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GEOMETRIC AND APPROXIMATE PROPERTIES OF CONVOLUTION POLYNOMIALS IN THE UNIT DISK

BY

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Abstract

The purpose of this paper is to prove several results in approximations through complex convolution polynomials with Jackson-type rate or with best approximation rate, having the quality of preservation of some properties in geometric function theory, like the preservation of: coefficients' bounds, positive real part, bounded turn, close-to-convexity, starlikeness, convexity, spirallikeness, α-convexity. Also, some sufficient conditions for starlikeness and univalence of analytic functions are preserved.

1. Introduction

Let us consider the open unit disk $\mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}$ and $A(\overline{\mathbb{D}}) = \{f : \overline{\mathbb{D}} \to \mathbb{C};\ f$ is analytic on $\mathbb{D}$, continuous on $\overline{\mathbb{D}}$, $f(0) = 0,\ f'(0) = 1\}$.

Recall that a function $f \in A(\overline{\mathbb{D}})$ is starlike if it is univalent on $\mathbb{D}$ and $f(\mathbb{D})$ is a starlike plane domain with respect to 0, and is convex if it is univalent on $\mathbb{D}$ and $f(\mathbb{D})$ is a convex plane domain.

Concerning the shape preserving complex approximation, firstly let us recall the following two known results.

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Theorem 1.1 ([12]). *Let*

$$\Omega_n(u) = \frac{(n!)^2}{(2n)!}\left(2\cos\frac{u}{2}\right)^{2n},$$

be the de la Vallée-Poussin kernel and let the complex convolution polynomials of f of degree $\leq n$ be defined as

$$P_n(f)(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})\Omega_n(x-t)dt=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(ze^{iu})\Omega_n(u)du, z=re^{ix} \in \overline{\mathbb{D}}.$$

If $f \in A(\overline{\mathbb{D}})$ is starlike (convex) then the polynomials $P_n(f)(z)$ are starlike (convex, respectively), for all $n \in \mathbb{N}$.

Theorem 1.2 ([14]). *Let us consider the nth Cesáro kernel of order $\alpha > 0$, given by*

$$K_n^{\alpha}(u) = \sum_{k=0}^{n}(A_{n-k}^{\alpha-1}/A_n^{\alpha})D_k(u),$$

$$D_k(u) = \frac{\sin\left(k+\frac{1}{2}\right)u}{\sin\frac{u}{2}}$$

and

$$A_m^{\alpha} = \binom{m+\alpha}{m} = \frac{(\alpha+1)\cdots(\alpha+m)}{m!}.$$

Define the convolution complex polynomial

$$s_n^{\alpha}(f)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(ze^{iu})K_n^{\alpha}(u)du,$$

$z = re^{ix} \in \overline{\mathbb{D}}$, $n \in \mathbb{N}$.

If $f \in A(\overline{\mathbb{D}})$ is convex then $s_n^{\alpha}(f)(z)$, $\alpha \geq 3$, $n \in \mathbb{N}$ are convex on $\mathbb{D}$.

Remarks. 1) In [16], the concept of n-starlikeness, $n = 0, 1, 2, \ldots$, is introduced by the condition $\mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)} > 0$, $z \in \mathbb{D}$, where $D^0(f)(z) = f(z)$, $D^1(f)(z) = zf'(z)$, $D^{n+1}(f)(z) = D[D^n(f)](z)$. For $n = 0$ we recapture the usual starlikeness and for $n = 1$ the usual convexity, respectively. It is known ([16, Corollary 3.2]) that f is n-starlike if and only if $zf'(z)$ is $(n-1)$-starlike.

Since the polynomials in Theorem 1.1 preserve the convexity of f, reasoning by recurrence as in [12], we immediately obtain that $P_n(f)(z)$ preserve the n-starlikeness, for all $n = 0, 1, 2, \ldots$, i.e.

$$\mathrm{Re}\frac{D^{n+1}(f)(z)}{D^n(f)(z)} > 0, \quad z \in \mathbb{D},$$

implies

$$\mathrm{Re}\frac{D^{n+1}(P_n(f))(z)}{D^n(P_n(f))(z)} > 0, \quad z \in \mathbb{D}.$$

2) Regarding the approximation error, in the case of Theorem 1.1 we have only the rather weak estimate

$$|f(z) - P_n(f)(z)| \leq 3\omega_1 \left(f; \frac{1}{\sqrt{n}}\right)_{\overline{\mathbb{D}}},$$

(see [4]), while in the case of Theorem 1.2 even a worst estimate can hold. Some related problems to Theorem 1.1 were solved in [15], but without connection to improve the approximation rate.

On the other hand, in the very recent papers [4-5], classes of convolution-type integral complex operators were considered and their approximation properties regarding rates, global smoothness preservation properties and some geometric properties were presented. Starting from Jackson kernels, Beatson [2] constructed new trigonometric kernels about which he observed (without proof, see [6] for a proof) that are bell-shaped. In [5], convolution-type complex polynomials based on these Beatson kernels were introduced, for which global smoothness preservation properties and Jackson-type estimates with respect to $\omega_1\left(f; \frac{1}{n}\right)_{\overline{\mathbb{D}}}$ were proved. It is also showed in the same paper that these operators transform the convex univalent functions only into close-to-convex univalent polynomials.

One of the main questions proposed to be solved by the present paper is to improve the rate of approximation in [12]. It is then natural to consider the following.

OPEN QUESTION. *Let $f \in A(\overline{\mathbb{D}})$. Can one construct a sequence of complex convolution polynomials $P_n(f)(z)$, $n = 1, 2, \ldots$, with the degree of*

$P_n(f)(z) \le n$, of the form

$$P_n(f)(z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(re^{it})Q_n(x-t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(ze^{iu})Q_n(u)du,$$

$z = re^{ix} \in \overline{\mathbb{D}}$, such that for some $p \ge 1$

$$|f(z) - P_n(f)(z)| \le C\omega_p\left(f; \frac{1}{n}\right)_{\overline{\mathbb{D}}},$$

and moreover, if f is starlike on $\mathbb{D}$ then all $P_n(f)(z)$ are starlike on $\mathbb{D}$ and if f is convex on $\mathbb{D}$ then all $P_n(f)(z)$ are convex on $\mathbb{D}$?

In this paper we give, among others, some partial answers to it, in the sense that the above Open Question is solved for some subclasses of starlike and convex functions. Thus, Section 2 contains new properties of the Beatson kernels and of the convolution polynomials introduced by [5] and based on them. In Section 3, we obtain many approximation results through convolution polynomials based on various trigonometric kernels (of Fejér, Jackson, Beatson, Cesáro, de la Vallée-Poussin mean), producing Jackson-type approximation rate or best approximation rate and preserving some properties in geometric function theory, like the coefficients' bounds, positive real part, bounded turn, close-to-convexity, starlikeness, convexity, spirallikeness, α-convexity. Also, some sufficient conditions for starlikeness and univalence of analytic functions are preserved.

2. New Properties of Beatson Kernels

Let $K_{n,r}(t)$ be the Jackson kernels given by

$$K_{n,r}(s) = \left(\frac{\sin\frac{ns}{2}}{\sin\frac{s}{2}}\right)^{2r}$$

and $c_{n,r}$ chosen such that $\frac{1}{\pi}\int_{-\pi}^{\pi} K_{n,r}(s)ds = c_{n,r}$, and let us consider the Beatson kernels

$$B_{n,r}(t) = \frac{n}{2\pi c_{n,r}}\int_{t-\pi/n}^{t+\pi/n} K_{n,r}(s)ds.$$

Firstly we prove the following lemma, which might be of independent interest.

Lemma 2.1. *For all n, $r \in \mathbb{N}$, n, $r \geq 2$ and $k \in \{0, 1, \dots, 2r-2\}$ we have*

$$\int_0^\pi t^k B_{n,r+1}(t)dt \leq Cn^{-k}.$$

Proof. If $k = 0$ then

$$\int_0^\pi t^k B_{n,r+1}(t)dt = \int_0^\pi B_{n,r+1}(t)dt \leq \int_0^{2\pi} B_{n,r+1}(t)dt = \pi.$$

Let $k \in \{1, 2, \dots, 2r-2\}$. Integrating by parts, we get

$$\begin{aligned}
\int_0^\pi t^k B_{n,r+1}(t)dt &= \frac{n}{2\pi c_{n,r+1}(k+1)}\pi^{k+1}\int_{\pi-\pi/n}^{\pi+\pi/n} K_{n,r+1}(u)du \\
&\quad -\frac{n}{2\pi c_{n,r+1}(k+1)}\int_0^\pi t^{k+1}K_{n,r+1}(t+\pi/n)dt \\
&\quad +\frac{n}{2\pi c_{n,r+1}(k+1)}\int_0^\pi t^{k+1}K_{n,r+1}(t-\pi/n)dt \\
&=: I_1 - I_2 + I_3.
\end{aligned}$$

We will estimate each integral I_1, I_2, I_3. For this purpose, the following relations (see e.g. [8, p.57]) are useful:

$$c_{n,r} \approx n^{2r-1}, \quad \int_0^\pi t^k K_{n,r}(t)dt \approx n^{2r-1-k}.$$

Firstly we have

$$\begin{aligned}
I_3 &\leq Cn^{-2r}\int_0^\pi t^{k+1}K_{n,r+1}(t-\pi/n)dt \\
&= Cn^{-2r}\int_{-\pi/n}^{\pi-\pi/n}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&= Cn^{-2r}\int_0^{\pi-\pi/n}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\quad +Cn^{-2r}\int_{-\pi/n}^0(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\leq 2Cn^{-2r}\int_0^\pi(v+\pi/n)^{k+1}K_{n,r+1}(v)dv \\
&\leq Cn^{-2r}n^{2(r+1)-1-(k+1)} = Cn^{-k}.
\end{aligned}$$

Secondly we obtain

$$\begin{aligned}
I_2 &\le Cn^{-2r}\int_0^{\pi} t^{k+1}K_{n,r+1}(t+\pi/n)dt\\
&= Cn^{-2r}\int_{\pi/n}^{\pi+\pi/n}(v-\pi/n)^{k+1}K_{n,r+1}(v)dv\\
&\le Cn^{-2r}\int_{\pi/n}^{\pi+\pi/n}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv\\
&= Cn^{-2r}\int_{\pi/n}^{\pi}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv\\
&\quad +Cn^{-2r}\int_{\pi}^{\pi+\pi/n}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv\\
&\le Cn^{-2r}\int_0^{\pi}(v+\pi/n)^{k+1}K_{n,r+1}(v)dv+Cn^{-2r}\int_{\pi}^{\pi+\pi/n}v^{k+1}K_{n,r+1}(v)dv\\
&\le Cn^{-k}+Cn^{-2r}\int_{\pi}^{\pi+\pi/n}v^{k+1}K_{n,r+1}(v)dv.
\end{aligned}$$

Denoting

$$J_2 = Cn^{-2r}\int_{\pi}^{\pi+\pi/n}v^{k+1}K_{n,r+1}(v)dv,$$

by the substitution $nv/2 = t$, we get

$$\begin{aligned}
J_2 &= Cn^{-2r}/n\int_{n\pi/2}^{n\pi/2+\pi/2}(2t/n)^{k+1}(\sin(t)/\sin(t/n))^{2(r+1)}dt\\
&= Cn^{-2r-k-2}\int_{n\pi/2}^{n\pi/2+\pi/2}t^{k+1}(\sin(t)/\sin(t/n))^{2(r+1)}dt.
\end{aligned}$$

But for $t\in[n\pi/2, n\pi/2+\pi/2]$, we have $t/n\in[\pi/2,\pi/2+\pi/(2n)]\subset[0,\pi]$, for all $n\ge 2$ and consequently

$$\sin(t/n)\ge\sin(\pi/2+\pi/(2n))=\cos(\pi/(2n))\ge\cos(\pi/4).$$

From this we obtain, by noting that $k\le 2r-2\le 2r+1$,

$$\begin{aligned}
J_2 &\le Cn^{-2r-k-2}\int_{n\pi/2}^{n\pi/2+\pi/2}t^{k+1}dt\le Cn^{-2r-k-2}(n\pi/2+\pi/2)^{k+1}\\
&\le Cn^{-2r-k-2+k+1}=Cn^{-2r-1}\le Cn^{-k}.
\end{aligned}$$

As a conclusion, we get $I_2 \le Cn^{-k}$. Finally, making the substitution $nu/2 = v$, we have

$$\begin{aligned} I_1 &\le Cn^{-2r}\int_{\pi-\pi/n}^{\pi+\pi/n} K_{n,r+1}(u)du \\ &= Cn^{-2r-1}\int_{n\pi/2-\pi/2}^{n\pi/2+\pi/2}(\sin(v)/\sin(v/n))^{2(r+1)}dv \\ &= Cn^{-2r-1}\int_{n\pi/2-\pi/2}^{n\pi/2}(\sin(v)/\sin(v/n))^{2(r+1)}dv \\ &\quad +Cn^{-2r-1}\int_{n\pi/2}^{n\pi/2+\pi/2}(\sin(v)/\sin(v/n))^{2(r+1)}dv =: J_1 + L_1. \end{aligned}$$

But $v \in [n\pi/2 - \pi/2, n\pi/2]$ is equivalent to $v/n \in [\pi/2 - \pi/(2n), \pi/2]$, it follows $\sin(v/n) \ge C(v/n)$ and

$$\begin{aligned} J_1 &\le Cn^{-2r-1}\int_{n\pi/2-\pi/2}^{n\pi/2}(\sin(v)/(v/n))^{2(r+1)}dv \\ &= Cn\int_{n\pi/2-\pi/2}^{n\pi/2}(\sin(v)/v)^{2(r+1)}dv \\ &\le Cn\int_{n\pi/2-\pi/2}^{n\pi/2}(1/v)^{2(r+1)}dv \\ &\le Cn(1/((n-1)\pi/2)^{2(r+1)} \le Cn^{-2r-1} \le Cn^{-k}, \end{aligned}$$

for all $n \ge 2$. Also, by the substitution $v/n = t$, we get

$$\begin{aligned} L_1 &= Cn^{-2r}\int_{\pi/2}^{\pi/2+\pi/(2n)}(\sin(nt)/\sin(t))^{2(r+1)}dt \\ &\le Cn^{-2r}\int_{\pi/2}^{\pi/2+\pi/(2n)}(1/\sin(t))^{2(r+1)}dt \\ &\le Cn^{-2r}(\pi/(2n))(1/\sin(\pi/2+\pi/(2n)))^{2(r+1)} \\ &\le Cn^{-2r-1}(1/\cos(\pi/(2n)))^{2(r+1)} \le Cn^{-2r-1} \le Cn^{-k}, \end{aligned}$$

since $\cos(\pi/(2n)) \ge \cos(\pi/4)$, for all $n \ge 2$.

Collecting all the estimates for I_1, I_2 and I_3, we get that $I_1 - I_2 + I_3 \le Cn^{-k}$, which proves the lemma. $\square$

As a consequence, we obtain the following:

Corollary 2.2. *Let $f \in A(\overline{\mathbb{D}})$. The convolution polynomials defined by*

$$P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(re^{it})B_{m,r}(x-t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{iu})B_{m,r}(u)du,$$

$z = re^{ix} \in \overline{\mathbb{D}}$, $m = [n/r]+1$, satisfy degree $P_n(f)(z) \le n$, and for all $r \ge 3$ we have

$$|f(z) - P_n(f)(z)| \le C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}},$$

for all $z \in \overline{\mathbb{D}}$ and all $n \in \mathbb{N}$, $n \ge 2$.

Here

$$\omega_p(f;\delta)_{\partial\mathbb{D}} = \sup\{|\Delta_u^p f(e^{ix})|;\ |x| \le \pi,\ |u| \le \delta\},$$

with $\Delta_u^p g(x) = \sum_{k=0}^{p}(-1)^{p-k}\binom{p}{k}g(x+ku)$.

Proof. Let $r \ge 3$ and $n \ge 2$ be fixed. Since $B_{m,r}(t)$ is even, reasoning as in the proof of Theorem 2 in [8, p.56] (see also [4, p.422]) we easily obtain

$$f(z) - P_n(f)(z) = \int_0^{\pi}[2f(z) - f(ze^{it}) - f(ze^{-it})]B_{m,r}(t)dt.$$

By the maximum modulus principle we can get on $|z| = 1$, and by passing to absolute value, (as in [8, p.56])

$$|f(z) - P_n(f)(z)| \le \omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}} \int_0^{\pi}(nt+1)^2 B_{m,r}(t)dt \le C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}},$$

where we also applied Lemma 2.1.

Indeed, for $|z| = 1$ by the above identity we get

$$\begin{aligned}|f(z) - P_n(f)(z)| &= \left|\int_0^{\pi}[2f(z) - f(ze^{it}) - f(ze^{-it})]B_{m,r}(t)dt\right| \\ &\le \int_0^{\pi}|2f(z) - f(ze^{it}) - f(ze^{-it})|B_{m,r}(t)dt.\end{aligned}$$

But

$$\begin{aligned}|2f(z)-f(ze^{it})-f(ze^{-it})| &\le \omega_2(f;t)_{\partial\mathbb{D}}=\omega_2\left(f;\frac{nt}{n}\right)_{\partial\mathbb{D}}\\ &\le C(nt+1)^2\omega_2\left(f;\frac{1}{n}\right)_{\partial\mathbb{D}},\end{aligned}$$

which together with Lemma 2.1 proves the corollary. □

Now, if we define, as in [5], the iterative Beatson kernels by recurrence as $B_{n,r,1}(t):=B_{n,r}(t)$,

$$\begin{aligned}B_{n,r,2}(t) &= \frac{n}{2\pi}\int_{t-\pi/n}^{t+\pi/n}B_{n,r,1}(s)ds,\ldots,B_{n,r,p}(t)\\ &= \frac{n}{2\pi}\int_{t-\pi/n}^{t+\pi/n}B_{n,r,p-1}(s)ds,\end{aligned}$$

$p=2,3,\ldots$, then the following generalization of Lemma 2.2 holds.

Lemma 2.3. *For all n, r, $p\in\mathbb{N}$ with $r\ge 2$, $n\ge p+1$ and $k\in\{0,1,\ldots,2r+2p-4\}$, we have*

$$\int_0^\pi t^kB_{n,r+p,p}(t)dt\le Cn^{-k}.$$

Proof. For $p=1$ we get Lemma 2.1. In what follows, for simplicity we prove the case $p=2$. We have

$$B_{n,r,2}(t)=\frac{n}{2\pi}\int_{t-\pi/n}^{t+\pi/n}B_{n,r}(s)ds=\frac{n^2}{(2\pi)^2c_{n,r}}\int_{t-\pi/n}^{t+\pi/n}\int_{x-\pi/n}^{x+\pi/n}K_{n,r}(s)dsdx.$$

If $k=0$ then obviously

$$\int_0^\pi t^kB_{n,r+2,2}(t)dt\le\int_0^{2\pi}t^kB_{n,r+2,2}(t)dt\le\pi.$$

Let $k = 1, 2, \ldots, 2r$. Integrating twice by parts, by simple calculation we get

$$\begin{aligned}
&\int_0^\pi t^k B_{n,r+2,2}(t)dt \\
&= B_{n,r+2,2}(\pi)\frac{\pi^{k+1}}{k+1} - B'_{n,r+2,2}(\pi)\frac{\pi^{k+2}}{(k+1)(k+2)} \\
&\quad + \frac{1}{(k+1)(k+2)}\int_0^\pi t^{k+2}B''_{n,r+2,2}(t)dt \\
&= \frac{n^2\pi^{k+1}}{(2\pi)^2(k+1)c_{n,r+2}}\int_{\pi-\pi/n}^{\pi+\pi/n}\int_{x-\pi/n}^{x+\pi/n} K_{n,r+2}(s)dsdx \\
&\quad - \frac{n^2\pi^{k+2}}{(2\pi)^2(k+1)(k+2)c_{n,r+2}}\Big[\int_\pi^{\pi+2\pi/n} K_{n,r+2}(s)ds - \int_{\pi-2\pi/n}^{\pi} K_{n,r+2}(s)ds\Big] \\
&\quad + \frac{n^2}{(2\pi)^2(k+1)(k+2)c_{n,r+2}}\int_0^\pi t^{k+2}[K_{n,r+2}(t+2\pi/n) - 2K_{n,r+2}(t) \\
&\quad + K_{n,r+2}(t-2\pi/n)dt] := I_1 - I_2 + I_3.
\end{aligned}$$

Taking into account that

$$c_{n,r} \approx n^{2r-1}, \quad \int_0^\pi t^k K_{n,r}(t)dt \approx n^{2r-1-k},$$

we immediately obtain

$$\frac{n^2}{c_{n,r+2}} \approx 1/n^{2r+1}, \quad \int_0^\pi t^{k+2}K_{n,r+2}(t)dt \approx n^{2r+1-k}.$$

Reasoning exactly as in the proof of Lemma 2.1 (as for I_2 and I_3 there), from the above relations we immediately get $I_3 \le Cn^{-k}$.

Again, reasoning as in the proof of Lemma 2.1 (as for I_1 there), we obtain $I_2 \le Cn^{-k}$.

It remains to estimate the integral

$$J = \int_{\pi-\pi/n}^{\pi+\pi/n}\int_{x-\pi/n}^{x+\pi/n} K_{n,r+2}(s)dsdx.$$

By the mean value theorem, there exists $\xi \in [\pi - \pi/n, \pi + \pi/n]$, such that

$$\begin{aligned} J &= \frac{2\pi}{n} \int_{\xi-\pi/n}^{\xi+\pi/n} K_{n,r+2}(s)ds \\ &= \frac{2\pi}{n} \int_{\xi-\pi/n}^{\xi} K_{n,r+2}(s)ds + \frac{2\pi}{n} \int_{\xi}^{\xi+\pi/n} K_{n,r+2}(s)ds \\ &= \frac{2\pi}{n^2} \int_{n\xi/2-\pi/2}^{n\xi/2} [\sin(t)/\sin(t/n)]^{2(r+2)} dt \\ &\quad + \frac{2\pi}{n^2} \int_{n\xi/2}^{n\xi/2+\pi/2} [\sin(t)/\sin(t/n)]^{2(r+2)} dt \\ &:= J_1 + J_2. \end{aligned}$$

We get

$$J_1 = \frac{2\pi}{n} \int_{\xi/2-\pi/(2n)}^{\xi/2} [\sin(nv)/\sin(v)]^{2(r+2)} dv$$

and

$$J_2 = \frac{2\pi}{n} \int_{\xi/2}^{\xi/2+\pi/(2n)} [\sin(nv)/\sin(v)]^{2(r+2)} dv.$$

But $|\sin(nv)| \leq 1$ and for all $\pi - \pi/n \leq \xi \leq \pi + \pi/n$, it follows $0 \leq \pi/2 - \pi/n \leq \xi/2 - \pi/(2n) \leq \xi/2 \leq \pi/2 + \pi/(2n) < \pi$, $0 < \pi/2 - \pi/n \leq \xi/2 \leq \xi/2 + \pi/(2n) \leq \pi/2 + \pi/n < \pi$.

Therefore, for $n \geq 3$, $|1/\sin(v)|$ is bounded for both integrals J_1 and J_2, which immediately implies $J_1 \leq Cn^{-2}$, $J_2 \leq Cn^{-2}$.

But

$$\begin{aligned} I_1 &= \frac{n^2}{(2\pi)^2} \frac{\pi^{k+1}}{(k+1)c_{n,r+2}} J \\ &\leq Cn^{-2}n^{-2r-1} = Cn^{-2r-3} \leq Cn^{-k}, \end{aligned}$$

because $k \leq 2r \leq 2r+3$.

As a conclusion we get the statement of Lemma 2.3 for the $p = 2$ case. For general p, the proof is similar, reasoning by recurrence and integrating p-times by parts the integral $\int_0^\pi t^k B_{n,r+p,p}(t)dt$. $\square$

As an immediate consequence, we obtain the following.

Corollary 2.4. *Let $f \in A(\overline{\mathbb{D}})$ and $p \in \mathbb{N}$ be fixed. The convolution polynomials defined by*

$$P_{n,r,p}(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{iu})B_{m,r,p}(u)du,$$

$z \in \overline{\mathbb{D}}$, $m = [n/r] + 1$, satisfy degree $P_{n,r,p}(f)(z) \leq n$, and moreover, for all m, $r \geq p + 2$, $z \in \overline{\mathbb{D}}$, we have

$$|f(z) - P_{n,r,p}(f)(z)| \leq C\omega_2\left(f; \frac{1}{n}\right)_{\partial\mathbb{D}}.$$

Proof. It is similar to the proof of Corollary 2.2, taking into account Lemma 2.3 and the fact that $B_{m,r,p}(u)$ are even. □

3. Approximation Preserving Geometric Properties

In this section, we consider approximations that preserve geometric properties of analytic functions, like the coefficients' bounds, real part positivity, bounded turn, close-to-convexity, starlikeness, convexity, α-convexity, spiralikeness and some sufficient conditions of starlikeness and univalence. The rates of approximation are of Jackson-type or of best approximation kind.

Concerning the coefficients of convolutions, we have the following

Theorem 3.1. (i) *If $f(x) = \sum_{k=0}^{\infty} a_k z^k$ is analytic on $\mathbb{D}$ and $O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n}\cos(kt)$, then for $P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt$, we have*

$$P_n(f)(z) = a_0 + \sum_{k=1}^{m_n} a_k\rho_{k,n}z^k$$

If $f(z) = z + \sum_{k=0}^{\infty} \frac{a_k}{z^k}$, $0 < |z| < 1$ is meromorphic, then

$$P_n(f)(z) = \rho_{1,n}z + a_0 + \sum_{p=1}^{m_n} \frac{a_p\rho_{p,n}}{z^p}.$$

(ii) *If $O_n(f) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n}\cos(kt) \geq 0$, $\forall t \in [0, \pi]$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} O_n(t)dt$*

$= 1$, *then*

$$|\rho_{k,n}| \leq 1, \text{ for all } k \in \{1, \ldots, m_n.\}$$

(iii) *Let* $F_n(t) = \frac{1}{2n}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^2$ *be the Fejér kernel,* $V_n(t) = 2F_{2n}(t) - F_n(t)$ *the de la Vallée-Poussin kernel,*

$$J_n(t) = \frac{3}{2n(2n^2+1)}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^4$$

the Jackson kernel, and

$$\begin{aligned}
B_{n,2,1}(t) &:= B_{n,2}(t) = \frac{n}{2\pi}\int_{t-\pi/n}^{t+\pi/n} J_n(t)dt, \\
B_{n,2,p}(t) &= \frac{n}{2\pi}\int_{t-\pi/n}^{t+\pi/n} B_{n,2,p-1}(t)dt, \quad p = 2, 3, \ldots,
\end{aligned}$$

the Beatson kernels. We have:

$$\begin{aligned}
F_n(t) &= \frac{1}{2} + \sum_{k=1}^{n-1}\frac{n-k}{n}\cos(kt), \\
V_n(t) &= \frac{1}{2} + \sum_{k=1}^{n}\cos(kt) + \sum_{k=n+1}^{2n}\frac{2n-k}{n}\cos(kt), \\
J_n(t) &= \frac{1}{2} + \sum_{k=1}^{2n-2}\lambda_{k,n}\cos(kt), \text{ where} \\
\lambda_{k,n} &= \frac{4n^3 - 6k^2n + 3k^3 - 3k + 2n}{2n(2n^2+1)}, \text{ if } 1 \leq k \leq n, \\
\lambda_{k,n} &= \frac{(k-2n) - (k-2n)^3}{2n(2n^2+1)}, \text{ if } n \leq k \leq 2n-2,
\end{aligned}$$

and for $p = 1, 2, \ldots,$

$$B_{n,2,p}(t) = \frac{1}{2} + \sum_{k=1}^{2n-2}\left[\frac{n}{k\pi}\sin(k\pi/n)\right]^p \cdot \lambda_{k,n}\cos(kt),$$

where $\lambda_{k,n}$ *are the coefficients in* $J_n(t)$.

Proof. (i) Reasoning exactly as in the proof of Theorem 1, (ii) in [4], the analytic case is immediate.

For the meromorphic case, we have

$$
\begin{aligned}
f(ze^{it})O_n(t) &= \Big[ze^{it} + \sum_{k=0}^{\infty} \frac{a_k}{z_k} \cdot e^{-ikt}\Big]\Big[\frac{1}{2} + \sum_{p=1}^{m_n} \rho_{p,n} \cos(pt)\Big] \\
&= \Big[ze^{it} + \sum_{k=0}^{\infty} \frac{a_k}{z^k} e^{-ikt}\Big]\Big\{\frac{1}{2} + \sum_{p=1}^{m_n} \rho_{p,n} \cdot \frac{1}{2}[e^{ipt} + e^{-ipt}]\Big\} \\
&= \Big[ze^{it} + \sum_{k=0}^{\infty} \frac{a_k}{z^k} e^{-ikt}\Big]\Big\{\frac{1}{2} + \frac{1}{2}\sum_{p=1}^{m_n} \rho_{p,n} e^{ipt} + \frac{1}{2}\sum_{p=1}^{m_n} \rho_{p,n} e^{-ipt}\Big\} \\
&= \frac{z}{2} e^{it} + \frac{z}{2}\sum_{p=1}^{m_n} \rho_{p,n} e^{it(p+1)} + \frac{z}{2}\sum_{p=1}^{m_n} \rho_{p,n} e^{it(1-p)} + \frac{1}{2}\sum_{k=0}^{\infty} \frac{a_k}{z^k} e^{-ikt} \\
&\quad + \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p=1}^{m_n} \frac{a_k \rho_{p,n}}{z^k} e^{it(p-k)} + \frac{1}{2}\sum_{k=0}^{\infty}\sum_{p=1}^{m_n} \frac{a_k \rho_{p,n}}{z^k} e^{-it(k+p)}.
\end{aligned}
$$

Integrating from $-\pi$ to π and reasoning as in [4, Theorem 1, (ii)], we immediately get

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt = \rho_{1,n} z + \Big[a_0 + \sum_{p=1}^{m_n} \frac{a_p \rho_{p,n}}{z^p}\Big].$$

(ii) We have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \cos(jt)O_n(t)dt = \rho_{j,n}, \quad \text{for all } j \in \{1, \dots, m_n\},$$

which implies

$$|\rho_{j,n}| = \frac{1}{\pi}\Big|\int_{-\pi}^{\pi} \cos(jt)O_n(t)dt\Big| \le \frac{1}{\pi}\int_{-\pi}^{\pi} |\cos jt| O_n(t)dt \le \frac{1}{\pi}\int_{-\pi}^{\pi} O_n(t)dt = 1.$$

(iii) The representation for $F_n(t)$ is well-known (see e.g. [3, p.339]).

Then,

$$\begin{aligned}
V_n(t) &= 2F_{2n}(t) - F_n(t)\\
&= 2\Big[\frac{1}{2} + \sum_{k=1}^{2n} \frac{2n-k}{2n}\cos(kt)\Big] - \Big[\frac{1}{2} + \sum_{k=1}^{n} \frac{n-k}{n}\cos(kt)\Big]\\
&= \frac{1}{2} + \sum_{k=1}^{n}\Big[\frac{2n-k}{n} - \frac{n-k}{n}\Big]\cos(kt) + \sum_{k=n+1}^{2n} \frac{2n-k}{n}\cos(kt)\\
&= \frac{1}{2} + \sum_{k=1}^{n}\cos(kt) + \sum_{k=n+1}^{2n} \frac{2n-k}{n}\cos(kt).
\end{aligned}$$

Concerning the Jackson kernel $J_n(t)$, by [9, Lemma 7, p.25-26], we have

$$(3!)2^{4-4-1}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^4 = \frac{1}{2}r_{0,n} + \sum_{k=1}^{2n-2} r_{k,n}\cos(kt),$$

where

$$r_{k,n} = \sum_{\nu=1}^{2}(-1)^{\nu+1}\binom{4}{2-\nu}(k-\nu n+1)(k-\nu n)(k-\nu n-1),$$

if $0 \le k \le n$, and

$$r_{k,n} = -(k-2n+1)(k-2n)(k-2n-1), \quad \text{if } n \le k \le 2n-2.$$

By $(k-\nu n+1)(k-\nu n)(k-\nu n-1) = (k-\nu n)^3 - (k-\nu n)$ and a simple calculation we have

$$\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{n}}\right)^4 = \frac{r_{0,n}}{6} + \sum_{k=1}^{2n-2}\frac{r_{k,n}}{3}\cos(kt),$$

where $r_{0,n} = 2n(2n^2+1)$, $r_{k,n} = 4(k-n)^3 - (k-2n)^3 - 3k + 2n = 4n^3 - 6k^2n + 3k^3 - 3k + 2n$, if $1 \le k \le n$, and $r_{k,n} = (k-2n) - (k-2n)^3$ if $n \le k \le 2n-2$.

Finally, by [6, relation (3.2) and Lemma 3.3,(i)] it follows

$$B_{n,2,1}(t) = \frac{1}{2} + \sum_{k=1}^{2n-2} \left[\frac{n}{2\pi}\sin(\frac{2\pi}{n})\right]\lambda_{k,n}\cos(kt),$$

and by iteration,

$$B_{n,2,p}(t) = \frac{1}{2} + \sum_{k=1}^{2n-2} \left[\frac{n}{2\pi}\sin\frac{2\pi}{n}\right]^p \lambda_{k,n}\cos(kt),$$

which proves the theorem. □

Remark. Note that while the kernels $F_n(t)$, $J_n(t)$ and $B_{n,2,p}(t)$ are ≥ 0, the kernel $V_n(t)$ is not nonnegative on $[0,\pi]$, but satisfying $\frac{1}{\pi}\int_{-\pi}^{\pi} V_n(k)dt = 1$. However, by Theorem 3.1, (iii), it easily follows from the expression of $V_n(t) = \frac{1}{2} + \sum_{k=1}^{2n} \mu_{k,n}\cos(kt)$, that we have $0 \leq \mu_{k,n} \leq 1$.

Also, if

$$B_{n,r,1}(t) := B_{n,r}(t) = \frac{n}{2\pi}\int_{t-\frac{\pi}{n}}^{t+\frac{\pi}{n}} c_{n,r}\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^{2r} dt,$$

$$B_{n,r,p}(t) = \frac{n}{2\pi}\int_{t-\frac{\pi}{n}}^{t+\frac{\pi}{n}} B_{n,r,p-1}(t)dt, \quad \text{where } \frac{1}{\pi}\int_{-\pi}^{\pi} c_n\left(\frac{\sin\frac{nt}{2}}{\sin\frac{t}{2}}\right)^{2r} dt = 1,$$

then $B_{n,r,p}(t) = \frac{1}{2} + \sum_{k=1}^{nr-n} \lambda_{k,n}^{(p)}\cos(kt) \geq 0$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} B_{n,r,p}(t)dt = 1$, i.e. by Theorem 3.1, (ii), $|\lambda_{k,n}^{(p)}| \leq 1$, for all $k \in \{0,\dots,nr-n\}, p \geq 1$.

Concerning the preservation of coefficients' bounds, we present

Theorem 3.2. *Let $f(z) = \sum_{k=0}^{\infty} a_k(f)z^k$ be analytic in $\mathbb{D}$.*

(i) *If $O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n}\cos(kt) \geq 0$ and $\frac{1}{\pi}\int_{-\pi}^{\pi} O_n(t)dt = 1$, then for*

$$P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt = \sum_{k=0}^{m_n} a_k(P_n(f))z^k,$$

we have $|a_k(P_n(f))| \leq |a_k(f)|$, for all $n \in \mathbb{N}$, $k \in \{0,\dots,m_n\}$.

(ii) *Denoting*

$$V_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})V_n(t)dt = \sum_{k=0}^{2n} a_k(V_n(f))z^k,$$

where $V_n(t)$ *is the kernel in the statement of Theorem* 3.1*,* (iii)*, we have*

$$|a_k(V_n(f))| \leq |a_k(f)|, \text{ for all } n \in \mathbb{N}, k \in \{0, \dots, 2n\}.$$

Proof. It is immediate from Theorem 3.1,(i),(ii) and the Remark after the proof of Theorem 3.1. □

Remark. We recall that according to [4], for $f \in A(\overline{\mathbb{D}})$ we get

$$||f - V_n(f)||_{\overline{\mathbb{D}}} \leq 4E_n(f), n = 1, 2, \dots,$$

while for $O_n(t) = J_n(t), n \in \mathbb{N}$, or $O_n(t) = B_{n,r,p}(t), n, r \geq p + 2, p \in \mathbb{N}$, (see Lemma 2.3) denoting $P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt$, we have

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2(f; \frac{1}{n})_{\partial\mathbb{D}} ,$$

where $C > 0$ is independent of f and n.

Now, let us denote by $S_1 = \{f : \mathbb{D} \to \mathbb{C}; f(z) = z + a_2z^2 + \dots$, analytic in $\mathbb{D}$, satisfying $\sum_{k=2}^{\infty} k|a_k| \leq 1\}$,

$S_2 = \{f : \mathbb{D} \to \mathbb{C}; f(z) = a_1z + a_2z^2 + \dots$, analytic in $\mathbb{D}$, satisfying $|a_1| \geq \sum_{k=2}^{\infty} |a_k|\}$.

According to e.g. [10, p.97, Exercise 4.9.1], if $f \in S_1$ then $|\frac{zf'(z)}{f(z)} - 1| < 1, z \in \mathbb{D}$ and therefore f is starlike (univalent) on $\mathbb{D}$.

Also, according to [1, p.22], if $f \in S_2$ then f is starlike (and univalent)in $\mathbb{D}$. Therefore $S_1, S_2 \subset S^*(\mathbb{D})$ -the class of univalent starlike functions on $\mathbb{D}$.

The next theorem shows that some approximation convolution polynomials preserve the classes S_1 and S_2.

Theorem 3.3. (i) *Let* $O_n(t)$ *be* $J_n(t), n \in \mathbb{N}$*, in Theorem* 3.1,(iii) *or* $B_{n,r,p}(t)$*,* $n, r \geq p + 2, p \in \mathbb{N}$*,* (*see Remark after the proof of Theorem* 3.1) *and denote* $P_n(f)(z) = \int_0^z Q_n(t)dt$, $Q_n(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{it})O_n(t)dt$. *Then* $f \in S_1$ *implies* $P_n(f) \in S_1$, *and if, in addition,* f' *is continuous on* $\overline{\mathbb{D}}$, *then*

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2(f'; \frac{1}{n})_{\partial\mathbb{D}} \ ,$$

where $C > 0$ *is independent of* f *and* n.

(ii) *If* $V_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})V_n(t)dt$, *where* $V_n(t)$ *is the kernel in the statement of Theorem* 3.1, (iii), *then* $f \in S_1$ *implies* $V_n(f) \in S_1$, *for all* $n \in \mathbb{N}$ *and if, in addition,* f *is continuous on* $\overline{\mathbb{D}}$, *then*

$$\|f - V_n(f)\|_{\overline{\mathbb{D}}} \leq 4E_n(f), n = 1, 2, \ldots, .$$

(iii) *For* $V_n(f)(z)$ *defined above,* $f \in S_2$ *implies* $V_n(f) \in S_2$, *for all* $n \in \mathbb{N}$.

(iv) *If the meromorphic function* $f(z) = z + \sum_{k=0}^{\infty} \frac{a_k(f)}{z^k}$ *is univalent on* $\{|z| > 1\}$, *then for*

$$P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt = \rho_{1,n}z + a_0 + \sum_{p=1}^{m_n} \frac{a_p(f)\rho_{p,n}}{z^p},$$

where

$$O_n(t) = \frac{1}{2} + \sum_{p=1}^{m_n} \rho_{p,n}\cos(pt)$$

is any from $J_n(t), V_n(t), n \in \mathbb{N}$ *or* $B_{n,r,p}(t), n, r \geq p + 2, p \in \mathbb{N}$ *we have,*

$$\sum_{k=1}^{\infty} k \cdot |a_k(P_k(f))|^2 \leq 1, \text{ with } a_k(P_n(f)) = a_k(f) \cdot \rho_{k,n}.$$

Proof. (i) Firstly, obviously $P_n(f)(0) = P_n'(f)(0) - 1 = 0$ and by [4] we get (see also Lemma 2.3, for $B_{n,r,p}(t)$)

$$|f(z) - P_n(f)(z)| = |\int_0^z f'(t)dt - \int_0^z Q_n(t)dt|$$
$$\leq |z| \cdot \|f' - Q_n\|_{\overline{\mathbb{D}}} \leq \|f' - Q_n\|_{\overline{\mathbb{D}}} \leq C\omega_2(f'; \frac{1}{n})_{\partial\mathbb{D}} \ .$$

Let $O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} \cos(kt)$ and $f(z) = z + a_2 z^2 + \cdots +$, i.e. $f'(z) = 1 + 2a_2 z + 3a_3 z^2 + \cdots +$. By Theorem 3.1 we have $Q_n(f)(z) = 1 + 2a_2\rho_{1,n} z + 3a_3\rho_{2,n} z^2 + \cdots +$, which implies $P_n(f)(z) = \int_0^z Q_n(t)dt = z + a_2\rho_{1,n} z^2 + a_3\rho_{2,n} z^3 + \cdots +$, therefore $a_k(P_n(f)) = a_k(f)\rho_{k-1,n}$. Therefore,

$$\sum_{k=2}^{m_n} k|a_k(P_n(f))| = \sum_{k=2}^{m_n} k \cdot |a_k(f)| \cdot |\rho_{k-1,n}| \leq \sum_{k=2}^{\infty} k|a_k(f)| \leq 1$$

(because $|\rho_{k-1,n}| \leq 1$), which implies that $P_n(f) \in S_1$.

(ii) Firstly and obviously $V_n(f)(0) = 0$. Let $f \in S_1, f(z) = z + a_2 z^2 + \ldots$. By Theorem 3.1, (i) and (iii), we get $V_n(f)(z) = z + A_2 z^2 + \cdots +$, (here $A_2 = a_2$ if $n \geq 2$) i.e. $V_n'(f)(0) = 1$.

By the Remark after the proof of Theorem 3.1, we get

$$\sum_{k=2}^{2n} k|a_k(V_n(f)| \leq \sum_{k=2}^{\infty} k|a_k(f)| \leq 1,$$

i.e. $V_n(f) \in S_1$.

By [4], for f continuous on $\overline{\mathbb{D}}$ we have

$$\|f - V_n(f)\|_{\overline{\mathbb{D}}} \leq 4E_n(f).$$

(iii) Let $f \in S_2, f(z) = \sum_{k=1}^{\infty} a_k(f) z^k$, where $|a_1(f)| \geq \sum_{k=2}^{\infty} |a_k(f)|$. By Theorem 3.1 we have

$$V_n(f)(z) = a_1(f)z + \sum_{k=2}^{2n} \rho_{k,n} a_n(f) \cdot z^k, 0 \leq \rho_{k,n} \leq 1,$$

therefore

$$\sum_{k=2}^{m_n} |a_k(V_n(f))| = \sum_{k=2}^{m_n} |a_k(f)| \cdot |\rho_{k,n}| \leq \sum_{k=2}^{\infty} |a_k(f)| \leq |a_1(f)| = a_1(V_n(f)),$$

which proves $V_n(f) \in S_2$.

(iv) By the Area theorem (see e.g. [7]) we have for f, $\sum_{k=1}^{\infty} k|a_k(f)|^2 \leq 1$.

Then,

$$\sum_{k=1}^{m_n} k\cdot|a_k(P_n(f))|^2=\sum_{k=1}^{m_n} k\cdot|a_k(f)|^2\cdot|\rho_{k,n}|^2\le\sum_{k=1}^{\infty} k\cdot|a_k(f)|^2\le 1,$$

which proves the theorem. □

Let us denote by $\mathcal{P}=\{f:\overline{\mathbb{D}}\to\mathbb{C}$; f is analytic on $\mathbb{D}$, $f(0)=1$, $Re[f(z)]>0$, $z\in\mathbb{D}\}$, $S_3=\{f:\overline{\mathbb{D}}\to\mathbb{C}$; f is analytic on $\mathbb{D}$, $f(0)=f'(0)-1=0$, $|f''(z)|\le 1$, $z\in\mathbb{D}\}$ and by $\mathcal{R}=\{f:\overline{\mathbb{D}}\to\mathbb{C}$; f is analytic on $\mathbb{D}$, $f(0)=f'(0)-1=0$, $Ref'(z)>0$, $z\in\mathbb{D}\}$.

It is well-known that $\mathcal{P}$ is the class of analytic functions with positive real part and $\mathcal{R}$ is called the class of functions with bounded turn (because $f\in\mathcal{R}$ is equivalent to $|\arg f'(z)|<\frac{\pi}{2}$, $z\in\mathbb{D}$). It is also known that $f\in\mathcal{R}$ implies the univalency of f on $\mathbb{D}$.

Also, by [11] it follows that $f\in S_3$ implies that f is starlike, univalent on $\mathbb{D}$.

Regarding the preservation of the classes $\mathcal{P},\mathcal{R}$ and S_3 through convolution polynomials, we present

Theorem 3.4. *Let $O_n(t)$ be $J_n(t)$, $n\in\mathbb{N}$ or $B_{n,r,p}(t)$, $n,r\ge p+2$, $p\in\mathbb{N}$.*

(i) *Denote*

$$P_n(f)(z)=\int_0^z O_n(t)dt, Q_n(z)=\frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{it})O_n(t)dt.$$

We have $P_n(\mathcal{R})\subset\mathcal{R}$, $P_n(S_3)\subset S_3$ and if, in addition, f' is continuous on $\overline{\mathbb{D}}$, then

$$\|f-P_n(f)\|_{\overline{\mathbb{D}}}\le C\omega_2(f';\frac{1}{n})_{\partial\mathbb{D}}\ ,$$

where $C>0$ is independent of f and n.

(ii) *Denoting*

$$P_n(f)(z)=\frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt,$$

we have $P_n(\mathcal{P}) \subset \mathcal{P}$ and if, in addition, f is continuous on $\overline{\mathbb{D}}$, then

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2(f; \frac{1}{n})_{\partial\mathbb{D}} \ ,$$

where $C > 0$ is independent of f and n.

Proof. (i) The error estimate follows as in the proof of Theorem 3.3, (i). Also, we immediately get $P_n(f)(0) = P_n'(f)(0) - 1 = 0$. Let $f(z) = U(r\cos t, r\sin t) + iV(r\cos t, r\sin t)$, $z = re^{it} \in \mathbb{D}$.

By $f'(z) = \frac{\partial U}{\partial x}(r\cos t, r\sin t) + i\frac{\partial V}{\partial x}(r\cos t, r\sin t)$, the hypothesis implies $\frac{\partial U}{\partial x}(r\cos t, r\sin t) > 0$, for all $z = re^{it} \in \mathbb{D}$.

But

$$\begin{aligned} P_n'(f)(z) &= Q_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{iu})O_n(u)du \\ = \frac{1}{\pi}\int_{-\pi}^{\pi} & \frac{\partial U}{\partial x}(r\cos(t+u), r\sin(t+u))O_n(u)du \\ & +i\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial V}{\partial x}(r\cos(t+u), r\sin(t+u))O_n(u)du, \end{aligned}$$

which implies

$$Re[P_n'(f)(z)] = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\partial U}{\partial x}(r\cos(t+u), r\sin(t+u)) \cdot O_n(u)du > 0,$$

because $O_n(u) > 0$ for all $u \in [-\pi, \pi]$ excepting a finite number of points.

Also, $|P_n''(f)(z)| = \frac{1}{\pi}|\int_{-\pi}^{\pi} e^{it}f''(ze^{it})O_n(t)dt| \leq \frac{1}{\pi}\int_{-\pi}^{\pi}|f''(ze^{it})|O_n(t)dt \leq 1$, $z \in \mathbb{D}$, for $f \in S_3$.

(ii) For $f = U + iV$, we easily get

$$Re[P_n(f)(z)] = \frac{1}{\pi}\int_{-\pi}^{\pi} U(r\cos(t+u), r\sin(t+u))O_n(u)du, z = re^{it} \in \mathbb{D},$$

which by the hypothesis $U(r\cos u, r\sin u) > 0$, for all $z = re^{iu} \in \mathbb{D}$, implies $Re[P_n(f)(z)] > 0, z \in \mathbb{D}$,

The error estimate is immediate by [4] for the $J_n(t)$ case and by Lemma 2.3 for $B_{n,r,p}(t)$. □

Remarks. 1) Theorem 3.4,(i) can be restated as follows : if $|\arg f'(z)| < \frac{\pi}{2}, z \in \mathbb{D}$, then

$$|\arg P_n'(f)(z)| < \frac{\pi}{2}, z \in \mathbb{D}$$

(and $P_n(f)(z)$ is univalent on $\mathbb{D}$).

2) The convolution polynomials $V_n(f)(z)$ based on the kernel $V_n(t)$ do not satisfy Theorem 3.4, because $V_n(t)$ is not nonnegative on $[0, \pi]$.

Now, let $M > 1$ and denote $S_M = \{f : \overline{\mathbb{D}} \to \mathbb{C};\ f$ is analytic on $\mathbb{D}$, $f(0) = f'(0) - 1 = 0,\ |f'(z)| < M,\ z \in \mathbb{D}\}$.

According to e.g. [10, p.111, Exercise 5.4.1], $f \in S_M$ implies that f is univalent on $\mathbb{D}_{\frac{1}{M}} = \{z \in \mathbb{C}; |z| < \frac{1}{M}\}$.

Concerning the preservation of the class S_M, we present

Theorem 3.5. *Let $O_n(t)$ be $J_n(t)$, $n \in \mathbb{N}$, or $B_{n,r,p}(t)$, $n, r \geq p+2$, $p \in \mathbb{N}$. Denote*

$$P_n(f)(z) = \int_0^z Q_n(t)dt, \qquad Q_n(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f'(ze^{it})O_n(t)dt.$$

We have $P_n(S_M) \subset S_M$ and if, in addition, f' is continuous on $\overline{\mathbb{D}}$, then

$$\|f - P_n(f)\|_{\overline{\mathbb{D}}} \leq C\omega_2(f'; \frac{1}{n})_{\partial\mathbb{D}}\ .$$

Proof. We have $P_n(f)(0) = P_n'(f)(0) - 1 = 0$,

$$|P_n'(f)(z)| = \frac{1}{\pi} \left| \int_{-\pi}^{\pi} f'(ze^{it})O_n(t)dt \right| \leq \frac{1}{\pi} \int_{-\pi}^{\pi} |f'(ze^{it})|O_n(t)dt < M,$$

$z \in \mathbb{D}$, if $|f'(z)| < M, z \in \mathbb{D}$. □

The next result concerning the convergence of the derivatives of convolution polynomials is useful.

Theorem 3.6. *Let*

$$O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} \cos(kt) \geq 0, t \in [0, \pi],$$

$\frac{1}{\pi}\int_{-\pi}^{\pi} O_n(t)dt = 1$ *and let us denote* $P_n(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(ze^{it})O_n(t)dt$. *If* f *is analytic on* $\mathbb{D}$ *with* f' *and* f'' *continuous on* $\overline{\mathbb{D}}$, *respectively, then*

$$\|f' - P_n'(f)\|_{\overline{\mathbb{D}}} \le C\omega_1(f'; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} + \|f'\|_{\overline{\mathbb{D}}} \cdot |1-\rho_{1,n}|, \quad n \in \mathbb{N}$$

and

$$\|f'' - P_n''(f)\|_{\overline{\mathbb{D}}} \le C\omega_1(f''; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} + \|f''\|_{\overline{\mathbb{D}}} \cdot |1-\rho_{2,n}|, \quad n \in \mathbb{N},$$

where $C > 0$ *is a constant independent of* f *and* n.

Proof. We have

$$P_n'(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} f'(ze^{it}) \cdot e^{it}O_n(t)dt,$$

where

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{it}O_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos tO_n(t)dt = \rho_{1,n}$$

(see the proof of Theorem 3.1, (ii)).

Reasoning as in the proof of Theorem 1, (i) in [5], we get

$$\begin{aligned} |P_n'(f)(z) - f'(z)| &= |P_n'(f)(z) - \rho_{1,n}f'(z) + \rho_{1,n}f'(z) - f'(z)| \\ &= |\frac{1}{\pi}\int_{-\pi}^{\pi} e^{it}O_n(t)[f'(ze^{it}) - f'(z)]dt + f'(z)[\rho_{1,n} - 1]| \\ &\le \frac{1}{\pi}\int_{-\pi}^{\pi} O_n(t)|f'(ze^{it}) - f'(z)|dt + \|f'\|_{\overline{\mathbb{D}}} \cdot |1-\rho_{1,n}| \\ &\le C\omega_1(f'; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} + \|f'\|_{\overline{\mathbb{D}}} \cdot |1-\rho_{1,n}|, z \in \overline{\mathbb{D}}. \end{aligned}$$

Also, by

$$P_n''(f)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{2it}O_n(t) \cdot f''(ze^{it})dt$$

and

$$\frac{1}{\pi}\int_{-\pi}^{\pi} e^{2it}O_n(t)dt = \frac{1}{\pi}\int_{-\pi}^{\pi} \cos(2t)O_n(t)dt = \rho_{2,n},$$

we get

$$P_n''(f)(z) - f''(z) = P_n''(f)(z) - \rho_{2,n}f''(z) + \rho_{2,n}f''(z) - f''(z),$$

and reasoning as abowe, we arrive at

$$\|P_n''(f) - f''\|_{\overline{\mathbb{D}}} \le C\omega_1(f''; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} + \|f''\|_{\overline{\mathbb{D}}} \cdot |1-\rho_{2,n}|. \qquad \square$$

Remarks. 1) Let $O_n(t)$ be the Jackson kernel $J_n(t)$, for example.

Then, by Theorem 3.1, (iii), we get

$$\rho_{1,n} = \frac{4n^3 - 4n}{2n(2n^2+1)} = \frac{2n^2-2}{2n^2+1}, 1 - \rho_{1,n} = \frac{3}{2n^2+1}$$

and

$$\rho_{2,n} = \frac{4n^3 - 22n + 18}{2n(2n^2+1}, 1 - \rho_{2,n} = \frac{24n-18}{4n^3+2n} = \frac{12n-9}{2n^3+n},$$

i.e. the order of convergence to zero of $|1-\rho_{1,n}|$ and $|1-\rho_{2,n}|$ is $\frac{1}{n^2}$.

Similar estimates of $|1-\rho_{1,n}|$ and $|1-\rho_{2,n}|$ hold for the Beatson kernels $B_{n,r,p}(t), n, r \ge p+2$ (see [6, Lemma 3.3]).

2) It is easy to show that $P_n(f)(0) = 0$, while $P_n'(f)(0) = \rho_{1,n}$. Supposing $\rho_{1,n} \ne 0$, the new polynomials,

$$R_n(f)(z) = \frac{1}{\rho_{1,n}} \cdot P_n(f)(z) = \frac{1}{\rho_{1,n}} \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) O_n(t) dt$$

have obviously the properties $R_n(f)(0) = R_n'(f)(0) - 1 = 0$. Also, for $z \in \overline{\mathbb{D}}$ and $f \in A(\overline{\mathbb{D}})$ we have

$$\begin{aligned} |R_n(f)(z) - f(z)| &= |\frac{1}{\rho_{1,n}} P_n(f) - \frac{1}{\rho_{1,n}} f(z) + \frac{1}{\rho_{1,n}} f(z) - f(z)| \\ &\le \frac{1}{|\rho_{1,n}|} \cdot \|P_n(f) - f\|_{\overline{\mathbb{D}}} + \|f\|_{\overline{\mathbb{D}}} \cdot |\frac{1}{\rho_{1,n}} - 1|. \end{aligned}$$

In the case when $O_n(t)$ is $J_n(t)$, we obtain

$$\begin{aligned} \|R_n(f) - f\|_{\overline{\mathbb{D}}} &\le \frac{2n^2+1}{2n^2-2} \cdot \|f - P_n(f)\|_{\overline{\mathbb{D}}} + \|f\|_{\overline{\mathbb{D}}} |\frac{2n^2+1}{2n^2-2} - 1| \\ &\le \frac{9}{6} \cdot C\omega_2(f; \frac{1}{n})_{\partial\mathbb{D}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{3}{2n^2-2} \\ &\le C[\omega_2(f; \frac{1}{n})_{\partial\mathbb{D}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n^2}], n \ge 2, \end{aligned}$$

i.e $\|R_n(f)-f\|_{\overline{\mathbb{D}}}\le C[\omega_2(f;\frac{1}{n})_{\partial\mathbb{D}}+\|f\|_{\overline{\mathbb{D}}}\cdot\frac{1}{n^2}]$, where $C>0$ is independent of f and n.

Similar estimate holds when $O_n(t)$ is

$$B_{n,r,p}(t), n, r\ge p+2, p\in\mathbb{N}.$$

3) Note that for $O_n(t)=V_n(t)$, Theorem 3.6 does not hold because $V_n(t)$ is not nonnegative.

Now, let us recall some classical definitions in geometric function theory.

Let

$$\begin{aligned}
S^*(\mathbb{D})=&\{f:\mathbb{D}\to\mathbb{C}; f \text{ is analytic on } \mathbb{D}, f(0)=f'(0)-1=0,\\
&Re[\frac{zf'(z)}{f(z)}]>0,\forall z\in\mathbb{D}\},\\
K(\mathbb{D})=&\{f:\mathbb{D}\to\mathbb{C}; f \text{ is analytic on } \mathbb{D}, f(0)=f'(0)-1=0,\\
&Re[\frac{zf''(z)}{f'(z)}]+1>0,\forall z\in\mathbb{D}\},\\
\mathcal{C}(\mathbb{D})=&\{f:\mathbb{D}\to\mathbb{C}; f \text{ is analytic on } \mathbb{D}, f(0)=f'(0)-1=0,\\
&\text{and there exists } h\in S^*(\mathbb{D}) \text{ such that } Re[\frac{zf'(z)}{h(z)}]>0,\forall z\in\mathbb{D}\},\\
M_\alpha(\mathbb{D})=&\{f:\mathbb{D}\to\mathbb{C}: f \text{ is analytic on } \mathbb{D}, f(0)=f'(0)-1=0, \frac{f(z)f'(z)}{z}\neq 0,\\
&\forall z\in\mathbb{D} \text{ and } Re[(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha(z\frac{f''(z)}{f'(z)}+1)]>0,\forall z\in\mathbb{D}\},
\end{aligned}$$

where $\alpha\in\mathbb{R}$, and

$$\begin{aligned}
S_\gamma(\mathbb{D})=&\{f:\mathbb{D}\to\mathbb{C}; f \text{ is analytic on } \mathbb{D}, f(0)=f'(0)-1=0, f(z)\neq 0,\\
&\forall z\in\mathbb{D} \text{ and } Re[e^{i\gamma}\frac{zf'(z)}{f(z)}]>0,\forall z\in\mathbb{D}\}, \text{ where } \gamma\in(\frac{-\pi}{2},\frac{\pi}{2}).
\end{aligned}$$

It is well-known that $S^*(\mathbb{D}), K(\mathbb{D}), \mathcal{C}(\mathbb{D}), M_\alpha(\mathbb{D})$ and $S_\gamma(\mathbb{D})$ are called the classes of starlike, convex, close-to-convex, α-convex and γ-spirallike functions, respectively. Note that all are subclasses of univalent functions.

In what follows we consider preservation by convolution polynomials, of the corresponding subclasses

$$S^*(\overline{\mathbb{D}}),\quad K(\overline{\mathbb{D}}),\quad \mathcal{C}(\overline{\mathbb{D}}),\quad M_\alpha(\overline{\mathbb{D}}) \text{ and } S_\gamma(\overline{\mathbb{D}}).$$

We present:

Theorem 3.7. *Let* $O_n(t) = \frac{1}{2} + \sum_{k=1}^{m_n} \rho_{k,n} \cos(kt)$ *be* $J_n(t), n \in \mathbb{N}$ *or*

$$B_{n,r,p}(t), n, r \geq p+2, p \in \mathbb{N}.$$

(i) *For* $f \in \mathcal{C}(\overline{\mathbb{D}})$, *let us define*

$$P_n(f)(z) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(ze^{it}) O_n(t) dt, R_n(f)(z) = \frac{1}{\rho_{1,n}} P_n(f)(z).$$

Then

$$\|R_n(f) - f\|_{\overline{\mathbb{D}}} \leq C[\omega_2(f; \frac{1}{n})_{\partial\mathbb{D}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n^2}],$$

where $C > 0$ *is independent of* f *and* n *and there exists* $n_0 = n_0(f)$, *such that* $R_n(f) \in \mathcal{C}(\overline{\mathbb{D}})$ *for all* $n \geq n_0$.

(ii) *For* $f(z) = zg(z)$, *let us define*

$$P_n(f)(z) = z \cdot \frac{1}{\pi} \int_{-\pi}^{\pi} g(ze^{it}) O_n(t) dt.$$

Then,

$$\|P_n(f) - f\|_{\overline{\mathbb{D}}} \leq C\omega_1(f; \frac{1}{n})_{\overline{\mathbb{D}}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n}$$

and there exists $n_0 = n_0(f)$ *such that for all* $n \geq n_0$ *we have*

$$P_n[S^*(\overline{\mathbb{D}})] \subset S^*(\overline{\mathbb{D}}), P_n[M_\alpha(\overline{\mathbb{D}})] \subset M_\alpha(\overline{\mathbb{D}}) \text{ and } P_n[S_\gamma(\overline{\mathbb{D}})] \subset S_\gamma(\overline{\mathbb{D}}).$$

Proof. (i) Let $f \in \mathcal{C}(\overline{\mathbb{D}})$. By Remark 2 of Theorem 3.6, we get $R_n(f)(0) = R_n'(f)(0) - 1 = 0$ and the error estimate of $\|f - R_n(f)\|_{\overline{\mathbb{D}}}$.

There exists $h \in S^*(\overline{\mathbb{D}})$, i.e. univalent on $\overline{\mathbb{D}}$, such that

$$Re[\frac{zf'(z)}{h(z)}] > 0, \quad \forall z \in \overline{\mathbb{D}}.$$

Denote $g(z) = \frac{z}{h(z)}$. Because $h(0) = 0$ and h is univalent, it follows $h(z) \neq 0, \forall z \in \overline{\mathbb{D}}, z \neq 0$ and $g(z)$ is analytic on $\overline{\mathbb{D}}$ (with $g(z) \neq 0, \forall z \in \overline{\mathbb{D}}$).

Therefore, $g(z)$ is continuous on $\overline{\mathbb{D}}$, which implies that there exists $M > 0$ with $|g(z)| \le M, \forall z \in \overline{\mathbb{D}}$.

But by Theorem 3.6 and by $\rho_{1;n} \to 1$, it follows that $R'_n(f) \to f'$, uniformly on $\overline{\mathbb{D}}$, i.e. $g(z) \cdot R'_n(f)(z) \to g(z)f'(z)$ uniformly on $\overline{\mathbb{D}}$. This implies that

$$Re[g(z)R'_n(f)(z)] \to Re[g(z)f'(z)] > 0,$$

uniformly on $\overline{\mathbb{D}}$, i.e. there exists $n_0 = n_0(f)$, such that

$$Re[g(z)R'_n(f)(z)] > 0,$$

for all $n \ge n_0$, i.e.

$$R_n(f)(z) \in \mathcal{C}(\overline{\mathbb{D}}),$$

for all $n \ge n_0$.

(ii) Let $f \in S^*(\overline{\mathbb{D}})$. Because $f(0) = 0$ and f is univalent on $\overline{\mathbb{D}}$, it follows $f(z) \ne 0$, for all $z \in \overline{\mathbb{D}}, z \ne 0$, i.e. $f(z) = z \cdot g(z), z \in \overline{\mathbb{D}}$, where g is analytic in $\overline{\mathbb{D}}$ and $g(z) \ne 0$, for all $z \in \overline{\mathbb{D}}$.

Denote $Q_n(g)(z) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(ze^{it})O_n(t)dt$ and $P_n(f)(z) = zQ_n(g)(z)$.

By [5, Theorem 1] and by [13] we get

$$\begin{aligned}
&|P_n(f)(z) - f(z)| = |zQ_n(g)(z) - zg(z)| \le |Q_n(g)(z) - g(z)| \\
&\le C\omega_1(g; (1-\rho_{1,n})^{1/2})_{\overline{\mathbb{D}}} \le C\omega_1(g; \frac{1}{n})_{\partial\mathbb{D}} \\
&= C\sup\{|\frac{f(z_1)}{z_1} - \frac{f(z_2)}{z_2}|; |z_1 - z_2| \le \frac{1}{n}, |z_1| = |z_2| = 1\} \\
&= C\sup\{|z_2 f(z_1) - z_1 f(z_2)|; |z_1 - z_2| \le \frac{1}{n}, |z_1| = |z_2| = 1\} \\
&\le C\sup\{|z_2| \cdot |f(z_1) - f(z_2)| + |z_1 - z_2| \cdot |f(z_2)|; |z_1 - z_2| \le \frac{1}{n}, |z_1| = |z_2| = 1\} \\
&\le C\omega_1(f; \frac{1}{n})_{\partial\mathbb{D}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n} \le C\omega_1(f; \frac{1}{n})_{\overline{\mathbb{D}}} + \|f\|_{\overline{\mathbb{D}}} \cdot \frac{1}{n}.
\end{aligned}$$

Also, by Theorem 3.6 we get $Q'_n(g) \to g'$ uniformly on $\overline{\mathbb{D}}$. Now, because $|g(z)| > 0, \forall z \in \overline{\mathbb{D}}$ and $Q_n(g) \to g$ uniformly on $\overline{\mathbb{D}}$, there exists $n_1 = n_1(g)$ and $m > 0$, such that for all $n \ge n_1$ we have

$$|Q_n(g)(z)| > m, \forall z \in \overline{\mathbb{D}} \text{ and therefore } Q_n(g)(z) \ne 0, \forall n \ge n_1, \forall z \in \overline{\mathbb{D}}.$$

Then, obviously $P'_n(f)(z) = zQ'_n(g)(z) + Q_n(g)(z) \to zg'(z) + g(z) = f'(z)$, uniformly on $\overline{\mathbb{D}}$, which implies

$$\begin{aligned}\frac{zP'_n(f)(z)}{P_n(f)(z)} &= \frac{z[zQ'_n(g)(z) + Q_n(g)(z)]}{zQ_n(g)(z)}\\ &= \frac{zQ'_n(g)(z) + Q_n(g)(z)}{Q_n(g)(z)} \to \frac{zg'(z) + g(z)}{g(z)}\\ &= \frac{f'(z)}{g(z)} = \frac{zf'(z)}{f(z)}, \text{ uniformly on } \overline{\mathbb{D}}.\end{aligned}$$

Therefore

$$Re\Big[\frac{zP'_n(f)(z)}{P_n(f)(z)}\Big] \to Re\Big[\frac{zf'(z)}{f(z)}\Big] > 0$$

uniformly on $\overline{\mathbb{D}}$, i.e. there is $n_0 = n_0(f) > n_1$ such that for all $n \geq n_0$ we have

$$Re\Big[\frac{zP'_n(f)(z)}{P_n(f)(z)}\Big] > 0, \text{ for all } z \in \overline{\mathbb{D}},$$

i.e. $P_n(f) \in S^*(\overline{\mathbb{D}})$, since $P_n(f)(0) = P'_n(f)(0) - 1 = 0$.

If $f \in S_\gamma(\overline{\mathbb{D}})$ then the reasonings are similar, so $P_n(f) \in S_\gamma(\overline{\mathbb{D}})$.

Now, let $f \in K(\overline{\mathbb{D}})$ and denote again $f(z) = z \cdot g(z)$, where the univalence of f on $\overline{\mathbb{D}}$, implies $g(z) \neq 0$, for all $z \in \overline{\mathbb{D}}$, with g analytic on $\overline{\mathbb{D}}$.

Because $f \in K(\overline{\mathbb{D}})$ if and only if $zf'(z) \in S^*(\overline{\mathbb{D}})$, it follows $f'(z) \neq 0$, for all $z \in \overline{\mathbb{D}}$, i.e. $|f'(z)| > 0$, for all $z \in \overline{\mathbb{D}}$.

By Theorem 3.6, we immediately get $P'_n(f) \to f'$ and $P''_n(f) \to f''$, uniformly on $\overline{\mathbb{D}}$. By reasoning as above, we get

$$Re\Big[\frac{zP''_n(f)}{P'_n(f)(z)}\Big] + 1 \to Re\Big[\frac{zf''(z)}{f'(z)}\Big] + 1,$$

uniformly on $\overline{\mathbb{D}}$. It follows that there is $n_0 = n_0(f)$, such that for all $n \geq n_0$ we have

$$Re\Big[\frac{zP''_n(f)}{P'_n(f)(z)}\Big] + 1 > 0, \quad \forall z \in \overline{\mathbb{D}},$$

i.e. $P_n(f) \in K(\overline{\mathbb{D}})$.

The proof of the inclusion $P_n[M_\alpha(\overline{\mathbb{D}})] \subset M_\alpha(\overline{\mathbb{D}})$ can be achieved in a similar way, which also proves the theorem completely. $\square$

Remarks. (1) It is an open question if the inclusions in Theorem 3.7 remain true if we consider the open unit disk $\mathbb{D}$, instead of $\overline{\mathbb{D}}$.

(2) A shortcoming of Theorem 3.7 is that the preservations of the classes hold begining with an index $n_0 = n_0(f)$. It is an open question if, in the case of Beatson kernels , due to their bell-shaped property , will these preservations hold true for all $n \in \mathbb{N}$?

(3) Another interesting question would be how will the convolution polynomials $P_n(f)(z)$ considered in Section 3 preserve the subordination and the distorsion of $f(z)$?

References

1. J. W. Alexander, Functions which map the interior of the unit circle upon simple regions, *Ann. of Math.*, **17**(1915), 12-22.

2. R. K. Beatson, *Bell-shape preserving convolution operators*, Technical Report, Department of Mathematics, University of Otago, Dunedin, New Zealand, 1978.

3. R. DeVore and G. G. Lorentz, *Constructive Approximation: Polynomials and Spline Approximation*, Springer-Verlag, New York, Berlin, Heidelberg, 1993.

4. S. G. Gal, Convolution-type integral operators in complex approximation, *Comput. Methods Funct. Theory*, **1** (2001), No.2, 417-432.

5. S. G. Gal, On Beatson convolution operators in the unit disk, *J. of Analysis*, **10**(2002), 101-106 .

6. H. H. Gonska and Jia-Ding Cao, Approximation by Boolean sums of positive linear operators, VI : monotone approximation and global smoothness preservation, *Rev. Anal. Numér. Théor. Approx.* (Cluj), **28** (1999), No.1, 37-61.

7. T. Gronwall, Some remarks on conformal representation, *Ann. of Math.*, **16**(1914-1915), No.2, 72-76.

8. G. G. Lorentz, *Approximation of Functions*, Chelsea, New York, 1987.

9. Y. Matsuoka, On the approximation of functions by some singular integrals, *Tohoku Math. J.*, **18** (1966), No.1, 13-43.

10. P. T. Mocanu, T. Bulboaca and Gr. St. Salagean, *Geometric Theory of Univalent Functions*(in Romanian), Casa Cartii de Stiinta, Cluj, 1999.

11. M. Obradovic, Simple sufficient conditions for univalence, *Mat. Vesnik*, **49** (1997), 241-244.

12. G. Pólya and I. J. Schoenberg, Remarks on de la Vallée-Poussin means and convex conformal maps of the circle, *Pacific J. Math.*, **8**(1958), 295-333.

13. L. A. Rubel, A. I. Shields and B. A. Taylor, Mergelyan sets and the modulus of continuity of analytic functions, *J. Approx. Theory*, **15**(1975), 23-40.

14. St. Ruscheweyh, Geometric properties of Cesàro means, *Results Math.*, **22**(1992), 739-748.

15. St. Ruscheweyh and L. C. Salinas, On the preservation of periodic monotonicity, *Constr. Approx.* **8**(1992), 129-140.

16. G. S. Salagean, *Subclasses of Univalent Functions*, Lectures Notes in Mathematics, 1013, Springer-Verlag, 1983, 362-372.

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