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APPROXIMATING FIXED POINTS OF NONEXPANSIVE MAPPINGS

BY

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Abstract

Let D be a subset of a normed space X and $T : D \to X$ be a nonexpansive mapping. In this paper we consider the following iteration method which generalizes Ishikawa iteration process:

$$\begin{aligned} x_{n+1} &= t_n^{(1)}T(t_n^{(2)}T(\cdots T(t_n^{(k)}Tx_n + (1-t_n^{(k)})x_n + u_n^{(k)}) + \cdots) \\ &\quad +(1-t_n^{(2)})x_n + u_n^{(2)}) + (1-t_n^{(1)})x_n + u_n^{(1)}, \end{aligned}$$

$n = 1, 2, 3 \ldots$, where $0 \le t_n^{(i)} \le 1$ for all $n \ge 1$ and $i = 1, \ldots, k$, and sequences $\{x_n\}$ and $\{u_n^{(i)}\}$, $i = 1, \ldots, k$, are in D.

We improve several results in [2], concerning approximation of fixed points of T.

1. Introduction

Let D be a subset of a normed space X. We say that a mapping $T : D \to X$ is nonexpansive if for all $x, y \in D$, $||Tx - Ty|| \le ||x - y||$ holds. During last four decades many authors have investigated nonexpansive mappings and the set of its fixed points. Browder [1] and Kirk [12] have shown that nonexpansive mapping T which maps a closed, bounded, convex subset C of a uniformly convex Banach space into itself has a nonempty fixed point set in C. In almost all papers authors used some iteration method for such investigations. However, in general, for arbitrary $x \in C$ the Picard iterates $T^n x$

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do not converge to a fixed point of T. Genel and Lindenstrauss [6] showed that there exists a nonexpansive mapping T defined on a closed, bouded and convex subset C of Hilbert space H such that the sequence $\{x_n\}$ defined by the recurrent formula $x_{n+1} = (x_n + Tx_n)/2$ does not converge. The sequence defined by $x_{n+1} = (1 - t_n)x_n + t_n Tx_n$, where $\{t_n\}$ is a real sequence whose terms belong to interval $[0, 1]$, has been investigated by Dotson [3], Edelstein [4], Groetsch [8], Ishikawa [9], Johnson [11], Krasnosel'skiǐ [13], Outlaw [16], Senter and Dotson [17] and others. They showed that these iterative methods may be used to find a fixed point of a nonexpansive mapping T mainly in a Hilbert space or a uniformly convex Banach space or a strictly convex Banach space. This sequence is considered as an iterative process of the type introduced by W. R. Mann [14]. Ishikawa [10] first used this iterative method for nonexpansive mappings without any assumption on convexity of the Banach space X. In [10] he proved the following theorem.

Theorem A. *Let D be a subset of a normed space X and $T : D \to X$ be a nonexpansive mapping. Given a sequence $\{x_n\}$ in D and a real sequence $\{t_n\}$, satisfying*

(a) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,
(b) $x_{n+1} = (1 - t_n)x_n + t_n Tx_n$, *for* $n = 1, 2, 3, \dots$,

if $\{x_n\}$ is bounded, then $||Tx_n - x_n|| \to 0$ as $n \to \infty$.

If the sequence $\{x_n\}$ in D is defined by the following recurrent formula

$$x_{n+1} = t_n T(s_n Tx_n + (1 - s_n)x_n) + (1 - t_n)x_n, \ x_1 \in D, \ n = 1, 2, 3 \dots,$$

where $\{t_n\}$ and $\{s_n\}$ are real sequences whose terms belong to the interval $[0, 1]$, we say that $\{x_n\}$ satisfies an Ishikawa iteration process (see [9]). In [2] Deng extends Theorem A to the Ishikawa iteration process. In this paper we consider the following iteration method:

$$\begin{aligned} x_{n+1} \ = \ & t_n^{(1)} T(t_n^{(2)} T(\cdots T(t_n^{(k)} Tx_n + (1 - t_n^{(k)})x_n + u_n^{(k)}) + \cdots) \\ & + (1 - t_n^{(2)})x_n + u_n^{(2)}) + (1 - t_n^{(1)})x_n + u_n^{(1)}, \end{aligned} \tag{1}$$

$n = 1, 2, 3 \dots$, where $0 \le t_n^{(i)} \le 1$ for all $n \ge 1$ and $i = 1, 2, \dots, k$.

This iteration process generalizes the Ishikawa iteration process. We prove an analogous theorem to Theorem A and Theorem 1 in [2]. These theorems will be consequences of our theorem. Also we generalize other results from [2].

2. Auxiliary Results

In this section we prove several auxiliary results which we will apply in the last section.

One can easily prove the following lemma.

Lemma 1. *Suppose that* $\{a_n\}$ *is a sequence of real numbers bounded from below, such that*

$$(\forall \varepsilon > 0)(\exists n_0 \in \mathbf{N})(\forall n \geq n_0)(\forall k \in \mathbf{N})\ a_{n+k} < a_n + \varepsilon.$$

Then the finite limit $\lim_{n\to\infty} a_n$ *exists.*

The next lemma is an easy consequence of Lemma 1.

Corollary 1.([22]) *Suppose that* $\{a_n\}$ *and* $\{b_n\}$ *are two sequences of nonnegative numbers such that* $a_{n+1} \leq a_n + b_n$ *for all* $n \geq 1$. *If* $\sum_{n=1}^{\infty} b_n < \infty$, *then the finite limit* $\lim_{n\to\infty} a_n$ *exists.*

The following lemma shows that the condition $\lim_{n\to\infty} ||a_n|| = d$ in [2, Lemma 2], may be replaced with $\liminf_{n\to\infty} ||a_n|| = d$.

Lemma 2. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of a normed space* X *and* $\{t_n\}$ *a sequence of real numbers. If the following conditions*

(a) $0 \leq t_n \leq t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,
(b) $a_{n+1} = (1 - t_n)a_n + t_n b_n$ *for all* $n \geq 1$,
(c) $\limsup_{n\to\infty} ||b_n|| < +\infty$,

are satisfied, then $\limsup_{n\to\infty} ||a_n|| \leq \limsup_{n\to\infty} ||b_n||$.

Proof. From (b) we obtain

$$a_n = a_1 \prod_{i=1}^{n-1}(1 - t_i) + \sum_{i=1}^{n-1} \prod_{j=i+1}^{n-1} (1 - t_j)t_i b_i.$$

Thus we have

$$\begin{aligned}||a_n|| &\leq ||a_1||\prod_{i=1}^{n-1}(1-t_i)+\sum_{i=1}^{n-1}\prod_{j=i+1}^{n-1}(1-t_j)t_i||b_i||\\ &= ||a_1||\prod_{i=1}^{n-1}(1-t_i)+\sum_{i=1}^{n-1}\Big(\prod_{j=i+1}^{n-1}(1-t_j)-\prod_{j=i}^{n-1}(1-t_j)\Big)||b_i||.\end{aligned}$$

From (c) we have that for each $\varepsilon>0$ there exists $n_1\in\mathbf{N}$ such that for $n\geq n_1$

$$||b_n||<d+\varepsilon,$$

holds, where $d=\limsup_{n\to\infty}||b_n||$.

On the other hand, from (a) we have that for each $\varepsilon>0$ there exists $n_0\in\mathbf{N}$ such that for $n\geq n_0+n_1$

$$\prod_{i=n_1}^{n-1}(1-t_i)\leq e^{-\sum_{i=n_1}^{n-1}t_i}<\varepsilon,$$

holds, here we use the following inequality $1+x\leq e^x,\ x\in\mathbf{R}$.

Thus for $n\geq n_0+n_1$ we have

$$\begin{aligned}&\sum_{i=1}^{n-1}\Big(\prod_{j=i+1}^{n-1}(1-t_j)-\prod_{j=i}^{n-1}(1-t_j)\Big)||b_i||\\ &\leq\ ||\{b_n\}||_\infty\Big(\prod_{j=n_1}^{n-1}(1-t_j)-\prod_{j=1}^{n-1}(1-t_j)\Big)+(d+\varepsilon)\Big(1-\prod_{j=n_1}^{n-1}(1-t_j)\Big),\\ &\leq\ 2\varepsilon||\{b_n\}||_\infty+(d+\varepsilon),\end{aligned}$$

where $||\{b_n\}||_\infty=\sup_{i\in N}||\{b_i\}||$. From (c) we have $||\{b_n\}||_\infty<\infty$. From all of the above we have

$$||a_n||\leq\varepsilon||a_1||+2\varepsilon||\{b_n\}||_\infty+(d+\varepsilon)$$

for $n\geq n_0+n_1$. Since $\varepsilon>0$ is arbitrary we obtain the result. □

Combining Lemma 2 and Lemma 2 in [2] we obtain the following lemma.

Lemma 3. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be two sequences of a normed space* X *and* $\{t_n\}$ *a sequence of real numbers. If the following conditions*

(a) $0 \le t_n \le t < 1$ *and* $\sum_{n=1}^{\infty} t_n = \infty$,
(b) $a_{n+1} = (1-t_n)a_n + t_n b_n$ *for all* $n \ge 1$,
(c) $\liminf_{n\to\infty} ||a_n|| = d$,
(d) $\limsup_{n\to\infty} ||b_n|| \le d$ *and* $\{\sum_{i=1}^{n} t_i b_i\}$ *is bounded,*

are satisfied, then $d = 0$.

Remark 1. Note that Lemma 3 improves Lemma 2 in [2].

Lemma 4. *Let* D *be a subset of a normed space* X *and* $T : D \to X$ *be a nonexpansive mapping with a nonempty fixed points set* $F(T)$ *in* D. *Let sequences* $\{x_n\}$ *and* $\{u_n^{(i)}\}$, $i = 1, \dots, k$, *in* D *satisfy the recurrent formula* (1). *Then*

$$\begin{aligned} ||x_{n+1} - p|| \;\le\; & ||x_n - p|| + ||u_n^{(1)}|| + ||u_n^{(2)}||t_n^{(1)} + ||u_n^{(3)}||t_n^{(1)}t_n^{(2)} + \cdots \\ & + ||u_n^{(k)}||t_n^{(1)}t_n^{(2)}\cdots t_n^{(k-1)} \end{aligned}$$

for all $n \ge 1$ *and all* $p \in F(T)$.

Proof. We prove this lemma by induction. Let $k = 1$, then we have

$$\begin{aligned} ||x_{n+1} - p|| &= ||t_n^{(1)}Tx_n + (1-t_n^{(1)})x_n + u_n^{(1)} - p|| \\ &\le ||t_n^{(1)}(Tx_n - Tp) + (1-t_n^{(1)})(x_n - p)|| + ||u_n^{(1)}|| \\ &\le t_n^{(1)}||Tx_n - Tp|| + (1-t_n^{(1)})||x_n - p|| + ||u_n^{(1)}|| \\ &\le t_n^{(1)}||x_n - p|| + (1-t_n^{(1)})||x_n - p|| + ||u_n^{(1)}|| \\ &= ||x_n - p|| + ||u_n^{(1)}||, \end{aligned}$$

as desired.

Let

$$y_n = t_n^{(2)}T(\cdots T(t_n^{(k)}Tx_n + (1-t_n^{(k)})x_n + u_n^{(k)}) + \cdots) + (1-t_n^{(2)})x_n + u_n^{(2)}.$$

By the inductive hypothesis we have

$$\begin{aligned} ||y_n - p|| \;\le\; & ||x_n - p|| + ||u_n^{(2)}|| + ||u_n^{(3)}||t_n^{(2)} + ||u_n^{(4)}||t_n^{(2)}t_n^{(3)} + \cdots \\ & + ||u_n^{(k)}||t_n^{(2)}t_n^{(3)}\cdots t_n^{(k-1)}. \end{aligned} \tag{2}$$

Since $x_{n+1} = t_n^{(1)} Ty_n + (1 - t_n^{(1)})x_n + u_n^{(1)}$ we have

$$\begin{aligned} ||x_{n+1} - p|| &= ||t_n^{(1)} Ty_n + (1 - t_n^{(1)})x_n + u_n^{(1)} - p|| \\ &\le ||t_n^{(1)}(Ty_n - Tp) + (1 - t_n^{(1)})(x_n - p)|| + ||u_n^{(1)}|| \\ &\le t_n^{(1)}||Ty_n - Tp|| + (1 - t_n^{(1)})||x_n - p|| + ||u_n^{(1)}|| \\ &\le t_n^{(1)}||y_n - p|| + (1 - t_n^{(1)})||x_n - p|| + ||u_n^{(1)}||. \end{aligned}$$

From this and (2) the result follows. □

Lemma 5. *Let D be a subset of a normed space X and $T : D \to X$ be a nonexpansive mapping. Let sequences $\{x_n\}$ and $\{u_n^{(i)}\}$, $i = 1, \dots, k$, in D satisfy recurrent formula* (1). *Then*

$$\begin{aligned} ||x_{n+1} - x_n|| &\le (t_n^{(1)} + t_n^{(1)}t_n^{(2)} + \cdots + t_n^{(1)}t_n^{(2)} \cdots t_n^{(k)})||Tx_n - x_n|| + ||u_n^{(1)}|| \\ &\quad + ||u_n^{(2)}||t_n^{(1)} + ||u_n^{(3)}||t_n^{(1)}t_n^{(2)} + \cdots + ||u_n^{(k)}||t_n^{(1)}t_n^{(2)} \cdots t_n^{(k-1)}, \end{aligned}$$

$n = 1, 2, 3 \dots$.

Proof. First, let $k = 1$. Then

$$||x_{n+1} - x_n|| = ||t_n^{(1)} Tx_n + (1 - t_n^{(1)})x_n + u_n^{(1)} - x_n|| \le t_n^{(1)}||Tx_n - x_n|| + ||u_n^{(1)}||,$$

as desired.

Let us suppose that statement is true for $k - 1$ and let y_n be defined as in Lemma 4. Then we have

$$\begin{aligned} ||x_{n+1} - x_n|| &= ||t_n^{(1)} Ty_n + (1 - t_n^{(1)})x_n + u_n^{(1)} - x_n|| \\ &\le t_n^{(1)}||Ty_n - x_n|| + ||u_n^{(1)}|| \\ &\le t_n^{(1)}(\,||Ty_n - Tx_n|| + ||Tx_n - x_n||\,) + ||u_n^{(1)}|| \\ &\le t_n^{(1)}(\,||y_n - x_n|| + ||Tx_n - x_n||\,) + ||u_n^{(1)}||. \end{aligned}$$

By the inductive hypothesis we obtain

$$\begin{aligned}
&||x_{n+1}-x_n||\\
\le\ & t_n^{(1)}\ (\ (t_n^{(2)}+t_n^{(2)}t_n^{(3)}+\cdots+t_n^{(2)}t_n^{(3)}\cdots t_n^{(k)})||Tx_n-x_n||\\
&+||u_n^{(2)}||+||u_n^{(3)}||t_n^{(2)}+||u_n^{(4)}||t_n^{(2)}t_n^{(3)}+\cdots\\
&+||u_n^{(k)}||t_n^{(2)}t_n^{(3)}\cdots t_n^{(k-1)}+||Tx_n-x_n||\ )+||u_n^{(1)}||\\
=\ & (t_n^{(1)}+t_n^{(1)}t_n^{(2)}+\cdots+t_n^{(1)}t_n^{(2)}\cdots t_n^{(k)})||Tx_n-x_n||\\
&+||u_n^{(1)}||+||u_n^{(2)}||t_n^{(1)}+||u_n^{(3)}||t_n^{(1)}t_n^{(2)}+\cdots+||u_n^{(k)}||t_n^{(1)}t_n^{(2)}\cdots t_n^{(k-1)}.
\end{aligned}$$

This completes inductive proof. □

Lemma 6. *Let D be a subset of a normed space X and $T:D\to X$ be a nonexpansive mapping. Let sequences $\{x_n\}$ and $\{u_n\}$ in D satisfy recurrent formula* (1)*. Then*

$$\begin{aligned}
&||Tx_{n+1}-x_{n+1}||\\
\le\ & (1+2\,(t_n^{(1)}t_n^{(2)}+\cdots+t_n^{(1)}t_n^{(2)}\cdots t_n^{(k)}))||Tx_n-x_n||\\
&+2(||u_n^{(1)}||+||u_n^{(2)}||t_n^{(1)}+||u_n^{(3)}||t_n^{(1)}t_n^{(2)}+\cdots+||u_n^{(k)}||t_n^{(1)}t_n^{(2)}\cdots t_n^{(k-1)}),
\end{aligned}$$

$n=1,2,3\ldots.$

Proof. Let us define y_n as in Lemma 4. Then we have

$$\begin{aligned}
&||Tx_{n+1}-x_{n+1}||\\
\le\ & ||Tx_{n+1}-Tx_n||+||Tx_n-x_{n+1}||\\
\le\ & ||x_{n+1}-x_n||+||t_n^{(1)}Ty_n+(1-t_n^{(1)})x_n+u_n^{(1)}-Tx_n||\\
\le\ & ||x_{n+1}-x_n||+t_n^{(1)}||Ty_n-Tx_n||+(1-t_n^{(1)})||x_n-Tx_n||+||u_n^{(1)}||\\
\le\ & ||x_{n+1}-x_n||+t_n^{(1)}||y_n-x_n||+(1-t_n^{(1)})||Tx_n-x_n||+||u_n^{(1)}||.
\end{aligned}$$

By Lemma 5 we obtain the desired inequality. □

3. Main Results

We are now in a position to formulate and prove the main results in this paper.

Theorem 1. *Let D be a subset of a normed space X and $T : D \to X$ be a nonexpansive mapping. Given a sequence $\{x_n\}$ in D satisfying the recurrent formula* (1), *where $u_n^{(i)} = 0$ for all $n \geq 1$ and for all $i \in \{1, \dots, k\}$, and real sequences $\{t_n^{(i)}\}$, $i = 1, 2, \dots, k$, satisfying*

(a) $0 \leq t_n^{(1)} \leq t < 1$ *and* $\sum_{n=1}^{\infty} t_n^{(1)} = \infty$,

(b) $0 \leq t_n^{(i)} \leq 1$, $i = 2, \dots, k$,

(c) $\sum_{n=1}^{\infty} (t_n^{(1)} t_n^{(2)} + \cdots + t_n^{(1)} t_n^{(2)} \cdots t_n^{(k)}) < \infty$

(d) $\lim_{n\to\infty} (t_n^{(2)} + t_n^{(2)} t_n^{(3)} + \cdots + t_n^{(2)} t_n^{(3)} \cdots t_n^{(k)}) = 0$,

if $\{x_n\}$ is bounded , then $||Tx_n - x_n|| \to 0$ as $n \to \infty$.

Proof. Let us define y_n as in Lemma 4. Since $\{||Tx_n - x_n||\}$ is bounded, by Corollary 1, Lemma 6 and (c) we conclude that there exists the finite limit $\lim_{n\to\infty} ||Tx_n - x_n||$, say d. Let $a_n = Tx_n - x_n$ and let the sequence $\{b_n\}$ satisfy the equality $a_{n+1} = (1 - t_n^{(1)}) a_n + t_n^{(1)} b_n$, where we assume that $b_n = 0$ if $t_n^{(1)} = 0$. Then $b_n = {t_n^{(1)}}^{-1} (Tx_{n+1} - Tx_n) + Tx_n - Ty_n$ and

$$
\begin{aligned}
||b_n|| &\leq {t_n^{(1)}}^{-1} ||Tx_{n+1} - Tx_n|| + ||Ty_n - Tx_n|| \\
&\leq {t_n^{(1)}}^{-1} ||x_{n+1} - x_n|| + ||y_n - x_n|| \\
&\leq (1 + 2(t_n^{(2)} + \cdots + t_n^{(2)} t_n^{(3)} \cdots t_n^{(k)})) ||Tx_n - x_n||.
\end{aligned}
$$

By (d) we have $\limsup ||b_n|| \leq d$.

On the other hand we have

$$
\begin{aligned}
\left|\left| \sum_{i=1}^{n} t_i^{(1)} b_i \right|\right| &= \left|\left| \sum_{i=1}^{n} (Tx_{i+1} - Tx_i + t_i^{(1)} (Tx_i - Ty_i)) \right|\right| \\
&\leq ||x_{n+1} - x_1|| + \sum_{i=1}^{n} t_i^{(1)} || y_i - x_i || \\
&\leq ||x_{n+1} - x_1|| + \sum_{i=1}^{n} t_i^{(1)} (t_i^{(2)} + \cdots + t_i^{(2)} t_i^{(3)} \cdots t_i^{(k)}) ||Tx_i - x_i||.
\end{aligned}
$$

By (c) we can conclude that the last expression is bounded. Therefore, by Lemma 3 we obtain the result. □

Remark 2. Theorem 1 above generalizes Theorem 1 of Deng [2]. It is not only a generalization in the sense of our new iterative method, it also generalizes this theorem in the case $k = 2$ since in our theorem we have the weaker condition $\sum_{n=1}^{\infty} t_n^{(1)} t_n^{(2)} < \infty$ instead of $\sum_{n=1}^{\infty} t_n^{(2)} < \infty$ which appears in Theorem 1 [2]. This weaker condition is supplied by Lemma 6 which provides a better estimate than the estimate from [22], which is applied in Theorem 1 [2].

The following theorem, analogous to the Theorem 1, refers to the case when the sequence $\{u_n\}$ is not zero.

Theorem 2. *Let D be a subset of a normed space X and $T : D \to X$ be a nonexpansive mapping. Given sequences $\{x_n\}$ and $\{u_n^{(i)}\}$, $i = 1, \dots, k$, in D which satisfy recurrent formula* (1) *and real sequences $\{t_n^{(i)}\}$, $i = 1, 2, \dots, k$, satisfying*

(a) $0 < a \le t_n^{(1)} \le b < 1$,
(b) $0 \le t_n^{(i)} \le 1, \quad i = 2, \dots, k$,
(c) $\sum_{n=1}^{\infty} (t_n^{(1)} t_n^{(2)} + \cdots + t_n^{(1)} t_n^{(2)} \cdots t_n^{(k)}) < \infty$
(d) $\lim_{n\to\infty} (t_n^{(2)} + t_n^{(2)} t_n^{(3)} + \cdots + t_n^{(2)} t_n^{(3)} \cdots t_n^{(k)}) = 0$,
(e) $\sum_{n=1}^{\infty} ||u_n^{(i)}|| < \infty$, $i = 1, \dots, k$,

if $\{x_n\}$ is bounded , then $||Tx_n - x_n|| \to 0$ as $n \to \infty$.

Proof. Let us define y_n as in Lemma 4. Since $\{||Tx_n - x_n||\}$ is bounded, by Corollary 1, Lemma 6, (c) and (e) we conclude that there exists the finite limit $\lim_{n\to\infty} ||Tx_n - x_n|| = d$. Let $a_n = Tx_n - x_n$ and suppose the sequence $\{b_n\}$ satisfies the equality $a_{n+1} = (1 - t_n^{(1)})a_n + t_n^{(1)} b_n + u_n^{(1)}$. Then $b_n = {t_n^{(1)}}^{-1} (Tx_{n+1} - Tx_n - u_n^{(1)}) + Tx_n - Ty_n$ and

$$\begin{aligned}
||b_n|| &\le {t_n^{(1)}}^{-1} ||Tx_{n+1} - Tx_n|| + ||Ty_n - Tx_n|| + {t_n^{(1)}}^{-1} ||u_n^{(1)}|| \\
&\le {t_n^{(1)}}^{-1} ||x_{n+1} - x_n|| + ||y_n - x_n|| + a^{-1} ||u_n^{(1)}|| \\
&\le (1 + t_n^{(2)} + \cdots + t_n^{(2)} \cdots t_n^{(k)}) ||Tx_n - x_n|| + {t_n^{(1)}}^{-1} (||u_n^{(1)}|| \\
&\quad + ||u_n^{(2)}|| t_n^{(1)} + ||u_n^{(3)}|| t_n^{(1)} t_n^{(2)} + \cdots + ||u_n^{(k)}|| t_n^{(1)} t_n^{(2)} \cdots t_n^{(k-1)}) \\
&\quad + (t_n^{(2)} + t_n^{(2)} t_n^{(3)} + \cdots + t_n^{(2)} t_n^{(3)} \cdots t_n^{(k)}) ||Tx_n - x_n|| \\
&\quad + ||u_n^{(2)}|| + ||u_n^{(3)}|| t_n^{(2)} + \cdots + ||u_n^{(k)}|| t_n^{(2)} t_n^{(3)} \cdots t_n^{(k-1)} + a^{-1} ||u_n^{(1)}|| \\
&\le (1 + 2(t_n^{(2)} + \cdots + t_n^{(2)} \cdots t_n^{(k)})) ||Tx_n - x_n|| + \frac{2}{a} \sum_{j=1}^{k} ||u_n^{(j)}||.
\end{aligned}$$

By (d) and (e) we obtain $\limsup_{n\to\infty} ||b_n|| \le d$.

On the other hand we have

$$\begin{aligned}
&\Big\|\sum_{i=1}^{n} t_i^{(1)} b_i\Big\| \\
&= \Big\|\sum_{i=1}^{n}(Tx_{i+1} - Tx_i - u_i^{(1)} + t_i^{(1)}(Tx_i - Ty_i))\Big\| \\
&\le ||x_{n+1} - x_1|| + \sum_{i=1}^{n} t_i^{(1)} ||\, y_i - x_i \,|| + \sum_{i=1}^{n} ||u_i^{(1)}|| \\
&\le ||x_{n+1} - x_1|| + \sum_{i=1}^{n} t_i^{(1)} (t_i^{(2)} + \cdots + t_i^{(2)} t_i^{(3)} \cdots t_i^{(k)}) ||Tx_i - x_i \,|| \\
&\quad + \sum_{i=1}^{n} t_i^{(1)} \Big(||u_i^{(2)}|| + ||u_i^{(3)}|| \, t_i^{(2)} + \cdots + ||u_i^{(k)}|| \, t_i^{(2)} t_i^{(3)} \cdots t_i^{(k-1)} \Big) + \sum_{i=1}^{n} ||u_i^{(1)}|| \\
&\le ||x_{n+1} - x_1|| + \sum_{i=1}^{n} (t_i^{(1)} t_i^{(2)} + \cdots + t_i^{(1)} t_i^{(2)} \cdots t_i^{(k)}) ||\, Tx_i - x_i \,|| + \sum_{i=1}^{n} \sum_{j=1}^{k} ||u_i^{(j)}||.
\end{aligned}$$

By (c) and (e) we can conclude that the last expression is bounded. Therefore, by Lemma 3 we obtain the result. □

Theorem 3. *Let D be a closed subset of a Banach space X, and $T : D \to X$ be a nonexpansive mapping from D into a compact subset of X. If $\{x_n\}$ is as in Theorem 1 or Theorem 2, then $\{x_n\}$ converges to a fixed point of T.*

Proof. Since $\{x_n\}$ is a subset of the set $\{x \in X \mid d(x, \overline{conv(T(D) \cup \{x_1\})}) \le \|\{u_n^{(1)}\}\|_\infty\}$, which is compact by well-known theorem of Mazur, we know that $\{x_n\}$ containes a subsequence $\{x_{n_k}\}$ which converges to some $p \in D$ since D is closed. By Theorem 1 (Theorem 2) we have $||Tx_{n_k} - x_{n_k}|| \to 0$ as $k \to \infty$. On the other hand

$$||Tp - p|| \le ||Tp - Tx_{n_k} + Tx_{n_k} - x_{n_k} + x_{n_k} - p|| \le 2||p - x_{n_k}|| + ||Tx_{n_k} - x_{n_k}||,$$

since T is nonexpansive, which implies that $p \in F(T)$.

By Lemma 4 we have

$$||x_{n+1} - p|| \le ||x_n - p|| + \sum_{i=1}^{k} ||u_n^{(i)}|| \tag{3}$$

for all $n \geq 1$. By Corollary 1 there exists $\lim_{n\to\infty} ||x_n - p|| = d$. Since $\lim_{k\to\infty} ||x_{n_k} - p|| = 0$ we have $d = 0$, as desired.

In [17] Senter and Dotson introduce the following definition:

Let D be a subset of a Banach space X. A mapping $T : D \to X$ with a nonempty fixed points set $F(T)$ in D will be said to satisfy *Condition I*, if there is a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0,\infty)$, such that $||x - Tx|| \geq f(d(x, F(T)))$ for all $x \in D$, where $d(x, F(T)) = \inf_{z\in F(T)} ||x - z||$. □

The following theorem generalize Theorem 2 in [10] and Theorem 4 in [2].

Theorem 4. *Let X, D and $\{x_n\}$ be as in Theorem 3. Let $T : D \to X$ be a nonexpansive mapping with a nonempty fixed points set $F(T)$ in D. If T satisfies Condition I, then $\{x_n\}$ converges to a member of $F(T)$.*

Proof. By Lemma 4 we have (3) and consequently

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)) + \sum_{i=1}^{k} ||u_n^{(i)}||.$$

Further, by Corollary 1 we can conclude that $\lim_{n\to\infty} d(x_n, F(T)) = r$ exists.

By (3) we easily obtain

$$||x_n|| \leq ||x_1-p||+||p||+\sum_{i=1}^{n-1}\sum_{j=1}^{k} ||u_i^{(j)}|| < ||x_1-p||+||p||+\sum_{i=1}^{\infty}\sum_{j=1}^{k} ||u_i^{(j)}|| < \infty,$$

hence, $\{x_n\}$ is bounded and consequently, by Theorem 1 (Theorem 2), $\lim_{n\to\infty} ||x_n - Tx_n|| = 0$.

From that and *Condition I*, we have

$$0 = \lim_{n\to\infty} ||x_n - Tx_n|| \geq \lim_{n\to\infty} f(d(x_n, F(T)))$$

which implies that $r = 0$. Let us show that $\{x_n\}$ converges to a member of $F(T)$. Since

$$\lim_{n\to\infty} d(x_n, F(T)) = 0 \quad \text{and} \quad \sum_{i=1}^{\infty}\sum_{j=1}^{k} ||u_i^{(j)}|| < \infty,$$

for any positive integer i there exists $N_i > 0$ and $p_i \in F(T)$ such that

$$||x_{N_i} - p_i|| < 2^{-(i+1)} \quad \text{and} \quad \sum_{i=N_i}^{\infty}\sum_{j=1}^{k}||u_i^{(j)}|| < 2^{-(i+1)},$$

which implies from (3) that $||x_n - p_i|| < 2^{-i}$, for all $n \geq N_i$. We may suppose that $N_{i+1} \geq N_i$ for all $i > 0$. Thus we have

$$\begin{aligned} ||p_i - p_j|| &\leq ||p_i - x_{N_{i+1}}|| + ||x_{N_{i+1}} - p_{i+1}|| + ||p_{i+1} - x_{N_{i+2}}|| \\ &\quad + \cdots + ||p_{j-1} - x_{N_j}|| + ||x_{N_j} - p_j|| \\ &\leq 2^{-i} + 2^{-(i+2)} + 2^{-(i+1)} + 2^{-(i+3)} + \cdots + 2^{-(j-1)} + 2^{-(j+1)} < 2^{-(i-1)}, \end{aligned}$$

which implies that $\{p_i\}$ is a Cauchy sequence. Thus there exists $p^* \in F(T)$, such that $\lim_{n\to\infty} p_n = p^*$, since $F(T)$ is closed. Since $||x_n - p_i|| < 2^{-i}$, for all $n \geq N_i$, we have $\lim_{n\to\infty} x_n = p^*$, completing the proof. □

By Theorem 1 or by Theorem 2 and the fixed point theorem of Gillespie and Williams [7], as in [2], it is easy to prove the following theorem. This theorem generalizes Theorem 1.8 of Veeramani [23] and Theorem 5 in [2].

Theorem 5. *Let D be a closed, bounded, convex subset of a Banach space X, and $T : D \to D$ be a nonexpansive mapping on D such that for some $\alpha > 0$ and for all $x, y \in D$*

$$||Tx - Ty|| \leq \alpha(||x - Tx|| + ||y - Ty||).$$

If $\{x_n\}$ is as in Theorem 1 or Theorem 2, then $\{x_n\}$ converges to the unique fixed point of T.

Recall that a Banach space X satisfies Opial's condition [15] if for each sequence $\{x_n\}$ in X, the condition $x_n \to x_0$ weakly implies $\liminf_{n\to\infty} ||x_n - x_0|| < \liminf_{n\to\infty} ||x_n - y||$ for all $y \in X, y \neq x_0$.

Theorem 6. *Suppose X is a Banach space that satisfies Opial's condition and D is weakly compact, and let T and $\{x_n\}$ be as in Theorem 1 or Theorem 2. Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Since D is a weakly compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a $p \in D$. By Theorem 1 or Theorem 2 we have $||Tx_n - x_n|| \to 0$ as $n \to \infty$. From that and the nonexpansivity of T we have

$$\liminf_{n\to\infty} ||x_n - p|| \geq \liminf_{n\to\infty} ||Tx_n - Tp|| = \liminf_{n\to\infty} ||x_n - Tp||.$$

Thus from Opial's condition we have $Tp = p$. Suppose that $\{x_n\}$ does not converges weakly to p. Then there are subsequence $\{x_{m_j}\}$ of $\{x_n\}$ and $q \neq p$ such that $x_{m_j} \to q$ weakly and $Tq = q$. By Lemma 4 and Corollary 1 we obtain that there exist finite limits

$$\lim_{n\to\infty} ||x_n - p|| \quad \text{and} \quad \lim_{n\to\infty} ||x_n - q||.$$

From that and Opial's condition we have

$$\begin{aligned}\lim_{n\to\infty} ||x_n - p|| &= \lim_{k\to\infty} ||x_{n_k} - p|| < \lim_{k\to\infty} ||x_{n_k} - q|| \\ &= \lim_{j\to\infty} ||x_{m_j} - q|| < \lim_{j\to\infty} ||x_{m_j} - p|| = \lim_{n\to\infty} ||x_n - p||,\end{aligned}$$

which is a contradiction. Hence the result follows. □

Remark 3. Iteration process (1) appeared for the first time in an earlier version of this paper titled "Approximating fixed points of nonexpansive mappings by a new iteration method" which was accepted for publication in the Far East Journal of Mathematical Sciences in 2002, and has already been cited in papers [18, 19, 20, 21]. However due to page charges the paper was not published.

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