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SOME MULTIVARIATE DISCRETE TIME SERIES MODELS FOR DEPENDENT MULTIVARIATE ZIPF COUNTS

BY

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Abstract

A family of models MD-AR(1), MD-AR(p), MD-MA(q), MD-ARMA(p,q) for multivariate discrete time series is developed and all their autocorrelation structures are investigated. Any fat-tailed dependent multivariate discrete random vectors can be fitted reasonably by these multivariate discrete time series models. In this article, four multivariate Zipf processes are mainly discussed. Some distributional properties of the Zipf processes, MZ-AR(1) and MZ-MA(1) including the joint distribution, time reversibility, expected run length, extreme order statistics, geometric minima are studied thoroughly in this paper. These multivariate Zipf processes provide potential models for multivariate discrete income time series data.

1. Introduction and Motivation

Some multivariate discrete time series models are developed in this paper. They can generate a stationary sequence of dependent multivariate discrete random vectors, which have a specified multivariate discrete marginal distribution, and their correlation structures are constructed. This work is extended from the result of Jacobs and Lewis (1978, 1983). There are

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three submodels included in this paper: (1) the multivariate discrete autoregressive process, MD-AR(p), (2) the multivariate discrete moving average process, MD-MA(q), (3) The mixed multivariate MD-ARMA(p, q) process. Their correlation structures are derived respectively.

These multivariate discrete time series models are suitable for any dependent stationary multivariate discrete random vectors, such as: multivariate Poisson, multivariate hypergeometric, multivariate multinomial, multivariate negative multinomial, etc., (Johnson and Kotz and Balakrishnan(1997)). However, the multivariate Zipf distribution was not in their book. Yeh (2002) developed six generalized multivariate Zipf distributions. The Zipf distribution is the discrete version of the Pareto distribution and it has been frequently used to model a wide variety of soci-economic integer variables such as the size of business firms, discrete income (Yule (1924), Arnold and Laguna (1977)). Asset returns are also some types of income and in reality, such kind of data may appear as multivariate forms which are fat upper tailed on each dimension and they are reasonably well fitted by multivariate Zipf distributions.

Motivation of this study is from the book of Tsay (2005) (Ch.3, 5, and 8), Tsay studied many properties of the financial time series data. The characteristics of asset returns are right fat-tailed. All conditional heteroscedastic models developed in recent years, such as ARCHBGARCHBEGARCH, etc. are all fat-tailed and continuous. Most financial time series analysis studies involve returns, in stead of prices of assets. Asset returns have more attractive statistical properties. Additionally, many such variables of random vectors are repeatedly observed over time and they possibly form a multivariate discrete Zipf time series data. Hence if the marginal distributions of the MD-AR(p), MD-MA(q), MD-ARMA(p, q) processes are specified as the multivariate Zipf (I), (II), (III), (IV) distributions, which will be introduced in subsection 2.3, then these models are defined as the multivariate Zipf processes and are denoted by MZ-AR(p), MZ-MA(q), and MD-ARMA(p, q) respectively. In this paper, some distributional properties of the particular MZ-AR(1) and MZ-MA(1) processes are studied detailedly.

As for the the statistical inferences and the identifiability of these multivariate Zipf processes will be studied in the near future by the author or some other researchers of time series analysis.

2. The Multivariate Discrete Time Series Models

The univariate discrete time series models proposed by Jacobs and Lewis (1978, 1983) can be easily extended to the multivariate versions of AR(p), MA(q), and ARMA(p,q). In the following subsections, the multivariate discrete autoregressive process MD-AR(1), and higher order autoregressive process MD-AR(p), and the multivariate discrete moving average process MD-MA(q) will be discussed. Finally, the mixed multivariate discrete MD-ARMA(p,q) processes are introduced.

2.1. The multivariate discrete autoregressive process MD-AR(1)

An m-variate discrete autoregressive process MD-AR(1) is defined as

$$A_n = V_n A_{n-1} + (I_m - V_n)Y_n, \tag{2.1}$$

where $\{V_n\} \overset{i.i.d.}{\sim}$ m-variate Bernoulli, i.e., $V_n = \begin{cases} I_m, & w.p. \quad \rho \\ \underline{0}_{m\times m}, & w.p. \quad 1-\rho \end{cases}$

I_m is the identity matrix of dimension $m \times m$ and $\underline{0}_{m\times m}$ is the $m \times m$ matrix of entry 0, and $\{Y_n\} \overset{i.i.d.}{\sim}$ any m-variate discrete random vector. The MD-AR(1) process is clearly Markovian and it is analogous to the Markov switching model proposed by Hamilton(1989).

Property 2.1. *In model* (2.1), (i) *if the initial value* A_0 *has the same distribution as the* Y_n*'s, then for each* $n = 1, 2, \ldots, A_n$ *has the same marginal distribution as the* Y_n*'s have, say* π. (ii) *If* A_0 *has an arbitrary* (*and possibly degenerate*) *distribution, then* A_n *converges in distribution to the same distribution as the* Y_n*'s distribution,* π.

In this paper, only the stationary case is considered, i.e. assume $A_0 \overset{d}{=} Y_n$. There are many m-variate discrete distributions introduced in Johnson and Kotz and Balakrishnan (1997), such as: multivariate Poisson, multivariate hypergeometric, multivariate multinomial, multivariate negative binomial, multivariate negative multinomial, and multivariate logarithmic series distributions. But m-variate discrete Zipf distributions are not in their book. Yeh (2002) developed six different m-variate discrete Zipf distributions. In this article, all the Y_n's and A_0 in Eq.(2.1) are chosen to have the m-variate

Zipf (I), (II) (III) and (IV) distributions and the sequence $\{A_n\}$ is defined as the multivariate Zipf autoregressive processes and denoted by MZ(I)-AR(1), MZ(II)-AR(1), MZ(III)-AR(1), MZ(IV)-AR(1) respectively. These multivariate Zipf processes will be studied in Subsections 2.3-2.6.

2.2. The correlation structure of the MD-AR(1) process

Although the A_n's in Eq.(2.1) have a stationary distribution π, the A_n's are not independent as the Y_n's. This is easily discerned by the autocovariance matrices between A_n and $A_{n-\ell}$.

For any two m-variate random vectors A_n and $A_{n-\ell}$ in Eq.(2.1), the autocovariance matrices between them are defined as $\Gamma(\ell) = \mathrm{Cov}\,(A_n, A_{n-\ell}) = E[(A_n - \underline{\mu})(A_{n-\ell} - \underline{\mu})']$, where $\underline{\mu} = E(A_n) = E(A_{n-\ell})$. In general, for any $\ell \in \{\pm 1, \pm 2, \pm 3, \dots\}$, it is verified that

$$\Gamma(\ell) = \Gamma(-\ell) = \rho^{|\ell|}\Gamma(0). \tag{2.2}$$

Thus, the MD-AR(1) process is time reversible.

2.3. The multivariate Zipf autoregressive process MZ-AR(1)

If the A_n's in Eq.(2.1) are marginally distributed as the multivariate Zipf distribution (Yeh (2002)), then this MD-AR(1) process is called the MZ-AR(1) process. Before we study the MZ-AR(1) process, it is necessary to introduce the four types of the multivariate Zipf (I), (II), (III), (IV) distributions. Their definitions are as follows.

Definition 2.1. *Multivariate Zipf Distribution*

Suppose $\underline{X} = (X_1, X_2, \dots, X_m)$ is an m-dimensional discrete random vector. Then $\underline{X}$ is said to have multivariate Zipf distribution.

(1) **of type (I)**, *if the joint survival function of $\underline{X}$ is*

$$\overline{F}_{\underline{X}}(k) = P(X \geq k) = \Big\{1 + \sum_{i=1}^{m}\Big(\frac{k_i}{\sigma_i}\Big)\Big\}^{-\alpha}, \tag{2.3}$$

for any $\underline{k} = (k_1, k_2, \ldots, k_m)$, $k_i \in \{0, 1, 2, \ldots\}$, $1 \leq i \leq m$ *where* $\alpha > 0$, $\sigma_i > 0$, $1 \leq i \leq m$, *denote* $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_m)$. *The distribution of* $\underline{X}$ *is denoted as* $\underline{X} \sim M^{(m)}$ *Zipf(I)*$(\underline{\sigma}, \alpha)$.

(2) **of type (II)**, *if the joint survival function of* $\underline{X}$ *is*

$$\overline{F}_{\underline{X}}(\underline{k}) = \Big\{1 + \sum_{i=1}^{m} \Big(\frac{k_i - \mu_i}{\sigma_i}\Big)\Big\}^{-\alpha}, \tag{2.4}$$

for any $\underline{k} = (k_1, k_2, \ldots, k_m)$, $k_i \geq \mu_i$ *and* k_i *is integer*, $1 \leq i \leq m$, *where* $\underline{\mu} = (\mu_i, \mu_2, \ldots, \mu_m)$, *each* μ_i *is an integer*, $1 \leq i \leq m$. *As for* α *and* $\underline{\sigma}$, *they are the same as in Eq.*(2.1). *The distribution of* $\underline{X}$ *is denoted as* $\underline{X} \sim M^{(m)}$ *Zipf(II)*$(\underline{\mu}, \underline{\sigma}, \alpha)$.

(3) **of type (III)**, *if the joint survival function of* $\underline{X}$ *is*

$$\overline{F}_{\underline{X}}(\underline{k}) = \Big\{1 + \sum_{i=1}^{m} \Big(\frac{k_i - \mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big\}^{-1}, \tag{2.5}$$

where $\underline{\gamma} = (\gamma_1, \gamma_2, \ldots, \gamma_m)$, $\gamma_i > 0$, *as for* $\underline{\mu}$, $\underline{\sigma}$, $\underline{k}$, *they are the same as in Eq.*(2.4).

The distribution of $\underline{X}$ *is denoted as* $\underline{X} \sim M^{(m)}$ *Zipf(III)*$(\underline{\mu}, \underline{\sigma}, \underline{\gamma})$.

(4) **of type (IV)**, *if the joint survival function of* $\underline{X}$ *is*

$$\overline{F}_{\underline{X}}(\underline{k}) = \Big\{1 + \sum_{i=1}^{m} \Big(\frac{k_i - \mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big\}^{-\alpha}, \tag{2.6}$$

where $\alpha > 0$, *and* $\underline{\mu}$, $\underline{\sigma}$, $\underline{\gamma}$, $\underline{k}$, *are the same as in Eq.*(2.5). *The distribution of* $\underline{X}$ *is denoted as* $\underline{X} \sim M^{(m)}$ *Zipf(IV)*$(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \alpha)$.

It is clear that the three multivariate Zipf(I), (II), (III) families can be identified as special cases of the multivariate Zipf(IV) family as follows:

$$\begin{aligned} M^{(m)}\text{Zipf(I)}(\underline{\sigma}, \alpha) &= M^{(m)}\text{Zipf(IV)}(\underline{0}, \underline{\sigma}, \underline{1}, \alpha), \\ M^{(m)}\text{Zipf(II)}(\underline{\mu}, \underline{\sigma}, \alpha) &= M^{(m)}\text{Zipf(IV)}(\underline{\mu}, \underline{\sigma}, \underline{1}, \alpha), \\ M^{(m)}\text{Zipf(III)}(\underline{\mu}, \underline{\sigma}, \underline{\gamma}) &= M^{(m)}\text{Zipf(IV)}(\underline{\mu}, \underline{\sigma}, \underline{\gamma}, \underline{1}). \end{aligned} \tag{2.7}$$

Under the constructions of Eqs.(2.3), (2.4), (2.5), (2.6), it is easily discerned that these four m-dim Zipf distributions are qualified as multivariate Zipf distributions by virture of having Zipf marginal variables.

2.4. The joint distributions and time reversibility of the MZ-AR(1) process

Let A_n and A_{n+1} be any two adjacent m-variate random vectors in the MZ-AR(1) process, then the joint survival function of A_n and A_{n+1} is derived from Eq.(2.1) as

$$\begin{aligned}
&\overline{F}_{A_n,A_{n+1}}(\underline{k}^1,\underline{k}^2)\\
&= \Big\{1+\sum_{i=1}^{m}\Big(\frac{\max(k_i^1,k_i^2)-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big\}^{-\alpha}\rho+\Big\{1+\sum_{i=1}^{m}\Big(\frac{k_i^1-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big\}^{-\alpha}\\
&\quad\cdot\Big\{1+\sum_{i=1}^{m}\Big(\frac{k_i^2-\mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big\}^{-\alpha}(1-\rho), \qquad (2.8)
\end{aligned}$$

for any two state vectors $\underline{k}^1=(k_1^1,k_2^1,\ldots,k_m^1)\in E^m=\{\underline{k}=(k_1,k_2,\ldots,k_m) \mid k_i\geq\mu_i,\ k_i$ is integer, $1\leq i\leq m\}$ and $\underline{k}^2=(k_1^2,k_2^2,\ldots,k_m^2)\in E^m$.

It is easily observed that Eq.(2.7) is symmetric in $\underline{k}^1$ and $\underline{k}^2$, and hence the MZ-AR(1) process is time reversible. The time reversibility is also confirmed by the correlation structure of the MD-AR(1) process in Subsection 2.2, $\Gamma(\ell)=\Gamma(-\ell)$, for any $\ell=1,2,\ldots$.

2.5. The distribution of the runs in the MZ-AR(1) process

From the previous Subsection 2.3, Eq.(2.7), it suffices to study the MZ(IV)-AR(1) process. Let $\{A_n\}$ be a sequence in the MZ(IV)-AR(1) process, i.e., A_n's satisfy

$$A_n=\begin{cases} A_{n-1}, & w.p.\quad \rho,\\ Y_n, & w.p.\quad 1-\rho.\end{cases}$$

Fix an m-variate state vector $\underline{i}=(i_1,i_2,\ldots,i_m)\in E^m$, where $E^m=\{\underline{k}=(k_1,k_2,\ldots,k_m)\mid k_i\geq\mu_i,\ k_i$ is integer, $1\leq i\leq m\}$ and, $\underline{\mu}=(\mu_1,\mu_2,\ldots,\mu_m)$, each μ_i is an integer, $1\leq i\leq m$, then the length of a run of $\underline{i}$ starting at time epoch one for $\{A_j\}$ in the MZ(IV)-AR(1) process is defined as $T_{\underline{i}}=\inf\{j: A_j\neq\underline{i}\}-1$. The probability mass function (pmf) of $T_{\underline{i}}$ is calculated as

$$P(T_{\underline{i}}=0)=P\{A_1\neq\underline{i}\}=1-\pi(\underline{i}), \qquad (2.9)$$

where $\pi(\underline{i})$ is the joint pmf of the multivariate Zipf(IV) distribution, it is obtained from Eq.(2.6) as

$$\pi(\underline{i}) = \Big\{1+\sum_{\ell=1}^{m}\Big(\frac{i_\ell-\mu_\ell}{\sigma_\ell}\Big)^{1/\gamma_\ell}\Big\}^{-\alpha} - \Big\{1+\sum_{\ell=1}^{m}\Big(\frac{i_\ell+1-\mu_\ell}{\sigma_\ell}\Big)^{1/\gamma_\ell}\Big\}^{-\alpha}. \qquad (2.10)$$

For general $\ell = 1, 2, 3, \ldots$, by the Markov property of MZ(IV)-AR(1), we have

$$P(T_{\underline{i}}=\ell) = P(A_1=\underline{i},\ A_2=\underline{i}, \ldots, A_\ell=\underline{i},\ A_{\ell+1}\neq\underline{i}) = \sum_{\underline{j}\in E^m-\{\underline{i}\}} \pi(\underline{i})P_{\underline{ii}}^{\ell-1}P_{\underline{ij}}, \qquad (2.11)$$

where $P_{\underline{ii}}$ and $P_{\underline{ij}}$ are the transition probabilities. They are evaluated as

$$P_{\underline{ij}}=P(A_{n+1}=\underline{j}\mid A_n=\underline{i})=\begin{cases}(1-\rho)\pi(\underline{j}), & \text{if } \underline{j}\neq\underline{i}\\ \rho+(1-\rho)\pi(\underline{i}), & \text{if } \underline{j}=\underline{i}\end{cases}, \text{ for any } \underline{i},\underline{j}\in E^m.$$

Thus, for any $\ell \in \{1, 2, \ldots\}$,

$$P(T_{\underline{i}} = \ell) = (1-\rho)\pi(\underline{i})(1-\pi(\underline{i}))\{\rho+(1-\rho)\pi(\underline{i})\}^{\ell-1}. \qquad (2.12)$$

From Eqs.(2.9) and (2.12), it is straightforward to check that $\sum_{\ell=0}^{\infty} P(T_{\underline{i}} = \ell) = 1$, and conclude that for any fixed state vector $\underline{i} \in E^m$, the length of a run of $\underline{i}$'s, $T_{\underline{i}}$ is a non-defective discrete random variable. The survival function of $T_{\underline{i}}$ is directly calculated from the pmf of $T_{\underline{i}}$ as

$$P(T_{\underline{i}} \geq \ell) = \begin{cases}1, & \text{if } \ell = 0,\\ \pi(\underline{i})\{\rho+(1-\rho)\pi(\underline{i})\}^{\ell-1}, & \text{if } \ell = 1, 2, \ldots.\end{cases} \qquad (2.13)$$

Property 2.2. *The expected run length for the MZ(IV)-AR(1) process is always greater than or equal to the expected run length of an i.i.d. sequence of multivariate Zipf(IV) random vectors.*

Note. This property is true not only for the MZ(IV)-AR(1) process, it is also true for any other multivariate discrete random vectors in the general

MD-AR(1) process defined in Subsection 2.1.

2.6. Extreme order statistics in the MZ-AR(1) process

Order statistics of discrete random vectors are usually intractable except for the extremes. Finite series minima and maxima in the univariate Z(III)-AR(1) process (Yeh (1990)) and the geometric minima in the multivariate Zipf(III) family (Yeh (2002)) are particularly well behaved. Owing to the similar forms between the multivariate MZ(III)-AR(1) process and the univariate Z(III)-AR(1) process, it is natually to study whether this closure properties for extreme order statistics in the univariate Z(III)-AR(1) process will also be true in the multivariate MZ(III)-AR(1) process.

2.6.1. Exact and asymptotic distributions of extremes

Let $A_1, A_2, \ldots, A_n$ be the first n random vectors from a MZ(III)-AR(1) or MZ(IV)-AR(1) process. Define $m_n = \min\{A_1, A_2, \ldots, A_n\}$ as the coordinatewise minima of $\{A_i\}_i^n$. By the Markov property of $\{A_i\}$, the survival function of m_n is calculated as

$$\begin{aligned}\overline{F}_{m_n}(\underline{k}) &= P(A_1 \geq k)\{P(A_2 \geq k \mid A_1 \geq k)\}^{n-1} \\ &= \frac{1}{\left\{1+\sum_{j=1}^{m}\left(\frac{k_j-\mu_j}{\sigma_j}\right)^{1/\gamma_j}\right\}^{\alpha}}\left\{\rho+\frac{1-\rho}{\left\{1+\sum_{j=1}^{m}\left(\frac{k_j-\mu_j}{\sigma_j}\right)^{1/\gamma_j}\right\}^{\alpha}}\right\}^{n-1},\end{aligned} \tag{2.14}$$

for any $\underline{k} \in E^m$. Eq.(2.14) is the exact survival function of m_n in the MZ(IV)-AR(1) process, if set $\alpha = 1$ in Eq.(2.14), then by Eq.(2.5), the exact survival function of m_n in the MZ(III)-AR(1) process is the following

$$\overline{F}_{m_n}(\underline{k}) = \frac{1}{\left(1+\sum_{j=1}^{m}\left(\frac{k_j-\mu_j}{\sigma_j}\right)^{1/\gamma_j}\right)}\left\{\frac{1+\rho\sum_{j=1}^{m}\left(\frac{k_j-\mu_j}{\sigma_j}\right)^{1/\gamma_j}}{1+\sum_{j=1}^{m}\left(\frac{k_j-\mu_j}{\sigma_j}\right)^{1/\gamma_j}}\right\}^{n-1}, \tag{2.15}$$

for any $\underline{k} \in E^m$. If we consider the particular state vector $\underline{k}_0 = (\mu_1 + \frac{\sigma_1}{n^{\gamma_1}}x_1, \mu_2 + \frac{\sigma_2}{n^{\gamma_2}}x_2, \ldots, \mu_m + \frac{\sigma_m}{n^{\gamma_m}}x_m)$ in Eq.(2.15) with all the $x_i > 0$, $1 \leq i \leq m$, then the limiting distribution of m_n is

$$\lim_{n\to\infty} \overline{F}_{m_n}(\underline{k}_0) = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\sum_{j=1}^{m} x_j^{1/\gamma_j}\right)} \left\{ \frac{1+\frac{\rho}{n}\sum_{j=1}^{m} x_j^{1/\gamma_j}}{1+\frac{1}{n}\sum_{j=1}^{m} x_j^{1/\gamma_j}} \right\}^{n-1}$$
$$= e^{-(1-\rho)\sum_{j=1}^{m} x_j^{1/\gamma_j}}. \tag{2.16}$$

If consider scale transformation coordinatewisely, then Eq.(2.16) becomes

$$\lim_{n\to\infty} P\left\{ \frac{(n(1-\rho))^{\underline{\gamma}}}{\underline{\sigma}}(m_n - \underline{\mu}) \geq \underline{x} \right\} = e^{-\sum_{j=1}^{m} x_j^{1/\gamma_j}}, \tag{2.17}$$

where $\frac{(n(1-\rho))^{\underline{\gamma}}}{\underline{\sigma}}(m_n - \underline{\mu}) = ((\frac{(n(1-\rho))^{\gamma_1}}{\sigma_1}(m_n^1 - \mu_1), \frac{(n(1-\rho))^{\gamma_2}}{\sigma_2}(m_n^2 - \mu_2), \ldots, \frac{(n(1-\rho))^{\gamma_m}}{\sigma_m}(m_n^m - \mu_m)))$, and $\underline{x} = (x_1, x_2, \ldots, x_m)$, $m_n = (m_n^1, m_2^n, \ldots, m_n^m)$. From Eq.(2.17), conclude that the limiting distribution of $\frac{(n(1-\rho))^{\underline{\gamma}}}{\underline{\sigma}}(m_n - \underline{\mu})$ is m's independent Weibull$(1/\gamma_j)$ random variables.

2.6.2. Geometric minima of the MZ(III)-AR(1) process

Yeh (2002) studied geometric random sampling of the multivariate Zipf (III) distributions and found that Zipf(III) is closed under the geometric minima. This interesting fact initiates Yeh into the study of the geometric minima closure properties for the auto-correlated MZ(III)-AR(1) process.

Suppose that $A_1, A_2, \ldots,$ is a sequence in the MZ(III)-AR(1) process and that N is a geometric random variable with pmf

$$P(N = n) = \begin{cases} pq^{n-1}, & \text{if } n = 1, 2, \ldots \quad (q = 1-p) \\ 0, & \text{o.w.} \end{cases}. \tag{2.18}$$

Assuming that N is independent of the A_i's, let $g = \min\{A_1, A_2, \ldots, A_N\}$. Since N and A_i's are independent, so $P\{g \geq k \mid N = n\} = P\{m_n \geq k\}$, for any $\underline{k} \in E^m$, where m_n and its survival function is defined in Eq.(2.15).

Thus,

$$
\begin{aligned}
P(g \geq \underline{k}) &= \sum_{n=1}^{\infty} P\{g \geq \underline{k} \mid N=n\} P(N=n) \\
&= \left\{1+\sum_{j=1}^{m}\Big(\frac{k_j-\mu_j}{\sigma_j\left(\rho+\frac{1-\rho}{p}\right)^{-\gamma_j}}\Big)^{1/\gamma_j}\right\}^{-1},
\end{aligned}
$$

it is discerned that the geometric minima is also an m-variate $M^{(m)}$Zipf(III) random vector, i.e.,

$$
g \sim M^{(m)}\text{Zipf(III)}\Big(\underline{\mu}, \Big(\rho+\frac{1-\rho}{p}\Big)^{-\underline{\gamma}}\underline{\sigma}, \underline{\gamma}\Big), \tag{2.19}
$$

where the parameter vectors are respectively defined as $\underline{\mu}=(\mu_1, \mu_2, \ldots, \mu_m)$, $\underline{\gamma}=(\gamma_1, \gamma_2, \ldots, \gamma_m)$, and

$$
\Big(\rho+\frac{1-\rho}{p}\Big)^{-\underline{\gamma}}\underline{\sigma}=\left(\Big(\rho+\frac{1-\rho}{p}\Big)^{-\gamma_1}\sigma_1, \Big(\rho+\frac{1-\rho}{p}\Big)^{-\gamma_2}\sigma_2, \ldots, \Big(\rho+\frac{1-\rho}{p}\Big)^{-\gamma_m}\sigma_m\right).
$$

3. The Higher Order Multivariate Discrete Autoregressive Process MD-AR(p)

A direct way of extending the MD-AR(1) process in Eq.(2.1) is to consider the following representation

$$
A_n = V_n A_{n-D_n} + (I_m - V_n) Y_n, \tag{3.1}
$$

where $\{D_n\}$ is a sequence of i.i.d. univariate discrete random variables with pmf

$$
P(D_n = i) = \alpha_i,
$$

$1 \leq i \leq p$, $\sum_{i=1}^{p} \alpha_i = 1$, and $\{V_n\} \overset{i.i.d.}{\sim}$ m-dim Bernoulli(ρ) random matrices.

$$
V_n = \begin{cases} I_m, & w.p. \quad \rho, \\ \underline{0}_{m\times m}, & w.p. \quad 1-\rho, \end{cases}
$$

and $\{V_n\} \overset{i.i.d.}{\sim}$ any m-variate discrete random vector. Expanding out Eq.(3.1), then each A_n is expressed as

$$A_n = \begin{cases} A_{n-1}, & w.p. \quad \rho\alpha_1, \\ A_{n-2}, & w.p. \quad \rho\alpha_2, \\ \vdots & \qquad\quad \vdots \\ A_{n-p}, & w.p. \quad \rho\alpha_p, \\ Y_n, & w.p. \quad 1-\rho \end{cases} \tag{3.2}$$

for $n = 1, 2, \ldots$, model (3.1) or (3.2) is denoted as MD-AR(p). The MD-AR(p) process is clearly a Markov p^{th}-order model and it is analogous to the model proposed by Lo and Mackinlay (1990).

Property 3.1. *In Eq.*(3.2) (i) *If the p's initial random vectors* $\{A_0, A_{-1}, \ldots, A_{-p+1}\}$ *have identical marginal distributions as the i.i.d. m-variate discrete random vectors* $\{Y_n\}$, *then all the* A_n*'s,* $n = 1, 2, \ldots$, *in the MD-AR(p) process have the same marginal distribution as the* Y_n*'s have.* (ii) *If* $\{A_0, A_{-1}, \ldots, A_{-p+1}\}$ *have arbitrary distribution, then* A_n *converges in distribution to the same distribution as the* Y_n*'s distribution,* π.

The proof of Property 3.1 is similar to Property 2.1 and hence is omitted. In the following, we will focus on the stationary case, i.e. condition (i) of Property 3.1 holds.

3.1. The correlation structure of the MD-AR(p) process

For any two m-variate random vectors A_n and $A_{n-\ell}$ in the MD-AR(p) process, the autocovariance matrices between A_n and $A_{n-\ell}$ are defined as $\Gamma(\ell) = \mathrm{Cov}\,(A_n, A_{n-\ell})$, by Eq.(3.2), then

$$\Gamma(\ell) = \rho\alpha_1\Gamma(\ell-1) + \rho\alpha_2\Gamma(\ell-2) + \cdots + \rho\alpha_p\Gamma(\ell-p) + (1-\rho)\Sigma_{Y,A}(\ell), \tag{3.3}$$

where $\Sigma_{Y,A}(\ell) = \mathrm{Cov}\,(Y_n, A_{n-\ell})$, consider the following cases:

$$\text{(i)}\ \ \ell = 0,\ \text{then}\ \Sigma_{Y,A}(0) = (1-\rho)\Sigma_Y. \tag{3.4}$$

$$\text{(ii)}\ \ \ell > 0,\ \Sigma_{Y,A}(\ell) = \underline{0}_{m\times m}. \tag{3.5}$$

$$\text{(iii)}\ \ \ell < 0,\ \Sigma_{Y,A}(\ell) = \rho \sum_{j=1}^{\min\{p,-\ell\}} \alpha_j \Sigma_{Y,A}(\ell + j). \tag{3.6}$$

Refer to Eq.(3.3), consider the following cases: for

(i) $\ell = 1, 2, \ldots, p$, the autocovariance matrices $\{\Gamma(\ell)\}_1^p$ of the MD-AR(p) process satisfy the multivariate version of the Yule-Walker equation which is

$$\begin{cases} \Gamma(1) = \rho\alpha_1\Gamma(0) + \rho\alpha_2\Gamma(-1) + \cdots + \rho\alpha_p\Gamma(-(p-1)) \\ \Gamma(2) = \rho\alpha_1\Gamma(1) + \rho\alpha_2\Gamma(0) + \cdots + \rho\alpha_p\Gamma(-(p-2)) \\ \qquad\vdots \\ \Gamma(p) = \rho\alpha_1\Gamma(p-1) + \rho\alpha_2\Gamma(p-2) + \cdots + \rho\alpha_p\Gamma(0) \end{cases}, \tag{3.7}$$

where $\Gamma(0) = \Sigma_Y$, for

(ii) $\ell > p$, then $\Gamma(\ell)$ is obtained recursively as

$$\Gamma(\ell) = \rho\alpha_1\Gamma(\ell-1) + \rho\alpha_2\Gamma(\ell-2) + \cdots + \rho\alpha_p\Gamma(\ell-p), \tag{3.8}$$

(iii) $\ell = 0$,

$$\Gamma(0) = \rho\sum_{j=1}^{p} \alpha_j\Gamma(-j) + (1-\rho)^2\Sigma_Y, \tag{3.9}$$

(iv) $\ell < 0$,

$$\Gamma(\ell) = \rho\sum_{j=1}^{p} \alpha_j\Gamma(\ell-j) + (1-\rho)\rho \sum_{j=1}^{\min\{p,-\ell\}} \alpha_j\Sigma_{Y,A}(\ell+j). \tag{3.10}$$

From Eqs.(3.7) and (3.8), it is observed that the autocovariance matrices $\Gamma(\ell)$ for the positive lag, $\ell = 1, 2, \ldots, p, p+1, \ldots$ of the MD-AR(p) process satisfy the system of the multivariate Yule-Walker equations (Janacek and Swift (1993)). This property is similar to that of the multivariate Gaussian AR(p) process (Tiao and Box (1981)).

For any two m-variate random vectors A_n and $A_{n+\ell}$, $\ell = 1, 2, \ldots$, in the MD-AR(p) process, we are interested in studying the probability that A_n and $A_{n+\ell}$ will choose the same random vector Y_k, $1 \leq k \leq n$. Let $P_A(\ell)$ denote such probability in the following.

Property 3.2. *In the MD-AR(p) process, the probability, $P_A(\ell)$, for $\ell = 1, 2, \ldots, p, p+1, \ldots$, will satisfy the univariate Yule-Walker equations.*

4. The Multivariate Discrete Moving Average Process MD-MA(p)

The MD-MA(q) process is defined as

$$X_n = V_n Y_{n-E_n} + (I_m - V_n) Y_n, \tag{4.1}$$

where $\{E_n\}$ is a sequence of univariate i.i.d. discrete random variables with pmf $P(E_n = i) = \beta_i$, $i = 1, 2, \ldots, q$, $\sum_{i=1}^{q} \beta_i = 1$, and $\{V_n\} \overset{i.i.d.}{\sim}$ m-dim. Bernoulli(ρ) random matrices, $V_n = \begin{cases} I_m, & w.p. \quad \rho \\ \underline{0}_{m\times m}, & w.p. \quad 1-\rho \end{cases}$, and $\{Y_n\} \overset{i.i.d.}{\sim}$ any m-variate discrete random vector.

Eq.(4.1) can be written explicitly as

$$X_n = \begin{cases} Y_{n-1}, & w.p. \quad \rho\beta_1 \\ Y_{n-2}, & w.p. \quad \rho\beta_2 \\ \vdots & \vdots \quad\quad \vdots \\ Y_{n-q}, & \vdots \quad\quad \rho\beta_q \\ Y_n, & w.p. \quad 1-\rho \end{cases} \tag{4.2}$$

for $n = 1, 2, \ldots$.

For any $\underline{k} \in E^m$ the survival function of X_n is calculated directly by Eq.(4.1), which is

$$\overline{F}_{X_n}(\underline{k}) = \Big\{\rho\beta_1 + \rho\beta_2 + \cdots + \rho\beta_q + (1-\rho)\Big\} P(Y_n \geq \underline{k}) = \overline{F}_{Y_n}(\underline{k}), \tag{4.3}$$

hence all the X_n's in the MD-MA(q) process have the same marginal m-variate discrete distribution as the Y_n's have.

4.1. The correlation structure of the MD-MA(q) process

For any two m-variate random vectors X_n and $X_{n+\ell}$ in the MD-MA(q) process, the autocovariance matrices between X_n and $X_{n+\ell}$ are defined as $\Gamma(\ell) = \mathrm{Cov}\,(X_n, X_{n+\ell})$, for any $\ell = 0, \pm 1, \pm 2, \ldots$, consider the following cases: for

(i) $\ell = 0$, $\quad \Gamma(0) = \mathrm{Cov}\,(X_n, X_n) \overset{\triangle}{=} \Sigma_X \overset{\triangle}{=} \Sigma_Y$,

(ii) $1 \leq \ell \leq q-1$, the autocovariance matrix of lag ℓ can be similarly calculated as

$$\Gamma(\ell) = \Gamma(-\ell) = \Big\{(1-\rho)\rho\beta_\ell + \rho^2 \sum_{i=1}^{q-\ell} \beta_i \beta_{i+\ell}\Big\}\Sigma_Y. \tag{4.4}$$

(iii) $\ell = q$

$$\Gamma(q) = (1-\rho)\rho\beta_q \Sigma_Y \tag{4.5}$$

(iv) $\ell > q$,

$$\Gamma(\ell) = \underline{0}_{m\times m} = \Gamma(-\ell). \tag{4.6}$$

Thus, the correlation structure of the MD-MA(q) process has a cut-off pattern after lag q, and by the relation $\Gamma(\ell) = \Gamma(-\ell)$ for any $\ell = 0, \pm 1, \pm 2, \ldots$, the MD-MA($q$) process is time reversible. It is easily observed from Eq.(4.2) and the structure of the autocovariance matrices $\{\Gamma(\ell)\}$ that the MD-MA(q) process is a q-dependent stationary multivariate time series model.

4.2. The multivariate Zipf moving average process MD-MA(q)

In this subsection, the MD-MA(1) process with the multivariate Zipf (I), (II), (III), (IV) as the marginal distribution is considered and it is called the MZ-MA(1) process. Unless otherwise indicated, we will restrict our attention to the MZ-MA(1) process in the remainder of this subsection; that is in Eq.(4.1), $E_n \equiv 1$ with probability 1, i.e., $P(E_n = 1) = 1$, then

$$X_n = \begin{cases} Y_n, & w.p. \quad 1-\rho \\ Y_{n-1}, & w.p. \quad \rho \end{cases}, \tag{4.7}$$

and $\{Y_n\} \overset{i.i.d.}{\sim}$ the m-variate Zipf (I), (II), (III), (IV) distributions which are introduced in Subsection 2.3. Note that, unlike the MZ-AR(1) process, the MZ-MA(1) process is not a Markov chain but is a one-dependent stationary process. From Eq.(2.7), it suffices to study the MZ(IV)-MA(1) process. Suppose the innovation process $\{Y_n\}$ are i.i.d. multivariate Zipf(IV) random vectors, then for each $n = 1, 2, \ldots$, X_n in the MZ-MA(1) process is identically distributed as the Y_n's. For any m-variate state vector $\underline{i} = (i_1, i_2, \ldots, i_m) \in E^m = \{\underline{k} = (k_1, k_2, \ldots, k_m) \mid k_i \geq \mu_i,\ k_i \text{ is integer},\ 1 \leq i \leq m\}$, the survival function of X_n is

$$\overline{F}_{X_n}(\underline{i}) = \Big\{1 + \sum_{\ell=1}^{m} \Big(\frac{i_\ell - \mu_\ell}{\sigma_\ell}\Big)^{1/\gamma_\ell}\Big\}^{-\alpha}, \tag{4.8}$$

Thus for all $n = 1, 2, \ldots$, X_n's are marginally distributed as the Y_n's which are multivariate Zipf(IV) random vectors. The correlation structure of the MZ-MA(1) process is followed from Section 4.1, which is

$$\begin{cases} \Gamma(0) = \Sigma_X = \Sigma_Y, \\ \Gamma(1) = (1-\rho)\rho\Sigma_Y = \Gamma(-1), \\ \Gamma(\ell) = \underline{0}_{m\times m} = \Gamma(-\ell), \text{ for } \ell \geq 2. \end{cases} \tag{4.9}$$

Therefore, the MZ-MA(1) process is a one-dependent stationary process and its autocorrelation structure has a cut-off pattern at lag one.

4.3. The joint distributions and time reversibility of the MD-MA(q) process

Let X_n and X_{n+1} be any two adjacent m-variate random vectors in the MZ-MA(1) process, then for any two state vectors $\underline{k}^1, \underline{k}^2 \in E^m$, the joint survival function of X_n and X_{n+1} is derived from Eq.(4.7) as

$$\begin{aligned}\overline{F}_{X_n,X_{n+1}}(\underline{k}^1, \underline{k}^2) \;=\; & \Big\{1 + \sum_{i=1}^{m}\Big(\frac{k_i^1 - \mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big\}^{-\alpha}\Big\{1 + \sum_{i=1}^{m}\Big(\frac{k_i^2 - \mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big\}^{-\alpha} \\ & \times\Big\{(1-\rho)^2 + \rho(1-\rho) + \rho^2\Big\} \\ & + \Big\{1 + \sum_{i=1}^{m}\Big(\frac{\max(k_i^1, k_i^2) - \mu_i}{\sigma_i}\Big)^{1/\gamma_i}\Big\}^{-\alpha}\rho(1-\rho). \end{aligned} \tag{4.10}$$

Clearly, this function is symmetric in $\underline{k}^1$ and $\underline{k}^2$, and hence the MZ-MA(1) process is time reversible. The time reversibility of the MZ-MA(1) process is also discerned from Eq.(4.9).

4.4. The distribution of the runs in the MZ-MA(1) process

Let $\{X_n\}$ be a sequence in the MZ(IV)-MA(1) process, i.e., X_n's satisfy Eq.(4.7). Fix an m-variate state vector $\underline{i} = (i_1, i_2, \ldots, i_m) \in E^m$, then the length of a run of $\underline{i}$ starting at time epoch one for $\{X_n\}$ in the MZ(IV).MA(1) process is defined as $T_{\underline{i}} = \inf\{n \geq 1 \mid X_n \neq \underline{i}\} - 1$. For any integer $n = 1, 2, \ldots$, the survival function of $T_{\underline{i}}$, $P(T_{\underline{i}} \geq n)$ is calculated as follows: let $a_n = P(T_{\underline{i}} \geq n)$, and $\pi(\cdot)$ be the marginal pmf of each X_n which is Eq.(2.10),

$$\pi(\underline{i}) = \Big\{1 + \sum_{\ell=1}^{m} \Big(\frac{i_\ell - \mu_\ell}{\sigma_\ell}\Big)^{1/\gamma_\ell}\Big\}^{-\alpha} - \Big\{1 + \sum_{\ell=1}^{m} \Big(\frac{i_\ell + 1 - \mu_\ell}{\sigma_\ell}\Big)^{1/\gamma_\ell}\Big\}^{-\alpha}.$$

From the definition of a_n, it is discerned that $a_0 = P(T_{\underline{i}} \geq 0) \equiv 1$, and

$$a_1 = P(T_{\underline{i}} \geq 1) = P(X_1 = \underline{i}) = \pi(\underline{i}) \tag{4.11}$$

By mathematical induction, the recursive relation among the survival probabilities $\{a_n\}_{n\geq 1}$ is derived as follows:

$$\begin{aligned} a_{n+1} &= \rho\pi(\underline{i})a_n + \rho(1-\rho)\pi(\underline{i})a_{n-1} + \rho(1-\rho)^2\pi^2(\underline{i})a_{n-2} + \cdots \\ &\quad + \rho(1-\rho)^n\pi^n(\underline{i})a_0 + (1-\rho)^{n+1}\pi^{n+1}(\underline{i}), \end{aligned} \tag{4.12}$$

for each $n = 1, 2, \ldots$. Therefore, the marginal pmf of the length of a run of $\underline{i}$ is $P(T_{\underline{i}} = n) = a_n - a_{n+1}$. The expected run length for the MZ(IV)-MA(1) process is calculated from Eq. (4.12). For each $n = 1, 2, \ldots$, summing Eq.(4.12) on both sides, and let $T_n = \sum_{\ell=1}^{n} a_\ell$, then the above equation is

$$\begin{aligned} T_{n+1} &= \pi(\underline{i}) + \rho\pi(\underline{i})T_n + \rho\sum_{j=1}^{n-1}(1-\rho)^j\pi^j(\underline{i})T_{n-j} + \rho\Big\{\sum_{j=1}^{n}(1-\rho)^j\pi^j(\underline{i})\Big\}a_0 \\ &\quad + \sum_{j=2}^{n+1}(1-\rho)^j\pi^j(\underline{i}). \end{aligned} \tag{4.13}$$

Because the run length of $\underline{i}$, $T_{\underline{i}}$ is a nonnegative discrete random variable, thus $E(T_{\underline{i}}) = \sum_{n=1}^{\infty} P(T_{\underline{i}} \geq n) = \sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} T_n$, and by letting $n \to \infty$ on both sides in Eq.(4.13), then $E(T_{\underline{i}})$ is solved as

$$E(T_{\underline{i}}) = \frac{\pi(\underline{i})\{1+\rho(1-\rho)(1-\pi(\underline{i}))\}}{(1-\pi(\underline{i}))\{1-\rho(1-\rho)\pi(\underline{i})\}} \triangleq \frac{P(\underline{i})}{1-P(\underline{i})}, \tag{4.14}$$

where $P(\underline{i}) = \pi(\underline{i}) + \rho(1-\rho)\pi(\underline{i})(1-\pi(\underline{i}))$.

It is found that $P(\underline{i}) \geq \pi(\underline{i})$ for any $0 \leq \rho \leq 1$ and $E(T_{\underline{i}}) \geq \frac{\pi(\underline{i})}{1-\pi(\underline{i})}$.

If $\rho = 0$ or $\rho = 1$, then according to Eq.(4.14), $\{X_n\}$ is reduced to a sequence of i.i.d. random vectors $\{Y_n\}$ with $E(T_{\underline{i}}) = \frac{\pi(\underline{i})}{1-\pi(\underline{i})}$ and in general, for $0 < \rho < 1$, $E(T_{\underline{i}}) \geq \frac{\pi(\underline{i})}{1-\pi(\underline{i})}$. Therefore, in the MZ-MA(1) process, the expected length of a run of $\underline{i}$ is greater than the expected run length of an i.i.d. sequence of random vectors. This property is analogous to that of the MZ-AR(1) process which is Property 2.2 in Subsection 2.5.

4.5. Exact distribution of the extremes in the MZ-MA(1) process

Give a sequence of random variables $X_1, X_2, \ldots, X_n$ from the MZ-MA(1) process, $n \in \{1, 2, \ldots\}$ define $m_n = \min\{X_1, X_2, \ldots, X_n\}$ as the coordinate-wise minima of $\{X_i\}_1^n$. The exact survival function of m_n is derived from the joint survival function of m_n and Y_n.

Let $\psi_n(,)$ be the joint survival function of m_n and Y_n, i.e. for any $\underline{k}^1$, $\underline{k}^2 \in E^m$, $\psi_n(\underline{k}^1, \underline{k}^2) = P(m_n \geq \underline{k}^1, Y_n \geq \underline{k}^2)$, by Eq.(4.7) m_n can be expressed as $m_n = \begin{cases} \min\{m_{n-1}, Y_n\}, & w.p. \quad (1-\rho) \\ \min\{m_{n-1}, Y_{n-1}\}, & w.p. \quad \rho \end{cases}$, then

$$\psi_n(\underline{k}^1, \underline{k}^2) = \psi_{n-1}(\underline{k}^1, \underline{0})\overline{F}_Y(\max(\underline{k}^1, \underline{k}^2))(1-\rho) + \psi_{n-1}(\underline{k}^1, \underline{k}^1)\overline{F}_Y(\underline{k}^2)\rho, \tag{4.15}$$

where $\psi_{n-1}(\underline{k}^1, \underline{k}^1)$ is obtained from Eq.(4.15) as

$$\psi_{n-1}(\underline{k}^1, \underline{k}^1) = \psi_{n-2}(\underline{k}^1, \underline{0})\overline{F}_Y(\underline{k}^1)(1-\rho) + \psi_{n-2}(\underline{k}^1, \underline{k}^1)\overline{F}_Y(\underline{k}^1)\rho, \tag{4.16}$$

and therefore, the exact survival function of m_n can be solved recursively from the following Eq.(4.17)

$$\overline{F}_{m_n}(\underline{k}^1) = \psi_n(\underline{k}^1, \underline{0}) = \psi_{n-1}(\underline{k}^1, \underline{0})\overline{F}_Y(\underline{k}^1)(1-\rho) + \psi_{n-1}(\underline{k}^1, \underline{k}^2)\rho, \quad (4.17)$$

for all $n \in \{2, 3, \ldots\}$ with initial conditions $\psi_1(\underline{k}^1, \underline{0}) = P(X_1 \geq \underline{k}^1) = \overline{F}_Y(\underline{k}^1)$, $\psi_0(\underline{k}^1, \underline{0}) = 1$ and $\psi_0(\underline{k}^1, \underline{k}^1) = \overline{F}_{\underline{Y}}(\underline{k}^1)$ and the exact survival functions of m_2,

$$\psi_2(\underline{k}^1, \underline{0}) = (\overline{F}_Y(\underline{k}^1))^2(1-\rho) + \overline{F}_Y(\underline{k}^1)(1-\rho)\rho + (\overline{F}_Y(\underline{k}^1))^2\rho^2.$$

A similar analysis is for the sample maxima $M_n = \max\{X_1, X_2, \ldots, X_n\}$. The exact cdf of M_n is derived from the joint cdf of M_n and Y_n. Let $\varphi_n(,)$ be the joint cdf of M_n and Y_n. By Eq.(4.7), M_n can be expressed as $M_n = \begin{cases} \min\{M_{n-1}, Y_n\}, & w.p. \quad (1-\rho) \\ \min\{M_{n-1}, Y_{n-1}\}, & w.p. \quad \rho \end{cases}$, then

$$\varphi_n(\underline{k}^1, \underline{k}^2) = \varphi_{n-1}(\underline{k}^1, \infty)F_Y(\min(\underline{k}^1, \underline{k}^2))(1-\rho) + \varphi_{n-1}(\underline{k}^1, \underline{k}^1)F_Y(\underline{k}^2)\rho, \quad (4.18)$$

The exact cdf of M_n is obtained by setting $\underline{k}^2 \to \infty$ coordinatewisely, then

$$F_{M_n}(\underline{k}^1) = \varphi_n(\underline{k}^1, \infty) = \varphi_{n-1}(\underline{k}^1, \infty)F_Y(\underline{k}^1)(1-\rho) + \varphi_{n-1}(\underline{k}^1, \underline{k}^1)\rho, \quad (4.19)$$

for all $n = 2, 3, \ldots$ with initial conditions $\varphi_1(\underline{k}^1, \infty) = F_Y(\underline{k}^1)$, and

$$\varphi_2(\underline{k}^1, \infty) = (F_Y(\underline{k}^1))^2(1-\rho) + F_Y(\underline{k}^1)(1-\rho)\rho + (F_Y(\underline{k}^1))^2\rho.$$

As for the geometric extremes, the MZ-MA(1) process is not closed under the geometric minima. This property is different from that of MZ-AR(1) process. The limiting distributions of m_n and M_n in the MZ-MA(1) process as $n \to \infty$ are not easy to derive from Eqs.(4.17) and (4.19) respectively.

5. The Multivariate Discrete Mixed Autoregressive Moving-Average Process MD-AR(p, q)

The mixed process MD-ARMA(p, q) model is constructed by coupling the two processes, MD-AR(p) and MD-MA(q), by means of a common m-

variate discrete innovation process, $\{Y_n\}$. The mixed MD-ARMA(p,q) process is generated by the probabilistic model

$$A_n = V_n A_{n-D_n} + (I_m - V_n) Y_{n-F_n} \tag{5.1}$$

where the sequences $\{V_n\}$, $\{Y_n\}$, and $\{D_n\}$ are defined as in the MD-AR(p) process Eq.(3.2), and $\{F_n\}$ is a sequence of i.i.d. univariate discrete random variables with pmf $P(F_n = i) = \beta_i$, $i = 0, 1, 2, \ldots, q$, $\sum_{i=0}^{q} \beta_i = 1$. Eq.(5.1) can be written explicitly as

$$A_n = \begin{cases} A_{n-1}, & w.p. \quad \rho\alpha_1, \\ A_{n-2}, & w.p. \quad \rho\alpha_2, \\ \vdots & \vdots \qquad \vdots \\ A_{n-p}, & \vdots \qquad \rho\alpha_p, \\ Y_n, & w.p. \quad (1-\rho)\beta_0, \\ Y_{n-1}, & w.p. \quad (1-\rho)\beta_1, \\ \vdots & \vdots \qquad \vdots \\ Y_{n-q}, & \vdots \qquad (1-\rho)\beta_q. \end{cases} \tag{5.2}$$

It is discerned that if $\rho = 0$ in Eq.(5.2), then the MD-ARMA(p,q) process is reduced to the MD-MA(q) process and if all the β_i's, $i = 1, 2, \ldots, q$, $\beta_i = 0$, while $\beta_0 = 1$, i.e., $P(F_n = 0) = 1$, then the MD-ARMA(p,q) process yields the MD-AR(p) process.

Analogous to Property 2.1 and 3.1, we will consider only the case of the random vectors $\{A_0, A_{-1}, \ldots, A_{-(p-1)}\}$ in the MD-ARMA(p,q) process identically distributed as the marginal distribution of the innovation process $\{Y_i\}$ for all $i \in \{-(q-1), \ldots, -1, 0, 1, \ldots\}$.

5.1. The correlation structure of the MD-ARMA(p,q) process

Given any lag ℓ, $\ell \in \{0, \pm 1, \pm 2, \ldots\}$, for any two m-variate random vectors A_n and $A_{n-\ell}$ in the MD-ARMA(p,q) process, the autocovariance matrices between A_n and $A_{n-\ell}$ are defined as $\Gamma(\ell) = \mathrm{Cov}\,(A_n, A_{n-\ell})$ and let

$\Sigma_{Y,A}(\ell - j) = \text{Cov}\,(Y_{n-j}, A_{n-\ell})$, for any $j \in \{0, 1, \ldots, q\}$, then

$$\Gamma(\ell) = \rho \sum_{j=1}^{p} \alpha_j \Gamma(\ell - j) + (1 - \rho) \sum_{j=0}^{q} \beta_j \Sigma_{Y,A}(\ell - j). \tag{5.3}$$

Consider the following cases: for

(i) $\ell = 0$, then

$$\Sigma_{Y,A}(0) = (1 - \rho)\beta_0 \Sigma_Y, \tag{5.4}$$

In general, for any ℓ assuming $p \geq q$,

(ii) $-q \leq \ell \leq -1$,

$$\Sigma_{Y,A}(\ell) = \rho \sum_{j=1}^{-\ell} \alpha_j \Sigma_{Y,A}(\ell + j) + (1 - \rho)\beta_{-\ell} \Sigma_Y. \tag{5.5}$$

(iii) $\ell < -q$, then $\Sigma_{Y,A}(\ell) = \rho \sum_{j=1}^{\min\{p,-\ell\}} \alpha_j \Sigma_{Y,A}(\ell + j)$,

(iv) $-q \leq \ell \leq -1$ and given $p < q$, then

$$\Sigma_{Y,A}(\ell) = \rho \sum_{j=1}^{\min\{-\ell,p\}} \alpha_j \Sigma_{Y,A}(\ell + j) + (1 - \rho)\beta_{-\ell} \Sigma_Y. \tag{5.6}$$

On the other hand, if $p < q$, then for any ℓ,

(v) $\ell \leq -q$,

$$\Sigma_{Y,A}(\ell) = \rho \sum_{j=1}^{p} \alpha_j \Sigma_{Y,A}(\ell + j), \tag{5.7}$$

(vi) $\ell > 0$,

$$\Sigma_{Y,A}(\ell) = \underline{0}_{m \times m}. \tag{5.8}$$

Refer to Eq.(5.4), consider the following cases:

(1) $\ell > q$, the autocovariance of the MD-ARMA(p, q) process is reduced to

$$\Gamma(\ell) = \rho\alpha_1 \Gamma(\ell - 1) + \rho\alpha_2 \Gamma(\ell - 2) + \cdots + \rho\alpha_p \Gamma(\ell - p). \tag{5.9}$$

This relation is the same as the Eqs.(3.7), (3.8).

(2) $1 \leq \ell \leq q$, the autocovariance of the MD-ARMA(p, q) process is

$$\Gamma(\ell) = \rho \sum_{j=1}^{p} \alpha_j \Gamma(\ell - j) + (1 - \rho) \sum_{j=0}^{q-\ell} \beta_{\ell+j} \Sigma_{Y,A}(-j). \tag{5.10}$$

From Eqs.(5.9) and (5.10), it is found that the autocovariance structure of the MZ-ARMA(p, q) process is analogous to that of the univariate continuous linear normal ARMA(p, q) process, i.e. after lag q. The set of the autocovariance matrices $\{\Gamma(\ell)\}_{\ell>q}$ of the MD-ARMA(p, q) process satisfies the multivariate version of the Yule-Walker equations which is the structure of the pure MD-AR(p) process.

6. Conclusions and Summary

Several models MD-AR(1), MD-AR(p), MD-MA(q), and MD-ARMA (p, q) for multivariate integer-valued stationary stochastic processes with any multivariate discrete random vector as marginal distribution are presented in this paper. These models are the multivariate extension of the univariate discrete time series model proposed by Jacobs and Lewis (1978). The correlation structure of each model is derived detailedly in this article. Yeh (2002) developed four generalized multivariate Zipf(I), (II), (III), (IV) distributions. Utilizing these four multivariate Zipf random vectors as the marginal distributions in the multivariate discrete time series models and thus the multivariate Zipf processes MZ-AR(1), MZ-AR(p), MZ-MA(q), MZ-ARMA(p, q) are constructed. Some distributional properties of the two submodels, MZ-AR(1) and MZ-MA(1) are studied thoroughly.

In summary of this paper, it is found that

(1) All the MD-AR(p), MD-MA(q), and MD-ARMA(p, q) processes are suitable for any multivariate discrete random vectors.

(2) The autocovariance matrices of the MD-AR(p) process satisfy the multivariate Yule-Walker equations. The autocovariance matrices of the MD-MA(q) process have a cut-off pattern after lag q. As for the mixed model MD-ARMA(p, q) process, the autocovariance matrices of lags after the order q also satisfy the multivariate Yule-Walker equations. All

these characteristics of the correlation structures in the multivariate discrete time series models are analogous to that of the linear multivariate Gaussian processes (Tiao and Box (1981)).

(3) All the multivariate Zipf processes, MZ-AR(1), MZ-AR(p), MZ-MA(q), MZ-ARMA(p, q) processes give a common generalized multivariate Zipf(I), (II), (III), (IV) marginal distributions under the stationary condition.

(4) The multivariate Zipf processes MZ-AR(1), MZ-MA(1), are time-reversible, and their expected run lengths are always greater than or equal to the expected run length for an i.i.d. sequence of the multivariate Zipf random vectors.

(5) The MZ-AR(1) process with multivariate Zipf(III) as marginal distribution is closed under geometric minima.

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APPENDIX

A.1. Proof of Property 2.1

Proof. For fixed $\underline{k} = (k_1, \ldots, k_m)$, let $a_n = \overline{F}_{A_n}(\underline{k})$, $b = \overline{F}_{Y_n}(\underline{k})$.

By model (2.1), an inductive argument can be represented as

$$a_n = \rho a_{n-1} + (1-\rho)b \tag{A.1}$$

If $A_0 \overset{d}{=} Y_n$, i.e. $a_0 = b$, then $a_1 = \rho b + (1-\rho)b = b$, by induction. Then all $a_n = b$, so $A_n \overset{d}{=} Y_n$ for any $n = 1, 2, \ldots$ and hence (i) is followed.

If A_0 has an arbitrary distribution, then by recursively using Eq(A.1) to yield

$$\begin{aligned} a_n &= \rho^n a_0 + (1-\rho)b\Big\{1 + \rho + \rho^2 + \cdots + \rho^{n-1}\Big\} = \rho^n a_0 + (1-\rho)b\Big\{\frac{1-\rho^n}{1-\rho}\Big\} \\ &= \rho^n a_0 + b(1-\rho^n), \end{aligned}$$

since $0 < \rho < 1$ so as $n \to \infty$ $\lim_{n\to\infty} a_n = b$ i.e., ${}^d A_n \to \pi$ and hence (ii) is followed.

A.2. Proof of Property 2.2

Proof. From Eq.(2.13), the expected run length

$$E(T_{\underline{i}}) = \sum_{\ell=1}^{\infty} P(T_{\underline{i}} \geq \ell) = \frac{\pi(\underline{i})}{(1-\rho)(1-\pi(\underline{i}))}. \tag{A.2}$$

Clearly, $E(T_{\underline{i}}) \geq \frac{\pi(\underline{i})}{1-\pi(\underline{i})}$, for $0 \leq \rho \leq 1$.

Consider two particular cases:

(i) $\rho = 0$, i.e., $V_n = \underline{0}_{m\times m}$ w.p. 1, then $A_n = Y_n$ for all $n = 1, 2, \ldots$, so the MZ(IV)-AR(1) process is reduced to the sequence of i.i.d. multivariate Zipf(IV) random vectors and in this case $E(T_{\underline{i}}) \geq \frac{\pi(\underline{i})}{1-\pi(\underline{i})}$.

(ii) If $\rho = 1$, i.e., $V_n = I_m$ w.p. 1, then $A_n = A_{n+1}$ for all $n = 1, 2, \ldots$, hence once $A_1 = \underline{i}$, then $A_1 = A_2 = A_3 = \cdots = \underline{i}$, so $E(T_{\underline{i}}) = \infty$.

A.3. Proof of Property 3.2

Proof. Let R_n be the random index of the Y_k, $1 \leq k \leq n$, that A_n chooses, i.e., $A_n = Y_{R_n}$. Referring to Eq.(3.2), know that A_n can be any one of the $\{Y_n, Y_{n-1}, \ldots, Y_1\}$. The possible value of R_n is $\{1, 2, \ldots, n\}$. Similar definition for the random variable $R_{n-\ell}$, it is the random index of the Y_k, $1 \leq k \leq n-\ell$, that $A_{n-\ell}$ chooses, i.e., $A_{n-\ell} = Y_{R_{n-\ell}}$. The possible value of $R_{n-\ell}$ is $\{1, 2, \ldots, n-\ell\}$. Clearly, the random variable R_n (or $R_{n-\ell}$) is independent of the $\{Y_k\}$. By definition, then

$$\begin{aligned} E(A_n A'_{n-\ell}) &= E(Y_{R_n} Y'_{R_{n-\ell}}) \\ &= \sum_{k=1}^{n-\ell} E(Y_k Y'_K) P(R_n = R_{n-\ell} = k) \\ &\quad + \sum_{k=1}^{n} \sum_{\substack{j=1 \\ (j\neq k)}}^{n-\ell} E(Y_k Y'_j) P(R_n = k, R_{n-\ell} = j). \end{aligned} \tag{A.3}$$

Note that all the Y_n's are i.i.d. m-variate discrete random vectors, hence the first term above becomes

$$E(Y_1Y_1')\sum_{k=1}^{n-\ell}P(R_n=R_{n-\ell}=k)=E(Y_1Y_1')\cdot P(R_n=R_{n-\ell}). \tag{A.4}$$

The second term becomes

$$\begin{aligned}&E(Y_1)(E(Y_2))'\sum_{k=1}^{n}\sum_{\substack{j=1\\(j\neq k)}}^{n-\ell}P(R_n=k,R_{n-\ell}=j)\\ &= E(Y_1)(E(Y_2))'P(R_n\neq R_{n-\ell})=E(Y_1)(E(Y_1))'\{1-P(R_n=R_{n-\ell})\}.\end{aligned} \tag{A.5}$$

Then

$$\begin{aligned}&E(A_n,A_{n-\ell}')\\ &= E(Y_1Y_1')P(R_n=R_{n-\ell})+E(Y_1)(E(Y_1))'\{1-P(R_n=R_{n-\ell})\}\\ &= \{E(Y_1,Y_1')-E(Y_1)(E(Y_1))'\}P(R_n=R_{n-\ell})+E(Y_1)(E(Y_1))'.\end{aligned} \tag{A.6}$$

From Property 3.1, conclude that the random vectors A_n's in the MD-AR(p) process are identically distributed as the Y_n's, so $E(Y_1Y_1')$ $-E(Y_1)(E(Y_1))'=\mathrm{Var}-\mathrm{Cov}\,(Y_1)=\mathrm{Var}-\mathrm{Cov}\,(A_n)=\mathrm{Var}-\mathrm{Cov}\,(A_{n-\ell})=$ $\Gamma(0)$ and $E(Y_1)(E(Y_1))'=E(A_n)(E(A_{n-\ell}))'$, refer to Eq.(A.6)

$$\begin{aligned}\Gamma(\ell)=\mathrm{Cov}\,(A_n,A_{n-\ell}) &= E(A_nA_{n-\ell}')-E(A_n)(E(A_{n-\ell}))'\\ &= \Gamma(0)P(R_n=R_{n-\ell})=\Gamma(0)P_A(\ell).\end{aligned} \tag{A.7}$$

It is discerned that Eq.(A.7) is true for any positive and negative lags, i.e., $\ell\in\{\pm1,\pm2,\ldots\}$ and $P_A(0)=1$. Plugging this relation $\Gamma(\ell)=\Gamma(0)P_A(\ell)$ into Eqs.(3.7) and (3.8), cancelling out the matrix $\Gamma(0)$ on both sides, then the univariate Yule-Walker equation for the probabilities $\{P_A(\ell)\}_1^\infty$ is obtained. It is $P_A(\ell)=\rho\sum_{j=1}^{p}\alpha_jP_A(\ell-j)$, for any $\ell\in\{1,2,\ldots\}$.

References

1. B. C. Arnold and L. Laguna, (1977). On generalized Pareto distributions with applications to income data, International Studies in Economics, Monograph. No. 10, Iowa State University, Ames, Iowa.

2. J. D. Hamilton, A new approach to the economic analysis of nonstionary time series and business cycle, *Econometrica*, **57**(1989), 357-384.

3. P. A. Jacobs and P. A. W. Lewis, Discrete time series generated by mixtures. I: correlational and runs properties, *J. Roy. Statist. Soc. Ser. B.*, **40**(1978), 94-105.

4. P. A. Jacobs and P. A. W. Lewis, Stationary discrete autoregressive-moving average time series generated by mixtures, *J. Time Ser. Anal.*, **4**(1983), No.1, 19-36.

5. G. Janacek and L. Swift, *Time Series, Forecasting, Simulation, Applications*, Ellis Horwood, 1993.

6. N. L. Johnson, S. Kotz and N. Balakrishnan, *Discrete Multivariate Distributions*, Wiley, New York, 1997.

7. A. Lo and A. C. Mackinlay, An econometric analysis of nonsynchronous trading, *J. Econometrics*, **45**, 181-212, 1990.

8. G. C. Tiao and G. E. P. Box, Modeling multiple time series with applications, *J. Amer. Statist. Assoc.*, **76**(1981), 802-816.

9. R. S. Tsay, *Analysis of Financial Time Series*, Wiley Series in Probability and Statistic, 2005.

10. H. C. Yeh, One discrete time series model for fat-tailed integer random variables: Zipf process, *Bull. Inst. Math. Acad. Sinica*, **18**(1990), No.1, 19-33.

11. H. C. Yeh, Six multivariate Zipf distributions and their related properties, *Statist. Probab. Lett.*, **56**, 131-141, 2002.

12. G. U. Yule, A mathematical theory of evolution based on the conclusions of Dr. J. C. Willis, *F.R.S. Philos., Trans. B.* **213**(1924), 21-87.

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