**Problem 1.** In Theorem 3, we have assumed that the functions $g : \mathbb{R} \to \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ are twice differentiable and $k : \mathbb{R} \to \mathbb{R}$ is four time differentiable. The proof of Theorem 3 heavily relies on this differentiability assumption. Thus we pose the following problem: Determine the general solution of the functional equation (3) without any regularity assumptions on the unknown functions $g, h, f$ and $k$.

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