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TAUTOLOGICAL EQUATIONS IN GENUS 2 VIA INVARIANCE CONSTRAINTS

BY

D. ARCARA AND Y.-P. LEE

Abstract

The main purpose of this work is to study genus two tautological equations. We verify the *Invariance Conjectures* of tautological equations [7] in genus two. In particular, a uniform derivation of all known genus two equations is given.

0. Introduction

The purpose of this paper is to verify the genus two case of *Invariance Conjectures* of tautological equations proposed in [7]. In particular, applying Theorem 5 in [8] (i.e. Conjecture 1 in [7]) and E. Getzler's Hodge numbers calculations [4], we are able to give a uniform derivation of all known genus two tautological equations: Mumford–Getzler's equation, Getzler's equation [4] and Belorousski–Pandharipande's equation [2]. This, combined with [5] in genus one and [1] in genus three, shows that this method generates and proves all known tautological equations.

0.1. Tautological rings

One reference for tautological rings, which is close to the spirit of the present paper, is R. Vakil's survey article [10].

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Let $\mathcal{M}_{g,n}$ be the moduli stacks of n-pointed smooth genus g curves. They have modular compactifications $\overline{\mathcal{M}}_{g,n}$, the moduli stacks of stable curves, introduced by P. Deligne, D. Mumford and F. Knudsen. $\overline{\mathcal{M}}_{g,n}$ are proper, irreducible, smooth Deligne–Mumford stacks. The Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ over $\mathbb{Q}$ are isomorphic to the Chow rings of their coarse moduli spaces. The tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ are subrings of $A^*(\overline{\mathcal{M}}_{g,n})$, or subrings of $H^{2*}(\overline{\mathcal{M}}_{g,n})$ via cycle maps, generated by some "geometric classes" which will be described below.

The first type of geometric classes are the *boundary strata*. $\overline{\mathcal{M}}_{g,n}$ have natural stratification by topological types. The second type of geometric classes are the Chern classes of tautological vector bundles. These includes cotangent classes ψ_i, Hodge classes λ_k and κ-classes κ_l.

To give a precise definition of the tautological rings, some natural morphisms between moduli stacks of curves will be used. The *forgetful morphisms*

$$\mathrm{ft}_i : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n} \tag{1}$$

forget one of the $n+1$ marked points. The *gluing morphisms*

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}, \quad \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}, \tag{2}$$

glue two marked points to form a curve with a new node. Note that the boundary strata are the images (of the repeated applications) of the gluing morphisms, up to factors in $\mathbb{Q}$ due to automorphisms.

Definition 1. The system of tautological rings $\{R^*(\overline{\mathcal{M}}_{g,n})\}_{g,n}$ is the smallest system of $\mathbb{Q}$-unital subalgebra (containing classes of type one and two, and is) closed under push-forwards via the forgetful and gluing morphisms.

The study of the tautological rings is one of the central problems in moduli of curves. The readers are referred to [10] and references therein for many examples and motivation. Note that the tautological rings are defined by generators and relations. Since the generators are explicitly given, *the study of tautological rings is the study of relations of tautological classes.*

0.2. Invariance conjectures

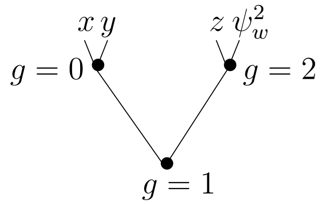
Here some ingredients in [7] and [8] will be briefly reviewed.

The strata can be conveniently presented by their (dual) graphs, which can be described as follows. To each stable curve C with marked points, one can associate a dual graph Γ. Vertices of Γ correspond to irreducible components. They are labeled by their geometric genus. Assign an edge joining two vertices each time the two components intersect. To each marked point, one draws an half-edge incident to the vertex, with the same label as the point. Now, the stratum corresponding to Γ is the closure of the subset of all stable curves in $\overline{\mathcal{M}}_{g,n}$ which have the same topological type as C. For each dual graph Γ, one can decorate the graph by assigning a monomial, or more generally a polynomial, of ψ to each half-edge and κ classes to each vertex. The tautological classes in $R^k(\overline{\mathcal{M}}_{g,n})$ can be represented by $\mathbb{Q}$-linear combinations of *decorated graphs. Since there is no κ, λ-classes involved in this paper, they will be left out of discussions below.*

For typesetting reasons, it is more convenient to denote a decorated graph by another notation, inspired by Gromov–Witten theory, called *gwi*. Given a decorated graph Γ.

- For the vertices of Γ of genus $g_1, g_2, \ldots$, assign a product of "brackets" $\langle\rangle_{g_1}\langle\rangle_{g_2}\ldots$. To simplify the notations, $\langle\rangle := \langle\rangle_0$.
- Assign each half-edge a symbol ∂^*. The external half-edges use super-indices $\partial^x, \partial^y, \ldots$, corresponding to their labeling. For each pair of half-edges coming from one and the same edge, the same super-index will by used, denoted by Greek letters $(\mu, \nu, \ldots.)$ Otherwise, all half-edges should use different super-indices.
- For each decoration to a half-edge a by ψ-classes ψ^k, assign a subindex to the corresponding half-edge ∂^a_k.
- For each a given vertex $\langle\rangle_g$ with m half-edges, n external half-edges, an insertion is placed in the vertex $\langle\partial^x_{k_1}\partial^y_{k_2}\ldots\partial^\mu_{k'}\partial^\nu_{k''}\ldots\rangle_g$.

Example. Let Γ be the following graph.



The corresponding gwi is:

$$\langle \partial^x \partial^y \partial^\mu \rangle \langle \partial^z \partial^w_2 \partial^\nu \rangle_2 \langle \partial^\mu \partial^\nu \rangle_1 .$$

The key tool employed in this paper is the *existence* of linear operators

$$\mathfrak{r}_l : R^k(\overline{\mathcal{M}}_{g,n}) \to R^{k-l+1}(\overline{\mathcal{M}}^{\bullet}_{g-1,n+2}), \quad l = 1, 2, \dots, \tag{3}$$

where the symbol $\bullet$ denotes the moduli of possibly disconnected curves. The existence is proved in [8] Theorem 5, originally Invariance Conjecture 1 in [7]. $\mathfrak{r}_l$ is defined as an operation on the decorated graphs. The output graphs have two more markings, which are denoted by i, j. In terms of gwis,

$$\begin{aligned}
& \mathfrak{r}_l \left(\langle \partial^\mu_{k'} \dots \rangle_{g'} \dots \langle \partial^\mu_{k''} \dots \rangle_{g''} \right) \\
= \ & \frac{1}{2} \Big(\langle \partial^i_{k'+l} \dots \rangle_{g'} \dots \langle \partial^j_{k''} \dots \rangle_{g''} + \langle \partial^j_{k'} \dots \rangle_{g'} \langle \partial^i_{k''+l} \dots \rangle_{g''} \Big) \\
& + \frac{1}{2} (-1)^{l-1} \Big(\langle \partial^j_{k'+l} \dots \rangle_{g'} \dots \langle \partial^i_{k''} \dots \rangle_{g''} + \langle \partial^i_{k'} \dots \rangle_{g'} \langle \partial^j_{k''+l} \dots \rangle_{g''} \Big) + \cdots \\
& + \frac{1}{2} \sum_{m=0}^{l-1} (-1)^{m+1} \langle \partial^i_{l-1-m} \partial^j_m \partial^\mu_{k'} \dots \rangle_{g'-1} \dots \langle \partial^\mu_{k''} \dots \rangle_{g''} + \cdots \\
& + \frac{1}{2} \sum_{m=0}^{l-1} (-1)^{m+1} \langle \partial^\mu_{k'} \dots \rangle_{g'} \dots \langle \partial^i_{l-1-m} \partial^j_m \partial^\mu_{k''} \dots \rangle_{g''-1} \\
& + \frac{1}{2} \Big(\sum_{m=0}^{l-1} (-1)^{m+1} \sum_{g=0}^{g'} \partial^\mu_{k'} \dots \big(\langle \partial^i_{l-1-m} \rangle_g \langle \partial^j_m \rangle_{g'-g} \big) \Big) \langle \partial^\mu_{k''} \dots \rangle_{g''} + \cdots \\
& + \frac{1}{2} \langle \partial^\mu_{k'} \dots \rangle_{g'} \Big(\sum_{m=0}^{l-1} (-1)^{m+1} \sum_{g=0}^{g''} \partial^\mu_{k''} \dots \big(\langle \partial^i_{l-1-m} \rangle_{g''} \langle \partial^j_m \rangle_{g''-g} \big) \Big),
\end{aligned} \tag{4}$$

where the notation $\partial^\mu_k \dots (\langle \partial^i_{l-1-m} \rangle_{g_1} \langle \partial^j_m \rangle_{g_2})$ means that the half-edge insertions $\partial^\mu_k \dots$ acts on the product of vertices $\langle \partial^i_{l-1-m} \rangle_{g_1} \langle \partial^j_m \rangle_{g_2}$ by Leibniz rule.

Note that $\langle \ldots \rangle_{-1} := 0$.

Remark. In terms of graphical operations, the first two lines stand for "cutting edges"; the middle two for "genus reduction"; the last two for "splitting vertices". These are explained in [7].

There are three Invariance Conjectures proposed in [7]. Invariance Conjecture 1 has been stated above. The remaining two conjectures are

Invariance Conjecture 2. *If* $\mathfrak{r}_l(E) = 0$ *for all* l, *then* $E = 0$ *is a tautological equation.*

Invariance Conjecture 3. *Invariance Conjecture 2 will produce all tautological equations inductively.*

0.3. The algorithm of finding tautological equations

A general algorithm of finding the tautological equations, based on Conjectures 2 and 3, is explained in [7] Section 2. Since these remain conjectural, one possible alternative to the general scheme is to

- Calculate the rank of tautological ring $R^k(\overline{\mathcal{M}}_{g,n})$ to see if there is any new equation. If so, write this new equation as

$$E = \sum c_m \Gamma_m = 0.$$

- Apply invariance equation (3)

$$\mathfrak{r}_1(E) = 0,$$

  to obtain the coefficients c_m.

Note that $\mathfrak{r}_1(\Gamma)$ lies in $R^k(\overline{\mathcal{M}}^{\bullet}_{g-1,n+1})$, whose structure is known by induction. In the case of $g = 2$, one has the comfort of knowing genus one tautological rings are completely determined by E. Getzler [3] and in preparation.

0.4. Main results

Theorem 1. *Invariance Conjectures hold for* $(g, n, k) = (2, 1, 2), (2, 2, 2), (2, 3, 2)$. *In particular, a uniform derivation of all known genus two tautological equations is given by invariance condition* (3).

Remark. As explained in [7] and [8], our calculation in terms of gwis can be translated verbatim into one for any (axiomatic) Gromov–Witten theories. Therefore, it completes (the write-up of) a proof of Virasoro conjecture in the semisimple case and of Witten's conjecture (on spin curves and Gelfand–Dickey hierarchies), both up to genus two.

1. Mumford–Getzler's Equation in $\overline{\mathcal{M}}_{2,1}$

In all calculations below, we will employ the "gwi" notations for decorated graphs. It is explained in [7] that gwis are equivalent to the decorated graphs, or a tautological class. The notations are obviously inspired by Gromov–Witten invariants.

1.1. Tautological classes of $R^2(\overline{\mathcal{M}}_{2,1})$

There are 8 boundary strata of codimension ≤ 2 in $\overline{\mathcal{M}}_{2,1}$: 1 stratum in codimension 0, 2 strata in codimension 1, and 5 strata in codimension 2. If we insert ψ classes, the 2 boundary strata in codimension 1 produce 5 different tautological elements in codimension 2. Note that the κ-classes can be expressed in terms of boundary and ψ-classes in genus two. So the only decoration one would need is the ψ-classes.

Here is a list of all the 11 strata with ψ classes in codimension 2:

$$\begin{aligned}
&\langle \partial_2^x \rangle_2,\ \langle \partial^x \partial_1^\mu \partial^\mu \rangle_1, \langle \partial_1^x \partial^\mu \partial^\mu \rangle_1,\ \langle \partial^x \partial_1^\mu \rangle_1 \langle \partial^\mu \rangle_1,\ \langle \partial^x \partial^\mu \rangle_1 \langle \partial_1^\mu \rangle_1,\ \langle \partial_1^x \partial^\mu \rangle_1 \langle \partial^\mu \rangle_1,\\
&\langle \partial^\mu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \partial^\nu \rangle,\ \langle \partial^x \partial^\mu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\nu \rangle,\ \langle \partial^\mu \partial^\nu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle,\\
&\langle \partial^\mu \rangle_1 \langle \partial^\nu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle, \langle \partial^x \partial^\mu \partial^\mu \partial^\nu \partial^\nu \rangle.
\end{aligned}$$

The 5 strata with ψ-classes can be written in terms of the 5 strata without ψ-classes using TRR's, and therefore we only have 6 terms which could be independent. A general element can be written as

$$\begin{aligned}
E \ = \ & c_1 \langle \partial_2^x \rangle_2 + c_2 \langle \partial^\mu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \partial^\nu \rangle + c_3 \langle \partial^x \partial^\mu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\nu \rangle \\
& + c_4 \langle \partial^\mu \partial^\nu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle + c_5 \langle \partial^\mu \rangle_1 \langle \partial^\nu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle + c_6 \langle \partial^x \partial^\mu \partial^\mu \partial^\nu \partial^\nu \rangle.
\end{aligned}$$

1.2. Calculating $\mathfrak{r}_1(E)$

Throughout this paper, the labelings i, j are assumed to be symmetrized for l odd, and anti-symmetrized for l even.

$$\langle \partial_2^x \rangle_2 \mapsto -\frac{1}{2}\langle \partial^\mu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^j \partial^\nu \rangle - \frac{1}{48}\langle \partial^x \partial^\mu \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^j \partial^\nu \rangle$$
$$-\frac{1}{24}\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^j \partial^\mu \partial^\nu \rangle - \frac{1}{24}\langle \partial^j \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^\mu \partial^\nu \rangle$$
$$\langle \partial^\mu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \partial^\nu \rangle \mapsto \frac{1}{24}\langle \partial^i \partial^\mu \partial^\mu \rangle \langle \partial^x \partial^j \partial^\nu \partial^\nu \rangle + \langle \partial^j \rangle_1 \langle \partial^i \partial^\mu \partial^\nu \rangle \langle \partial^x \partial^\mu \partial^\nu \rangle$$
$$-\frac{1}{2}\langle \partial^i \partial^j \partial^\mu \rangle \langle \partial^x \partial^\mu \partial^\nu \partial^\nu \rangle - \langle \partial^\mu \rangle_1 \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle$$
$$\langle \partial^x \partial^\mu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\nu \rangle \mapsto \langle \partial^\mu \rangle_1 \langle \partial^i \partial^x \partial^\mu \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle + \frac{1}{24}\langle \partial^i \partial^x \partial^\mu \partial^\mu \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle$$
$$-\frac{1}{2}\langle \partial^i \partial^j \partial^x \partial^\mu \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle - \langle \partial^i \rangle_1 \langle \partial^j \partial^x \partial^\mu \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle$$
$$\langle \partial^\mu \partial^\nu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle \mapsto 2\langle \partial^\mu \rangle_1 \langle \partial^i \partial^\mu \partial^\nu \rangle \langle \partial^x \partial^j \partial^\nu \rangle + \frac{1}{12}\langle \partial^i \partial^\mu \partial^\mu \partial^\nu \rangle \langle \partial^x \partial^j \partial^\nu \rangle$$
$$-\frac{1}{2}\langle \partial^i \partial^j \partial^\mu \partial^\nu \rangle \langle \partial^x \partial^\mu \partial^\nu \rangle - \langle \partial^i \rangle_1 \langle \partial^j \partial^\mu \partial^\nu \rangle \langle \partial^x \partial^\mu \partial^\nu \rangle$$
$$\langle \partial^\mu \rangle_1 \langle \partial^\nu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle \mapsto \frac{1}{12}\langle \partial^\nu \rangle_1 \langle \partial^i \partial^\mu \partial^\mu \rangle \langle \partial^x \partial^j \partial^\nu \rangle - \langle \partial^\nu \rangle_1 \langle \partial^i \partial^j \partial^\mu \rangle \langle \partial^x \partial^\mu \partial^\nu \rangle$$
$$\langle \partial^x \partial^\mu \partial^\mu \partial^\nu \partial^\nu \rangle \mapsto 4\langle \partial^i \partial^\mu \partial^\nu \rangle \langle \partial^x \partial^j \partial^\mu \partial^\nu \rangle - 2\langle \partial^i \partial^x \partial^\mu \partial^\mu \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle$$

1.3. Setting $\mathfrak{r}_1(E) = 0$

Now we will pick a basis, and set its coordinates to zero.

$$\langle \partial^\mu \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^j \partial^\nu \rangle \ : \ -\frac{1}{2}c_1 + 2c_4 - c_5 = 0. \tag{5}$$
$$\langle \partial^\mu \rangle_1 \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle \ : \ -c_2 + c_3 + \frac{1}{12}c_5 = 0. \tag{6}$$
$$\langle \partial^i \rangle_1 \langle \partial^j \partial^x \partial^\mu \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle \ : \ -\frac{1}{24}c_1 + c_2 - c_3 - c_4 = 0. \tag{7}$$
$$\langle \partial^x \partial^i \partial^\mu \partial^\mu \rangle \langle \partial^j \partial^\nu \partial^\nu \rangle \ : \ \frac{1}{24}c_2 + \frac{1}{24}c_3 - 2c_6 = 0. \tag{8}$$

The remaining terms are related to each via WDVV as follows:

$$\langle \partial^x \partial^i \partial^j \partial^\mu \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle \ = \ \langle \partial^x \partial^\mu \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^j \partial^\nu \rangle + 2\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^j \partial^\mu \partial^\nu \rangle$$
$$-2\langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\mu \partial^\nu \partial^\nu \rangle,$$

$$\begin{aligned}\langle\partial^x\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle &= \langle\partial^x\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\nu\rangle + \langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\mu\partial^\nu\rangle\\ &\quad -\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\partial^\nu\rangle.\end{aligned}$$

Therefore, among the 5 vectors, only 3 of them are linearly independent.

$$\langle\partial^x\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\nu\rangle \;:\; -\frac{1}{48}c_1 - \frac{1}{2}c_2 - \frac{1}{2}c_3 + 4c_6 = 0. \tag{9}$$

$$\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\mu\partial^\nu\rangle \;:\; -\frac{1}{24}c_1 - \frac{1}{2}c_4 - c_3 + 4c_6 = 0. \tag{10}$$

$$\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\partial^\nu\rangle \;:\; c_3 + \frac{1}{12}c_4 - 4c_6 = 0. \tag{11}$$

The system of equations (5), (6), (7), (8), (9), (10) and (11) has a unique solution (up to scaling)

$$c_2 = -\frac{13}{240}c_1,\; c_3 = \frac{1}{240}c_1,\; c_4 = -\frac{1}{10}c_1,\; c_5 = -\frac{7}{10}c_1,\; c_6 = -\frac{1}{960}c_1$$

We therefore obtain that, if we let $c_1 = -1$,

$$\begin{aligned}&-\langle\partial_2^x\rangle_2 + \frac{13}{240}\langle\partial^\mu\rangle_1\langle\partial^x\partial^\mu\partial^\nu\partial^\nu\rangle - \frac{1}{240}\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\partial^\nu\rangle\\ &+\frac{1}{10}\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle + \frac{7}{10}\langle\partial^\mu\rangle_1\langle\partial^\nu\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle + \frac{1}{960}\langle\partial^x\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle = 0,\end{aligned}$$

which is Mumford–Getzler's equation.

1.4. Checking $\mathfrak{r}_2(E) = 0$

Let us now calculate $\mathfrak{r}_2(E)$.

$$\begin{aligned}\langle\partial_2^x\rangle_2 &\mapsto -\frac{1}{576}\langle\partial^i\partial^\mu\partial^\mu\rangle\langle\partial^x\partial^\alpha\partial^j\rangle\langle\partial^\alpha\partial^\nu\partial^\nu\rangle\\ \langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^\nu\rangle_1 &\mapsto \frac{1}{24}\langle\partial^j\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^x\partial^\alpha\partial^\nu\rangle\\ &\quad -\frac{1}{24}\langle\partial^j\partial^x\partial^\mu\rangle\langle\partial^i\partial^\nu\partial^\nu\rangle\langle\partial^\alpha\partial^\alpha\partial^\mu\rangle\\ \langle\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle &\mapsto -\frac{1}{24}\langle\partial^j\partial^\nu\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\alpha\rangle\langle\partial^x\partial^\nu\partial^\mu\rangle\\ \langle\partial^x\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle &\mapsto -\langle\partial^j\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^x\partial^\alpha\partial^\nu\rangle\\ &\quad -\langle\partial^j\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^x\partial^\alpha\partial^\nu\rangle.\end{aligned}$$

The other graphs all have $\mathfrak{r}_2(\Gamma) = 0$.

Therefore, $\mathfrak{r}_2(E) = 0$ as

$$\frac{1}{576}c_1 + \frac{1}{24}c_3 + \frac{1}{24}c_3 + \frac{1}{24}c_4 - c_6 - c_6 = 0.$$

2. Getzler's Equation in $\overline{\mathcal{M}}_{2,2}$

2.1. Tautological classes in $\overline{\mathcal{M}}_{2,2}$ of codimension 2

A general linear combination of codimension 2 tautological classes in $\overline{\mathcal{M}}_{2,2}$ is, after removing the linearly dependent classes from the induced equations (TRR's and Mumford–Getzler's),

$$\begin{aligned} E \;=\; & c_1\langle\partial_1^x\partial_1^y\rangle_2 + c_2\langle\partial_1^\mu\rangle_2\langle\partial^x\partial^y\partial^\mu\rangle + c_3\langle\partial^x\partial^y\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle \\ & + c_4\langle\partial^\mu\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^y\partial^\nu\rangle + c_5\langle\partial^x\partial^y\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1 \\ & + c_6\langle\partial^x\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^\nu\rangle_1 + c_7\langle\partial^y\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^\nu\rangle_1 \\ & + c_8\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^y\partial^\nu\rangle_1 + c_9\langle\partial^x\partial^\mu\partial^\nu\rangle_1\langle\partial^y\partial^\mu\partial^\nu\rangle \\ & + c_{10}\langle\partial^y\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle + c_{11}\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^y\partial^\mu\partial^\nu\rangle \\ & + c_{12}\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^\nu\rangle_1 + c_{13}\langle\partial^x\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^\mu\rangle_1\langle\partial^\nu\rangle_1 \\ & + c_{14}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\rangle_1\langle\partial^y\partial^\nu\rangle_1 + c_{15}\langle\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^\mu\rangle_1\langle\partial^x\partial^\nu\rangle_1. \end{aligned}$$

2.2. Setting $\mathfrak{r}_1(E) = 0$

The routine calculation of $\mathfrak{r}_1(E)$ is omitted. Again, a basis will be chosen and the components set to zero.

$$\langle\partial^x\partial^i\rangle_1\langle\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^j\partial^\mu\rangle : \; c_7 - c_8 - c_9 = 0. \tag{12}$$

$$\langle\partial^y\partial^i\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^j\partial^\mu\rangle : \; c_6 - c_8 - c_{10} = 0. \tag{13}$$

$$\langle\partial^x\partial^\mu\rangle_1\langle\partial^y\partial^i\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle : \; -c_7 + c_8 + \frac{1}{24}c_{15} = 0. \tag{14}$$

$$\langle\partial^y\partial^\mu\rangle_1\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle : \; -c_6 + c_8 + \frac{1}{24}c_{14} = 0. \tag{15}$$

$$\langle\partial^i\partial^\mu\rangle_1\langle\partial^x\partial^y\partial^j\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle: \ -\frac{1}{240}c_2 - c_8 = 0. \tag{16}$$

$$\langle\partial^i\partial^\mu\rangle_1\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle: \ -c_4 + \frac{1}{24}c_{12} = 0. \tag{17}$$

$$\langle\partial^i\rangle_1\langle\partial^\mu\rangle_1\langle\partial^x\partial^y\partial^\nu\rangle\langle\partial^\nu\partial^j\partial^\mu\rangle: \ -2c_1 - c_2 + 2c_{13} - c_{14} - c_{15} = 0. \tag{18}$$

$$\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^y\partial^i\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle: \ \frac{1}{10}c_2 + c_4 + c_4 - c_{11} = 0. \tag{19}$$

$$\langle\partial^\mu\rangle_1\langle\partial^\nu\rangle_1\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^y\partial^j\partial^\nu\rangle: \ -c_1 - 2c_{13} + c_{14} + c_{15} = 0. \tag{20}$$

$$\langle\partial^\mu\rangle_1\langle\partial^\nu\rangle_1\langle\partial^x\partial^y\partial^i\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle: \ \frac{7}{10}c_2 + c_{12} - c_{13} = 0. \tag{21}$$

$$\langle\partial^i\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^y\partial^j\rangle\langle\partial^\nu\rangle_1: \ \frac{13}{240}c_2 + c_4 - c_5 + \frac{1}{24}c_{12} = 0. \tag{22}$$

$$\langle\partial^i\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^y\partial^\nu\rangle\langle\partial^\nu\rangle_1: \ -\frac{1}{24}c_1 - c_5 + c_6 + \frac{1}{24}c_{15} = 0. \tag{23}$$

$$\langle\partial^i\partial^y\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^x\partial^\nu\rangle\langle\partial^\nu\rangle_1: \ -\frac{1}{24}c_1 - c_5 + c_7 + \frac{1}{24}c_{14} = 0. \tag{24}$$

$$\langle\partial^i\partial^\mu\partial^\mu\rangle\langle\partial^x\partial^y\partial^j\partial^\nu\rangle\langle\partial^\nu\rangle_1: \ -c_5 + c_8 + \frac{1}{12}c_{13} = 0. \tag{25}$$

$$\langle\partial^x\partial^y\partial^i\rangle\langle\partial^j\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle: \ \frac{1}{960}c_2 - c_3 + \frac{1}{24}c_4 = 0. \tag{26}$$

$$\langle\partial^x\partial^i\partial^\mu\partial^\mu\rangle\langle\partial^y\partial^j\partial^\nu\partial^\nu\rangle: \ -\frac{1}{576}c_1 - 2c_3 + \frac{1}{24}c_6 + \frac{1}{24}c_7 = 0. \tag{27}$$

$$\langle\partial^x\partial^y\partial^i\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle: \ -2c_3 + \frac{1}{24}c_5 + \frac{1}{24}c_8 = 0. \tag{28}$$

The 7 vectors

$$\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^y\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\rangle_1, \ \langle\partial^x\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^j\rangle_1,$$
$$\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\rangle_1, \ \langle\partial^x\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\rangle_1,$$
$$\langle\partial^x\partial^y\partial^i\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\rangle_1, \ \langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\rangle_1,$$
$$\langle\partial^x\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^i\partial^\nu\rangle\langle\partial^j\rangle_1$$

are related by WDVV equations. There are 4 independent vectors.

$$\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^y\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\rangle_1 : \; -\frac{1}{12}c_1 - c_7 + 2c_5 - c_{11} - c_6 = 0. \tag{29}$$

$$\langle\partial^x\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^j\rangle_1 : \; -\frac{1}{12}c_1 - c_9 + 2c_5 - c_{11} + c_{10} - 2c_6 = 0. \tag{30}$$

$$\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\rangle_1 : \; -\frac{1}{24}c_2 - c_4 - c_5 + \frac{1}{24}c_{12} + c_{11} - c_{10} + c_6 = 0. \tag{31}$$

$$\langle\partial^x\partial^y\partial^i\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\rangle_1 : \; -c_8 - c_{10} + c_6 = 0. \tag{32}$$

All of the other remaining terms are related to each other via WDVV, TRR's and Getzler's genus one equation. After applying the above equations, one can write them in terms of a basis.

$$\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^y\partial^j\partial^\nu\rangle\langle\partial^\mu\partial^\nu\rangle_1 :$$
$$c_1 + c_2 + 20c_4 - 24c_5 + 24c_7 + 2c_9 + 26c_{10} - 2c_{11} = 0. \tag{33}$$

$$\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^j\partial^\nu\rangle\langle\partial^\mu\partial^\nu\rangle_1 :$$
$$c_1 + \frac{1}{2}c_2 + 12c_4 - 12c_5 + 12c_7 + 12c_{10} - \frac{1}{2}c_{12} = 0. \tag{34}$$

$$\langle\partial^y\partial^i\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^\nu\rangle_1 :$$
$$-c_1 - \frac{1}{2}c_2 - 10c_4 + 12c_5 - 12c_7 + 2c_9 - 12c_{10} - \frac{1}{2}c_{15} = 0. \tag{35}$$

$$\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^\nu\rangle_1 :$$
$$-c_1 - \frac{1}{2}c_2 - 10c_4 + 12c_5 - 12c_7 - 10c_{10} - \frac{1}{2}c_{14} = 0. \tag{36}$$

$$\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\nu\rangle_1 :$$
$$-4c_1 - 2c_2 - 20c_4 + 24c_5 - 24c_7 - 2c_9 - 26c_{10} + 2c_{11} = 0. \tag{37}$$

$$\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^y\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1 :$$
$$-3c_1 - c_2 - 8c_4 + 12c_5 - 12c_7 + 2c_9 - 10c_{10} - \frac{1}{2}c_{12} - c_{13} = 0. \tag{38}$$

$$\langle\partial^y\partial^i\partial^j\partial^\mu\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1 :$$
$$\frac{3}{2}c_1 + \frac{1}{2}c_2 + 10c_4 - 12c_5 + 12c_7 - 2c_9 + 12c_{10} + c_{13} - \frac{1}{2}c_{14} = 0. \tag{39}$$

$\langle\partial^y\partial^i\partial^\mu\rangle\langle\partial^x\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1$:

$$-\frac{3}{2}c_1 - \frac{1}{2}c_2 + 2c_4 + 2c_9 - \frac{1}{2}c_{12} - \frac{1}{2}c_{15} = 0. \tag{40}$$

$\langle\partial^x\partial^y\partial^i\partial^\mu\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1$:

$$c_1 + c_2 + 8c_4 - 12c_5 + 12c_7 + 12c_{10} + 2c_{11} + \frac{1}{2}c_{12} - c_{13} = 0. \tag{41}$$

$$\langle\partial^x\partial^y\partial^\nu\partial^\mu\rangle\langle\partial^\mu\partial^i\partial^j\partial^\nu\rangle : \quad \frac{1}{24}c_2 + \frac{5}{6}c_4 - \frac{5}{12}c_{11} = 0. \tag{42}$$

$\langle\partial^x\partial^i\partial^\nu\partial^\mu\rangle\langle\partial^\mu\partial^y\partial^j\partial^\nu\rangle$:

$$-\frac{1}{8}c_1 - \frac{1}{24}c_2 + 4c_3 - \frac{5}{6}c_4 - c_7 - c_{10} + \frac{1}{12}c_{11} = 0. \tag{43}$$

$\langle\partial^x\partial^\nu\partial^\nu\partial^\mu\rangle\langle\partial^\mu\partial^y\partial^i\partial^j\rangle$:

$$-\frac{1}{24}c_1 - \frac{1}{24}c_2 - \frac{5}{6}c_4 + c_5 - \frac{1}{2}c_6 - \frac{1}{2}c_7 - \frac{1}{12}c_{11} = 0. \tag{44}$$

$\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^\mu\partial^y\partial^j\partial^\nu\partial^\nu\rangle$:

$$-\frac{1}{16}c_1 - \frac{1}{48}c_2 - \frac{5}{12}c_4 - \frac{5}{12}c_{10} = 0. \tag{45}$$

$\langle\partial^y\partial^i\partial^\mu\rangle\langle\partial^\mu\partial^x\partial^j\partial^\nu\partial^\nu\rangle$:

$$-\frac{5}{48}c_1 - \frac{1}{16}c_2 - \frac{5}{4}c_4 + c_5 - c_7 + \frac{1}{12}c_9 - \frac{1}{2}c_{10} - \frac{1}{12}c_{11} = 0. \tag{46}$$

$\langle\partial^y\partial^\nu\partial^\mu\rangle\langle\partial^\mu\partial^x\partial^i\partial^j\partial^\nu\rangle$:

$$\frac{1}{24}c_1 + \frac{1}{24}c_2 + \frac{5}{6}c_4 - c_5 + c_7 - \frac{1}{2}c_9 + \frac{1}{2}c_{10} + \frac{1}{12}c_{11} = 0. \tag{47}$$

$\langle\partial^i\partial^\nu\partial^\mu\rangle\langle\partial^\mu\partial^x\partial^y\partial^j\partial^\nu\rangle$:

$$\frac{1}{12}c_1 + \frac{1}{12}c_2 + 4c_3 + \frac{5}{3}c_4 - 2c_5 + c_7 + c_{10} + \frac{1}{12}c_{11} = 0. \tag{48}$$

$\langle\partial^\nu\partial^\nu\partial^\mu\rangle\langle\partial^\mu\partial^x\partial^y\partial^i\partial^j\rangle$:

$$-\frac{1}{24}c_1 - \frac{1}{24}c_2 - \frac{5}{6}c_4 + c_5 - \frac{1}{2}c_7 - \frac{1}{2}c_8 - \frac{1}{2}c_{10} - \frac{1}{12}c_{11} = 0. \tag{49}$$

Solving equations (12)-(49), we can write all of the coefficients in terms of c_1:

$$\begin{aligned} c_2 &= -3c_1,\ c_3 = -\frac{1}{576}c_1,\ c_4 = \frac{1}{30}c_1,\ c_5 = -\frac{23}{240}c_1,\ c_6 = -\frac{1}{48}c_1, \\ c_7 &= -\frac{1}{48}c_1,\ c_8 = \frac{1}{80}c_1,\ c_9 = -\frac{1}{30}c_1,\ c_{10} = -\frac{1}{30}c_1,\ c_{11} = -\frac{7}{30}c_1, \end{aligned}$$

$$c_{12} = \frac{4}{5}c_1,\ c_{13} = -\frac{13}{10}c_1,\ c_{14} = -\frac{4}{5}c_1,\ c_{15} = -\frac{4}{5}c_1$$

and these are the coefficients of Getzler's equation in $\overline{\mathcal{M}}_{2,2}$.

2.3. Checking $\mathfrak{r}_2(E) = 0$

Again, one has to pick a basis and check all components vanish.

$$\langle\partial^\mu\rangle_1\langle\partial^j\partial^x\partial^y\rangle\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle:\ \frac{1}{20}c_2 + c_4 - c_5 - c_8 + \frac{1}{24}c_{12} = 0.$$

$$\langle\partial^j\partial^x\partial^y\rangle\langle\partial^i\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^\mu\partial^\alpha\partial^\alpha\rangle:\ \frac{1}{1152}c_2 - c_3 + \frac{1}{24}c_4 - \frac{1}{24}c_8 = 0.$$

$$\langle\partial^j\partial^x\partial^y\rangle\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\nu\partial^\alpha\partial^\alpha\rangle:\ \frac{1}{480}c_2 - 2c_3 + \frac{1}{12}c_4 = 0.$$

$$\langle\partial^j\rangle_1\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle:$$
$$-\frac{1}{12}c_1 - \frac{1}{24}c_2 - c_4 + c_5 - c_8 - c_9 - c_{10} + \frac{1}{24}c_{12} = 0.$$

$$\langle\partial^\mu\rangle_1\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^\mu\partial^\alpha\rangle\langle\partial^x\partial^y\partial^\alpha\rangle:$$
$$\frac{1}{12}c_1 + \frac{1}{24}c_2 - c_4 - c_5 + c_8 + \frac{1}{24}c_{12} + \frac{1}{24}c_{14} + \frac{1}{24}c_{15} = 0.$$

$$\langle\partial^\mu\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^x\partial^\nu\partial^\alpha\rangle:\ -\frac{1}{24}c_1 - c_5 + c_8 + c_{10} + \frac{1}{24}c_{15} = 0.$$

$$\langle\partial^\mu\rangle_1\langle\partial^j\partial^x\partial^\mu\rangle\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^y\partial^\nu\partial^\alpha\rangle:\ -\frac{1}{24}c_1 - c_5 + c_8 + c_9 + \frac{1}{24}c_{14} = 0.$$

$$\langle\partial^j\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^y\partial^\nu\partial^\alpha\rangle:\ -\frac{1}{576}c_1 - 2c_3 + \frac{1}{24}c_6 + \frac{1}{24}c_8 + \frac{1}{24}c_9 = 0.$$

$$\langle\partial^j\partial^y\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^x\partial^\nu\partial^\alpha\rangle:\ -\frac{1}{576}c_1 - 2c_3 + \frac{1}{24}c_7 + \frac{1}{24}c_8 + \frac{1}{24}c_{10} = 0.$$

The 7 vectors

$$\langle\partial^i\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^y\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle, \langle\partial^x\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^y\partial^\mu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle,$$

$$\langle\partial^y\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^x\partial^\mu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle, \langle\partial^x\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle,$$

$$\langle\partial^i\partial^x\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle, \langle\partial^i\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle,$$

$$\langle\partial^i\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle$$

are related by WDVV equations. 4 of them are linear independent.

$$\langle\partial^i\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^y\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle \ : \quad \frac{1}{288}c_1+\frac{1}{576}c_2-2c_3+\frac{1}{12}c_8+\frac{1}{24}c_9+\frac{1}{24}c_{10}=0.$$

$$\langle\partial^x\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^y\partial^\mu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle \ : \ \frac{1}{24}c_6-\frac{1}{24}c_8-\frac{1}{24}c_9=0.$$

$$\langle\partial^y\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^x\partial^\mu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle \ : \ \frac{1}{24}c_7-\frac{1}{24}c_8-\frac{1}{24}c_{10}=0.$$

$$\langle\partial^x\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle \ : \quad \frac{1}{288}c_1-4c_3+\frac{1}{6}c_8+\frac{1}{24}c_9+\frac{1}{24}c_{10}+\frac{1}{24}c_{11}=0.$$

All of the remaining strata are related by WDVV equations. Only 3 of them are linearly independent.

$$\langle\partial^x\partial^j\partial^\mu\rangle\langle\partial^\mu\partial^y\partial^i\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle \ : \quad -4c_3-\frac{5}{6}c_4+c_8+\frac{1}{6}c_9+\frac{1}{6}c_{10}-\frac{1}{12}c_{11}=0.$$

$$\langle\partial^x\partial^j\partial^\alpha\partial^\mu\rangle\langle\partial^\mu\partial^i\partial^\nu\rangle\langle\partial^\nu\partial^y\partial^\alpha\rangle \ : \ -\frac{5}{6}c_4-\frac{5}{6}c_9=0.$$

$$\langle\partial^y\partial^j\partial^\alpha\partial^\mu\rangle\langle\partial^\mu\partial^i\partial^\nu\rangle\langle\partial^\nu\partial^x\partial^\alpha\rangle \ : \ -\frac{5}{6}c_4-\frac{5}{6}c_{10}=0.$$

Therefore, $\mathfrak{r}_2(E)=0$.

2.4. Calculating $\mathfrak{r}_3(E)$

Since $l=3$ case is new, the calculation is presented.

$$\begin{aligned}\langle\partial_1^x\partial_1^y\rangle_2 \ \mapsto \ & -\frac{1}{24}\langle\partial^y\partial^\nu\partial^j\rangle\langle\partial^x\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\nu\partial^i\rangle\langle\partial^\mu\partial^\beta\partial^\beta\rangle\\ & -\frac{1}{24}\langle\partial^y\partial^\nu\partial^\mu\rangle\langle\partial^x\partial^\alpha\partial^j\rangle\langle\partial^\alpha\partial^\nu\partial^i\rangle\langle\partial^\mu\partial^\beta\partial^\beta\rangle\\ & -\frac{1}{24}\langle\partial^x\partial^\alpha\partial^j\rangle\langle\partial^\alpha\partial^i\partial^\mu\rangle\langle\partial^y\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle\\ & -\frac{1}{24}\langle\partial^y\partial^\alpha\partial^j\rangle\langle\partial^\alpha\partial^i\partial^\mu\rangle\langle\partial^x\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle\end{aligned}$$

$$-\frac{1}{24}\langle\partial^i\partial^\mu\partial^j\rangle\langle\partial^y\partial^\beta\partial^\nu\rangle\langle\partial^x\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle$$
$$-\frac{1}{24}\langle\partial^i\partial^\mu\partial^j\rangle\langle\partial^y\partial^\beta\partial^\nu\rangle\langle\partial^x\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle$$
$$+\frac{1}{48}\langle\partial^\alpha\partial^y\partial^\beta\rangle\langle\partial^\beta\partial^x\partial^\mu\rangle\langle\partial^\mu\partial^j\partial^\nu\rangle\langle\partial^\nu\partial^i\partial^\alpha\rangle$$
$$+\frac{1}{48}\langle\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^\nu\partial^y\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^j\rangle\langle\partial^\nu\partial^i\partial^\alpha\rangle$$
$$+\frac{1}{48}\langle\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^\mu\partial^j\partial^\nu\rangle\langle\partial^\alpha\partial^y\partial^\beta\rangle\langle\partial^\beta\partial^\nu\partial^i\rangle$$
$$+\frac{1}{48}\langle\partial^\alpha\partial^y\partial^\beta\rangle\langle\partial^\beta\partial^j\partial^\nu\rangle\langle\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^\mu\partial^\nu\partial^i\rangle$$
$$+\frac{1}{48}\langle\partial^j\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^\beta\rangle\langle\partial^\beta\partial^x\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^i\rangle$$
$$+\frac{1}{48}\langle\partial^j\partial^\nu\partial^\alpha\rangle\langle\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^i\partial^y\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle$$
$$+\frac{1}{576}\langle\partial^y\partial^\mu\partial^\alpha\rangle\langle\partial^\mu\partial^x\partial^\nu\rangle\langle\partial^\nu\partial^i\partial^\alpha\rangle\langle\partial^j\partial^\beta\partial^\beta\rangle$$
$$+\frac{1}{576}\langle\partial^x\partial^\nu\partial^\alpha\rangle\langle\partial^y\partial^\mu\partial^\alpha\rangle\langle\partial^\mu\partial^\nu\partial^i\rangle\langle\partial^j\partial^\beta\partial^\beta\rangle$$
$$+\frac{1}{576}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^i\partial^\nu\rangle\langle\partial^y\partial^\alpha\partial^\beta\rangle\langle\partial^\alpha\partial^j\partial^\beta\rangle$$

$$\langle\partial_1^\mu\rangle_2\langle\partial^x\partial^y\partial^\mu\rangle \mapsto -\frac{1}{5760}\langle\partial^i\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^y\partial^j\rangle$$
$$+\frac{1}{960}\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\mu\partial^\mu\rangle\langle\partial^x\partial^y\partial^j\rangle$$
$$-\frac{1}{24}\langle\partial^\mu\partial^j\partial^\beta\rangle\langle\partial^\beta\partial^i\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\langle\partial^x\partial^y\partial^\mu\rangle$$
$$-\frac{1}{24}\langle\partial^i\partial^\nu\partial^j\rangle\langle\partial^\alpha\partial^\mu\partial^\beta\rangle\langle\partial^\beta\partial^\nu\partial^\alpha\rangle\langle\partial^x\partial^y\partial^\mu\rangle$$
$$+\frac{1}{48}\langle\partial^\mu\partial^\beta\partial^\alpha\rangle\langle\partial^\beta\partial^j\partial^\nu\rangle\langle\partial^\nu\partial^i\partial^\alpha\rangle\langle\partial^x\partial^y\partial^\mu\rangle$$
$$+\frac{1}{48}\langle\partial^j\partial^\nu\partial^\alpha\rangle\langle\partial^\mu\partial^\beta\partial^\alpha\rangle\langle\partial^\beta\partial^\nu\partial^i\rangle\langle\partial^x\partial^y\partial^\mu\rangle$$
$$+\frac{1}{576}\langle\partial^\mu\partial^\nu\partial^\alpha\rangle\langle\partial^\nu\partial^i\partial^\alpha\rangle\langle\partial^j\partial^\beta\partial^\beta\rangle\langle\partial^x\partial^y\partial^\mu\rangle$$

$$\langle\partial^x\partial^y\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle \mapsto +4\langle\partial^i\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\alpha\partial^j\rangle\langle\partial^\alpha\partial^x\partial^y\rangle$$
$$-2\langle\partial^i\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^x\partial^y\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle$$
$$-4\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^x\partial^y\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle$$

$$-4\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^x\partial^\mu\rangle\langle\partial^j\partial^y\partial^\nu\rangle$$
$$-4\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^y\partial^\mu\rangle\langle\partial^j\partial^x\partial^\nu\rangle$$
$$-\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^x\partial^y\rangle$$
$$+4\langle\partial^i\partial^\mu\partial^\alpha\rangle\langle\partial^\alpha\partial^x\partial^y\rangle\langle\partial^j\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle$$
$$+2\langle\partial^i\partial^\mu\partial^\alpha\rangle\langle\partial^\alpha\partial^x\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^y\partial^\nu\rangle$$
$$+4\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^x\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^y\partial^\mu\rangle$$

$$\begin{aligned}
\langle\partial^\mu\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^y\partial^\nu\rangle \mapsto\ & +\frac{1}{12}\langle\partial^i\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^j\rangle\langle\partial^\mu\partial^\alpha\partial^\alpha\rangle\langle\partial^x\partial^y\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^i\partial^\mu\partial^\beta\rangle\langle\partial^\beta\partial^\nu\partial^\mu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\langle\partial^x\partial^y\partial^j\rangle\\
&-\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^j\partial^\mu\rangle\langle\partial^x\partial^y\partial^\nu\rangle\\
&+\frac{1}{2}\langle\partial^i\partial^\beta\partial^\mu\rangle\langle\partial^j\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^x\partial^y\partial^\nu\rangle\\
&+\frac{1}{2}\langle\partial^i\partial^\beta\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^x\partial^y\partial^\nu\rangle\\
&-\frac{1}{12}\langle\partial^i\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^y\partial^\nu\rangle\\
&-\frac{1}{24}\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^\mu\partial^\mu\rangle\langle\partial^x\partial^y\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^i\partial^\alpha\partial^\alpha\rangle\langle\partial^j\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\mu\partial^\mu\rangle\langle\partial^x\partial^y\partial^\nu\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^y\partial^\nu\rangle_1 \mapsto\ & +\frac{1}{24}\langle\partial^\mu\partial^\mu\partial^j\rangle\langle\partial^i\partial^y\partial^\beta\rangle\langle\partial^\beta\partial^\nu\partial^x\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\\
&-\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^y\rangle\langle\partial^\beta\partial^j\partial^x\rangle\\
&+\frac{1}{2}\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\beta\partial^y\rangle\langle\partial^j\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^x\rangle\\
&+\frac{1}{2}\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\beta\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^y\rangle\langle\partial^\alpha\partial^\beta\partial^x\rangle\\
&-\frac{1}{24}\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^x\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^y\partial^\nu\rangle\\
&-\frac{1}{24}\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^y\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^x\partial^\nu\rangle\\
&-\frac{1}{24}\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^x\partial^y\rangle\\
&+\frac{1}{24}\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^\alpha\rangle\langle\partial^j\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^x\partial^y\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\partial^x\partial^\mu\partial^\nu\rangle_1\langle\partial^y\partial^\mu\partial^\nu\rangle \ \mapsto\ & +\frac{1}{12}\langle\partial^i\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^x\rangle\langle\partial^\mu\partial^\alpha\partial^\alpha\rangle\langle\partial^y\partial^j\partial^\nu\rangle\\
&-\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^j\partial^x\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{2}\langle\partial^i\partial^\beta\partial^\mu\rangle\langle\partial^j\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^x\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{2}\langle\partial^i\partial^\beta\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^x\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\\
&-\frac{1}{24}\langle\partial^i\partial^\alpha\partial^x\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\\
&-\frac{1}{12}\langle\partial^i\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^x\partial^\nu\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^i\partial^\alpha\partial^\alpha\rangle\langle\partial^j\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^x\partial^\mu\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\partial^y\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle \ \mapsto\ & +\frac{1}{12}\langle\partial^i\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^y\rangle\langle\partial^\mu\partial^\alpha\partial^\alpha\rangle\langle\partial^x\partial^j\partial^\nu\rangle\\
&-\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^j\partial^y\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{2}\langle\partial^i\partial^\beta\partial^\mu\rangle\langle\partial^j\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^y\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{2}\langle\partial^i\partial^\beta\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^y\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle\\
&-\frac{1}{24}\langle\partial^i\partial^\alpha\partial^y\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle\\
&-\frac{1}{12}\langle\partial^i\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^y\partial^\nu\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^i\partial^\alpha\partial^\alpha\rangle\langle\partial^j\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^y\partial^\mu\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle
\end{aligned}$$

The other graphs all have $\mathfrak{r}_3(\Gamma) = 0$.

2.5. Checking $\mathfrak{r}_3(E) = 0$

Now the $\mathfrak{r}_3(E)$ has only a few independent strata.

$$\langle\partial^i\partial^\mu\partial^\mu\rangle : \quad \frac{1}{288}c_1+\frac{1}{576}c_2-2c_3+\frac{1}{12}c_8+\frac{1}{24}c_9+\frac{1}{24}c_{10} = 0.$$

$$\langle\partial^i\partial^x\partial^y\rangle : \quad \frac{1}{1152}c_2-c_3+\frac{1}{24}c_4-\frac{1}{24}c_8 = 0.$$

$$\langle\partial^i\partial^x\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\partial^y\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle : \frac{1}{576}c_1+2c_3-\frac{1}{12}c_8-\frac{1}{24}c_9-\frac{1}{24}c_{10} = 0.$$

The remaining strata, which are all equivalent as codimension 4 classes in $\overline{\mathcal{M}}_{1,4}$, have coefficient

$$-\frac{1}{8}c_1 - \frac{1}{24}c_2 = 0.$$

3. The Belorousski-Pandharipande Equation in $\overline{\mathcal{M}}_{2,3}$

3.1. Strata in $\overline{\mathcal{M}}_{2,3}$ of codimension 2

Throughout this section, we will always assume that the three external labelings x, y, z are symmetrized.

Using Mumford–Getzler's and Getzler's equations for genus 2, all genus 2 terms with more than one descendent can be rewritten in terms of the others. Also, using TRR's, all genus 0 or 1 terms with a descendent can be rewritten in terms of the others. We are therefore left with 21 strata which could be independent.[1]

A general linear combination is

$$\begin{aligned}
E = \sum_{S_3(x,y,z)} & c_1\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^\nu\rangle_2 + c_2\langle\partial^x\partial^y\partial^z\partial^\mu\rangle\langle\partial_1^\mu\rangle_2 \\
& +c_3\langle\partial_1^x\partial^\mu\rangle_2\langle\partial^\mu\partial^y\partial^z\rangle + c_4\langle\partial^x\partial_1^\mu\rangle_2\langle\partial^\mu\partial^y\partial^z\rangle \\
& +c_5\langle\partial^x\partial^y\partial^z\partial^\mu\partial^\nu\rangle\langle\partial^\mu\rangle_1\langle\partial^\nu\rangle_1 + c_6\langle\partial^x\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^\mu\rangle_1\langle\partial^\nu\partial^z\rangle_1 \\
& +c_7\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\rangle_1\langle\partial^\nu\partial^y\partial^z\rangle_1 + c_8\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^y\partial^\nu\rangle\langle\partial^\nu\partial^z\rangle_1 \\
& +c_9\langle\partial^x\partial^y\partial^z\partial^\mu\rangle\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^\nu\rangle_1 + c_{10}\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^\nu\partial^y\partial^z\rangle \\
& +c_{11}\langle\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^\nu\rangle_1\langle\partial^\nu\partial^y\partial^z\rangle + c_{12}\langle\partial^x\partial^y\partial^z\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1 \\
& +c_{13}\langle\partial^x\partial^y\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^z\rangle_1 + c_{14}\langle\partial^x\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^y\partial^z\rangle_1 \\
& +c_{15}\langle\partial^x\partial^y\partial^z\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\partial^\nu\rangle + c_{16}\langle\partial^x\partial^y\partial^z\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\nu\rangle_1 \\
& +c_{17}\langle\partial^x\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\nu\partial^z\rangle_1 + c_{18}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\nu\partial^y\partial^z\rangle_1 \\
& +c_{19}\langle\partial^x\partial^y\partial^z\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle_1 + c_{20}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\partial^z\rangle_1 \\
& +c_{21}\langle\partial^x\partial^y\partial^z\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle = 0,
\end{aligned}$$

[1] We rewrote the strata here in the same order as in the Belorousski-Pandharipande equation, with the extra strata $\langle\partial^x\partial^y\partial^z\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle$, which does not appear in the equation, at the end.

3.2. Setting $\mathfrak{r}_1(E) = 0$

Again a basis is chosen for the output graphs of $\mathfrak{r}_1(E)$, and the coefficients are set to zero.

$$\langle\partial^x\partial^\mu\partial^j\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^i_1\rangle_2 \ : \ c_1 + c_2 = 0. \tag{50}$$

$$\langle\partial^x\partial^y\partial^i\rangle\langle\partial^z\partial^\mu\partial^j\rangle\langle\partial^\mu_1\rangle_2 \ : \ -3c_2 + 3c_3 + c_4 = 0. \tag{51}$$

$$\langle\partial^x\partial^y\partial^i\rangle_1\langle\partial^z\partial^j\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle \ : \ c_{14} - 3c_{15} - c_{18} = 0. \tag{52}$$

$$\langle\partial^x\partial^y\partial^\mu\rangle_1\langle\partial^z\partial^i\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle \ : \ \frac{1}{24}c_7 - c_{14} + c_{15} + c_{15} + c_{15} = 0. \tag{53}$$

$$\langle\partial^x\partial^i\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^y\partial^z\partial^j\rangle \ : \ -\frac{1}{80}c_3 - \frac{1}{240}c_4 - 3c_{15} = 0. \tag{54}$$

$$\langle\partial^x\partial^i\partial^\mu\rangle_1\langle\partial^y\partial^z\partial^\mu\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle \ : \ -c_{20} + \frac{1}{24}c_{11} = 0. \tag{55}$$

$$\langle\partial^x\partial^\mu\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^z\partial^j\rangle \ : \ \frac{1}{30}c_3 + \frac{1}{10}c_4 - c_{17} + c_{20} + c_{20} = 0. \tag{56}$$

$$\langle\partial^i\partial^\mu\partial^\nu\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^z\partial^j\rangle \ : \ \frac{1}{30}c_3 - c_{18} = 0. \tag{57}$$

$$\langle\partial^i\partial^\mu\partial^\mu\rangle_1\langle\partial^x\partial^y\partial^\nu\rangle\langle\partial^\nu\partial^z\partial^j\rangle \ : \ c_{19} - c_{20} = 0. \tag{58}$$

$$\langle\partial^\mu\partial^\nu\partial^\nu\rangle_1\langle\partial^x\partial^i\partial^\mu\rangle\langle\partial^y\partial^z\partial^j\rangle \ : \ -\frac{1}{30}c_3 - 2c_{19} - c_{19} + c_{20} = 0. \tag{59}$$

$$\langle\partial^x\partial^i\rangle_1\langle\partial^\mu\rangle_1\langle\partial^y\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^j\partial^\mu\rangle \ : \ -c_4 + c_6 - 2c_7 + c_{10} - c_{11} = 0. \tag{60}$$

$$\langle\partial^x\partial^\mu\rangle_1\langle\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^z\partial^j\rangle \ : \ \frac{4}{5}c_3 + \frac{7}{5}c_4 - c_6 + c_{10} + c_{11} = 0. \tag{61}$$

$$\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^y\partial^i\rangle\langle\partial^\nu\rangle_1\langle\partial^\nu\partial^z\partial^j\rangle \ : \ -2c_6 + c_7 + c_7 + 2c_8 = 0. \tag{62}$$

$$\langle\partial^i\partial^\mu\rangle_1\langle\partial^\nu\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^z\partial^j\rangle \ : \ \frac{4}{5}c_3 - c_7 = 0. \tag{63}$$

$$\langle\partial^i\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^y\rangle\langle\partial^\nu\rangle_1\langle\partial^\nu\partial^z\partial^j\rangle \ : \ -c_3 + c_{10} - c_{11} = 0. \tag{64}$$

$$\langle\partial^i\partial^\mu\rangle_1\langle\partial^\mu\rangle_1\langle\partial^x\partial^y\partial^\nu\rangle\langle\partial^\nu\partial^z\partial^j\rangle \ : \ c_9 - c_{11} = 0. \tag{65}$$

$$\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^\mu\rangle_1\langle\partial^x\partial^i\partial^\nu\rangle\langle\partial^y\partial^z\partial^j\rangle \ : \ -\frac{4}{5}c_3 - 2c_9 - c_9 + c_{11} = 0. \tag{66}$$

$$\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^y\partial^z\partial^j\rangle \ : \ \frac{1}{10}c_2 - c_{16} + c_{19} + c_{19} = 0. \tag{67}$$

$$\langle\partial^i\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^y\partial^z\partial^j\rangle \;:\; -\frac{1}{240}c_2 - c_{15} = 0. \tag{68}$$

$$\langle\partial^x\partial^\mu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^y\partial^z\partial^j\rangle \;:\; \frac{1}{48}c_3+\frac{13}{240}c_4+\frac{1}{24}c_{10}-c_{13}+c_{20} = 0. \tag{69}$$

$$\langle\partial^\mu\rangle_1\langle\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^y\partial^z\partial^j\rangle \;:\; \frac{7}{10}c_2 - c_5 + c_9 = 0. \tag{70}$$

$$\langle\partial^\mu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^y\partial^z\partial^j\rangle \;:\; \frac{13}{240}c_2 + \frac{1}{24}c_9 - c_{12} + c_{19} = 0. \tag{71}$$

$$\langle\partial^x\partial^y\partial^i\rangle\langle\partial^j\partial^z\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle \;:\; \frac{1}{576}c_3 + \frac{1}{960}c_4 + \frac{1}{24}c_{20} + 3c_{21} = 0. \tag{72}$$

$$\langle\partial^\mu\partial^\mu\partial^i\rangle\langle\partial^j\partial^x\partial^y\partial^z\partial^\nu\partial^\nu\rangle \;:\; \frac{1}{24}c_{12} + \frac{1}{24}c_{15} + 2c_{21} = 0. \tag{73}$$

$$\langle\partial^x\partial^y\partial^z\partial^i\rangle\langle\partial^j\partial^\mu\partial^\mu\partial^\nu\partial^\nu\rangle \;:\; \frac{1}{960}c_2 + \frac{1}{24}c_{19} + c_{21} = 0. \tag{74}$$

$$\langle\partial^x\partial^\mu\partial^\mu\partial^i\rangle\langle\partial^j\partial^y\partial^z\partial^\nu\partial^\nu\rangle \;:\; \frac{1}{24}c_{13} + \frac{1}{24}c_{14} + 6c_{21} = 0. \tag{75}$$

The coefficients of the Belorousski-Pandharipande equation are the only solution of the equations (50)-(75).

3.3. Checking $\mathfrak{r}_2(E) = 0$

Since the whole output graphs are far too numerous, we shall present the the coefficients of the following four (disconnected) graphs are zero:

$$\langle\partial^i\partial^x\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle, \quad \langle\partial^i\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle$$
$$\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^z\rangle, \quad \langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle$$

In the following, the graphs Γ appearing in BP equation such that $\mathfrak{r}_2(\Gamma)$ contain any of the above four graphs will be listed.

$$\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^\nu\rangle_2 \;\mapsto\; \frac{1}{10}\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^z\rangle + \cdots$$

$$\langle\partial_1^x\partial^\mu\rangle_2\langle\partial^\mu\partial^y\partial^z\rangle \;\mapsto\; \frac{1}{24}\langle\partial^i\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle + \cdots$$

$$\langle\partial^x\partial_1^\mu\rangle_2\langle\partial^\mu\partial^y\partial^z\rangle \;\mapsto\; \frac{1}{24}\langle\partial^i\partial^x\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle+\cdots$$

$$\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^y\partial^\nu\rangle\langle\partial^\nu\partial^z\rangle_1 \;\mapsto\; -\frac{1}{12}\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle+\cdots$$

$$\begin{aligned}\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^\nu\partial^y\partial^z\rangle \;\mapsto\; & -\frac{1}{24}\langle\partial^i\partial^x\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\\ & -\frac{1}{24}\langle\partial^i\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle+\cdots\end{aligned}$$

$$\begin{aligned}\langle\partial^x\partial^y\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^z\rangle_1 \;\mapsto\; & -\langle\partial^i\partial^x\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\\ & +2\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle+\cdots\end{aligned}$$

$$\begin{aligned}\langle\partial^x\partial^y\partial^z\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\partial^\nu\rangle \;\mapsto\; & -6\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\\ & +3\langle\partial^i\partial^x\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle+\cdots\end{aligned}$$

$$\langle\partial^x\partial^y\partial^z\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\nu\rangle_1 \;\mapsto\; \langle\partial^\mu\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^z\rangle+\cdots$$

$$\begin{aligned}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\nu\partial^y\partial^z\rangle_1 \;\mapsto\; & 2\langle\partial^i\partial^x\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\\ & -2\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle+\cdots\end{aligned}$$

$$\begin{aligned}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\partial^z\rangle_1 \;\mapsto\; & -2\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^z\rangle\\ & +\langle\partial^i\partial^x\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\\ & +\langle\partial^i\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle+\cdots\end{aligned}$$

Checking the coefficients:

$$\langle\partial^i\partial^x\rangle_1\langle\partial^j\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\ :$$
$$\frac{1}{24}c_4-\frac{1}{24}c_{10}-c_{13}+3c_{15}+2c_{18}+c_{20}=0.$$

$$\langle\partial^i\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\ :\ \frac{1}{24}c_3-\frac{1}{24}c_{10}+c_{20}=0.$$

$$\langle\partial^\mu\partial^\nu\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^z\rangle\ :\ \frac{1}{10}c_{10}+c_{16}-2c_{20}=0.$$

$$\langle\partial^x\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^y\rangle\langle\partial^j\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\ :\ -\frac{1}{12}c_8+2c_{13}-6c_{15}-2c_{18}=0.$$

3.4. Checking $\mathfrak{r}_3(E) = 0$

In the same spirit as the case $l = 2$, only the following four disconnected output graphs will be presented here.

$$\langle \partial^x \partial^y \partial^z \partial^i \rangle \langle \partial^j \partial^\alpha \partial^\mu \rangle \langle \partial^\mu \partial^\alpha \partial^\nu \rangle \langle \partial^\nu \partial^\beta \partial^\beta \rangle,$$

$$\langle \partial^i \rangle_1 \langle \partial^x \partial^y \partial^\mu \rangle \langle \partial^\mu \partial^z \partial^\nu \rangle \langle \partial^\nu \partial^i \partial^\alpha \rangle \langle \partial^\alpha \partial^\beta \partial^\beta \rangle,$$

$$\langle \partial^i \partial^x \partial^\mu \partial^\mu \rangle \langle \partial^y \partial^z \partial^\nu \rangle \langle \partial^\nu \partial^j \partial^\alpha \rangle \langle \partial^\alpha \partial^\beta \partial^\beta \rangle,$$

$$\langle \partial^\mu \rangle_1 \langle \partial^\mu \partial^i \partial^x \rangle \langle \partial^y \partial^z \partial^\nu \rangle \langle \partial^\nu \partial^j \partial^\alpha \rangle \langle \partial^\alpha \partial^\beta \partial^\beta \rangle.$$

$$\begin{aligned}
&\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \partial^y \partial^z \rangle \langle \partial^\nu \rangle_2 \mapsto \\
&\qquad -\frac{1}{24} \langle \partial^j \rangle_1 \langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \partial^y \partial^z \rangle \langle \partial^i \partial^\alpha \partial^\nu \rangle \langle \partial^\alpha \partial^\beta \partial^\beta \rangle + \cdots
\end{aligned}$$

$$\begin{aligned}
\langle \partial^x \partial^y \partial^z \partial^\mu \rangle \langle \partial^\mu_1 \rangle_2 \mapsto\ & -\frac{1}{5760} \langle \partial^x \partial^y \partial^z \partial^j \rangle \langle \partial^i \partial^\alpha \partial^\mu \rangle \langle \partial^\alpha \partial^\beta \partial^\beta \rangle \langle \partial^\mu \partial^\nu \partial^\nu \rangle \\
& + \frac{1}{960} \langle \partial^x \partial^y \partial^z \partial^j \rangle \langle \partial^i \partial^\alpha \partial^\nu \rangle \langle \partial^\alpha \partial^\beta \partial^\nu \rangle \langle \partial^\beta \partial^\mu \partial^\mu \rangle \\
& + \cdots
\end{aligned}$$

$$\begin{aligned}
\langle \partial^x_1 \partial^\mu \rangle_2 \langle \partial^\mu \partial^y \partial^z \rangle \mapsto\ & -\frac{1}{24} \langle \partial^j \rangle_1 \langle \partial^x \partial^\mu \partial^\beta \rangle \langle \partial^\beta \partial^i \partial^\nu \rangle \langle \partial^\nu \partial^\alpha \partial^\alpha \rangle \langle \partial^\mu \partial^y \partial^z \rangle \\
& -\frac{1}{24} \langle \partial^j \rangle_1 \langle \partial^i \partial^\nu \partial^\mu \rangle \langle \partial^\alpha \partial^x \partial^\beta \rangle \langle \partial^\beta \partial^\nu \partial^\alpha \rangle \langle \partial^\mu \partial^y \partial^z \rangle \\
& -\frac{1}{24} \langle \partial^x \partial^\nu \partial^j \rangle \langle \partial^\nu \rangle_1 \langle \partial^i \partial^\alpha \partial^\mu \rangle \langle \partial^\alpha \partial^\beta \partial^\beta \rangle \langle \partial^\mu \partial^y \partial^z \rangle \\
& -\frac{1}{576} \langle \partial^x \partial^\nu \partial^\nu \partial^j \rangle \langle \partial^i \partial^\alpha \partial^\mu \rangle \langle \partial^\alpha \partial^\beta \partial^\beta \rangle \langle \partial^\mu \partial^y \partial^z \rangle \\
& + \cdots
\end{aligned}$$

$$\begin{aligned}
\langle \partial^x \partial^\mu_1 \rangle_2 \langle \partial^\mu \partial^y \partial^z \rangle \mapsto\ & -\frac{1}{24} \langle \partial^j \rangle_1 \langle \partial^\mu \partial^x \partial^\beta \rangle \langle \partial^\beta \partial^i \partial^\nu \rangle \langle \partial^\nu \partial^\alpha \partial^\alpha \rangle \langle \partial^\mu \partial^y \partial^z \rangle \\
& -\frac{1}{24} \langle \partial^j \rangle_1 \langle \partial^i \partial^\nu \partial^x \rangle \langle \partial^\alpha \partial^\mu \partial^\beta \rangle \langle \partial^\beta \partial^\nu \partial^\alpha \rangle \langle \partial^\mu \partial^y \partial^z \rangle \\
& +\frac{1}{24} \langle \partial^\mu \rangle_1 \langle \partial^\mu \partial^i \partial^x \rangle \langle \partial^\nu \partial^\beta \partial^\alpha \rangle \langle \partial^\beta \partial^j \partial^\alpha \rangle \langle \partial^\nu \partial^y \partial^z \rangle \\
& +\frac{1}{576} \langle \partial^i \partial^\mu \partial^\mu \partial^x \rangle \langle \partial^\nu \partial^\beta \partial^\alpha \rangle \langle \partial^\beta \partial^j \partial^\alpha \rangle \langle \partial^\nu \partial^y \partial^z \rangle \\
& + \cdots
\end{aligned}$$

$$\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\rangle_1\langle\partial^\nu\partial^y\partial^z\rangle_1 \mapsto \frac{1}{24}\langle\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^j\rangle\langle\partial^i\partial^z\partial^\alpha\rangle\langle\partial^\alpha\partial^\nu\partial^y\rangle\langle\partial^\nu\partial^\beta\partial^\beta\rangle + \cdots$$

$$\begin{aligned}\langle\partial^\mu\rangle_1\langle\partial^\mu\partial^x\partial^\nu\rangle_1\langle\partial^\nu\partial^y\partial^z\rangle \mapsto\ & \frac{1}{24}\langle\partial^j\rangle_1\langle\partial^i\partial^\nu\partial^\mu\rangle\langle\partial^\mu\partial^\alpha\partial^x\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^\nu\partial^y\partial^z\rangle\\ & -\frac{1}{24}\langle\partial^\mu\rangle_1\langle\partial^j\partial^x\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^\nu\partial^y\partial^z\rangle\\ & + \cdots\end{aligned}$$

$$\begin{aligned}\langle\partial^x\partial^y\partial^z\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1 \mapsto\ & \langle\partial^j\rangle_1\langle\partial^i\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^z\rangle\langle\partial^\beta\partial^x\partial^y\rangle\\ & -3\langle\partial^\mu\rangle_1\langle\partial^j\partial^x\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^y\partial^z\rangle\\ & + \cdots\end{aligned}$$

$$\langle\partial^x\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^y\partial^z\rangle_1 \mapsto \frac{1}{24}\langle\partial^j\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^\alpha\partial^y\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle + \cdots$$

$$\begin{aligned}\langle\partial^x\partial^y\partial^z\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\partial^\nu\rangle \mapsto\ & -\frac{1}{24}\langle\partial^x\partial^y\partial^z\partial^j\rangle\langle\partial^i\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\\ & -\langle\partial^j\rangle_1\langle\partial^i\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^z\rangle\langle\partial^\beta\partial^x\partial^y\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\\ & +3\langle\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^x\rangle\langle\partial^j\partial^\beta\partial^\alpha\rangle\langle\partial^\beta\partial^y\partial^z\rangle\langle\partial^\alpha\partial^\nu\partial^\nu\rangle\\ & +\frac{1}{8}\langle\partial^i\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^\beta\partial^\alpha\rangle\langle\partial^\beta\partial^y\partial^z\rangle\langle\partial^\alpha\partial^\nu\partial^\nu\rangle\\ & + \cdots\end{aligned}$$

$$\begin{aligned}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\nu\partial^y\partial^z\rangle_1 \mapsto\ & -\langle\partial^j\rangle_1\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^z\rangle\langle\partial^\alpha\partial^\beta\partial^y\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle\\ & +2\langle\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^y\rangle\langle\partial^x\partial^\alpha\partial^\nu\rangle\langle\partial^j\partial^\beta\partial^z\rangle\langle\partial^\beta\partial^\alpha\partial^\nu\rangle\\ & +\frac{1}{12}\langle\partial^i\partial^y\partial^\mu\partial^\mu\rangle\langle\partial^x\partial^\alpha\partial^\nu\rangle\langle\partial^j\partial^\beta\partial^z\rangle\langle\partial^\beta\partial^\alpha\partial^\nu\rangle\\ & + \cdots\end{aligned}$$

$$\langle\partial^x\partial^y\partial^z\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle_1 \mapsto \frac{1}{24}\langle\partial^x\partial^y\partial^z\partial^j\rangle\langle\partial^i\partial^\nu\partial^\mu\rangle\langle\partial^\mu\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle + \cdots$$

$$\begin{aligned}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\partial^z\rangle_1 \mapsto\ & -\frac{1}{24}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^j\partial^\nu\partial^\nu\partial^z\rangle\\ & -\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^j\rangle_1\langle\partial^i\partial^\alpha\partial^z\rangle\langle\partial^\alpha\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle\\ & +\langle\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^z\rangle\langle\partial^j\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\nu\partial^\alpha\rangle\langle\partial^x\partial^y\partial^\alpha\rangle\\ & +\frac{1}{24}\langle\partial^i\partial^z\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\nu\partial^\alpha\rangle\langle\partial^x\partial^y\partial^\alpha\rangle\\ & + \cdots\end{aligned}$$

Checking the coefficients:

$$\langle\partial^x\partial^y\partial^z\partial^i\rangle\langle\partial^j\partial^\alpha\partial^\mu\rangle\langle\partial^\mu\partial^\alpha\partial^\nu\rangle\langle\partial^\nu\partial^\beta\partial^\beta\rangle : \frac{1}{1152}c_2 - \frac{1}{24}c_{15} + \frac{1}{24}c_{19} = 0.$$

$$\langle\partial^i\rangle_1\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^i\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle :$$
$$-\frac{1}{24}c_1 - \frac{1}{12}c_3 - \frac{1}{12}c_4 + \frac{1}{24}c_{11} + c_{12} - c_{15} - c_{18} - c_{20} = 0.$$

$$\langle\partial^i\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^y\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^j\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle :$$
$$-\frac{1}{576}c_3 + \frac{1}{576}c_4 + \frac{1}{24}c_{14} + \frac{1}{8}c_{15} + \frac{1}{12}c_{18} = 0.$$

$$\langle\partial^\mu\rangle_1\langle\partial^\mu\partial^i\partial^x\rangle\langle\partial^y\partial^z\partial^\nu\rangle\langle\partial^\nu\partial^j\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle :$$
$$-\frac{1}{24}c_3 + \frac{1}{24}c_4 + \frac{1}{24}c_7 - \frac{1}{24}c_{11} - 3c_{12} + 3c_{15} + 2c_{18} + c_{20} = 0.$$

3.5. Checking $\mathfrak{r}_4(E) = 0$

Since $l = 4$ case is new, the calculation is presented.

$$\begin{aligned}
\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^\nu\rangle_2 \mapsto \\
&-\frac{1}{5760}\langle\partial^x\partial^\mu\partial^j\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^i\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle \\
&+\frac{1}{960}\langle\partial^x\partial^\mu\partial^j\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^i\partial^\beta\partial^\alpha\rangle\langle\partial^\beta\partial^\gamma\partial^\alpha\rangle\langle\partial^\gamma\partial^\nu\partial^\nu\rangle \\
&+\frac{1}{576}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^i\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\partial^x_1\partial^\mu\rangle_2\langle\partial^\mu\partial^y\partial^z\rangle \mapsto &-\frac{1}{1920}\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^x\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^y\partial^z\rangle \\
&+\frac{1}{720}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^y\partial^z\rangle \\
&+\frac{1}{576}\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\gamma\partial^\mu\rangle\langle\partial^\gamma\partial^x\partial^\mu\rangle\langle\partial^j\partial^y\partial^z\rangle \\
&+\frac{1}{576}\langle\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^\alpha\partial^i\partial^\beta\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle \\
&+\frac{1}{576}\langle\partial^i\partial^\beta\partial^\mu\rangle\langle\partial^\gamma\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\gamma\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle \\
&+\frac{1}{576}\langle\partial^i\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^x\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^j\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\partial^x\partial^\mu_1\rangle_2\langle\partial^\mu\partial^y\partial^z\rangle \mapsto\ & -\frac{1}{5760}\langle\partial^i\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\partial^y\partial^z\rangle\\
&+\frac{1}{960}\langle\partial^i\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\gamma\partial^\nu\rangle\langle\partial^\gamma\partial^\mu\partial^\mu\rangle\langle\partial^j\partial^y\partial^z\rangle\\
&+\frac{1}{576}\langle\partial^\mu\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^i\partial^\beta\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle\\
&+\frac{1}{576}\langle\partial^i\partial^\beta\partial^x\rangle\langle\partial^\gamma\partial^\mu\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\gamma\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle\\
&+\frac{1}{576}\langle\partial^i\partial^\beta\partial^x\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^\mu\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^j\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\partial^x\partial^y\partial^z\partial^\mu\rangle_1\langle\partial^\mu\partial^\nu\partial^\nu\rangle \mapsto & +\frac{1}{24}\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\nu\partial^\beta\partial^\alpha\rangle\langle\partial^\beta\partial^\gamma\partial^z\rangle\langle\partial^\gamma\partial^x\partial^y\rangle\langle\partial^j\partial^\mu\partial^\mu\rangle\\
&-\frac{1}{8}\langle\partial^i\partial^y\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^x\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^z\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\\
&-\frac{1}{8}\langle\partial^i\partial^\mu\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^x\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^y\partial^z\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\\
&+\frac{1}{8}\langle\partial^i\partial^\beta\partial^x\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\alpha\partial^\mu\rangle\langle\partial^\alpha\partial^y\partial^z\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^i\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\alpha\partial^z\rangle\langle\partial^\alpha\partial^x\partial^y\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^i\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^\gamma\partial^z\rangle\langle\partial^\gamma\partial^x\partial^y\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^\nu\partial^y\partial^z\rangle_1 \mapsto & -\frac{1}{24}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^y\partial^z\rangle\\
&-\frac{1}{24}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^x\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^y\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{12}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\beta\partial^y\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\alpha\partial^z\rangle\langle\partial^\alpha\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^\beta\partial^z\rangle\langle\partial^\beta\partial^\gamma\partial^y\rangle\langle\partial^\gamma\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle
\end{aligned}$$

$$\begin{aligned}
\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\partial^z\rangle_1 \mapsto & +\frac{1}{24}\langle\partial^x\partial^y\partial^j\rangle\langle\partial^i\partial^\alpha\partial^\nu\rangle\langle\partial^\nu\partial^\beta\partial^\alpha\rangle\langle\partial^\beta\partial^\gamma\partial^z\rangle\langle\partial^\gamma\partial^\mu\partial^\mu\rangle\\
&-\frac{1}{24}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^z\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\nu\partial^\nu\rangle\\
&-\frac{1}{24}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\mu\partial^z\rangle\\
&+\frac{1}{24}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\beta\partial^\mu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\alpha\partial^z\rangle\langle\partial^\alpha\partial^\nu\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\beta\partial^z\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\alpha\partial^\nu\rangle\langle\partial^\alpha\partial^\mu\partial^\nu\rangle\\
&+\frac{1}{24}\langle\partial^x\partial^y\partial^\mu\rangle\langle\partial^i\partial^\beta\partial^z\rangle\langle\partial^\beta\partial^\gamma\partial^\nu\rangle\langle\partial^\gamma\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle
\end{aligned}$$

The other graphs all have $\mathfrak{r}_4(\Gamma) = 0$.

Now let's check all the coefficients:

$$\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^i\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle \ :$$
$$\frac{1}{576}c_1 + \frac{1}{288}c_3 + \frac{1}{288}c_4 + \frac{1}{12}c_{15} + \frac{1}{24}c_{18} = 0.$$

$$\langle\partial^i\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^x\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^\mu\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^y\partial^z\rangle \ :$$
$$\frac{1}{384}c_3 + \frac{1}{1152}c_4 - \frac{1}{8}c_{15} - \frac{1}{24}c_{18} + \frac{1}{24}c_{20} = 0.$$

$$\langle\partial^x\partial^\mu\partial^j\rangle\langle\partial^\mu\partial^y\partial^z\rangle\langle\partial^i\partial^\beta\partial^\nu\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle \ :$$
$$\frac{1}{1152}c_1 + \frac{1}{24}c_{15} - \frac{1}{24}c_{20} = 0.$$

$$\langle\partial^j\partial^x\partial^\mu\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^y\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^z\rangle\langle\partial^\beta\partial^\gamma\partial^\gamma\rangle \ :$$
$$\frac{1}{576}c_3 - \frac{1}{576}c_4 - \frac{1}{4}c_{15} - \frac{1}{8}c_{18} = 0.$$

3.6. Conclusion

By Lemma 1 in [7], $\mathfrak{r}_l(E) = 0$ for $l \geq 3$ (respectively $4, 5$) for $(g, n, k) = (2, 1, 2)$, (respectively $(2, 2, 2), (2, 3, 2)$). Therefore, Invariance Conjectures 1 and 2 are proved. By a Betti number calculation of E. Getzler [4], they are the only tautological equations for the corresponding (g, n, k). Therefore, Invariance Conjecture 3 also holds.

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Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, U.S.A.

E-mail: arcara@math.utah.edu

E-mail: yplee@math.utah.edu