Bulletin of the Institute of Mathematics
Academia Sinica (New Series)
Vol. **1** (2006), No. 4, pp. 523-535

OSCILLATION OF SEMILINEAR ELLIPTIC EQUATIONS WITH INTEGRABLE COEFFICIENTS

BY

ZHITING XU

Abstract

.

Some oscillation criteria for the second order semilinear elliptic differential equation

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_jy] + p(x)f(y) = 0, \quad x \in \Omega(r_0), \tag{E}$$

are established. Particularly, Hille's theorem [Trans. Amer. Math. Soc. 64, 234-252(1948)] is extended to (E).

1. Introduction and Preliminaries

In this paper we treat the oscillation problem of the second order semilinear elliptic differential equation of the form

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_jy] + p(x)f(y) = 0 \tag{1.1}$$

in an exterior domain $\Omega(r_0) \subseteq \mathbb{R}^N$, where $x = (x_1, \cdots, x_N) \in \Omega(r_0)$, $N \geq 2$, $D_iy = \partial y/\partial x_i$ for all i, $\Omega(r_0) = \{x \in \mathbb{R}^N : |x| \geq r_0\}$ for some $r_0 > 0$, $|\cdot|$ is the usual Euclidean norm in $\mathbb{R}^N$.

Throughout this paper, we assume that the following conditions hold.

Received October 6, 2004.

AMS Subject Classification: 35B05, 35J60, 34C10.

Key words and phrases: Oscillation, semilinear elliptic equations, second order, integrable coefficient.

(A1) $A = (a_{ij})$ is a real symmetric positive definite matrix function with $a_{ij} \in C_{loc}^{1+\nu}(\Omega(r_0), \mathbb{R})$ for all i, j, and $\nu \in (0,1)$.
Denote by $\lambda_{\max}(x) \in C(\Omega(r_0), \mathbb{R}^+)$ the largest eigenvalue of the matrix A. We suppose that there exists a function $\lambda \in C([r_0, \infty), \mathbb{R}^+)$ such that

$$\lambda(r) \geq \max_{|x|=r} \lambda_{\max}(x) \quad \text{for} \quad r \geq r_0, \quad \text{and} \quad \int_{r_0}^{\infty} \frac{s^{1-N}}{\lambda(s)} ds = \infty;$$

(A2) $p \in C_{loc}^{\nu}(\Omega(r_0), \mathbb{R})$, $p(x)$ does not eventually vanish;

(A3) $f \in C(\mathbb{R}, \mathbb{R}) \cup C^1(\mathbb{R} - \{0\}, \mathbb{R})$, $yf(y) > 0$ and $f'(y) \geq k > 0$ for all $y \neq 0$.

As usual, a function $y \in C_{loc}^{2+\nu}(\Omega(r_0), \mathbb{R})$ is called a solution of (1.1) if $y(x)$ satisfies (1.1) for all $x \in \Omega(r_0)$. We restrict our attention only the nontrivial solution of (1.1), i.e., to the solution $y(x)$ satisfying $\sup\{|y(x)| : x \in \Omega(r)\} > 0$ for every $r \geq r_0$. Regarding the question of existence of solution of (1.1) we refer the reader to the monograph [2]. A nontrivial solution $y(x)$ of (1.1) is said to be oscillatory in $\Omega(r_0)$ if the set $\{x \in \Omega(r_0) : y(x) = 0\}$ is unbounded, otherwise it is said to be nonoscillatory. (1.1) called oscillatory if all its nontrivial solutions are oscillatory.

For many years, a great deal of attention has been paid to the oscillation of (1.1) with variable coefficient $p(x)$, and various approaches have evolved. One of the most effective methods of procedure is that which seeks to reduce the problem to the one-dimensional Riccati inequality. Particularly, with outstanding contributions from Noussair and Swanson [7], some classical oscillation theorems (such as Fite [1], Kamenev [5, 6] and others) for second order linear ordinary differential equation

$$y''(t) + p(t)y(t) = 0, \quad p \in C([t_0, \infty), \mathbb{R}) \tag{1.2}$$

have been extended to (1.1) (see, for example, [7, 9-15] and the references cited therein). However, as we know, the oscillatory theory for (1.1) has not yet been elaborated unlike that of (1.2) (see, [8]). In view of this fact, it is therefore of interest to find new oscillation results for the semilinear elliptic equation (1.1).

In [7], Noussair and Swanson discussed the oscillation of (1.1) and gave some oscillation theorems, one of which is as follows.

Theorem 1.1. *If*

$$\int_{\Omega(r_0)} p(x)dx = \infty,$$

then (1.1) *is oscillatory.*

This result is given in [7] in a more general form. But the above particular form of Noussair and Swanson's theorem is base one.

The case

$$\int_{\Omega(r_0)} p(x)dx < \infty$$

remains of interest and can produce either oscillatory or nonoscillatory behavior for (1.1).

The motivation for present work has come chiefly from the idea due to Hille [4] and Noussair and Swanson [7]. The aim of this paper is to study oscillation properties of (1.1) via Riccati technique and derive new oscillation criteria for this equation under the assumption

$$0 < \int_{\Omega(r)} p(x)dx < \infty, \quad \text{for } r \geq r_0 \tag{1.3}$$

Especially, thereby extending Hille's Theorem to (1.1).

The following notations will be used throughout this paper.

$$P_M(r) = \int_{S_r} p(x)d\sigma, \quad P(r) = \int_{\Omega(r)} p(x)dx,$$

and

$$\varphi(r) = \frac{kr^{1-N}}{\omega_N \lambda(r)}, \quad \Psi(r) = \int_{r_0}^{r} \varphi(s)ds,$$

where $S_r = \{x \in \mathbb{R}^N : |x| = r\}$ for $r > 0$, $d\sigma$ and ω_N denote the spherical integral element in $\mathbb{R}^N$ and the surface of the $N-$dimensional unit sphere, respectively.

The following Lemma will be useful for establishing oscillation criteria for (1.1). It is similar to Hartman's Lemma [3].

Lemma 1.1. *Let* (1.3) *hold. Suppose that* (1.1) *has a nonoscillatory solution* $y(x) \neq 0$ *for* $x \in \Omega(r_1)$, $(r_1 \geq r_0)$, *and let*

$$W(x) = \frac{1}{f(y)}(A\nabla y)(x), \quad \text{and} \quad Z(r) = \int_{S_r} W(x) \cdot \mu(x) d\sigma,$$

then $Z(r) > 0$, *and satisfies*

$$Z'(r) + \varphi(r) Z^2(r) + P_M(r) \leq 0 \quad \text{for} \quad r \geq r_1. \tag{1.4}$$

Furthermore,

$$Z(r) = \int_{\Omega(r)} f'(y)(W^T A^{-1} W)(x) d\sigma + P(r) \tag{1.5}$$

$$\geq \int_r^{\infty} \varphi(s) Z^2(s) ds + P(r), \tag{1.6}$$

where $\nabla y = (D_1 y, \cdots, D_N y)^T$, $\mu(x) = x/|x|$, $(x \neq 0)$, *denotes the outward unit normal.*

Proof. Differentiating $W(x)$ and making use of (1.1), we have

$$\operatorname{div} W(x) = -p(x) - f'(y)(W^T A^{-1} W)(x). \tag{1.7}$$

Then, by Green's formula, we get

$$Z'(r) = \int_{S_r} \operatorname{div} W(x) d\sigma = -P_M(r) - \int_{S_r} f'(y)(W^T A^{-1} W)(x) d\sigma. \tag{1.8}$$

In view of (A1), we find that

$$(W^T A^{-1} W)(x) \geq \lambda_{max}^{-1}(x) |W(x)|^2.$$

The Schwartz inequality gives that

$$\int_{S_r} |W(x)|^2 d\sigma \geq \frac{r^{1-N}}{\omega_N} \left[\int_{S_r} W(x) \cdot \mu(x) d\sigma \right]^2 = \frac{r^{1-N}}{\omega_N} Z^2(r).$$

Thus, by (1.8), we obtain

$$Z'(r) + \frac{kr^{1-N}}{\omega_N \lambda(r)} Z^2(r) + P_M(r) \le 0, \tag{1.9}$$

which follows that (1.4) holds.

On the other hand, using integrating (1.8) from b to r, $(b \ge r_1)$, we have

$$Z(r) - Z(b) + \int_b^r P_M(s)ds + \int_b^r d\tau \int_{S_\tau} f'(y)(W^T A^{-1} W)(x) d\sigma = 0. \tag{1.10}$$

By virtue of (1.3), if $u(r)$ is defined by

$$u(r) = \int_b^r d\tau \int_{S\tau} f'(y)(W^T A^{-1} W)(x) d\sigma,$$

then

$$\lim_{r\to\infty} [Z(r) + u(r)] = C, \tag{1.11}$$

where C is a finite constant.

Now, we show that

$$\lim_{r\to\infty} u(r) < \infty. \tag{1.12}$$

Otherwise, $\lim_{r\to\infty} u(r) = \infty$, then, by (1.11),

$$\lim_{r\to\infty} Z(r)u^{-1}(r) = -1.$$

Thus, there exists $r^* > b$ such that for $r \ge r^*$,

$$Z(r)u^{-1}(r) \le -\frac{1}{2}. \tag{1.13}$$

Note that

$$u(r) \ge \frac{k}{\omega_N} \int_b^r \frac{s^{1-N}}{\lambda(s)} Z^2(s) ds. \tag{1.14}$$

By (1.13), we have

$$\frac{k^2}{4\omega_N^2} \frac{r^{1-N}}{\lambda(r)} \le \frac{k^2}{\omega_N^2} \frac{r^{1-N}}{\lambda(r)} \frac{Z^2(r)}{u^2(r)} \le \frac{r^{1-N}}{\lambda(r)} Z^2(r) \left[\int_b^r \frac{s^{1-N}}{\lambda(r)} Z^2(s) ds\right]^{-2},$$

consequently,

$$\begin{aligned}\frac{k^2}{4\omega_N}\int_{r^*}^{r}\frac{s^{1-N}}{\lambda(s)}ds &\le \int_{r^*}^{r}\frac{r^{1-N}}{\lambda(r)}Z^2(r)\left[\int_{b}^{r}\frac{s^{1-N}}{\lambda(r)}Z^2(s)ds\right]^{-2}ds\\ &\le \left[\int_{b}^{r^*}\frac{s^{1-N}}{\lambda(r)}Z^2(s)ds\right]^{-2},\end{aligned}$$

which contradicts (A1). So (1.12) hold. From (1.11), it follows that $\lim_{r\to\infty} Z(r)$ exists. If $\lim_{r\to\infty} Z(r) = d \neq 0$, there exists a sufficiently large r_2 such that $Z^2(r) \ge d^2/2$ for $r \ge r_2$, then, by (1.12) and (1.14),

$$\frac{kd^2}{2\omega_N}\int_{r_2}\frac{s^{1-N}}{\lambda(s)}ds \le \lim_{r\to\infty} u(r) < \infty.$$

This contradicts (A1), which implies that $\lim_{r\to\infty} Z(r) = 0$. Taking limit in (1.10) as $r\to\infty$, we get (1.5) holds. Noting (1.5) and (1.3), we get $Z(r) > 0$ for $r \ge r_1$. From (1.5) and (1.14), we establish (1.6). Thus, the proof is compete. □

2. Main Results

In this section, we will give new oscillation criteria for (1.1). First of all, we establish Hille-type oscillation theorem [4] for (1.1). Throughout this paper we always assume that condition (1.3) holds without further mentioning.

Theorem 2.1. *If*

$$\liminf_{r\to\infty}\Psi(r)P(r) > \frac{1}{4}, \tag{2.1}$$

then (1.1) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of (1.1). Without loss of generality, let us consider that $y = y(x) > 0$ on $\Omega(r_1)$, $(r_1 \ge r_0)$, then it follows from Lemma 1.1 that (1.6) holds. Moreover, by (2.1), there exist an $\alpha > 1/4$ and $r_2 \ge r_1$ such that

$$P(r) \ge \frac{\alpha}{\Psi(r)} \quad \text{for } r \ge r_2.$$

Hence, in view of (1.6), we find that $Z(r) \geq \alpha/\Psi(r)$ for $r \geq r_2$. Applying the same step, we get

$$\begin{aligned} Z(r) &\geq \int_r^\infty \frac{\alpha^2}{\Psi^2(r)}\varphi(s)ds + P(r) \\ &= \frac{\alpha^2+\alpha}{\Psi(r)} \quad \text{for } r \geq r_2. \end{aligned}$$

Repeating the above procedure n-times, we conclude that

$$Z(r) \geq \frac{\beta_n}{\Psi(r)} \quad \text{for } r \geq r_2,$$

where $\beta_1 = \alpha$ and $\beta_{n+1} = \beta_n^2 + \alpha$ for $n = 1, 2, \cdots$.

As we see, the sequence $\{\beta_n\}$ is nondecreasing and bounded, while $\lim_{r\to\infty} \beta_n = \beta$ is a solution the quadratic equation $\beta^2 - \beta + \alpha = 0$. This implies that $1 - 4\alpha \geq 0$ which contradicts $\alpha > 1/4$. □

Theorem 2.2. *If*

$$\liminf_{r\to\infty} \frac{\int_r^\infty \varphi(s)P^2(s)ds}{P(r)} > \frac{1}{4}, \tag{2.2}$$

then (1.1) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of (1.1). Without loss of generality, let us consider that $y = y(x) > 0$ on $\Omega(r_1)$, $(r_1 \geq r_0)$, then it follows from Lemma 1.1 that (1.6) holds. By (2.2), there exist an $\alpha > 1/4$ and $r_2 \geq r_1$ such that

$$\int_r^\infty \varphi(s)P^2(s)ds \geq \alpha P(r) \quad \text{for } r \geq r_2.$$

Using this inequality and (1.6), as in the proof of Theorem 2.1, we get

$$Z(r) \geq c_n P(r) \quad \text{for } r \geq r_2,$$

where $c_1 = 1$ and $c_{n+1} = \alpha c_n^2 + 1$ for $n = 1, 2, \cdots$.

It is easy to see that the sequence $\{c_n\}$ is nondecreasing and bounded, while $\lim_{r\to\infty} c_n = c$ is a solution the quadratic equation $\alpha c^2 - c + 1 = 0$. This implies that $1 - 4\alpha \geq 0$ which contradicts $\alpha > 1/4$. □

Theorem 2.3. *If*

$$\int_{r_0}^{\infty} \Psi^{\alpha}(s)P_M(s)ds = \infty \quad \text{for some } \alpha \in (0,1), \tag{2.3}$$

then (1.1) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of (1.1). Without loss of generality, let us consider that $y = y(x) > 0$ on $\Omega(r_1)$, ($r_1 \geq r_0$). By Lemma 1.1, (1.4) has a positive solution $Z(r)$ on $[r_1, \infty)$. Let $h(r) = \Psi^{\alpha}(r)Z(r)$ for $r \geq r_1$, then, by (1.4), for $r \geq r_1$,

$$\begin{aligned} h'(r) &\leq -\Psi^{\alpha}(r)P_M(r) - \Psi^{\alpha}(r)\varphi(r)\left[Z(r) - \frac{\alpha}{2\Psi(r)}\right]^2 + \frac{\alpha^2}{4}\Psi^{\alpha-2}(r)\varphi(r) \\ &\leq -\Psi^{\alpha}(r)P_M(r) + \frac{\alpha}{4}\Psi^{\alpha-2}(r)\varphi(r). \end{aligned}$$

Integrating this inequality and using (2.3), we lead to a contradiction. □

Theorem 2.4. *If*

$$\int_{r_0}^{\infty} \left[\Psi(s)P_M(s) - \frac{\varphi(s)}{4\Psi(s)}\right] ds = \infty, \tag{2.4}$$

then (1.1) *is oscillatory.*

Proof. Let $y = y(x)$ be a nonoscillatory solution of (1.1). Without loss of generality, let us consider that $y = y(x) > 0$ on $\Omega(r_1)$, ($r_1 \geq r_0$). By Lemma 1.1, (1.4) has a positive solution $Z(r)$ on $[r_1, \infty)$. Set $h(r) = \Psi(r)Z(r) - \frac{1}{2}$ for $r \geq r_1$, then, by (1.4), for $r \geq r_1$,

$$\begin{aligned} h'(r) &\leq \varphi(r)Z(r) + \Psi(r)[-\varphi(r)Z^2(r) - P_M(r)] \\ &= -\frac{\varphi(r)}{\Psi(r)}h^2(r) + \frac{\varphi(r)}{4\Psi(r)} - \Psi(r)P_M(r). \end{aligned}$$

Integrating this inequality and using (2.4), we conclude that there exists $b_1 \geq r_1$ such that $h(r) \leq -1$ for $r \geq b_1$. This implies $Z(r) < 0$ for $r \geq b_1$, which is a contradiction. □

Theorem 2.5. *If*

$$\lim_{r\to\infty}\left[\int_{r_0}^{r}P(s)ds\right]\left[(1+\Phi(s))\int_{r_0}^{r}\frac{\lambda(s)}{s^{2-N}}ds\right]^{-1/2}=\infty, \tag{2.5}$$

where

$$\Phi(r)=\int_{r_0}^{r}\exp\left[-\frac{4k}{\omega_N}\int_{r_0}^{s}\frac{\tau^{1-N}}{\lambda(\tau)}P(\tau)d\tau\right]ds,$$

then (1.1) *is oscillatory.*

Proof. Let $y=y(x)$ be a nonoscillatory solution of (1.1). Without loss of generality, let us consider that $y=y(x)>0$ on $\Omega(r_0)$. By Lemma 1.1, we have

$$Z(r)=u(r)+P(r),$$

where

$$u(r)=\int_{\Omega(r)}f'(y)(W^TA^{-1}W)(x)d\sigma.$$

Hence

$$\begin{aligned}-u'(r) &= \int_{S_r}f'(y)(W^TA^{-1}W)(x)d\sigma\geq\frac{k}{\omega_N}\frac{r^{1-N}}{\lambda(r)}Z^2(r)\\ &= \frac{k}{\omega_N}\frac{r^{1-N}}{\lambda(r)}[u(r)+P(r)]^2\geq\frac{4k}{\omega_N}\frac{r^{1-N}}{\lambda(r)}P(r)u(r).\end{aligned}$$

This implies that

$$u(r)\leq u(r_0)\exp\left(-\frac{4k}{\omega_N}\int_{r_0}^{r}\frac{s^{1-N}}{\lambda(s)}P(s)ds\right).$$

Thus

$$\int_{r_0}^{r}ds\int_{s}^{\infty}\frac{\tau^{1-N}}{\lambda(\tau)}Z^2(\tau)d\tau\leq k_1\Phi(r),\quad k_1>0,$$

and consequently,

$$\int_{r_0}^{r}(s-r_0)\frac{s^{1-N}}{\lambda(s)}Z^2(s)ds+(r-r_0)\int_{r}^{\infty}\frac{s^{1-N}}{\lambda(s)}Z^2(s)ds\leq k_1\Phi(r).$$

So

$$\int_{r_0}^{r} \frac{s^{2-N}}{\lambda(s)} Z^2(s)ds \le k_2^2[1+\Phi(r)], \quad k_2 > 0.$$

From this and Schwarz's inequality, we have

$$\begin{aligned} \left(\int_{r_0}^{r} Z(s)ds\right)^2 &\le \left(\int_{r_0}^{r} \frac{s^{2-N}}{\lambda(s)} Z^2(s)ds\right)\left(\int_{r_0}^{r} \frac{\lambda(s)}{s^{2-N}}ds\right) \\ &\le k_2^2[1+\Phi(r)]\int_{r_0}^{r} \frac{\lambda(s)}{s^{2-N}}ds. \end{aligned} \tag{2.6}$$

It from (1.6) and (2.6) that

$$\int_{r_0}^{r} P(s)ds \le \int_{r_0}^{r} Z(s)ds \le k_2\left[(1+\Phi(r))\int_{r_0}^{r} \frac{\lambda(s)}{s^{2-N}}ds\right]^{1/2},$$

i.e.,

$$\left[\int_{r_0}^{r} P(s)ds\right]\left[(1+\Phi(r))\int_{r_0}^{r} \frac{\lambda(s)}{s^{2-N}}ds\right]^{-1/2} \le k_2,$$

this contradicts (2.5). □

Corollary 2.1. *If* $\Phi(r) < \infty$ *and*

$$\lim_{r\to\infty}\left[\int_{r_0}^{r} P(s)ds\right]\left[\int_{r_0}^{r} \frac{\lambda(s)}{s^{2-N}}ds\right]^{-1/2} = \infty, \tag{2.7}$$

then (1.1) *is oscillatory.*

Example 2.1. Consider the semilinear elliptic equation

$$\frac{\partial}{\partial x_1}\left(\frac{1}{|x|^2}\frac{\partial y}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{|x|^2}\frac{\partial y}{\partial x_2}\right) + \frac{\nu}{|x|^4}(y+y^3) = 0, \tag{2.8}$$

where $x \in \Omega(1)$, $N = 2$, and $\nu > 1$. Clearly

$$\lambda(r) = \frac{1}{r^2}, \quad p(x) = \frac{\nu}{|x|^4},$$

then

$$\varphi(r) = \frac{r}{2\pi},\ \Psi(r) = \frac{r^2-1}{4\pi},\ P(r) = \frac{\pi\nu}{r^2}.$$

Thus

$$\liminf_{r\to\infty} \Psi(r)P(r) = \frac{\nu}{4},$$

or

$$\liminf_{r\to\infty} \frac{\int_r^\infty \varphi(s)P^2(s)ds}{P(r)} = \frac{\nu}{4}.$$

Thus, by Theorem 2.1 or Theorem 2.2, (2.8) is oscillatory for $\nu > 1$.

Example 2.2. Consider the semilinear elliptic equation

$$\frac{\partial}{\partial x_1}\left(\frac{1}{|x|^2}\frac{\partial y}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{|x|^2}\frac{\partial y}{\partial x_2}\right) + \frac{1+k\sin|x|}{|x|^\gamma}(y+y^5) = 0, \qquad (2.9)$$

where $x \in \Omega(1)$, $N = 2$, $k \in \mathbb{R}$, and $2 < \gamma \leq 3$. Clearly

$$\lambda(r) = \frac{1}{r^2}, \quad p(x) = \frac{1+k\sin|x|}{|x|^\gamma},$$

then

$$\varphi(r) = \frac{r}{2\pi},\ \Psi(r) = \frac{r^2-1}{4\pi},\ P_M(r) = \frac{2\pi(1+k\sin r)}{r^{\gamma-1}}.$$

Thus

$$\int_1^\infty \Psi^{1/2}(r)P_M(r)dr = \sqrt{\pi}\int_1^\infty \frac{(1+k\sin r)(r^2-1)^{1/2}}{r^{\gamma-1}}dr = \infty \quad \text{for } \gamma \leq 3.$$

Thus, by Theorem 2.3, (2.9) is oscillatory for $2 < \gamma \leq 3$.

Example 2.3. Consider the semilinear elliptic equation

$$\frac{\partial}{\partial x_1}\left(\frac{1}{|x|}\frac{\partial y}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{|x|}\frac{\partial y}{\partial x_2}\right) + \frac{3-2|x|\cos\sqrt{|x|}-\sqrt{|x|}\sin\sqrt{|x|}}{|x|^3}y = 0, \qquad (2.10)$$

where $x \in \Omega(1)$, $N = 2$, and $0 < \gamma \leq 1$. Clearly

$$\lambda(r) = \frac{1}{r}, \quad p(x) = \frac{3-2|x|\cos\sqrt{|x|}-\sqrt{|x|}\sin\sqrt{|x|}}{|x|^3}$$

then

$$\begin{aligned} P(r) &= \int_{\Omega(r)} p(x)dx = \frac{2\pi(3-2\cos\sqrt{r})}{r} \ge \frac{2\pi}{r}, \\ \Phi(r) &= \int_1^r \exp\left[-\frac{4k}{\omega_N}\int_1^s \frac{\tau^{1-N}}{\lambda(\tau)}P(\tau)d\tau\right] ds \\ &\le \int_1^r \exp\left(-4\int_1^s \frac{1}{\tau}d\tau\right) ds \\ &\le \int_1^r \frac{1}{s^4}ds < \infty \quad \text{as } r\to\infty, \end{aligned}$$

and

$$\begin{aligned} \lim_{r\to\infty}\left[\int_1^r P(s)ds\right]\left[\int_1^r \frac{\lambda(s)}{s^{2-N}}ds\right]^{-1/2} &\ge \lim_{r\to\infty}\left[\int_1^r \frac{2\pi}{s}ds\right]\left[\int_1^r \frac{1}{s}ds\right]^{-1/2} \\ &= \lim_{r\to\infty} 2\pi\sqrt{\ln r} = \infty. \end{aligned}$$

Thus, by Corollary 2.1, (2.10) is oscillatory.

References

1. W. B. Fite, Concerning the zeros of the solutions of certain differential equations, *Trans. Amer. Math. Soc.*, **19**(1918), 341-352.
2. D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Now York. 1983.
3. P. Hartman, *Ordinary Differential Equations*, Wiley, Now York. 1982.
4. E. Hille, Nonoscillation theorems, *Trans. Amer. Math. Soc.*, **64**(1948), 234-252.
5. I. V. Kamenev, Oscillation of solution of a second order differential equation with an “integrally small” coefficient, *Differencial’nye Uravnenija.*, **13**(1977), 2141-2148. (In Russian)
6. I. V. Kamenev, An integral criterion for oscillation of linear differential equations, *Math. Zamtki.*, **23** (1978), 249-251. (In Russian)
7. E. S. Noussair and C. A. Swanson, Oscillation of semilinear elliptic inequalities by Riccati transformation, *Canad. J. Math.*, **32**(4)(1980), 908-923.
8. C. A. Swanson, *Comparison and Oscillatory Theory of Linear Differential Equations*, Academic Press, New York. 1968.
9. C. A. Swanson, Semilinear second order elliptic oscillation, *Cand. Math. Bull.*, **22**(1979), 139-157.
10. Z. T. Xu, Oscillation of second order elliptic partial differential equations with a “weakly integrally small” coefficient, *J. Systems Sci. Math. Sci.*, **18**(4)(1998), 478-484. (In Chinese)

11. Z. T. Xu, Oscillation of second order nonlinear differential elliptic equations, *Appl. Math. J. Chinese Univ. Ser. (A).*, **17**(1)(2002), 29-36. (In Chinese)

12. Z. T. Xu, Riccati technique and oscillation of semilinear elliptic equations, *Chinese J. Contemp. Math.*, **24**(4)(2003), 329-340.

13. Z. T. Xu, D. K. Ma and B. G. Jia, Oscillation theorems for elliptic equations of second order, *Acta Math. Sci. Ser. A Chin. Ed.*, **24**(2)(2004), 144-151. (In Chinese)

14. Z. T. Xu and H. Y. Xing, Kamenev-type oscillation criteria for semilinear elliptic differential equations, *Northeast. Math. J.*, **22**(2)(2004), 153-160.

15. B. G. Zhang, T. Zhao and B. S. Lalli, Oscillation criteria for nonlinear second order elliptic differential equations, *Chinese Ann. Math. Ser. B*, **17**(1)(1996), 89-102.

School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, P. R. China.

E-mail: xztxhyyj@pub.guangzhou.gd.cn