THE MORSE THEORY AND THE MASLOV-TYPE INDEX THEORY

BY

CHUN-GEN LIU

Abstract

In this lecture note, we give a brief introduction to the classic Morse theory, the Morse homology theory and the Maslov-type index theory and its iteration theory. It based on some lectures in the seminar of dynamical systems in Institute of Mathematics, Academia Sinica, Taipei.

1. Introduction

1.1. The finite dimensional and infinite dimensional Morse theory

The topological properties of a manifold usually is global information of the manifold. The shapes of the neighborhoods of critical points of a Morse function defined on the manifold are usually local information of this manifold. Morse theory connects the two aspects of analytic information and topological information of a manifold.

The Morse theory became an important method in the studies of nonlinear analysis, specially in the studies of the closed geodesic theory on a Riemannian manifold. It had many developments in the fields of analysis and geometry. For example, the Morse (co)homological theory, the infinite dimensional Morse theory(the critical group theory), and the Floer homological theory were developed on the fundamental of the classical Morse theory.

These theories are very useful tools today in the study of differential equations, minimal surfaces theory, harmonic maps, symplectic geometry and symplectic topology, etc. In this lecture note, we will give a brief introduction on the classical Morse theory and some of its developments. The main material comes from the references [3]−[5], [6], [46], [47] and [52]. The first part contains the Morse Lemma, deformation theory, and the Morse inequalities. The second part contains the Morse homology theory and then the infinite dimensional Morse theory, i.e., the theory of critical groups.

1.2. Morse index and Maslov-type index theory

Morse index theory has played a very important role in the nonlinear problems, including nonlinear differential equations, such as existence, multiplicity and stability problems of closed geodesics in a Riemannian manifold. Early in 1930's, this index theory was developed by M. Morse in his work on closed geodesics on Riemannian manifolds. With the iteration formula established by R. Bott, many deep results for closed geodesics have been obtained via Morse theory. During the past almost 30 years, the study of existence and multiplicity of periodic solutions of nonlinear Hamiltonian has been one of the important directions in the field of Hamiltonian dynamics. In this period a great number of research papers have appeared in this and related areas, and many aspects of critical point theory have been applied to the variational study of Hamiltonian systems. It is natural to apply the Morse theoretical method to the problems involving various solutions of nonlinear Hamiltonian systems. It is well known that all critical points of the associated variational functional of a first order Hamiltonain system possess infinite Morse indices. The have been a number of attempts in finding finite representations of Morse indices for periodic orbits of Hamiltonian systems. The so called Maslov-type index theory is one of the successful attempts. Since 1980, two different index theories for periodic solutions of nonlinear Hamiltonian systems have appeared. One index theory was developed by I. Ekeland in 1980's for convex Hamiltonian systems. A beautiful systematic treatment of his index theory was given in his celebrated book [10]. Another index theory is a classification of general linear Hamiltonian system with periodic coefficients(without convexity). This index theory began with the

work of H. Amann and E. Zehnder in [1]. They established the corresponding index theory for linear Hamiltonian systems with constant coefficients. After that many mathematicians worked on this problem (cf. [6] and [23]).

The linearized system of a nonlinear Hamiltonian system

$$\dot{x}(t) = JH'(t, x(t)) \tag{1.2.1}$$

at a solution $x(t)$ is a linear Hamiltonian system

$$\dot{z}(t) = JB(t)z(t), \tag{1.2.2}$$

where $B(t) = H''(t, x(t))$, $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ and I_n is $n \times n$ identity matrix. The fundamental solution of (1.2.2) is a path in the symplectic group $\mathrm{Sp}(2n)$ starting from the identity. Here

$$\mathrm{Sp}(2n) = \{M \in GL(\mathbb{R}^{2n}) \,|\, M^T JM = J\}$$

and M^T denotes the transpose of M. We define the set of symplectic paths by

$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0, \tau], \mathrm{Sp}(2n) \,|\, \gamma(0) = I_{2n})\}.$$

For any $\gamma \in \mathcal{P}_\tau(2n)$, we define $\nu_\tau(\gamma) = \dim \ker_{\mathbb{C}}(\gamma(1) - I)$. If $\nu_\tau(\gamma) = 0$, we say that the symplectic path γ is non-degenerate, and degenerate otherwise.

In their celebrated paper [9], C. Conley and E. Zehnder defined an index $i(\gamma) \in \mathbb{Z}$ for any non-degenerate path $\gamma \in \mathcal{P}_\tau(2n)$ with $n \geq 2$, i.e., the so called Conley-Zehnder index. For $n = 1$ this index was studied in [43]. The index theory for degenerate linear Hamiltonians was established by Y.Long in [37] and C. Viterbo in [54]. Then in [38] this index theory was further extended to any paths in $\mathcal{P}_\tau(2n)$. For any path $\gamma \in \mathcal{P}_\tau(2n)$, we call the index pair

$$(i_\tau(\gamma), \nu_\tau(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}$$

the Maslov-type index. If $\gamma(t)$ is the fundamental solution of the linear system (1.2.2), we denote the index pair of γ also by $(i_\tau(B), \nu_\tau(B))$. It is a classification of the linear Hamiltonian systems. If $x(t)$ is a τ-periodic solution of the nonlinear Hamiltonian system (1.2.1) with $H(t + \tau, x) = H(t, x)$ for any (t, x) and $B(t) = H''(t, x(t))$, we denote $(i_\tau(x), \nu_\tau(x)) =$

$(i_\tau(B), \nu_\tau(B))$ in this case. In this lecture, we will give a brief introduction to the Maslov-type index theory and its iteration theory.

Acknowledgements

2. An Induction to the Morse Theory

2.1. Morse theory of differentiable functions on a manifold

Let M be a smooth compact differentiable manifold, $f \in C^2(M, \mathbb{R})$. A critical point of f is a point $p \in M$ such that $df(p) = 0$. The Morse index of f at a critical point p is defined by

$$\mu(p) = \text{number of negative eigenvalue of } f''(p).$$

A critical piont is non-degenerate if

$$Hf = f''(p) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right), \quad x = (x_1, \ldots, x_n) \text{ local coordinates at } p$$

is not degenerate, i.e., its nullity $n(p) = 0$. The function f is called Morse function, if all of its critical points are non-degenerate. We denote the level set of f behind the value a by

$$f_a = \{x \in M | f(x) \le a\}.$$

Lemma 2.1.1.(Morse Lemma) *If $p \in M$ is a non-degenerate critical point of f, we can choose local coordinates (U, φ) at p such that*

$$\begin{aligned} f(x) &= f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_n^2, \\ x &= (x_1, \ldots, x_n) \in U, \ p = (0, \ldots, 0), \ k = \mu(p). \end{aligned}$$

Theorem 2.1.2. *If $df(p) \neq 0$ for all points $p \in M$ with $a \leq f(p) \leq b$, then f_a and f_b are diffeomorphic, i.e., $f_a \cong f_b$. Furthermore, f_a is a deformation retract of f_b, so that the inclusion map $f_a \to f_b$ is a homotopy equivalence.*

*Proof.* The idea of the proof is to push f_b down to f_a along to the orthogonal trajectories of the hypersurfaces f =constant, i.e. along the negative flow of f. By the condition $\nabla f(p) \neq 0$ for $p \in f^{-1}([a,b])$, we can make a deformation via the negative gradient flow line such that any flow line starting from the boundary of f_b will intersect the boundary of f_a among a bounded time interval. See the figure below. $\qquad\square$
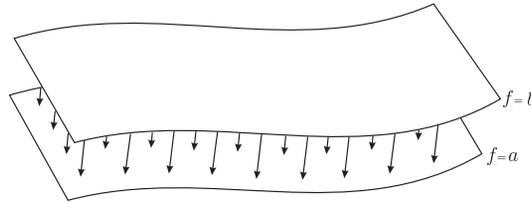
Figure 2.1.1. The negative gradient flow.

Example. If M is a compact manifold and f is a differentiable function on M with only two critical points, both of which are non-degenerate, then M is homeomorphic to a sphere S^n.

Claim. One of the critical points must be minimum and another must be maximum. At the minimum point, say p, the Morse index $\mu(p) = 0$. At the maximum point, say q, the Morse index $\mu(q) = n = \dim M$. Thus M is the union of two closed n-cells. It is easy to construct a homeomorphim between M and S^n.
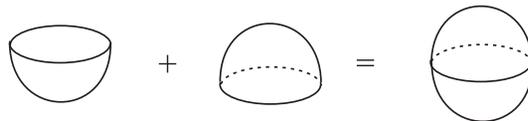
Figure 2.1.2. Gluing the neighborhoods of the two critical points.

Let $e_r = \{(x_1, \ldots, x_r) \in \mathbb{R}^r \mid x_1^2 + \cdots x_r^2 \leq 1\}$ be the r-cell. $\partial e_r =$

$\{(x_1, \ldots, x_r) \in \mathbb{R}^r \mid x_1^2 + \cdots x_r^2 = 1\}$ the boundary of e_r. $\varphi : \partial e_r \to X$ a continuous map, X is a topological space. The topological space $X \cup_\varphi e_r$ is said attached e_r to X by φ if

$$X \cup_\varphi e_r = X \cup e_r / \sim, \quad \varphi(p) \sim p.$$

Theorem 2.1.3. *Let M be a compact differential manifold. $f : M \to \mathbb{R}$ as defined before. If $f^{-1}([a,b])$ contains exactly one non-degenerate critical point p of index k, $a < f(p) < b$, then f_b has the homotopy type of $f_a \cup_\varphi e_k$.*

Proof. We may assume that $f(p) = 0$. It is sufficient to prove there is a small number $\varepsilon > 0$ such that

$$f_\varepsilon \sim f_a \cup_\varphi e_k.$$

By the Morse lemma, there is a neighborhood U of p and local coordinates $y_1, \ldots, y_n$ in U such that f is given by

$$f = -y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2.$$

We set

$$A^\varepsilon = \{y \in f_\varepsilon \cap U \mid y_1^2 + \cdots + y_k^2 \le \rho\}$$

for some small $\rho > 0$, and

$$f_\varepsilon^* = \overline{f_\varepsilon \setminus A^\varepsilon},$$

then $f_\varepsilon = A^\varepsilon \cup f_\varepsilon^*$. For suitable chosen $\varepsilon > 0$ and $\rho > 0$, this just means that f_ε is obtained from f_ε^* by attaching a product $e_k \times I_{n-k}$ with $I_s = \{(x_1, \ldots, x_s) \in \mathbb{R}^s \mid 0 \le x_j \le 1\}$. Namely we have

$$f_\varepsilon = f_\varepsilon^* \cup_\phi (e_k \times I_{n-k}) \sim f_\varepsilon^* \cup_\psi e_k.$$

We can show that

$$f_\varepsilon^* \cong f_a$$

by the same method as in the proof of Theorem 2.1.1. See the figure below.   □
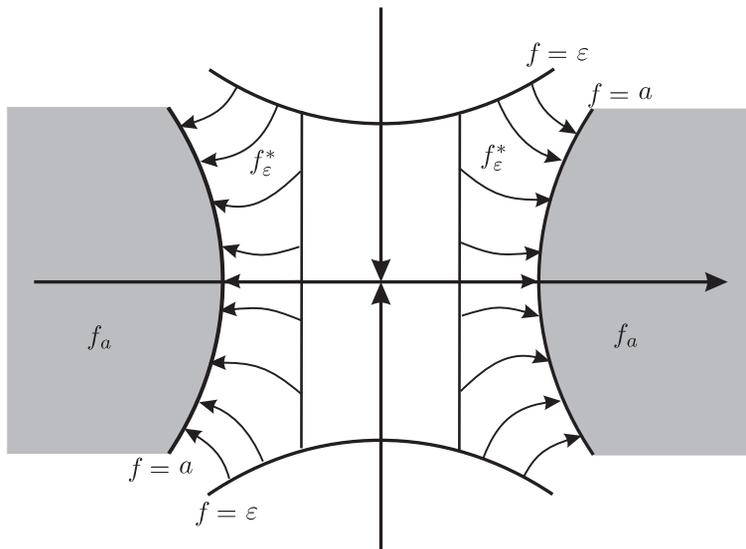
Figure 2.1.3. Make the homotopy.

Example. Let T^2 be a 2-dimensional torus, resting on its tangent plane. The function $f : T^2 \to \mathbb{R}$ is defined by the distance of the points on T^2 from the tangent plane V. It is smooth (real analytic). The set of critical points= $\{s, r, q, p\}$. Suppose $f(s) = c_1$, $f(r) = c_2$, $f(q) = c_3$, $f(p) = c_4$. Then $\mu(s) = 2$, $\mu(r) = 1$, $\mu(q) = 1$, $\mu(p) = 0$.
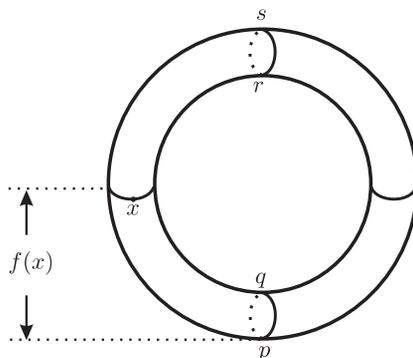
Figure 2.1.4. The function defined on the torus.
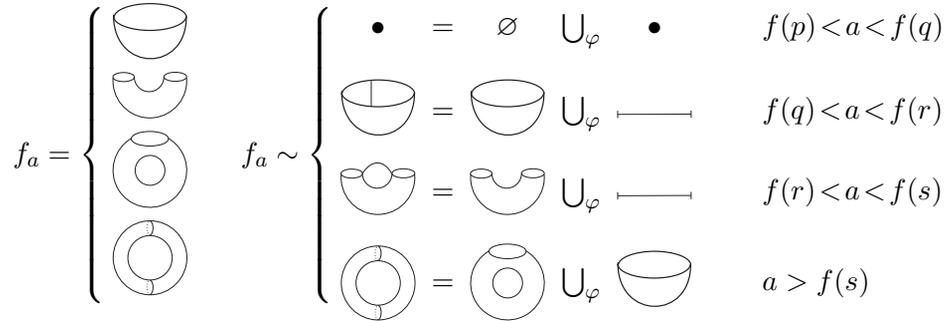
Figure 2.1.5. The diffeomorphim and homotopy.

2.2. The Morse inequlities

Let M be a compact manifold, which can be built up by successively attaching cells, in the way described in the section 2.1. Then there is a CW-complex K, such that its cells are in dimension preserving 1-1 correspondence with the attached cells, and the homology of K is the homology of M(with respect to any group of coefficients). We may take $\mathbb{R}$ domain of coefficients. Let

$$C = \sum_{i=0}^{n} C_i$$

the (naturally graded) vector space of chains of K.

$$Z = \sum_{i=0}^{n} Z_i$$

the space of cycles,

$$B = \sum_{i=0}^{n} B_i$$

the space of the boundary and

$$H = \sum_{i=0}^{n} H_i$$

the real homology group of K. By definition, we have the exact sequence

(1) $0 \to Z \xrightarrow{i} C \xrightarrow{p} B \xrightarrow{\delta} 0$

(2) $0 \to B \to Z \to H \to 0$

where δ reduced the degree by 1. Setting $\dim C_i = c_i$, $\dim Z_i = z_i$, $\dim B_i = b_i$ and $\dim H_i = h_i$. From (1) and (2), we have

$$c_i = b_{i-1} + z_i, \ z_i = h_i + b_i.$$

So there holds

$$c_i - h_i = b_i + b_{i-1}, \ i = 0, 1, \ldots, \ b_{-1} = 0. \tag{2.2.1}$$

Since $b_i \geq 0$, the relations lead to the following sequence of inequalities

$$
\begin{array}{ll}
c_0 & \geq \ h_0 \\
c_1 - c_0 & \geq \ h_1 - h_0 \\
c_2 - c_1 + c_0 & \geq \ h_2 - h_1 + h_0 \\
\cdots\cdots & \quad \cdots\cdots
\end{array}
\tag{2.2.2}
$$

Let f be a non-degenerate differentiable function on the compact differentiable manifold M. By slightly perturbation, we can assume that any pair of critical points $p, q \in M$ of f satisfy $f(p) \neq f(q)$. By Theorem 2.1.2 and the above arguments, we have the following result.

Theorem 2.2.1. *Let c_i, $i = 0, 1, \ldots$ be the number of critical points of index i of f, and h_i the i th-Betti number of M, i.e., $h_i = H_i(M; \mathbb{R})$. Then there exist a sequence of non-negative integers b_i such that*

$$c_i - h_i = b_{i-1} + b_i, i = 0, 1, \ldots. \tag{2.2.3}$$

Therefore (2.2.2) holds in this case. We set

$$\mathcal{M}_t(f) = \sum_{i=0}^{\infty} c_i t^i,$$

$$\mathcal{P}_t(M) = \sum_{i=0}^{\infty} h_i t^i,$$

$$\mathcal{Q}_t(f) = \sum_{i=0}^{\infty} b_i t^i.$$

The inequalities of (2.2.2) *can be written as*

$$\mathcal{M}_t(f) - \mathcal{P}_t(M) = (1+t)\mathcal{Q}_t(f). \tag{2.2.4}$$

Corollary 2.2.2. (Morse Lacunary Principle) *Suppose that no consecutive powers of* t *occur in* $\mathcal{M}_t(f)$. *Then* $\mathcal{Q}_t(f) \equiv 0$. *So that*

$$\mathcal{M}_t(f) = \mathcal{P}_t(M) \tag{2.2.5}$$

for any coefficient $\mathbb{K}$. *In particular,* M *is then free of torsion.*

Proof. If $\mathcal{Q}_t(f) = \sum_{i=0}^{\infty} b_i t^i \neq 0$, for example $b_i \neq 0$, then the right should contains the consecutive powers: t^i and t^{i+1}. Thus $\mathcal{M}_t(f)$ also contains this two consecutive terms. $\qquad\square$

Example. Consider the unit sphere

$$S^{2n+1} = \{(z_0, \dots, z_n) \in \mathbb{C}^{n+1} | \sum_{i=0}^{n} |z_i|^2 = 1\},$$

and on it defines the function

$$f(z) = \frac{1}{2} \sum_{i=0}^{n} \lambda_i |z_i|^2,$$

where $\lambda_0 < \lambda_1 < \cdots < \lambda_n$ are sequence of distinct real numbers. It is clear that f is invariant under the S^1 action on S^{2n+1}:

$$e^{\sqrt{-1}\theta} : (z_0, \dots, z_n) \to (e^{\sqrt{-1}\theta} z_0, \dots, e^{\sqrt{-1}\theta} z_n).$$

Thus the function can descends to the projective space $\mathbb{C}P^n$. Around the point $e_i = (0, \dots, 0, 1, 0 \cdots 0)$, the i th element is 1 and others are zero, we have a local coordinate of $\mathbb{C}P^n$. Since $\sum_{i=0}^{n} |z_j| = 1$, we can write

$$f(z) = \frac{1}{2} \Big(\sum_{j \neq i} (\lambda_j - \lambda_i) |z_j|^2 + \lambda_i \Big).$$

The point e_i is the critical point. $f''(e_i)$ has eigenvalues

$$\lambda_0 - \lambda_i, \lambda_1 - \lambda_i, \ldots, \lambda_{i-1} - \lambda_i, \lambda_{i+1} - \lambda_i, \ldots, \lambda_n - \lambda_i.$$

Every eigenvalue has real dimension 2. So $\mu(e_i) = 2i$ and by definition

$$\mathcal{M}_t(f) = 1 + t^2 + t^4 + \cdots + t^{2n}.$$

The lacunary principle applies and we conclude that

$$\mathcal{P}_t(\mathbb{C}P^n) = 1 + t^2 + t^4 + \cdots + t^{2n}. \tag{2.2.6}$$

Corollary 2.2.3. *Let M be a compact differentiable manifold, $f \in C^2(M, \mathbb{R})$. If all critical points are non-degenerate, then the number of critical points of f satisfies*

$$^\natural crit(f) \geq \dim H_*(M, \mathbb{R}) = \sum_{i=0}^{n} h_i. \tag{2.2.7}$$

In the general case, the number of critical points of f satisfies

$$^\natural crit(f) \geq cup\ length(M), \tag{2.2.8}$$

where the cup length of M is defined by

$$cup\ length(M) = \max\{k \in \mathbb{N} | \exists \omega_1, \ldots, \omega_{k-1} \in H^*(M, \mathbb{R})$$
$$with\ \dim \omega_i > 0\ and\ \omega_1 \cup \cdots \cup \omega_{k-1} \neq 0\}.$$

Proof. (2.2.7) is a direct consequence of Theorem 2.2.1 with $t = 1$. (2.2.8) is a Lyusternik-Schnirelmann type estimate on the critical points of f. If there exist $\omega_1 \cup \cdots \cup \omega_m \neq 0$ with $\dim \omega_i > 0$, and f possesses only m critical points. By the deformation lemma, one can cover M by m open contractible sets M_i, $i = 1, \ldots, m$. Since $\dim \omega_i > 0$, one can choose representaions of ω_i which come from $\tilde{\omega}_i \in H^*(M, M_i)$. But the cup product $\tilde{\omega}_1 \cup \cdots \cup \tilde{\omega}_m \in H^*(M, \cup_1^m M_i) = 0$. This is a contradiction. $\qquad\square$

2.3. Infinite dimensional Morse theory

In this section, we give a brief introduction to the infinite dimensional Morse theory. One can see this theory is a natural generalization of the classical finite dimensional Morse theory. All the material comes from [6].

Let M be a (infinite dimensional) Hilbert manifold, $f \in C^1(M, \mathbb{R})$. We say that f satisfies the $(PS)_c$ condition if any sequence $\{x_n\} \subset M$ along with $f(x_n) \to c \in \mathbb{R}$ and $df(x_n) \to \theta$ (strongly) possesses a convergent subsequence. We say that f satisfies the (PS) condition if it satisfies $(PS)_c$ condition for all $c \in \mathbb{R}$.

Lemma 2.3.1.(Deformation Lemma). *Suppose $f \in C^1(M, \mathbb{R})$ satisfies $(PS)_c$ for $c \in [a, b]$.*

(1) If there is no critical point in $f^{-1}((a, b])$, then f_a is a strong deformation retract of f_b.

(2) If a is the only critical value of f in $[a, b)$, and the critical set K_a corresponding to the critical value a is only isolated points. Then f_a is a strong deformation retract of $f_b \setminus K_b$.

Proof. The idea of the proof is similar to that in the proof of Theorem 2.1.1. The $(PS)_c$ condition in some sense is a condition such that one can do every things as in the compact cases.   □

Definition 2.3.2.([6]) Let $p \in M$ be an isolated critical point of f with $f(p) = c$. We define the qth critical group with coefficient group G at p by

$$C_q(f, p) = H_q(f_c \cap U_p, (f_c \setminus \{p\}) \cap U_p; G), \qquad (2.3.1)$$

where U_p is a neighborhood of p such that the critical point set K satisfying $K \cap (f_c \cap U_p) = \{p\}$. According to the excision property of the singular homology theory, $C_q(f, p)$ is well defined for $q = 0, 1, \ldots$, i.e., they do not depend on the choice of U_p.

As in the finite dimensional case, the critical group $C_q(f, p)$ is the local property of the critical point p. The following result says something on the relation between the Morse index and the critical groups.

Theorem 2.3.3. *Suppose that $f \in C^2(M, \mathbb{R})$ and p is a nondegenerate critical point of f with index k, then*

$$C_q(f, p) = \begin{cases} G, & q = k, \\ 0, & q \neq k. \end{cases} \tag{2.3.2}$$

In general, if p is a critical point with finite Morse index k and nullity j, then

$$C_q(f, p) = 0, \ q \notin [k, k + j]. \tag{2.3.3}$$

Proof. We prove (2.3.2) here. Suppose the Hilbert manifold is modulo on the Hilbert space H. So we can treat the local neighborhood of p in M as the Hilbert space and write the function f in this neighborhood as

$$f(x) = \frac{1}{2}(Ax.x),$$

where A is a bounded, invertible (by the non-degenerateness), self-adjoint operator. We write $H = H^+ \oplus H^-$ with $H^\pm$ the positive and negative space with respect to the spectral decomposition of the operator A. Define $P^\pm : H \to H^\pm$ the orthogonal projection. By the Morse lemma, we have

$$f(x) = \frac{1}{2}\left(\|(AP^+)^{1/2}x\|^2 - \|(-AP^-)^{1/2}x\|^2\right), \ x \in B_\varepsilon = \{\|x\| \leq \varepsilon\}.$$

As the finite dimensional case, one can push the set $B_\varepsilon \cap f_0$ into the set $H^- \cap f_0$. Thus

$$
\begin{aligned}
C_q(f, p) &\cong H_q(f_0 \cap B_\varepsilon, (f_0 \setminus \{\theta\}) \cap B_\varepsilon) \\
&\cong H_1(H^- \cap B_\varepsilon, (H^- \setminus \{\theta\}) \cap B_\varepsilon) \\
&\cong H_q(B^k, S^{k-1}) \cong \begin{cases} G, & q = k \\ 0, & q \neq k. \end{cases}
\end{aligned}
$$

(2.3.3) follows from (2.3.2) and a so call shifting theorem. See [6] for details. □

From (2.3.2) and (2.3.3) we know that, if the Morse index and nullity at a critical point are finite, the critical group can be calculated in some sense. But it make no sense in study of the periodic solutions of first order nonlinear

systems, since at every critical point, its Morse index is infinite. Recently, W. Kryszewski and A. Szulkin [22] defined an infinite dimensional cohomology theory and a Morse theory for this kind of "strong infinite" functional. The main idea of this Morse theory is to combine the critical group theory and an index theory (it is known as Maslov-type index theory, we will give a brief introduction in section 4 below), specially use the information from the Galerkin approximation formulae in Theorem 4.1.11 below. They defined the ε-approximation cohomology group sequence

$$c_\varepsilon^q(f, p) = H_\varepsilon^q(W, W^-)$$

for an admissible pair (W, W^-) of p, which in some sense is a kind of Gromoll-Meyer pair with respect to some gradient-like vector field. ε is a sequence comes from the Galerkin approximation scheme and the Maslov-type index in Theorem 4.1.11. It is in fact the Morse indices of the functional in the finite dimensional truncated spaces.
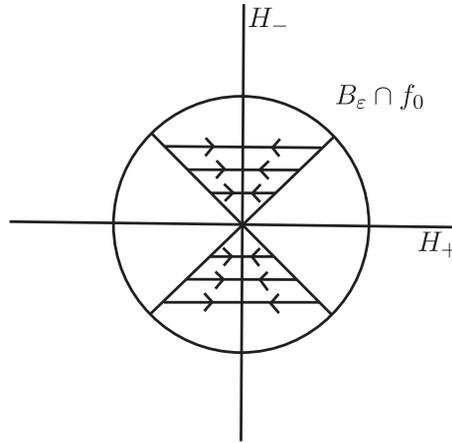
Figure 2.3.1. The process of homotopy.

Suppose f has only isolated critical values $\{c_i\}$.

Definition 2.3.4. For a pair of regular values $a < b$, we call

$$M_q(a, b) = \sum_{a < c_i < b} \operatorname{rank} H_q(f_{c_i + \varepsilon_i}, f_{c_i - \varepsilon_i}; G)$$

the qth Morse type number of the function f in the interval (a, b), $q = 0, 1, \ldots$.

By the exactness of the singular homology theory, we have

Theorem 2.3.5.

$$H_*(f_{c+\varepsilon}, f_{c-\varepsilon}; G) \cong H_*(f_c, f_c \setminus K_c) \cong \bigoplus_{j=1}^{m} C_*(f, z_j), \ K_c = \{z_j\}_{j=1}^{m}. \quad (2.3.4)$$

Thus we have

$$M_q(a, b) = \sum_{a < c_i < b} \sum_{j=1}^{m_i} \text{rank} C_q(f, z_j^i). \quad (2.3.5)$$

Setting $\beta_q(a, b) = \text{rank} H_q(f_b, f_a; G)$, the Morse inequalities now turn into the following form

Theorem 2.3.6. *Suppose f satisfies the $(PS)_c$ condition for $a < c < b$, and the critical points in $f^{-1}([a, b])$ are isolated, a and b are regular values, then there holds*

$$\sum_{q=0}^{\infty} M_q(a, b) t^q - \sum_{q=0}^{\infty} \beta_q(a, b) t^q = (1 + t) Q(t), \quad (2.3.6)$$

where $Q(t)$ is a formal series with non-negative coefficients.

Proof. By the exactness of the singular homology theory, the proof is standard. We omit it here. One can refer [6] for a complete proof.   □

The critical group theory is an important tool in studying the existence and multiple problems of nonlinear problems. We refer the celebrated book [6] and the references therein for various applications of the critical group theory.

3. The Morse Homology Theory

3.1. The negative gradient flow and connecting orbits

Let M^n be an n-dimensional compact Riemannian manifold. $f \in C^\infty$

$(M, \mathbb{R})$ is a Morse function. We consider the negative gradient flow

$$\psi : \mathbb{R} \times M \to M$$
$$\frac{\partial}{\partial t}\psi(t, x) = -\nabla f(\psi(t, x)), \quad \psi(0, x) = x, \quad \forall x \in M. \qquad (3.1.1)$$

For a critical point $x \in \mathrm{Crit}(f)$, by definition the stable and unstable manifolds of x are the submanifolds of M:

$$W^s(x) \;=\; \{p \in M| \lim_{t \to +\infty} \psi(t, p) = x\}$$
$$W^u(x) \;=\; \{p \in M| \lim_{t \to -\infty} \psi(t, p) = x\}$$

One can always find a generic Riemannian metric on M such that the stable and unstable manifolds intersect transversally, i.e.,

$$W^u(x) \pitchfork W^s(y), \forall x, y \in \mathrm{Crit} f.$$

The connecting orbit of the critical points x and y is defined by

$$\mathcal{M}^f_{x,y} = W^u(x) \cap W^s(y) = \{\gamma : \mathbb{R} \to M | \dot{\gamma} = -\nabla f(\gamma), \gamma(-\infty) = x, \gamma(+\infty) = y\}.$$

Claim: For $x \neq y \in \mathrm{Crit} f$

(1) $\mathcal{M}^f_{x,y} \approx W^u(x) \cap W^s(x)$, by $\gamma \to \gamma(0)$.

(2) $\mathcal{M}^f_{x,y}$ is a submanifold of M with dimension

$$\dim \mathcal{M}^f_{x,y} = \mu(x) - \mu(y).$$

(3) The group $\mathbb{R}$ acts on $\mathcal{M}^f_{x,y}$ by $\gamma \cdot \tau = \gamma(\tau + \cdot)$ for any $\gamma \in \mathcal{M}^f_{x,y}$ and $\tau \in \mathbb{R}$. We denote the quotient space by

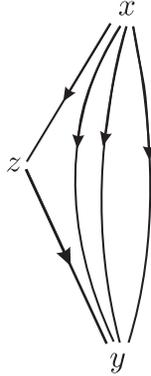$$\widehat{\mathcal{M}}^f_{x,y} = \mathcal{M}^f_{x,y}/\mathbb{R}.$$

So we have

$$\dim \widehat{\mathcal{M}}^f_{x,y} = \mu(x) - \mu(y) - 1$$

(4) If $\mu(x) - \mu(y) = 1$, the manifold $\widehat{\mathcal{M}}^f_{x,y}$ is compact and hence only finite many points. If $\mu(x) - \mu(y) = 2$, then $\widehat{\mathcal{M}}^f_{x,y}$ is compact up to broken

trajectories of order two. i.e.,

$$\partial \widehat{\mathcal{M}}_{x,y}^f = \bigcup_{\mu(x)-\mu(z)=1} \widehat{\mathcal{M}}_{x,z}^f \times \widehat{\mathcal{M}}_{z,y}^f \qquad (3.1.2)$$

Figure 3.1.1. Boundary of the moduli space.

(5) There is a coherent orientation for all moduli spaces $\mathrm{cl}(\widehat{\mathcal{M}}_{x,y}^f)$, the closure of the space $\widehat{\mathcal{M}}_{x,y}^f$. Furthermore, if $\mu(x) - \mu(y) = 1$, then the coherent orientation for every connecting orbit $\gamma \in \mathcal{M}_{x,y}^f$, is compatible with the formula (3.1.2) above, and this orbit itself has a natural orientation induced by the flow time t. We therefore can defined a function $\tau_0 : \widehat{\mathcal{M}}_{x,y}^f \to \{+1, -1\}$ by $\tau_0(\gamma(0)) = 1$ if these two orientations for γ coincide, and $\tau_0(\gamma(0)) = -1$ otherwise.

Remark. The dimensional formula $\dim \mathcal{M}_{x,y}^f = \mu(x) - \mu(y)$ follows from the Fredholm theory. We linearized the equation in (3.1.1), and get a linear equation

$$(F_A z)(t) := \left(\frac{d}{dt} + A(t) \right) z(t) = 0, \ z \in W^{1,2}(\mathbb{R}, \mathbb{R}^n), \ A(t) = f''(\gamma(t)).$$

It is clear that $A^- := A(-\infty) = f''(x)$, $A^+ := A(+\infty) = f''(y)$. By some subtle analysis and calculations, we conclude that the differential operator $F_a : H^{1,2}(\mathbb{R}, \mathbb{R}^n) \to L^2(\mathbb{R}, \mathbb{R}^n)$ is a Fredholm operator, and the Fredholm index

$$\mathrm{ind} F_a = \ker F_A - \mathrm{coker} F_A = \mu(A^-) - \mu(A^+) = \mu(x) - \mu(y).$$

By the transversal condition, we have $\text{coker} F_A = 0$, so

$$\dim \mathcal{M}^f_{x,y} = \ker F_A = \mu(x) - \mu(y).$$

3.2. Morse-Witten complex and Morse homology

In this section, we still assume that M is a closed manifold with a generical Riemannian metric, and $f : M \to \mathbb{R}$ is a Morse function with critical point set $\text{Crit} f$. We denote by

$$\text{Crit}_k f = \{x \in \text{Crit} f \,|\, \mu(x) = k\}.$$

We want to explain how to prove the following classical result again for Morse function f.

Claim: $\natural \text{Crit}_k f \geq b_k$, $b_k = H_k(M, \mathbb{Z})$ for $k = 0, 1, \ldots, n = \dim M$.

For this purpose, we give a brief introduction of the Morse homology. For the details, we refer the book [52].

The Morse-Witten complex is defined by

$$C_k(f) = \text{Crit}_k f \otimes \mathbb{Z} = \sum_i \lambda_i x_i, \ \lambda_i \in \mathbb{Z}, \ x_i \in \text{Crit}_k(f)$$

with the boundary operator $\partial_k : C_k(f) \to C_{k-1}(f)$ defined by

$$\partial_k x = \sum_{y \in \text{Crit}_{k-1} f} n(x, y) y, \ \ \forall x \in \text{Crit}_k(f), \tag{3.2.1}$$

where $n(x, y)$ is defined by

$$n(x, y) = \sum_{\gamma \in \mathcal{M}^f_{x,y}} \tau_0(\gamma(0)).$$

Remark. If we define the Morse complex by the coefficient field $\mathbb{Z}_2$, we can ignore the orientation, which is very complicated to defined, and in this

case $n(x, y)$ is the modulo 2 number

$$n(x, y) = \sum_{\gamma \in \mathcal{M}^f_{x,y}} \tau_0(\gamma(0)) \mod 2.$$

Theorem 3.2.1. $\partial^2 = 0$, *so we can define the Morse homology by*

$$H_k(C_*(f), \partial) = \ker \partial_k / \mathrm{im} \partial_{k-1}.$$

Proof. For any $x \in C_k(f)$, without loss of generality, we assume $x \in \mathrm{Crit}_k f$, $k \geq 1$.

$$\partial^2 x = \sum_{z \in \mathrm{Crit}_{k-2} f} \sum_{y \in \mathrm{Crit}_{k-1} f} n(x, y) n(y, z) z$$

$$= \sum_{z \in \mathrm{Crit}_{k-2} f} \sum_{(u,v)} \tau_0(u) \tau_0(v) z,$$

where (u, v) is a 2 braked orbit starting from x and ending at z, by gluing in the middle at a critical point y with $\mu(x) - \mu(y) = 1$. This braked orbit is a part of the boundary of $\mathrm{cl}\hat{\mathcal{M}}^f_{x,z}$. But by (3.1.2) we know that each components of $\mathrm{cl}\hat{\mathcal{M}}^f_{x,z}$ contains exact two parts of the braked orbits as its boundary since every 1 dimension manifold with boundary homeomorphic to an interval $[0, 1]$, see the following figure.
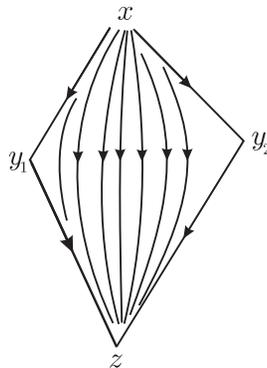
Figure 3.1.2. Boundaries of the moduli space.

In the coefficient $\mathbb{Z}_2$ case, the number

$$\sum_{(u,v)} \tau_0(u)\tau_0(v) = \sum_{(u,v)} 1 \equiv 0 \bmod 2.$$

In the general case, the situation is much complicated, but the idea is simple. We know that the braked orbit (u, v) occurs in pair, and by the coherence orientation, the pair of braked orbits possesses opposite orientations. Thus the summation

$$\sum_{(u,v)} \tau_0(u)\tau_0(v) = 0$$

still holds in the coefficient $\mathbb{Z}$ case. This proves that $\partial^2 = 0$.   □

The Morse homology is defined by the generic choice of the function and Riemannian metric, but it is independent of these choices essentially. We have the following result.

Theorem 3.2.2.

$$H_*(C_*(f), \partial) \cong H_*(M, \mathbb{Z}).$$

Remark. The details of the proof of the theorem is not so simple to give here. The idea is that the homology theory is the unique one to satisfy some axioms (Eilenberg-Steenrod Axioms: the existence of a long exact homology sequence; the homotopy axiom; the excision axiom; and the dimension axiom) and the functorial properties. So in order to prove this theorem, one should verify the Morse homological theory possesses these axioms. We refer the book [52] for details. We remind that the coefficient group $\mathbb{Z}$ can be replaced by $\mathbb{Z}_2$ or other domain.

This theorem tell us something between topology and analysis. The left side of the above equality come from critical points of a Morse function which belongs to the analytic category, and the right side of it belongs to topology category. It give a direct proof of the above claim about the relation of the number of critical points and the Beti number of the manifold. Namely we have the following result.

Corollary 3.2.3.

$$\sharp Crit_k f \geq b_k, = \mathrm{rank} H_k(M, \mathbb{Z}).$$

This result is exactly that in Morse inequalities.

3.3. Cup product

Next we choose three functions f, g, h on M such that $f - g$, $g - h$ and $f - h$ are Morse functions. We are going to define a map

$$C_*(f - g) \otimes C_*(g - h) \to C_*(f - h).$$

For this end, let $p \in \mathrm{Crit}(f - g)$, $q \in \mathrm{Crit}(g - h)$ and $r \in \mathrm{Crit}(f - h)$, we consider the following moduli space

$$
\begin{aligned}
&\mathcal{M}(p, q, r) \\
&= \left\{ (l_1, l_2, l_3) \middle| \begin{array}{l} \dot{l}_1 = -\nabla(f - g), \ \dot{l}_2 = -\nabla(g - h), \ \dot{l}_3 = -\nabla(f - h), \\ l_1(0) = l_2(0) = l_3(0), \ l_1(-\infty) = p, l_2(-\infty) = q, \ l_3(+\infty) = r \end{array} \right\}
\end{aligned}
$$

Namely we consider the moduli space of the following configurations:
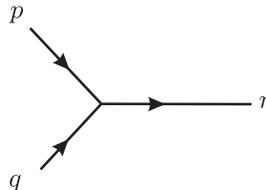
Figure 3.3.1. Cup product: an element in the moduli space.

Claim: If f, g and h and the metric are generic, then $\mathcal{M}(p, q, r)$ is a manifold with

$$\dim \mathcal{M}(p, q, r) = \mu(p) + \mu(q) - \mu(r) - n. \tag{3.3.1}$$

Here the number n is the dimension of M. Using the moduli space $\mathcal{M}(p, q, r)$, we have the following definition.

Definition 3.3.1. We defined a map $\eta_2 : C_k(f-g) \otimes C_l(g-h) \to$ $C_{k+l-n}(f-h)$ by

$$\eta_2(p \otimes q) = \sum_{\mu(r)=\mu(p)+\mu(q)-n} {}^{\natural}\mathcal{M}(p,q,r)r, \forall p \in Crit_k(f-g),\ q \in Crit_l(g-h).$$

$$(3.3.2)$$

Remark The moduli space can be compactified such that

$$\partial \mathcal{M}(p,q,r) = \bigcup_{p'} \widehat{\mathcal{M}}_{p,p'}^{f-g} \times \mathcal{M}(p',q,r) \cup \bigcup_{q'} \widehat{\mathcal{M}}_{q,q'}^{g-h} \times \mathcal{M}(p,q',r)$$

$$\cup \bigcup_{r'} \widehat{\mathcal{M}}_{r',r}^{f-h} \times \mathcal{M}(p,q,r'). \tag{3.3.3}$$
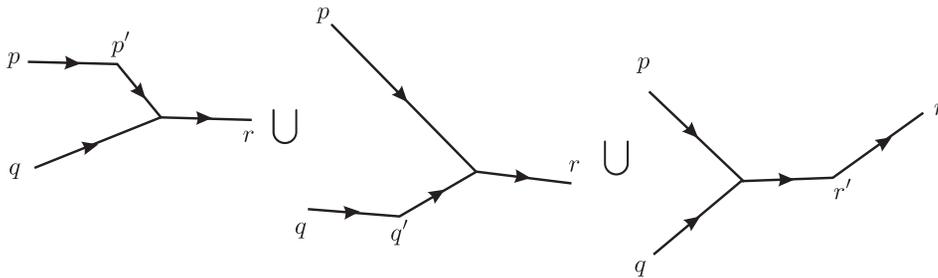
Figure 3.3.2. Boundary of the moduli space.

So the summation (3.3.2) is well defined, and by (3.3.3), we have

$$\partial \eta_2 = \eta_2 \partial.$$

i.e., the map η_2 is a chain map. Thus η_2 induces a cup product of the Morse homology

$$(\eta_2)_* : H_k(C_*(f-g)) \otimes H_l(C_*(g-h)) \to H_{k+l-n}(f-g).$$

By Theorem 3.2.2, we have

$$(\eta_2)_* : H_k(M, \mathbb{Z}) \otimes H_l(M, \mathbb{Z}) \to H_{k+l-n}(M, \mathbb{Z})$$

and there holds

$$(\eta_2)_*(x \otimes y) = \mathrm{PD}(\mathrm{PD}(x) \cup \mathrm{PD}(y)), \qquad (3.3.4)$$

where PD is the Poincaré duality.

By a map η_3 defined in [13], we can see that this cup product is associative, namely, there holds

$$(\eta_2)_*((\eta_2)_*(x \otimes y) \otimes z) = (\eta_2)_*(x \otimes (\eta_2)_*(y \otimes z)). \qquad (3.3.5)$$

(3.3.5) can also follow from the Poincaré duality relation (3.3.4).

Using the gradient flow of the Morse function f to replace the negative gradient flow as in section 3.2, we can get the Morse cohomology theory and the cup product structure.(c.f. [13] and [14]).

Following the basic ideas of the definition of the Morse homology and combining the Gromov psudo-holomorphic theory, Floer constructed a so called *Floer (co)homological* theory for compact symplectic manifolds, and ultimately conduced to the proofs of the Arnold conjecture about the number of fixed points of a Hamiltonian diffeomorphism (non-degenerate cases) on closed symplectic manifolds. Unfortunately, we do not have enough space to explain this beautiful theory here. We refer the papers [11], [12], [15], [36], [50] and the references therein for details. One can apply the Morse homology theory to the study of degenerate Arnold conjecture(the general cases). For this topic, we refer the papers [53], [27] and the references therein.

4. The Maslov-Type Index Theory

4.1. The definition of the Maslov-type index for a symplectic path

We first recall some properties of the symplectic group.

Let $(\mathbb{R}^{2n}, \omega)$ be the linear symplectic space, where $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ is the standard symplectic form. A map $\varphi : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is called symplectic if $\varphi^*\omega = \omega$. The matrix M corresponding to a linear symplectic map is called symplectic matrix. Any symplectic matrix M should satisfy $M^T J M = J$. For any symplectic matrix $M \in \mathrm{Sp}(2n)$, we have the following results.

(1) If $\lambda \in \sigma(M)$, then $\bar{\lambda}$, λ^{-1} and $\bar{\lambda}^{-1} \in \sigma(M)$ with the same multiplicity of λ.

(2) $\det M = 1$, i.e., the any symplectic map φ preserving the the "area" in $\mathbb{R}^{2n}$.

(3) There exist unique orthogonal symplectic matrix U and positively definite symmetric symplectic matrix P such that $M = PU$. We call it the polar decomposition.

(4) A positively definite symmetric matrix $P \in \mathcal{L}(\mathbb{R}^{2n})$ is symplectic, if and only if it has the form

$$P = \exp(Q) = I + Q + \frac{1}{2!}Q^2 + \cdots + \frac{1}{k!}Q^k + \cdots,$$

where $Q = \begin{pmatrix} A & B \\ B & -A \end{pmatrix}$, A and B are symmetric $n \times n$ matrices.

(5) An orthogonal matrix $U \in \mathcal{L}(\mathbb{R}^{2n})$ is symplectic, if and only if it has the form

$$U = \begin{pmatrix} A & -B \\ B & A \end{pmatrix},$$

where $A^T B$ is symmetric, and $A^T A + B^T B = I$. These conditions are also the necessary and sufficient conditions such that $A \pm \sqrt{-1}B$ are unitary matrices.

(6) The space $\mathrm{Sp}(2n)$ is path connected and its fundamental group $\pi_1(\mathrm{Sp}(2n)) = \mathbb{Z}$. So for any closed path in $\mathrm{Sp}(2n)$ there is a natural way to defined an integer as its index.

Claim. For a proof of the result (1), we notice that the coefficients of the characteristic polynomial $f_M(\lambda) = \det(M - \lambda I)$ are all real, so if $\lambda \in \sigma(M)$, then $\bar{\lambda} \in \sigma(M)$. Thus we need to show that $\lambda^{-1} \in \sigma(M)$. By the condition $M^T J M = J$ we know that $(\det M)^2 = 1$, in fact by the result (2), there holds $\det M = 1$, thus

$$f_M(\lambda) = \det M \det(I - \lambda J^{-1} M^T J) = 0 \Rightarrow \det(I - \lambda M^T) = 0.$$

It implies

$$\lambda^{2n} f_M(\lambda^{-1}) = 0.$$

If we using $\det M = 1$, the characteristic polynomial can be written in the symmetric form

$$f_M(\lambda) = \lambda^n \sum_{k=0}^{n} a_k(\lambda^k + \lambda^{-k}),\ a_n = 1,\ \ a_k \in \mathbb{R}.$$

The proof of the result (2) follows from the definition $\varphi^* \omega = \omega$ and $\omega^n = \omega \wedge \cdots \wedge \omega$ is the volume form of $\mathbb{R}^{2n}$. The reason is that $\varphi^*(\omega^n) = \det M \cdot \omega^n = \omega^n$.

We now give a proof of the result (3). Since MM^T is a positive definite matrix, we define a symmetric positive definite matrix

$$P = (MM^T)^{1/2},$$

and set

$$U = P^{-1}M.$$

We have

$$UU^T = P^{-1}MM^TP^{-1} = P^{-1}P^2P^{-1} = I,$$

so U is orthogonal.

If M possesses two polar decompositions $M = P_1U_1 = P_2U_2$, then $M^T = U_1^T P_1 = U_2^T P_2$. Thus

$$P_1^2 = P_1U_1U_1^TP_1 = MM^T$$

and

$$P_2^2 = P_2U_2U_2^TP_2 = MM^T.$$

So $P_1 = P_2$, and so $U_1 = U_2$.

Since $M = J^{-1}(M^T)^{-1}J$, we have

$$M = J^{-1}(U^TP)^{-1}J = J^{-1}P^{-1}JJ^{-1}(U^T)^{-1}J = P_1U_1$$

is a polar decomposition. Thus we have

$$J^{-1}P^{-1}J = P,\ \ J^{-1}(U^T)^{-1}J = U.$$

It implies that P and U are symplectic.

Proof of the result (4).

Necessity. Since $\exp(Q) = P$ is symplectic,

$$\exp(Q) = J^{-1}(\exp(Q))^{-1}J = \exp(-J^{-1}QJ).$$

Since Q and $-J^{-1}QJ$ are symmetric, by the uniqueness, we have $Q = -J^{-1}QJ$. Setting $Q = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, we have $D = -A$ and $B = C$. By $Q^T = Q$, we have $B = B^T$ and $A = A^T$.

Sufficiency. $P = \exp(Q)$ with Q as defined above, it only need to invert the above computations.

Proof of (5).

The sufficiency follows from direct computations.

Necessity. By the given conditions it holds that $U^T K U = K$ with $K = I$ or J. Thus it also holds for $K = I + \sqrt{-1}J$. Let

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then

$$U^T(I + \sqrt{-1}J)U = \begin{pmatrix} I & \sqrt{-1}I \\ -\sqrt{-1}I & I \end{pmatrix}.$$

Comparing both sides we obtain

$$E^*E = F^*F = I, \ \ E^*F = \sqrt{-1}I,$$

where $E = A + \sqrt{-1}C$ and $F = B + \sqrt{-1}D$, E^* denotes the complex conjugate of E^T. This implies

$$E^*(E + \sqrt{-1}F) = 0.$$

But E^* is non-singular. Thus $E + \sqrt{-1}F = 0$. This implies

$$D = A, \ \ C = -B.$$

The symmetry of $A^T B$ and $A^T A + B^T B = I$ follow from the orthogonality of U.

The second part of the result (5) is easy to prove.

The result (6) follows from the results (4) and (5). We can write

$$\text{Sp}(2n) = \text{U}(n) \times (\text{positively definite symmetric symplectic matrices}).$$

It is well known that $\pi_1(\text{U}(n)) = \pi_1(S^1) = \mathbb{Z}$. The space of positively definite symmetric symplectic matrices is contractible, this follows from that any $P = \exp(Q)$ as defined in (4) can be deformed in this space to the identity I by $\exp(tQ)$, $t \in [0, 1]$.

In the particular case $n = 1$, the orthogonal symplectic matrix $U \in \text{Sp}(2) \cap O(2)$ has the form

$$U(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

The positively definite symmetric symplectic matrix has the form

$$P(r, z) = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix}, \ r > 0, \ z \in \mathbb{R}.$$

Thus any symplectic matrix $M \in \text{Sp}(2)$ possesses the polar decomposition

$$M = M(r, \theta, z) = \begin{pmatrix} r & z \\ z & \frac{1+z^2}{r} \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

We can take $(r, \theta, z) \in \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}$ as the coordinates of the space $\text{Sp}(2)$. Thus $\text{Sp}(2) \cong \mathbb{R}^3 \setminus \{the \ z \ axis\}$. We denote by $\text{Sp}(2n)^* = \{M \in \text{Sp}(2n) : \det(M - I) \neq 0\}$, and $\text{Sp}(2n)^0 = \text{Sp}(2n) \setminus \text{Sp}(2n)^*$. The following is the figure of $\text{Sp}(2)^0$.
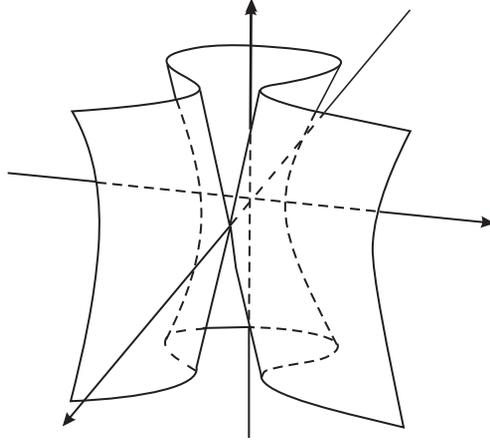
Figure 4.1.1. The surface $\mathrm{Sp}(2)^0$.

We now turn to the definition of the Maslov-type index for a symplectic path starting from the identity.

A τ-periodic solution of a nonlinear Hamiltonian system is a solution of the following problem

$$\begin{cases} \dot{z}(t) = JH'(t, z(t)) \\ z(0) = z(\tau), \end{cases} \tag{4.1.1}$$

where $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ with $H(t, z) = H(t + \tau, z)$ for all $(t, z) \in \mathbb{R} \times \mathbb{R}^{2n}$. $H'(t, z)$ is the gradient of H with respect to the $2n$-dimensional variable z. The linearized system of the above system (4.1.1) at a τ-periodic solution $z(t)$ is the following linear Hamiltonian system

$$\dot{y}(t) = JB(t)y(t), \tag{4.1.2}$$

where $B(t) = H''(t, z(t))$ is a symmetric $2n \times 2n$ matrix function. The fundamental solution $\gamma(t)$ is a $2n \times 2n$ matrix function satisfying

$$\begin{cases} \dot{\gamma}(t) = JB(t)\gamma(t) \\ \gamma(0) = I. \end{cases}$$

It is well known that the fundamental solution γ of a linear Hamiltonian system is a symplectic path starting from the identity. We denote the set of

all the symplectic paths starting from the identity by

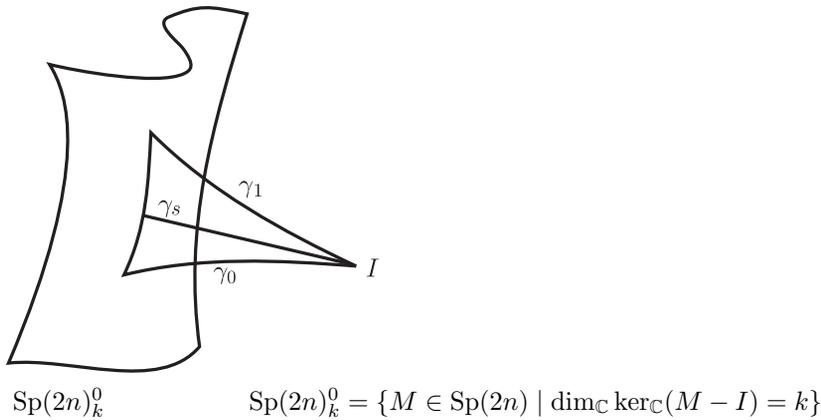$$\mathcal{P}_\tau(2n) = \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) : \gamma(0) = I\}.$$

We denote the unit circle in the complex plane by $\mathbf{U} = \{x \in \mathbb{C} : |x| = 1\}$.

Definition 4.1.1. For $\omega \in \mathbf{U}$, $\tau > 0$, $\gamma \in \mathcal{P}_\tau(2n)$, we define

$$\nu_\omega(\gamma) = \dim_{\mathbb{C}} \ker_{\mathbb{C}}(\gamma(\tau) - \omega I).$$

A path $\gamma \in \mathcal{P}_\tau(2n)$ is called ω-*degenerated* if $\nu_{\tau,\omega}(\gamma) > 0$, otherwise it is called ω-*non-degenerated*.

Definition 4.1.2. For $\omega \in U$, given two paths γ_1 and $\gamma_2 \in \mathcal{P}_\tau(2n)$, we say γ_1 and γ_2 are ω-homotopic on $[0,\tau]$ and write $\gamma_1 \sim_\omega \gamma_2$, if there exists a map $\delta \in C([0,1] \times [0,\tau], \mathrm{Sp}(2n))$ such that $\delta(0,\cdot) = \gamma_1(\cdot)$, $\delta(1,\cdot) = \gamma_2(\cdot)$, $\delta(s,0) = I$ and $\nu_\omega(\delta(s,\cdot))$ is constant for $s \in [0,1]$.

$$\mathrm{Sp}(2n)_k^0 \qquad\qquad \mathrm{Sp}(2n)_k^0 = \{M \in \mathrm{Sp}(2n) \mid \dim_{\mathbb{C}} \ker_{\mathbb{C}}(M - I) = k\}$$

Figure 4.1.2. A case of homotopy of two symplextic paths.

4.1.1. Non-degenerated paths in $\mathrm{Sp}(2n)$

We denoted the subset of ω-non-degenerated paths in $\mathcal{P}_\tau(2n)$ by $\mathcal{P}_{\tau,\omega}^*(2n)$, and $\mathcal{P}_{\tau,\omega}^0(2n) = \mathcal{P}_\tau(2n) \setminus \mathcal{P}_{\tau,\omega}^*(2n)$. We recall that $D(a) = \mathrm{diag}(a, 1/a)$ for $a \neq 0$ and define $M_n^+ = D(2)^{\diamond n}$, $M_n^- = D(-2) \diamond D(2)^{\diamond(n-1)}$. For two symplectic matrices $M_i = \begin{pmatrix} A_i & B_i \\ C_i & D_i \end{pmatrix} \in \mathrm{Sp}(2n_i)$ with A_i, B_i, C_i, $D_i \in \mathcal{L}(\mathbb{R}^{n_i}, \mathbb{R}^{n_i})$

the set of $n_i \times n_i$ matrices, $i = 1, 2$, we recall that the symplectic direct sum (or $\diamond$-product) of M_1 and M_2 are defined by

$$M_1 \diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

We denoted by $M^{\diamond k}$ the k-fold symplectic direct sum $M \diamond M \diamond \cdots \diamond M$. Note that the symplectic direct summation of two symplectic matrices is still symplectic matrix. The two symplectic matrices $M_n^{\pm}$ defined above are located in the different connected components of $\mathrm{Sp}_\omega^*(2n) = \{M \in \mathrm{Sp}(2n) : \det(M - \omega I) \neq 0\}$.

It is well known that every $M \in \mathrm{Sp}(2n)$ has unique polar decomposition $M = AU$ with $A = (MM^T)^{1/2}$, $U = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix}$ and $u = u_1 + \sqrt{-1}u_2$ is a unitary matrix. If $\gamma(t) = A(t)U(t)$, $t \in [0, 1]$ is a continuous symplectic path starting from the identity, there exists a continuous real function $\Delta(t)$ satisfying $\det u(t) = \exp(\sqrt{-1}\Delta(t))$. We define $\Delta_\tau(\gamma) = \Delta(\tau) - \Delta(0) \in \mathbb{R}$ which is depends only on γ. Particularly, if $\gamma(\tau) \subseteq \{M_n^+, M_n^-\}$, we have $\frac{1}{\pi}\Delta_\tau(\gamma) \in \mathbb{Z}$.

Lemma 4.1.3.([41], [39]) *If γ_0 and $\gamma_1 \in \mathcal{P}_n(2n)$ possess common end point $\gamma_0(\tau) = \gamma_1(\tau)$, then $\Delta_\tau(\gamma_0) = \Delta_\tau(\gamma_1)$ if and only if $\gamma_0 \sim_\omega \gamma_1$ on $[0, \tau]$ with fixed end points for some $\omega \in \mathbf{U}$.*

For any $\omega \in \mathbf{U}$, and $\gamma \in \mathcal{P}_{\tau,\omega}^*(2n)$, we can connect $\gamma(\tau)$ to M_n^+ or M_n^- by a path $\beta : [0, \tau] \to \mathrm{Sp}(2n)_\omega^*$. The adjoining path $\beta * \gamma$ is defined by

$$\beta * \gamma(t) = \begin{cases} \gamma(2t), & t \in [0, 1/2], \\ \beta(2t-1), & t \in [1/2, 1]. \end{cases}$$

It is easy to see that the integer $k = \frac{1}{\pi}\Delta_\tau(\beta * \gamma)$ is independent of the choice of the path β.
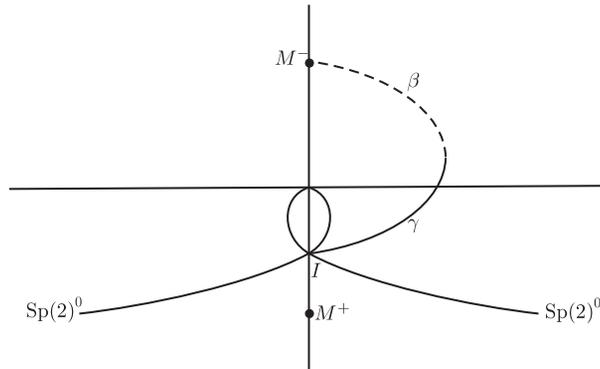
Figure 4.1.3. The adjoint symplectic path.

Definition 4.1.4.([41], [39]) For $\tau > 0$, $\omega \in U$ and $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$, we define

$$i_{\tau,\omega}(\gamma) = k = \frac{1}{\pi}\Delta_\tau(\beta * \gamma).$$

We call the number $i_{\tau,\omega}(\gamma)$ the ω-index of γ. Particularly, we call the number $i_\tau(\gamma) := i_{\tau,1}(\gamma)$ (with $\omega = 1$) the Maslov-type index of the symplectic path γ.

4.1.2. Degenerated paths in $\mathbf{Sp}(2n)$

For a degenerated symplectic path $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$, we can choose a non-degenerated symplectic path $\beta_0 \in \mathcal{P}^*_{\tau,\omega}(2n)$, sufficiently C^0-close to γ in $\mathcal{P}_\tau(2n)$ such that

$$i_{\tau,\omega}(\beta_0) = \inf\{i_{\tau,\omega}(\beta) \mid \beta \in P^*_{\tau,\omega}(2n), \beta \text{ is sufficiently } C^0\text{-close to } \gamma \text{ in } \mathcal{P}_\tau(2n)\},$$

and we can also choose a non-degenerated symplectic path $\beta_1 \in \mathcal{P}^*_{\tau,\omega}(2n)$, sufficiently C^0-close to γ in $\mathcal{P}_\tau(2n)$ such that

$$i_{\tau,\omega}(\beta_1) = \sup\{i_{\tau,\omega}(\beta) \mid \beta \in P^*_{\tau,\omega}(2n), \beta \text{ is sufficiently } C^0\text{-close to } \gamma \text{ in } \mathcal{P}_\tau(2n)\}.$$

In this case, we have the following result.

Lemma 4.1.5.([41], [39]) *With the above notations, there holds*

$$i_{\tau,\omega}(\beta_1) - i_{\tau,\omega}(\beta_0) = \nu_{\tau,\omega}(\gamma).$$

Definition 4.1.6.([41], [39]) For any $\tau > 0$, $\omega \in \mathbf{U}$, $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$, we define

$$i_{\tau,\omega}(\gamma) = \inf\{i_{\tau,\omega}(\beta) \,|\, \beta \in P^*_{\tau,\omega}(2n), \beta \text{ is sufficiently } \ C^0 close to \ \gamma \text{ in } \mathcal{P}_\tau(2n)\}$$

For any symplectic path $\gamma \in \mathcal{P}_\tau(2n)$, $\omega \in \mathbf{U}$, the index pair

$$(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega}(\gamma)) \in \mathbb{Z} \times \{0, 1, \ldots, 2n\}$$

is well defined. We call the index pair the index function of γ at ω. We also call $i_{\tau,\omega}(\gamma)$ the ω-index of γ, and $\nu_{\tau,\omega}(\gamma)$ the nullity of γ. If $\omega = 1$, the index pair is simply denoted by $(i_\tau(\gamma), \nu_\tau(\gamma))$, which is the so called Maslov-type index of γ.

For a linear Hamiltonian system

$$\dot{y}(t) = JB(t)y(t), \ y \in \mathbb{R}^{2n} \tag{4.1.3}$$

with $B(t)$ the symmetric τ-periodic $2n \times 2n$ continuous matrix function. Its fundamental solution γ_B is a symplectic path, i.e., $\gamma_B \in \mathcal{P}_\tau(2n)$. In this case, we denote the index function of the linear Hamiltonian system (or of the matrix function B) by

$$(i_{\tau,\omega}(B), \nu_{\tau,\omega}(B)) := (i_{\tau,\omega}(\gamma_B), \nu_{\tau,\omega}(\gamma_B)). \tag{4.1.4}$$

As usual, the eigenvalues of $\gamma_B(\tau)$ are called *Floquet multipliers* of the linear Hamiltonian system (4.1.3)(or B).

Let $H \in C^1(\mathbb{R}/(\tau\mathbb{Z}) \times \mathbb{R}^{2n}, \mathbb{R})$. Suppose x is a τ-periodic solution of the Hamiltonian system

$$\dot{x}(t) = JH'(t, x(t)), \tag{4.1.5}$$

such that H is C^2 along the orbit $x(\mathbb{R})$ of x. The associated symplectic path of x is defined to be the fundamental solution $\gamma_x = \gamma_B$ of the linearized Hamiltonian system (4.1.3) with $B(t) = H''(t, x(t))$ for all t. In this case, we define the index function of the periodic solution x by

$$(i_{\tau,\omega}(x), \nu_{\tau,\omega}(x)) := (i_{\tau,\omega}(\gamma_x), \nu_{\tau,\omega}(\gamma_x)). \tag{4.1.6}$$

As usual, the eigenvalues of the symplectic matrix $\gamma_x(\tau)$ are called *Floquet multipliers* of the periodic solution x of the Hamiltonian system (4.1.5).

4.1.3. Properties of the index function

Lemma 4.1.7.(Symmetry)([41], [39]) *For any $\omega \in \mathbf{U}$ and $\gamma \in \mathcal{P}_\tau(2n)$, there holds*

$$(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega}(\gamma)) = (i_{\tau,\bar{\omega}}(\gamma), \nu_{\tau,\bar{\omega}}(\gamma)). \qquad (4.1.7)$$

Theorem 4.1.8.(Symplectic additivity)([41], [39]) *For any $\omega \in \mathbf{U}$, suppose $\gamma_i \in \mathcal{P}_\tau(2n_i)$ for $i = 0, 1$, then $\gamma_0 \diamond \gamma_1 \in \mathcal{P}_\tau(2n_0 + 2n_1)$ and*

$$(i_{\tau,\omega}(\gamma_0\diamond\gamma_1), \nu_{\tau,\omega}(\gamma_0\diamond\gamma_1)) = (i_{\tau,\omega}(\gamma_0), \nu_{\tau,\omega}(\gamma_0)) + (i_{\tau,\omega}(\gamma_1), \nu_{\tau,\omega}(\gamma_1)). \ (4.1.8)$$

Theorem 4.1.9.(Homotopic invariant)([41], [39]) *For any $\omega \in \mathbf{U}$, suppose γ_0 and $\gamma_1 \in \mathcal{P}_\tau(2n)$ satisfying $\gamma_0 \sim_\omega \gamma_1$. Then there holds*

$$(i_{\tau,\omega}(\gamma_0), \nu_{\tau,\omega}(\gamma_0)) = (i_{\tau,\omega}(\gamma_1), \nu_{\tau,\omega}(\gamma_1)). \qquad (4.1.9)$$

Theorem 4.1.10.(Local constant)([41]) *The index function $(i_{\tau,\omega}(\gamma), \nu_{\tau,\omega}(\gamma))$ is local constant. The discontinuous points appear only at the ω which is the eigenvalues of $\gamma(\tau)$, i.e., at the points of Floquet multiplier of γ.*

For the monotonicity of the Maslov-type index, we have the following result. We denote by $\mathcal{L}_s(\mathbb{R}^{2n})$ the set of symmetric $2n \times 2n$ matrices.

Theorem 4.1.11.(Monotonicity)([28]) *For any two matrix functions $B_j \in C(S^1, \mathcal{L}_s(\mathbb{R}^{2n}))$ with $B_0(t) < B_1(t)$ for all $t \in \mathbb{R}$, we have*

$$i(B_1) - i(B_0) = I(B_0, B_1),$$

where $I(B_0, B_1) = \sum_{s \in [0,1)} \nu((1-s)B_0 + sB_1)$, and $\nu((1-s)B_0 + sB_1) = \dim \ker(\gamma_s(1) - I)$, $\gamma_s(t)$ is the fundamental solution of the linear systems $\dot{z} = JB_s(t)z$ with $B_s(t) = (1-s)B_0(t) + sB_1(t)$. Particularly, there holds

$$i(B_0) + \nu(B_0) \leq i(B_1).$$

By noting that $i(B_0) = -n$ and $\nu(B_0) = 2n$ for $B_0 \equiv 0$, it implies that for any symmetric positively definite (convex case) matrix function $B(t) > 0$, there holds $i(B) \geq n$.

4.1.4. The relation of Maslov-type index and Morse index: Galerkin approximation

We consider the following problem

$$\begin{cases} \dot{x}(t) = JH'(t, x(t)) \\ x(\tau) = x(0) \end{cases} \tag{4.1.10}$$

In the following, we always suppose that the Hamiltonian function H satisfies the following conditions:

(H1) $H \in C^2(\mathbb{R} \times \mathbb{R}^{2n}, \mathbb{R})$ and

$$H(t + \tau, x) = H(t, x), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2n}. \tag{4.1.11}$$

(H2) There exist constants $a > 0$ and $p > 1$ such that

$$|H''(t, x)| \leq a(1 + |x|^p), \quad \forall (t, x) \in \mathbb{R} \times \mathbb{R}^{2n}. \tag{4.1.12}$$

Recall that $W = W^{1/2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ is the subspace of $L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^{2n})$ which consists of all elements

$$z(t) = \sum_{k \in \mathbb{Z}} \exp(2k\pi t J) a_k, \quad a_k \in \mathbb{R}^{2n},$$

satisfying

$$\|z\|_{1/2,2}^2 := \sum_{k \in \mathbb{Z}} (1 + |k|)|a_k|^2 < +\infty.$$

This space is a Hilbert space with the norm $\| \cdot \|_{1/2,2}$ and the inner product $\langle \cdot, \cdot \rangle_{1/2,2}$. We define an operator $A : W \to W$ such that

$$\langle Ax, y \rangle = \int_0^\tau (-J\dot{x}(t), y(t)) \, dt, \quad \forall x, \, y \in W. \tag{4.1.13}$$

A is a bounded self-adjoint operator with finite dimensional kernel N, and the restriction $A|_{N^\perp}$ is invertible. Define the functional on W by

$$f(x) = \frac{1}{2}\langle Ax, x\rangle - \int_0^\tau H(t, x)\, dt, \ \forall x \in W. \qquad (4.1.14)$$

Then $f \in C^2(W, \mathbb{R})$ and a critical point of f corresponds to a solution of the problem (4.1.10). If $x = x(t)$ is a critical point of f, the second variation of f at x is given by

$$(f''(x)h, h) = \int_0^\tau \left[(-J\dot{h}, h) - (H''(t, x)h, h)\right] dt = \langle (A - B)h, h\rangle \ \forall h \in W, \qquad (4.1.15)$$

where $B : W \to W$ defined by

$$\langle Bz_1, z_2\rangle = \int_0^\tau (B(t)z_1(t), z_2(t))\, dt, \ \forall z_1, \ z_2 \in W \ \ B(t) = H''(t, x(t)). \qquad (4.1.16)$$

It is well known that both the dimensions of the positive and the negative eigen-subspaces of the quadratic form (4.1.15)(i.e., the Morse index of $-f$ and f at x) are infinite. Let $W_m = \sum_{k=-m}^{m} \exp(2k\pi tJ)\mathbb{R}^{2n}$, and $P_m :$ $W \to W_m$ the projection operator. Then the sequence $\Gamma = \{P_m \,|\, m \in \mathbb{N}\}$ is a Galerkin approximation scheme with respect to A, i.e., it satisfies the following three conditions:

1° $W_m = P_mW$ is finite dimensional space for all $m \in \mathbb{N}$.

2° $P_m \to I$ strongly as $m \to \infty$.

3° $[P_m, A] := P_mA - AP_m = 0$.

With this $\Gamma = \{P_m \,|\, m \in \mathbb{N}\}$, we get a finite dimensional approximation $P_m(A - B)P_m$ of the operator $A - B$. The domain of the operator $P_m(A - B)P_m$ is W_m. For this finite dimensional approximation operator, we have the following result.

Theorem 4.1.12.([16], [39]) *Let* $\{P_m\}$ *be a Galerkin approximation scheme with respect to* A. *Then there exist* $d > 0$ *sufficiently small and*

$m_0 > 0$ *such that for all* $m \geq m_0$,

$$\dim m_d^+(P_m(A - B)P_m) = m + n - i_\tau(B) - \nu_\tau(B),$$
$$\dim m_d^0(P_m(A - B)P_m) = \nu_\tau(B), \tag{4.1.17}$$
$$\dim m_d^-(P_m(A - B)P_m) = m + n + i_\tau(B).$$

where $m_d^+(\cdot)$, $m_d^-(\cdot)$, $m_d^0(\cdot)$ *denote the eigenspaces corresponding to the eigenvalue* λ *belonging to* $[d, +\infty)$, $(-\infty, d]$ *and* $(-d, d)$ *respectively.*

4.1.5. The relation of Maslov-type index and Morse index: the saddle point reduction

Let $S^1 = \mathbb{R}/(\tau\mathbb{Z})$. We equip the Hilbert space $L = L^2(S^1, \mathbb{R}^{2n})$ with the usual norm

$$\|x\|_{L^2} = \left(\int_0^\tau |x(t)|^2 \right)^{1/2}, \quad \forall x \in L^2.$$

In the Hilbert space L we define an operator A by

$$\langle Ax, y \rangle_L = \int_0^\tau (-J\dot{x}, y)\, dt, \quad \forall x, y \in L. \tag{4.1.18}$$

Then the domain of A is $\mathrm{dom}A = W^{1,2}(S^1, \mathbb{R}^{2n})$. The range of A is closed and the resolution of A is compact. The spectrum of the operator A is $\sigma(A) = 2\pi\mathbb{Z}$. It is a point spectrum, i.e., it contains only eigenvalues, and the multiplicity of every eigenvalue is $2n$. The eigen-subspace of A belonging to the eigenvalue $2k\pi$ is

$$E_k = \exp(2k\pi tJ)\mathbb{R}^{2n} = ((\cos 2k\pi t)I + (\sin 2k\pi t)J)\mathbb{R}^{2n}.$$

Especially, $\ker(A) = \mathbb{R}^{2n}$. We consider the Hamiltonian systems (4.1.10) with the condition (H1) in the above subsection 4.1.4 and

(H3) There exists a constant $C(H) > 0$ such that

$$|H''(t, x)| \leq C(H), \quad \forall (t, x) \in [0, \tau] \times \mathbb{R}^{2n}. \tag{4.1.19}$$

Define a functional on the space L by

$$g(x) = \int_0^\tau H(t, x(t))\, dt. \tag{4.1.20}$$

By the conditions (H1) and (H3), we have $g \in C^1(L, \mathbb{R})$, and

$$g'(x) = H'(t, x). \tag{4.1.21}$$

$g'(x)$ is Gadeaux differentiable, its Gadeaux derivative is

$$dg'(x)y = H''(t, x(t))y, \tag{4.1.22}$$

and there exists a constant $c(H) > 0$ such that

$$\|dg'(x)\|_{\mathcal{L}(L)} \leq c(H). \tag{4.1.23}$$

Define the action functional by

$$f(x) = \frac{1}{2}\langle Ax, x\rangle_L - g(x), \quad \forall x \in \mathrm{dom}A = W^{1,2}(S^1, \mathbb{R}^{2n}), \tag{4.1.24}$$

Under the conditions (H1) and (H3), $f \in C^1(W, \mathbb{R})$, f' is Gadeaux differentiable. The critical points of f are solutions of the problem (4.1.10).

Let $P_0 : L \to E_0 = \mathbb{R}^{2n}$ be the projection map. Define

$$A_0 x = Ax + P_0 x, \quad \forall x \in W. \tag{4.1.25}$$

Without loss of generality, we suppose the constant in (4.1.23) satisfies $c(H) \notin \sigma(A_0)$ and $c(H) > 1$. Denote by $\{E_\lambda\}$ the spectral resolution of the selfadjoint operator A_0, we define the projections on the Hilbert space L by

$$P = \int_{c(H)}^{c(H)} dE_\lambda, \quad P^+ = \int_{c(H)}^{+\infty} dE_\lambda, \quad P^- = \int_{-\infty}^{-c(H)} dE_\lambda. \tag{4.1.26}$$

Then the Hilbert space L possesses an orthogonal decomposition

$$L = L^+ \oplus L^- \oplus Z, \tag{4.1.27}$$

where $Z = PL$ is a finite dimensional space, and $L^\pm = P^\pm L$. With standard arguments as in [39], [1] and [6], we have the following result.

Theorem 4.1.13.([1], [6]) *Suppose the function H satisfies the conditions* (H1) *and* (H3). *Then there exists a functional $a \in C^2(Z, \mathbb{R})$ and an*

injection map $u \in C^1(Z, L)$ *such that* $u : Z \to W$ *satisfies the following conditions:*

$1°$ *The map* u *has the form* $u(z) = w(z) + z$, *where* $\mathcal{P}w(z) = 0$.

$2°$ *The functional* a *satisfies*

$$a(z) = f(u(z))$$
$$a'(z) = Az - Pg'(u(z)) = Au(z) - g'(u(z)),$$
$$a''(z) = (AP - Pdg'(u(z)))u'(z) = [A - dg'(u(z))]u'(z).$$

And a' *is globally Lipschitz continuous.*

$3°$ $z \in Z$ *is a critical point of* a, *i.e.,* $a'(z) = 0$, *if and only if* $u(z)$ *is a critical point of* f.

$4°$ *If* $g(u) = \langle Bu, u \rangle_L := \int_0^\tau (B(t)u(t), u(t)) \, dt$, $u \in L$, *then* $a(z) = \frac{1}{2}\langle (A - B)z, z \rangle_L$.

$5°$ $\dim \ker a''(z) = \nu\tau(\gamma)$, *where* γ *is the fundamental solution of the linear Hamiltonian systems* $\dot{y} = JH''(t, u(z)(t))y$.

Particularly, for the symmetric matrix continuous function $B(t)$ satisfying $B(t + \tau) = B(t)$, we define a symmetric operator B on L by

$$\langle Bx, y \rangle_L = \int_0^\tau (B(t)x(t), y(t)) \, dt, \quad \forall x, y \in L \qquad (4.1.28)$$

and define

$$f(x) = \frac{1}{2}\langle (A - B)x, x \rangle_L, \quad \forall x \in W. \qquad (4.1.29)$$

The critical points of f are solutions of the following problem

$$\begin{cases} \dot{x} = JB(t)x \\ x(1) = x(0). \end{cases} \qquad (4.1.30)$$

By Theorem 4.1.13, we obtain a subspace

$$Z = \{x | x(t) = \sum_{|k| \leq k_0} \exp(2k\pi tJ)a_k, \ a_k \in \mathbb{R}^{2n}\}$$

with a sufficiently large $k_0 \in \mathbb{N}$, an injection map $u \in C^\infty(Z, L)$, and a

smooth functional $a \in C^\infty(Z, \mathbb{R})$ defined by

$$a(z) = f(u(z)), \quad \forall z \in Z \tag{4.1.31}$$

Let $2d = \dim Z$. Note that the origin of Z as a critical point of a corresponds to the origin of L as a critical point of f. Denote by m^* for $* = +, 0$ and $-$ the positive, null, and negative Morse indices of the functional a at the origin respectively, i.e., the total multiplicities of positive, zero, and negative eigenvalues of the $2d \times 2d$ matrix $a''(0)$ respectively. We have the following result.

Theorem 4.1.14.([9], [43], [37]$-$[41]) *There hold*

$$\begin{cases} m^- = d + i_\tau(B), \\ m^0 = \nu_\tau(B), \\ m^+ = d - i_\tau(B) - \nu_\tau(B). \end{cases} \tag{4.1.32}$$

We consider Problem (4.1.10) with H satisfying condition (H1) in subsection 4.1.4 and (H3) above. Recall that the functional $f(x)$ is defined in (4.1.24). By Theorem 4.1.13, there exist the corresponding functional $a : Z \to \mathbb{R}$ and an injection $u : Z \to L$ such that $a(z) = f(u(z))$. Suppose $z = z(t) \in Z$ is a critical point of a. Then $x = x(t) = u(z)(t)$ is a solution of problem (4.1.10). Suppose $m^*(z)$ with $* = 0, \pm$ are the Morse index of a at z. We have the following result.

Theorem 4.1.15. *Under the above conditions and notations, there hold*

$$\begin{cases} m^-(z) = d + i_\tau(x), \\ m^0(z) = \nu_\tau(x), \\ m^+(z) = d - i_\tau(x) - \nu_\tau(x). \end{cases} \tag{4.1.33}$$

4.1.6. The relation of Maslov-type index and Morse index: the second order Hamiltonian systems

We consider the following problem

$$\begin{cases} \ddot{x} + \nabla V(t, x) = 0, \\ x(\tau) = x(0), \quad \dot{x}(\tau) = \dot{x}(0), \end{cases} \tag{4.1.34}$$

where $V \in C^2(\mathbb{R} \times \mathbb{R}^n, \mathbb{R})$ satisfying $V(t + \tau, x) = V(t, x)$ for all $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. Let $W = W^{1,2}(S^1, \mathbb{R}^n)$ with the inner product

$$\langle x, y \rangle_W = \int_0^1 [(x(t), y(t)) + (\dot{x}(t), \dot{y}(t))] \, dt.$$

We define a functional $F : W \to \mathbb{R}$ by

$$F(x) = \int_0^1 [\frac{1}{2}(\dot{x}(t), \dot{x}(t)) - V(t, x(t))] \, dt, \quad \forall x \in W. \tag{4.1.35}$$

The critical points of F are solutions of problem (4.1.34). Suppose $x \in W$ is a critical point of F. The Hessian of F at x is given by

$$(F''(x)y, z) = \int_0^1 [(\dot{y}(t), \dot{z}(t)) - (\Delta V(t, x(t))y(t), z(t))] \, dt. \tag{4.1.36}$$

The linearized system of (4.1.34) at x is given by the linear second order systems

$$\ddot{y} + \Delta V(t, x(t))y = 0. \tag{4.1.37}$$

We rewrite this systems into a first order linear Hamiltonian systems

$$\dot{z} = JB(t)z, \quad z \in \mathbb{R}^{2n}, \tag{4.1.38}$$

where $z = (\dot{y}, y)^T$ and $B(t) = \begin{pmatrix} -I & 0 \\ 0 & -\Delta V(t, x(t)) \end{pmatrix}$. Suppose $\gamma_x = \gamma(t)$ is the fundamental solution of (4.1.38). Then the index $(i_\tau(\gamma_x), \nu_\tau(\gamma_x))$ is well defined. We denote the Morse index and nullity of F at x by $m^-(x)$ and $m^0(x)$, i.e., the total multiplicities of all the negative eigenvalues and zeros of $F''(x)$ respectively.

Theorem 4.1.16.([2], [39]) *Under the above conditions, there hold*

$$m^-(x) = i_\tau(\gamma_x), \quad m^0(x) = \nu_\tau(\gamma_x), \tag{4.1.39}$$

We refer the paper [34] for the Morse index theory and its iteration

theory of the closed geodesics on Riemannian (Finsler) manifolds.

4.1.7. Calculate the index by dual variational methods

Let $B \in C(S^1, \mathcal{L}_s(\mathbb{R}^{2n}))$, where $S^1 = \mathbb{R}/\mathbb{Z}$, and $\mathcal{L}_s(\mathbb{R}^{2n})$ be the set of symmetric $2n \times 2n$ metrics. Consider the linear Hamiltonian system

$$\dot{z} = JB(t)z, \ z \in \mathbb{R}^{2n}.$$

We first give a brief introduction to the dual Morse index theory of the above system. Let $W^{1,2} = W^{1,2}(S^1, \mathbb{R}^{2n})$, and $L = L^2(S^1, \mathbb{R}^{2n})$. The embedding $j : W^{1,2} \to L$ is compact. Both $W^{1,2}$ and L are Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_2$ respectively. We define an operator $A : L \to L$ with domain $W^{1,2}$ by $Ax = -J\frac{d}{dt}$. The spectrum of A is isolated. In fact $\sigma(A) = 2\pi\mathbb{Z}$. Let $k \notin \sigma(A)$ be so large such that $B(t) + kI > 0$. Then the operator $\Lambda_k = A + kI : W^{1,2} \to L$ is invertible, and its inverse is compact. We define a quadratic form in L by

$$Q^*_{k,B}(v, u) = \int_0^1 [(C_k(t)v(t), u(t)) - (\Lambda_k^{-1}v(t), u(t))] \, dt, \ \forall v, u \in L,$$

where $C_k(t) = (B(t) + kI)^{-1}$. Denote $Q^*_{k,B}(v) = Q^*_{k,B}(v, v)$. Then

$$\langle C_k v, v \rangle_2 = \int_0^1 (C_k(t)v(t), v(t)) \, dt$$

define a Hilbert structure in L. $C_k^{-1}\Lambda_k^{-1}$ is a self-adjoint and compact operator under this inter product. By the spectral theory, there exists a basis e_j, $j \in \mathbb{N}$ of L, and an eigenvalue sequence $\lambda_j \to 0$ in $\mathbb{R}$ such that

$$\langle C_k e_i, e_j \rangle_2 = \delta_{ij},$$
$$\langle \Lambda_j^{-1} e_j, v \rangle_2 = \langle C_k \lambda_j e_j, v \rangle_2, \ \ \forall v \in L.$$

For any $v \in L$ with $v = \sum_{j=1}^{\infty} \xi_j e_j$, there holds

$$Q^*_{k,B}(v) = -\int_0^1 (\Lambda_k^{-1}v(t), v(t)) - (C_k(t)v(t), v(t)) \, dt = \sum_{j=1}^{\infty} (1 - \lambda_j)\xi_j^2.$$

Define

$$L_k^-(B) = \Big\{ \sum_{j=1}^{\infty} \xi_j e_j \,|\, \xi_j = 0 \quad \text{if} \quad 1 - \lambda_j \geq 0 \Big\}$$

$$L_k^0(B) = \Big\{ \sum_{j=1}^{\infty} \xi_j e_j \,|\, \xi_j = 0 \quad \text{if} \quad 1 - \lambda_j \neq 0 \Big\}$$

$$L_k^+(B) = \Big\{ \sum_{j=1}^{\infty} \xi_j e_j \,|\, \xi_j = 0 \quad \text{if} \quad 1 - \lambda_j \leq 0 \Big\}.$$

Observe that $L_k^-(B)$, $L_k^0(B)$ and $L_k^+(B)$ are $Q_{k,B}^*$-orthogonal, and $L = L_k^-(B)$ $\oplus L_k^0(B) \oplus L_k^+(B)$. Since $\lambda_j \to 0$ as $j \to \infty$, both $L_k^-(B)$ and $L_k^0(B)$ are finite dimensional subspaces. We define the k-dual Morse index of B by

$$i_k^*(B) = \dim L_k^-(B), \ \nu_k^*(B) = \dim L_k^0(B).$$

We have the following result for the relation of k-dual Morse index and the Maslov-type index.

Theorem 4.1.17.([28]) *There hold*

$$i_k^*(B) = i(B) + n + 2n\Big[\frac{k}{2\pi}\Big], \ \ \nu_k^*(B) = \nu(B),$$

where $[a] = \max\{j \in \mathbb{Z} \,|\, j \leq a\}$.

The Maslov-type index defined by spectral flow was studied in [45]. The Maslov-type index theory with Lagrangian boundary condition was studied by the author in [29], and that with other non-periodic boundary condition was studied in [30] recently.

4.2. Iteration theory of the Maslov-type index

In this section we consider the iteration theory of the Maslov-type index. Namely, we will give brief introduction to the theory which related the iterated indices $(i_{k\tau}(\gamma^k), \nu_{k\tau}(\gamma^k))$ for the iterated symplectic path $\gamma^k \in \mathcal{P}_{k\tau}(2n)$ of γ with the index $(i_\tau(\gamma), \nu_\tau(\gamma))$ of the symplectic path $\gamma \in \mathcal{P}_\tau(2n)$, where the k-th iterated path γ^k of γ is defined by

$$\gamma^k(t) = \gamma(t - k\tau)\gamma(\tau)^k, \ \ \forall \, k\tau \leq t \leq (k+1)\tau. \tag{4.2.1}$$

To understand this iterated path, we consider a τ-periodic solution $x : S^1 = \mathbb{R}/(\tau\mathbb{Z}) \to \mathbb{R}^{2n}$ of the nonlinear Hamiltonian system

$$\dot{x}(t) = JH'(t, x(t)). \tag{4.2.2}$$

Here we suppose H is C^2 along the orbit $x(\mathbb{R})$ and τ-periodic in time t. We define the iteration of the τ-periodic function x by

$$x^k(t) = x(t - j\tau), \;\; j\tau \le t \le (j+1)\tau, \; 0 \le j \le k - 1.$$

Then x^k becomes a $k\tau$-periodic solution of the system (4.2.2). But geometrically it is the same as x. We set $B_k(t) = H(t, x^k(t))$ for $k \in \mathbb{N}$. Then $\gamma^k \in \mathcal{P}_{k\tau}(2n)$ is the fundamental solution of the linearized system

$$\dot{y}(t) = JB_k(t)y(t). \tag{4.2.3}$$

Definition 4.2.1. The mean index of a symplectic path $\gamma \in \mathcal{P}_\tau(2n)$ is defined by

$$\hat{i}_\tau(\gamma) = \lim_{k\to\infty} \frac{i_{k\tau}(\gamma^k)}{k}. \tag{4.2.4}$$

From Theorem 4.1.10, we have the following definition.

Definition 4.2.2. For any $M \in \mathrm{Sp}(2n)$ and $\omega \in U$, the following number

$$S_M^\pm(\omega) = \lim_{\varepsilon\to 0^+} i_{\tau, \exp(\pm\varepsilon\sqrt{-1})\omega}(\gamma) - i_{\tau,\omega}(\gamma)$$

with $\gamma \in \mathcal{P}_\tau(2n)$ satisfying $\gamma(\tau) = M$ do not depend on the choice of γ. We call it the *splitting number* of M at ω.

4.2.1. Precise iteration formulae

With the index function defined in section 1, we are able to introduce the following Bott-type formula.

Theorem 4.2.3.([41], [39]) *For any* $\gamma \in \mathcal{P}_\tau(2n)$ *and* $m \in \mathbb{N}$, *there hold*

$$
\begin{aligned}
i_{m\tau}(\gamma^m) &= \sum_{\omega^m=1} i_{\tau,\omega}(\gamma), \\
\nu_{m\tau}(\gamma^m) &= \sum_{\omega^m=1} \nu_{\tau,\omega}(\gamma).
\end{aligned}
\tag{4.2.5}
$$

A direct consequence is the following formula, which tells us that the mean index is well defined.

Corollary 2.4.2.([41], [39]) *For any* $\gamma \in \mathcal{P}_\tau(2n)$, *there hold*

$$
\hat{i}_\tau(\gamma) = \frac{1}{2\pi} \int_0^{2\pi} i_{\tau,\exp(\sqrt{-1}\theta)}(\gamma)\, d\theta.
\tag{4.2.6}
$$

For the hyperbolic case, the following iteration formula is very simple.

Corollary 4.2.5. *For any* $\gamma \in \mathcal{P}_\tau(2n)$, *if* $\sigma(\gamma(\tau)) \cap U = \emptyset$, *in this case we call* γ *hyperbolic, and there hold*

$$
\hat{i}_\tau(\gamma) = i_\tau(\gamma), \quad i_{m\tau}(\gamma^m) = m i_\tau(\gamma), \ \forall\, m \in \mathbb{N}.
\tag{4.2.7}
$$

The splitting numbers are determined by the end matrix $\gamma(\tau)$, so the following iteration formula tells us that the iteration properties are only dependent on the matrix $M = \gamma(\tau)$.

Theorem 4.2.6.([44], [39]) *For any* $\gamma \in \mathcal{P}_\tau(2n)$, *let* $M = \gamma(\tau)$. *Then for any* $m \in \mathbb{N}$ *we have*

$$
\begin{aligned}
i_{m\tau}(\gamma^m) =\,& m(i_\tau(\gamma) + S_M^+(1) - C(M)) + \\
& + 2 \sum_{\theta \in (0,2\pi)} E\left(\frac{m\theta}{2\pi}\right) S_M^-(e^{\sqrt{-1}\theta}) - (S_M^+(1) + C(M)).
\end{aligned}
\tag{4.2.8}
$$

Here $E(a) = \min\{k \in \mathbb{Z} \mid k \geq a\}$ *and* $C(M)$ *is defined by*

$$
C(M) = \sum_{\theta \in (0,2\pi)} S_M^-(e^{\sqrt{-1}\theta}).
$$

4.2.2. Iteration inequalities

Theorem 4.2.7.([31], [32]) *For any $\gamma \in \mathcal{P}_\tau(2n)$ and $\omega \in U \setminus \{1\}$, it always holds that*

$$i_\tau(\gamma) + \nu_\tau(\gamma) - n \leq i_{\tau,\omega}(\gamma) \leq i_\tau(\gamma) + n - \nu_{\tau,\omega}(\gamma). \qquad (4.2.9)$$

Theorem 4.2.8.([31], [32], [39]) *For any $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbb{N}$, it always holds that*

$$m\hat{i}_\tau(\gamma) - n \leq i_{m\tau}(\gamma^m) \leq m\hat{i}_\tau(\gamma) + n - \nu_{m\tau}(\gamma^m). \qquad (4.2.10)$$

Theorem 4.2.9.([31], [32], [39]) *For any $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbb{N}$, it always holds that*

$$m(i_\tau(\gamma)+\nu_\tau(\gamma)-n)+n-\nu_\tau(\gamma) \leq i_{m\tau}(\gamma^m) \leq m(i_\tau(\gamma)+n)-(\nu_{m\tau}(\gamma^m)-\nu_\tau(\gamma)). \qquad (4.2.11)$$

Remark. All the estimates (4.2.9)−(4.2.11) are optimal in the sense of the left equality and the right equality can be achieved by some suitable symplectic paths $\gamma \in \mathcal{P}_\tau(2n)$. For the iteration formulae of Morse index theory of closed geodesic on a Riemannian (or Finsler) manifold, we refer the readers to the paper [34]. For the further applications of the iteration theory of Maslov-type index theory, we refer the readers to the papers [16], [24]−[26], [32]−[33], [35], [37]−[41, [43] and [44]

4.3. Application to nonlinear Hamiltonian systems

We will give two examples to explain how to apply this index theory to the study of the nonlinear Hamiltonian systems. One is the study of Rabinowitz conjecture about the existence of the minimal periodic solution for some nonlinear Hamiltonian systems. The reason we choose this problem is that it is easy to explain and in some sense, the calculation is not so complicated. Another is the multiplicity problems of closed characteristics on some hypersurfaces in $\mathbb{R}^{2n}$. The latter is complicated to explain it clearly, so we only state some results and give some references for details.

4.3.1. The minimal periodic problems

We now apply the results obtained above to autonomous Hamiltonian systems

$$-J\dot{x} = Bx + H'(x), \qquad x \in \mathbb{R}^{2n}, \tag{4.3.1}$$

where $n \in \mathbb{N}$ and $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$, and B is a $2n \times 2n$ symmetric semi-positively definite matrix whose operator norm is denoted by $\|B\|$.

Theorem 4.3.1. *Suppose $B \in \mathcal{L}(\mathbb{R}^{2n})$ is a symmetric semi-positively definite matrix, and the Hamiltonian function H satisfies the conditions: (H1) $H \in C^2(\mathbb{R}^{2n}, \mathbb{R})$. (H2) there are constants $\mu > 2$ and $r_0 > 0$ such that*

$$0 < \mu H(x) \le H'(x) \cdot x, \quad \forall |x| \ge r_0.$$

(H3) $H(x) = o(|x|^2)$ at $x = 0$. (H4) $H(x) \ge 0 \quad \forall x \in \mathbb{R}^{2n}$.

Then for every $0 < \tau < \frac{2\pi}{\|B\|}$, the system (4.3.1) possesses a non-constant τ-periodic solution x satisfying

$$i_\tau(x) \le n + 1. \tag{4.3.2}$$

Moreover, suppose this solution x further satisfies the following condition: (HX) $H''(x(t)) \ge 0 \ \forall t \in \mathbb{R}$ and $\int_0^\tau H''(x(t))\,dt > 0$. Then τ is the minimal period of x.

Proof. Fix $\tau \in (0, \frac{2\pi}{\|B\|})$. By conditions (H1)−(H4), we can find a non-constant τ-periodic solution x of (4.3.1) via the saddle point theorem such that (4.3.2) holds. For reader's convenience, we sketch the proof here and refer the reader to Theorem 4.3.5 of [24] or Theorem 4.23 of [51] for details.

In fact, following P. Rabinowitz' pioneering work [48], choose $K > 0$ and $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $\chi(t) = 1$ if $t \le K$, $\chi(t) = 0$ if $t \ge K + 1$, and $\chi'(t) < 0$ if $y \in (K, K+1)$. Set

$$\hat{H}_K(z) = \frac{1}{2}Bz \cdot z + H_K(z),$$

with

$$H_K(z) = \chi(|z|)H(z) + (1 - \chi(|z|))R_K|z|^4,$$

where the constant R_K satisfies

$$R_K \geq \max_{K \leq |z| \leq K+1} \frac{H(z)}{|z|^4}.$$

Let $E = W^{1/2,2}(\mathbb{R}/(\tau\mathbb{Z}), \mathbb{R}^{2n})$ be the Sobolev space with the usual norm. Define a functional f_K on E by

$$f_K(z) = \frac{1}{2} \int_0^\tau (\dot{z} \cdot Jz - \hat{H}_K(z)) \, dt, \quad \forall z \in E. \tag{4.3.3}$$

For $m \in \mathbb{N}$, define $E^0 = \mathbb{R}^{2n}$,

$$E_m = \left\{ z \in E \mid z(t) = \sum_{k=-m}^{m} \exp(\frac{2k\pi t}{\tau} J) a_k, \ a_k \in \mathbb{R}^{2n} \right\},$$

$$E^\pm = \left\{ z \in E \mid z(t) = \sum_{\pm k > 0} \exp(\frac{2k\pi t}{\tau} J) a_k, \ a_k \in \mathbb{R}^{2n} \right\},$$

and $E_m^+ = E_m \cap E^+$, $E_m^- = E_m \cap E^-$. We have $E_m = E_m^- \oplus E^0 \oplus E_m^+$. Let P_m be the projection $P_m : E \to E_m$. Then $\{E_m, P_m\}_{m \in \mathbb{N}}$ form a Galerkin approximation scheme of the operator $-Jd/dt$ on E. Denote by $f_{K,m} = f_K|_{E_m}$. Set $Q_m = \{re : 0 \leq r \leq r_1\} \oplus \{B_{r_1}(0) \cap (E_m^- \oplus E_m^0)\}$ with some $e \in \partial B_1(0) \cap E_m^+$ for large $r_1 > 0$ and small $\rho > 0$. Then ∂Q_m and $B_\rho(0) \cap E_m^+$ form a homological link (cf. P.84 of [6] or p.167 of [18]). By the definition of τ, we obtain a constant $\delta = \delta(K) > 0$ such that

$$f_{K,m}(z) \geq \delta > 0, \qquad \forall z \in \partial B_\rho(0) \cap E_m^+,$$

and

$$f_{K,m}(z) \leq 0, \qquad \forall z \in \partial Q_m.$$

It is well known that f_K satisfies the usual (P.S)* condition on E, i.e. a sequence $\{x_m\}$ with $x_m \in E_m$ possesses a convergent subsequence in E, provided $f'_{K,m}(x_m) \to 0$ as $m \to \infty$ and $|f_{K,m}(x_m)| \leq b$ for some $b > 0$ and all $m \in \mathbb{N}$. Thus by the saddle point theorem (cf. [49]), the Galerkin approximation method, and Theorem 4.2.1 of [51], we obtain a critical point $x_K \in E$ of f_K such that $0 < c_K \equiv f_K(x_K) \leq M_1$, where M_1 is a constant independent of K and there holds $i_\tau(x_K) \leq n + 1$.

Now the arguments in the section 6 of [49] yields a constant M_2 independent of K such that $\|x_K\|_C \leq M_2$. Choose $K > M_2$. Then $x \equiv x_K$ is a non-constant τ-periodic solution of the system (4.3.1) satisfying (4.3.2).

Denote the minimal period of this solution x by τ/m for some $m \in \mathbb{N}$. By the condition (HX) and B being semi-positively definite, using (9.17) of [39], we obtain

$$i_{\tau/m}(x) \geq n. \tag{4.3.4}$$

Since the system (4.3.3) is autonomous, we have

$$\nu_{\tau/m}(x) \geq 1. \tag{4.3.5}$$

Therefore, by (4.3.2), (4.3.4), (4.3.5), and Theorem 4.2.9, we obtain $m = 1$ and complete the proof. $\qquad\square$

Remark. If $B = 0$, Theorem 4.3.1 holds for every $\tau > 0$.

The following corollary gives more accessible sufficient conditions for the existence of solutions with prescribed minimal period.

Corollary 4.3.2. *Under the conditions of Theorem* 4.3.1 *except* (HX), *which is replaced by the following two conditions:* (H5) $H''(x) \geq 0$ *for all* $x \in \mathbb{R}^{2n}$. (H6) *The set* $D = \{x \in \mathbb{R}^{2n} | H'(x) \neq 0, \ 0 \in \sigma(H''(x))\}$ *is hereditarily disconnected, i.e. every connected component of* D *contains only one point.*

Then the system (4.3.1) *possesses a* τ*-periodic solution* x *with minimal period* τ.

Proof. Since conditions (H5) and (H6) imply the condition (HX) holds for every non-constant periodic solution of (4.3.1), the corollary follows from Theorem 4.3.1. $\qquad\square$

Similarly, we consider the existence of non-constant periodic solutions with prescribed minimal period for the following autonomous second order Hamiltonian systems

$$\ddot{x} + V'(x) = 0, \qquad x \in \mathbb{R}^n, \tag{4.3.6}$$

where $n \in \mathbb{N}$ and $V : \mathbb{R}^n \to \mathbb{R}$ is a function. In this paper, we consider the following conditions on V:

(V1) $V \in C^2(\mathbb{R}^n, \mathbb{R})$.

(V2) There exist constants $\mu > 2$ and $r_0 > 0$ such that

$$0 < \mu V(x) \leq V'(x) \cdot x, \qquad \forall |x| \geq r_0.$$

(V3) $V(x) \geq V(0) = 0 \ \forall x \in \mathbb{R}^n$.

(V4) $V(x) = o(|x|^2)$, at $x = 0$.

(V5) There exist constants $b > 0$ and $r_1 > 0$ such that

$$V(x) \leq \frac{b}{2} |x|^2, \ \ \forall |x| \leq r_1.$$

(V6) $V''(x) \geq 0, \ \ \forall x \in \mathbb{R}$.

(V7) $D = \{x \in \mathbb{R}^n | V'(x) \neq 0, \ 0 \in \sigma(V''(x))\}$ is hereditarily disconnected.

Theorem 4.3.3. *Suppose V satisfies the condition* (V1)$-$(V4), (V6) *and* (V7). *Then for every $\tau > 0$, the system (4.3.6) possesses a non-constant τ-periodic solution with minimal period τ.*

Proof. For the system (4.3.6), we consider the following functional

$$\psi(x) = \int_0^\tau (\frac{1}{2} |\dot{x}|^2 - V(x)) \, dt, \ \ \forall x \in W^{1,2}(\mathbb{R}/(\tau\mathbb{Z}), \mathbb{R}^n).$$

By using the saddle point theorem (cf. Theorem 4.4 of [49], here we choosing $E = W^{1,2}(\mathbb{R}/(\tau\mathbb{Z}), \mathbb{R}^n)$, $X = \mathbb{R}^n$, $Y = L_\tau \equiv \{x \in E | x(0) = 0\}$), under the conditions (V1)-(V4) it is well known that there exists a crictal point $x \in E$ of ψ such that its Morse index satisfying $m^-(x, \tau) \leq n + 1$. From [1] and [54], we know the Morse index $m^-(x, \tau)$ of x is just the Maslov-type index $i_\tau(B)$ of the matrix $B(t) = \begin{pmatrix} I & 0 \\ 0 & V''(x(t)) \end{pmatrix}$. Thus we have

$$i_\tau(x) = i_\tau(B) \leq n + 1. \tag{4.3.7}$$

When x further satisfies (V6) and (V7), denote the minimal period of x by τ/m for some $m \in \mathbb{N}$. By (V6), (V7) we have

$$i_{\tau/m}(x) = i_{\tau/m}(B) \geq n. \tag{4.3.8}$$

Note that the system is autonomous, we have

$$\nu_{\tau/m}(x) = \nu_{\tau/m}(B) \geq 1. \tag{4.3.9}$$

Therefore, by Theorem 4.2.9 we have $m = 1$. $\hspace{2cm}$ $\square$

By the same argument as above, we have the following result whose proof is omitted.

Theorem 4.3.4. *Suppose V satisfies the conditions* (V1)$-$(V3) *and* (V5)$-$(V7). *Then for every* $0 < \tau < 2\pi/\sqrt{b}$, (4.3.6) *possesses a non-constant τ-periodic solution x with minimal period τ.*

In his pioneering work [48], P. Rabinowitz proposed a conjecture: whether a superquadratic Hamiltonian system possesses a periodic solution with a prescribed minimal period. This conjecture has been deeply studied by many mathematicians. We refer to [9], [16], [32], [39], and references therein for survey of the study on this problem. Our Theorem 4.3.1 and Theorem 4.3.3 follow the idea of [39], generalize corresponding results in [10] and are different from that of [16].

For Rabinowitz' conjecture on the second order Hamiltonian systems, similar results under various convexity conditions have been proved (cf. [10] and reference therein). In [32] and [39] under precisely the conditions (V1)$-$(V4) of Rabinowitz, Y. Long proved that for any $\tau > 0$ the system (4.3.6) possesses a τ-periodic solution x whose minimal period is at least $\tau/(n+1)$.

We refer the papers [9], [16], [24]$-$[26], [31]$-$[34], [35], [37]$-$[42], [43] and [44] for further applications of the Maslov-type index theory to the study of periodic solutions for nonlinear Hamiltonian systems. In the follow, we list some recent development in this direction.

4.3.2. The fixed energy problems for autonomous nonlinear Hamiltonian systems

We consider a compact hypersurface Σ in $\mathbb{R}^{2n}$. For $x \in \Sigma$, let $N_\Sigma(x)$ be the outward normal vector of Σ at x. We consider the given energy problem of finding $\tau > 0$ and an absolutely continuous curve $x : [0, \tau] \to \mathbb{R}^{2n}$ such that

$$\begin{cases} \dot{x}(t) = JN_\Sigma(x(t)), \ x(t) \in \Sigma, \ \forall t \in \mathbb{R}, \\ x(\tau) = x(0), \end{cases} \qquad (4.3.10)$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the standard symplectic matrix on $\mathbb{R}^{2n}$ with I_n being the identity matrix in $\mathbb{R}^n$. We call any such curve x solving the above problem with minimal period $\tau > 0$ the *closed characteristic* on Σ. Any two closed characteristics x_1 and x_2 on Σ are geometrically distinct if $x_1(\mathbb{R}) \neq x_2(\mathbb{R})$. Let $\mathcal{J}(\Sigma)$ be the set of all closed characteristics on Σ, $[(x, \tau)]$ be the set of all closed characteristics on Σ which are geometrically the same as (x, τ), and $\tilde{\mathcal{J}}(\Sigma)$ be the set of all geometrically distinct ones on Σ. For any C^2 function $H : \mathbb{R}^{2n} \to \mathbb{R}$ satisfying $H^{-1}(1) = \Sigma$ with $\nabla H(x) \neq 0$ for all $x \in \Sigma$, we can turn the problem (4.3.10) into the following nonlinear Hamiltonian system with fixed energy

$$\begin{cases} \dot{x}(t) = J\nabla H(x(t)), \\ H(x(t)) = 1, \\ x(\tau) = x(0). \end{cases} \qquad (4.3.11)$$

For a periodic solution x with period $\tau > 0$ of any Hamiltonian system, we linearized the Hamiltonian system at x, and get a linear Hamiltonian system

$$\dot{z}(t) = JB(t)z(t), \quad B(t) \text{ is symmetric for all } t \in [0, \tau]. \qquad (4.3.12)$$

Suppose $\gamma_x(t)$ is the fundamental solution of the linear Hamiltonian system (4.3.12), i.e., $\gamma_x(t)$ solves the equation (4.3.12) with $\gamma_x(0) = I_{2n}$. It is well known that $\gamma_x(t)$ is a symplectic matrix for all $t \in \mathbb{R}$.

We call the eigenvalues of matrix $\gamma_x(\tau)$ the *Floquet multipliers* of (x, τ). If all Floquet multipliers of (x, τ) lie on the unit circle $\mathbf{U} = \{z \in \mathbb{C}| \, |z| = 1\}$, we say that the closed characteristic (x, τ) is *elliptic*. If no Floquet multiplier lies on the unit circle except 1, and the algebraic multiplicity of $1 \in \sigma(\gamma_x(\tau))$

is 2, we say that the closed characteristic (x, τ) is *hyperbolic*, otherwise, (x, τ) is *non-hyperbolic*. About the multiplicity of closed characteristics on a convex hypersurface, we introduce the following surprised results. We denote by $\mathcal{J}(\Sigma)$ the number of geometrical distinct closed characteristics lying on Σ.

Theorem 4.3.5.([44]) *For any convex C^2 hypersurface Σ in $\mathbb{R}^{2n}$, there holds*

$$\mathcal{J}(\Sigma) \geq [n/2] + 1.$$

Furthermore, if $\mathcal{J}(\Sigma) < \infty$, there exists at least one elliptic closed characteristic.

Theorem 4.3.6.([35]) *For any centrical symmetric convex C^2 hypersurface Σ in $\mathbb{R}^{2n}$, there holds*

$$\mathcal{J}(\Sigma) \geq n.$$

For a star-shaped hypersuface, we introduce the following results.

Theorem 4.3.7.([33]) *For a star-shaped hypersurface Σ in $\mathbb{R}^{2n}$, either there exist infinitely many closed characteristics, or there exists at least one nonhyperbolic closed characteristic, provided every closed characteristic on Σ possesses its Maslov-type mean index greater than 2 when n is odd, and greater than 1 when n is even.*

Theorem 4.3.8.([21]) *A non-degenerate hypersurface Σ in $\mathbb{R}^{2n}$ possesses at least two geometrically distinct closed characteristics for $n \geq 2$. Moreover, for $n = 2$, such a hypersurface Σ either possesses infinitely many closed characteristics, or possesses at least two geometrically distinct elliptic closed characteristics.*

Here we mean that a hypersuface Σ is non-degenerate if all closed characteristics and their iterations are non-degenerate. We mention that in [19] and [20] the results in Theorem 4.3.7 and Theorem 4.3.8 have been extended to a somewhat more general cases.

References

1. H. Amann and E. Zehnder, Nontrivial solutions for a class of non-resonance problems and applications to nonlinear differential equations, *Ann. Scoula Norm. Sup. Pisa. Cl. Sci. Series*, **4.7** (1980), 539-603.

2. T. An and Y.Long, Index theories of second order Hamiltonian systems, *Nonlinear Anal.*, **34** (1998), 585-592.

3. R. Bott, *Lectures on Morse Theory*, Als Manuskript vervielfältigt im Mathematischen Institut der Universität Bonn, 1960.

4. R. Bott, Morse theory indomitable, *Publications mathématiques de l'I.H.E.S.*, **68** (1988), 99-114.

5. R. Bott, Lectures on Morse theory, old and new, *Bull. Amer. math. Soc.* **7**(1982), no.2, 331-358.

6. K. C. Chang, *Infinite Dimensional Morse Theory And The Multiple Solution Problems*, Birkhäuser, Boston, 1991.

7. K. C. Chang, J. Q. Liu and M. J. Liu, Nontrivial periodic solutions for strong resonance Hamiltonian systems, *Ann. Inst. H.Poincaré Anal. non. linéaire*, **14**(1997), no.1, 103-117.

8. C. Conley, *Isolated Invariant Sets and the Morse Index*, CBMS Reg. Conf. Series in math. 38, Amer. Math. Soc. 1978.

9. C. Conley and E. Zehnder, Maslov-type index theory for flows and periodic solutions for Hamiltonian systems, *Comm. Pure Appl. Math.*, **37** (1984), 207-253.

10. I. Ekeland, *Convexity Methods in Hamiltonian Mechanics*, Springer-Verlag, Berlin, Heidelberg, New York, 1990.

11. A. Floer, The unregularized gradient flow of the symplectic action, *Comm. Pure Appl. Math.*, **41**(1988), 775-813.

12. A. Floer, Symplectic fixed point and holomorphic shperes, *Comm. Pure Appl. Math.*, **120**(1989), 575-611.

13. K. Fukaya, Morse homotopy, A^∞-category, and Floer homologyies, Proceedings of GARC Workshop on geometry and topology, ed. by H.J.Kim, Seou National University, Korea, 1993.

14. K. Fukaya and Y.-G. Oh, Zero-loop open strings in the cotangentbundle and Morse homotopy, *Asian J. Math.* **1**(1997), no.1, 96-180.

15. K. Fukaya and K. Ono, Arnold conjecture and Gromov-Witten invariants, *Topology*, **38**(1999), no. 5, 933-1048.

16. G. Fei and Q. Qiu, Periodic solutions of asymptotically linear Hamiltonian systems, *China Ann. Math.*, **18B**(1997), no.3, 359-372.

17. M. Gromov, Pseudo holomorphic curves in symplectic manifolds, *Invent. Math.* **82**(1985), 307-347.

18. N. Ghoussoub, *Duality and Perturbation Methods in Critical Point Theory*, Cambridge University Press, Cambridge, 1993.

19. F. Guo and C. Liu, *The Stability of The Closed Characteristics on Normal Linear Positive-type Hypersurfaces,* Preprint.

20. F. Guo and C. Liu, *The Closed Characteristics on Normal Linear Positive-type Hypersurfaces,* Preprint.

21. X. Hu and Y. Long, Colsed characteristics on non-degenerate star-shaped hypersurfacers in $\mathbf{R}^{2n}$, *Sci. China Ser. A*, **45**(2002), no. 8,1038-1052.

22. W. Kryszewski and A. Szulkin, An infinite dimensional Morse theory with applications, *Trans. Amer. Math. Soc.*, **349**(1997), no.8, 3181-3234.

23. S. Li and J. Liu, Morse theory and asymptotically linear Hamiltonian systems, *J. Difference Equations*, **78**(1989), 53-73.

24. C. Liu, Subharmonic solutions of Hamiltonian systems, *Nonlinear Anal.*, **42** (2000), 185-198.

25. C. Liu, The stability of subharmonic solutions for Hamiltonian systems, *J. Math. Anal. Appl.*, **240**(1999), 491-504.

26. C. Liu, The stability of the periodic solutions of second order Hamiltonian systems, *Chinese Ann. Math. Ser. B*, **21**(2000), no.2, 225-232.

27. C. Liu, Cup-length estimate for Lagrangian intersections, *J. Differential Equations*, **209**(2005), no. 1, 57-76.

28. C. Liu, A note on the monotonicity of Maslov-type index of linear Hamiltonian systems with applications, *Proc. Roy. Soc. Edinburgh Sect. A*, **135**(2005), 1263-1277.

29. C. Liu, Maslov-type index theory for symplectic paths with Lagrangian boundary conditions, Preprint.

30. C. Liu, Maslov *P*-index theory for a symplectic path with applications, *Chinese Ann. Math. Ser. B*, **27**(2006), no. 2, 1-18.

31. C. Liu and Y. Long, An optimal increasing estimate for iterated Maslov-type indices, *Chinese Sci. Bull.*, **42**(1997), 2275-2277.

32. C. Liu and Y. Long, Iteration inequalities of the Maslov-type index theory with application, *J. Differential Equations*, **165**(2000), 355-376.

33. C. Liu and Y. Long, Hyperbolic characteristics on star-shaped hypersurfaces, *Ann. Inst. H. Poinceré Anal. Non Linéaire*, **16** (1999), 725-746.

34. C. Liu and Y. Long, Iterated index formulae for closed geodesics with application, *Sci. China*, **45**(2002), no.1, 9-28.

35. C. Liu, Y. Long and C. Zhu, Multiplicity of closed characteristics on symmetric convex hypersurfaces in $\mathbb{R}^{2n}$, *Math. Ann.*, **323**(2002), 201-215.

36. G. Liu and T. Tian, Floer homology and Arnold conjecture, *J. Differential Geom.*, **49**(1998), no. 1, 1-74.

37. Y. Long, Maslov-type index, degenerate critical points, and asymptotically linear Hamiltonian systems, *Sci. China*, **33** (1990), 1409-1419.

38. Y. Long, A Maslov-type index theory for symplectic paths, *Topol. Methods Nonlinear Anal.* **10**(1997), 47-78.

39. Y. Long, *Index Theory for Symplectic Paths with Applications*, Progress in Mathematics, Volume 207, Birkhäuser Verlag, 2002.

40. Y. Long, *Index Theory of Hamiltonian systems with Applications* (In Chinese), Science Press, Beijin, 1993.

41. Y. Long, Bott formula of the Maslov-type index theory, *Pacific J. Math.*, **187** (1999), 113-149.

42. Y. Long, Multiple periodic points of the Poincaré map of Lagrangian systems on tori, *Math. Z.* **233**(2000), no.3, 443-470.

43. Y. Long and E. Zehnder, Morse index theory for forced oscillations of asymptotically linear Hamiltonian systems, In *Stocastic Process. Phys. and Geom.*, S. Albeverio et al. ed. World Sci. (1990), 528-563.

44. Y. Long and C. Zhu, Closed characteristics on compact convex hypersurfaces in $\mathbb{R}^{2n}$, *Ann. Math.*, **155**(2002), no.2, 317-368.

45. Y. Long and C. Zhu, Maslov-type index theory for symplectic paths and spectral flow (II), *Chinese Ann. Math. Ser. B*, **21**(2000), no. 1, 89-108.

46. J. W. Milnor, Morse Theory. Based on lecture notes by M. Spivak and R. Wells. Princeton, N. J., Princeton University Press, 1963.

47. M. Morse, *The Calculus of Variations in the Large*, New York, The American Mathematical Society, 1934.

48. P. Rabinowitz, Periodic solutions of Hamiltonian systems, *Comm. Pure Appl. Math.*, **31**(1978), 157-184.

49. P. H. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, *CBMS Regional Conf. Ser. in Math. Amer. Math. Soc.*, **65**(1986).

50. Y. Ruan, Virtual neighborhood and pseudo-holomorphic curves, Proceedings of 6th Gokova Geometry-Topology conference, *Turkish J. Math.*, **23**(1999), no.1, 161-231.

51. T. Wang and G. Fei, Subharmonics for superquadratic Hamiltonian systems via the iteration method of the Maslov-type index theory, Preprint.

52. M. Schwarz, *Morse Homology*, Birkhäuser Verlag, Basel, Boston (1993).

53. M. Schwarz, A quantum cup-length estimate for symplectic fixed points, *Invent. Math.*, **133**(1998), 353-397.

54. C. Viterbo, A new obstruction to embedding Lagrangian tori, *Invent. Math.*, **100**(1990), 301-320.

School of Mathematics and LPMC, Nankai University, Tianjin 300071, People's Republic of China.

E-mail: liucg@nankai.edu.cn